Mathematical Programming 66 (1994) 137–144

A new method for a class of linear variational inequalities[1]

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Received 19 January 1993; revised manuscript received 18 October 1993

Abstract

In this paper we introduce a new iterative scheme for the numerical solution of a class of linear variational inequalities. Each iteration of the method consists essentially only of a projection to a closed convex set and two matrix-vector multiplications. Both the method and the convergence proof are very simple.

Keywords: Linear variational inequality; Linear complementarity problem; Projection

1. Introduction

We consider a class of linear variational inequalities

$$\text{(LVI)} \quad u \in \Omega, \qquad (v-u)^{\mathrm{T}}(Mu+q) \geqslant 0, \quad \text{for all } v \in \Omega, \tag{1}$$

where $M \in \mathbb{R}^{n \times n}$ is a positive semidefinite matrix (not necessarily symmetric), $q \in \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is a closed convex set. The linear complementarity problem

$$\text{(LCP)} \quad u \geqslant 0, \qquad (Mu+q) \geqslant 0, \qquad u^{\mathrm{T}}(Mu+q) = 0 \tag{2}$$

is a special (LVI) when $\Omega = \{u \in \mathbb{R}^n \mid u \geqslant 0\}$. Variational inequalities, linear complementarity problems have played a significant role in mathematical programming. These subjects have been studied since the mid 1960's starting with the works of Cottle, Dantzig [3], Lemke [9, 10] and developed by many others. There is already a substantial number of algorithms for the numerical solution of linear complementarity problems and variational inequalities [1, 4–8, 12–15]. Our objective in this paper is to offer a new alternative iterative method for solving problem (1). Let Ω^* denote the solution set of (LVI) and $P_\Omega(\cdot)$ denote

[1]This work is supported by the National Natural Science Foundation of the P.R. China and NSF of Jiangsu.

SSDI 0025-5610(93)E0104-M

the projection to Ω. Throughout this paper we assume that $\Omega^* \neq \emptyset$ and the projection to Ω is simple to carry out (e.g. when Ω is a general orthant, a box, a sphere, a cylinder or a subspace).

2. The method

It is well known [2], that the linear variational inequality (1) is equivalent to the following linear projection equation

$$\text{(LPE)} \quad u = P_\Omega[u-(Mu+q)], \tag{3}$$

i.e., to solve (LVI) is equivalent to finding a zero point of the continuous nonsmooth function

$$e(u) := u - P_\Omega[u-(Mu+q)]. \tag{4}$$

We state our algorithm as follows:

Projection and Contradiction Algorithm (PC Algorithm).
Given $u^0 \in \mathbb{R}^n$. For $k=0,1,\ldots$, if $u^k \notin \Omega^*$, then

$$u^{k+1} = u^k - \rho(u^k)d(u^k), \tag{5}$$

where

$$d(u^k) = (M^{\mathrm{T}} + I)e(u^k) \tag{6}$$

and

$$\rho(u^k) = \frac{\|e(u^k)\|^2}{\|d(u^k)\|^2}. \tag{7}$$

Obviously, each iteration of the method consists essentially of only a projection to Ω and the computation of Mu and $M^{\mathrm{T}}e(u)$. We call it a projection and contraction method because in each iteration a projection has to be carried out and the Euclidean distance of the iterates to the solution set monotonically converges to zero, which will be proved in the next section.

3. Convergence

Theorem 1. *Let $u^* \in \Omega^*$. Then*

$$(u-u^*)^{\mathrm{T}}(I+M^{\mathrm{T}})e(u) \geqslant \|e(u)\|^2, \quad \textit{for all } u \in \mathbb{R}^n. \tag{8}$$

Proof. Since $\Omega \subset \mathbb{R}^n$ is a closed convex set and $u^* \in \Omega$, we know by the properties of a projection on a closed convex set [11, Appendix B] that

$$\{u^* - P_\Omega(v)\}^{\mathrm{T}}\{v - P_\Omega(v)\} \leqslant 0, \quad \text{for all } v \in \mathbb{R}^n.$$

By setting $v := u - (Mu+q)$ we obtain

$$\{u^* - P_\Omega[u-(Mu+q)]\}^{\mathrm{T}}\{e(u)-(Mu+q)\} \leqslant 0.$$

It follows

$$\{P_\Omega[u-(Mu+q)]-u^*\}^{\mathrm{T}} e(u) \geqslant \{P_\Omega[u-(Mu+q)]-u^*\}^{\mathrm{T}}(Mu+q)$$

and

$$(u-u^*)^{\mathrm{T}} e(u) \geqslant \|e(u)\|^2 + \{P_\Omega[u-(Mu+q)]-u^*\}^{\mathrm{T}}(Mu+q). \tag{9}$$

Note that

$$(u-u^*)^{\mathrm{T}} M^{\mathrm{T}} e(u) = e(u)^{\mathrm{T}}(Mu+q) - e(u)^{\mathrm{T}}(Mu^*+q). \tag{10}$$

Adding (9) and (10) and using $M \geqslant 0$, we get

$$\begin{aligned}
&(u-u^*)^{\mathrm{T}}(I+M^{\mathrm{T}})e(u) \\
&\quad \geqslant \|e(u)\|^2 + (u-u^*)^{\mathrm{T}}(Mu+q) - e(u)^{\mathrm{T}}(Mu^*+q) \\
&\quad = \|e(u)\|^2 + \{P_\Omega[u-(Mu+q)]-u^*\}^{\mathrm{T}}(Mu^*+q) + (u-u^*)^{\mathrm{T}} M(u-u^*) \\
&\quad \geqslant \|e(u)\|^2 + \{P_\Omega[u-(Mu+q)]-u^*\}^{\mathrm{T}}(Mu^*+q).
\end{aligned}$$

Because $P_\Omega[u-(Mu+q)] \in \Omega$ and $u^* \in \Omega^*$, it follows that

$$\{P_\Omega[u-(Mu+q)]-u^*\}^{\mathrm{T}}(Mu^*+q) \geqslant 0$$

and the proof is complete. □

Theorem 2. *The sequence* $\{u^k\}$ *generated by the PC Algorithm for* (LVI) *satisfies*

$$\|u^{k+1}-u^*\|^2 \leqslant \|u^k-u^*\|^2 - \rho(u^k)\|e(u^k)\|^2 \quad \textit{for all } u^* \in \Omega^*. \tag{11}$$

Proof. Using (5)–(8) we get

$$\begin{aligned}
\|u^{k+1}-u^*\|^2 &= \|u^k-u^*-\rho(u^k)d(u^k)\|^2 \\
&= \|u^k-u^*\|^2 - 2\rho(u^k)(u^k-u^*)^{\mathrm{T}} d(u^k) + \rho^2(u^k)\|d(u^k)\|^2 \\
&\leqslant \|u^k-u^*\|^2 - \rho(u^k)\|e(u^k)\|^2. \qquad \square
\end{aligned}$$

From (7), $\rho(u) \geqslant 1/\|M^{\mathrm{T}}+I\|^2 := c > 0$. Then from (11) we get

$$\|u^{k+1}-u^*\|^2 \leqslant \|u^k-u^*\|^2 - c\|e(u^k)\|^2, \quad \textit{for all } u^* \in \Omega^*. \tag{12}$$

The function $\|e(u)\|$ measures how much u fails to be in Ω^*. Eq. (12) states that we get a 'big' profit from an iteration, if $\|e(u)\|$ is not too small; conversely, if we get a very small profit from an iteration, then $\|e(u)\|$ is already very small and u^k is a 'sufficiently good' approximation of a $u^* \in \Omega^*$.

Theorem 3. *The sequence* $\{u^k\}$ *generated by the PC Algorithm for* (LVI) *converges to a solution point* u^*.

Proof. Let $\hat{u}$ be a solution of (LVI). First, from (12) we have

$$\|u^k - \hat{u}\| \leqslant \|u^0 - \hat{u}\|$$

and the sequence $\{u^k\}$ is bounded. Also from (12) we get

$$c\sum_{k=0}^{\infty} \|e(u^k)\|^2 \leqslant \|u^0 - \hat{u}\|^2$$

and it follows that

$$\lim_{k\to\infty} e(u^k) = 0.$$

Let u^* be a cluster point of $\{u^k\}$ and the subsequence $\{u^{k_j}\}$ converges to u^*. Because $e(u)$ is continuous, then

$$e(u^*) = \lim_{j\to\infty} e(u^{k_j}) = 0$$

and u^* is a solution of (LVI). Since $u^* \in \Omega^*$ and

$$\|u^{k+1} - u^*\| \leqslant \|u^k - u^*\|,$$

the sequence $\{u^k\}$ has exactly a cluster point and

$$\lim_{k\to\infty} u^k = u^*. \qquad \square$$

The following theorem will be proved under the assumption that Ω is a positive orthant (in this case the special LVI is a LCP).

Theorem 4. *If $\Omega = \{u \mid u \geqslant 0\}$, then the sequence $\{u^k\}$ generated by the PC Algorithm converges to a $u^* \in \Omega^*$ globally linearly.*

Proof. It is easy to see that Ω^* is a closed convex set. Theorem 3 shows that $\{u^k\}$ converges to a solution point u^* and

$$\{u^k\} \subset \{u \in \mathbb{R}^n \mid \|u - u^*\| \leqslant \|u^0 - u^*\|\}. \tag{13}$$

From (12) we only need to prove that there exists an $\eta > 0$ so that

$$\frac{\|e(u)\|}{\|u - u^*\|} \geqslant \eta, \quad \text{for all } u \in \{u^k\}. \tag{14}$$

If $M = I$, then it follows that $u^* = P_\Omega[-q]$ and $e(u) = u - u^*$, so (14) is trivially true. Therefore, in the following, we only need to consider the case when $M \neq I$. For $\bar{u} \in \Omega^*$, let

$$T(\bar{u}) := \{u \in \mathbb{R}^n \mid \|u - \bar{u}\| < \|\bar{u} - u^*\|\}$$

and

$$T := \bigcup_{\bar{u} \in \Omega^*} T(\bar{u}).$$

Because (12) is true for every solution point in Ω^* and $\{u^k\}$ converges to u^*, it follows that

$$\{u^k\} \cap T = \emptyset. \tag{15}$$

We prove the assertion (15) by contradiction. Assume $u^j \in \{u^k\}$ and $u_j \in T(\bar{u})$ for some $\bar{u} \in \Omega^*$, say, $\|u^j - \bar{u}\| = \|\bar{u} - u^*\| - 2\epsilon_0$, where $\epsilon_0 > 0$. Since $\{u^k\}$ converges to u^*, then there exists a $p > 0$, such that $u^{j+p} \in \{u^k\}$ and $\|u^{j+p} - u^*\| \leqslant \epsilon_0$. It follows that

$$\begin{aligned} \|u^{j+p} - \bar{u}\| &\geqslant \|\bar{u} - u^*\| - \|u^{j+p} - u^*\| \\ &\geqslant \|u^j - \bar{u}\| + 2\epsilon_0 - \epsilon_0 \\ &= \|u^j - \bar{u}\| + \epsilon_0. \end{aligned}$$

Because $\bar{u} \in \Omega^*$, the above fact contradicts (12). Therefore,

$$\{u^k\} \subset S := \{u \in \mathbb{R}^n \mid \|u - u^*\| \leqslant \|u^0 - u^*\|\} \setminus T.$$

Let

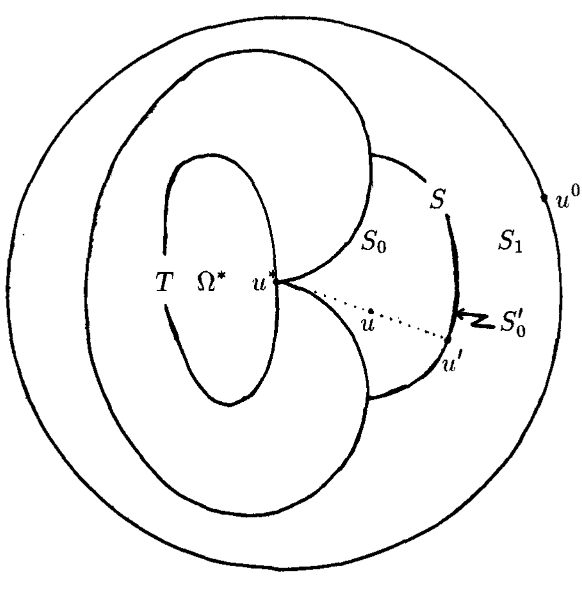
$$\Sigma := [i \mid |[u^* - (Mu^* + q)]_i| > 0\},$$

$$\delta := \begin{cases} \min\{|[u^* - (Mu^* + q)]_i| \mid i \in \Sigma\} & \text{if } \Sigma \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

$$S_0 := \{u \in \Omega \mid \|u - u^*\|_\infty < \delta / \|I - M\|_\infty\} \cap S$$

and

$$S_0' := \{u \in \Omega \mid \|u - u^*\|_\infty = \delta / \|I - M\|_\infty\} \cap S.$$



Without loss of generality we can assume $S_0 \subset S$. Further let

$$S_1 := S \setminus S_0,$$

and for any $u \in S \setminus \{u^*\}$

$$w(u) := \frac{\|e(u)\|}{\|u-u^*\|}.$$

It is clear that S_1 is bounded and closed. Since S_1 is compact and $w(u)$ is a continuous function on S_1,

$$\min\{w(u) \mid u \in S_1\} := \eta > 0.$$

For any $u \in S_0 \setminus \{u^*\}$, there exists and $u' \in S_0' \subset S_1$, such that

$$u - u^* = r(u' - u^*)$$

with an $r \in (0, 1)$. For this u, we distinguish the following cases:

(1) $u_i^* = 0$ and $(Mu^* + q)_i = 0$. Note that in this case $[u^* - (Mu^* + q)]_i = 0$ and

$$[u - (Mu + q)]_i = r \cdot [u' - (Mu' + q)]_i.$$

It follows that

$$\begin{aligned} &\{P_\Omega[u - (Mu + q)] - P_\Omega[u^* - (Mu^* + q)]\}_i \\ &\quad = r \cdot \{P_\Omega[u' - (Mu' + q)] - P_\Omega[u^* - (Mu^* + q)]\}_i. \end{aligned}$$

(2) $u_i^* \geqslant \delta$ and $(Mu^* + q)_i = 0$. In this case $[u^* - (Mu^* + q)]_i \geqslant \delta$. According to the definitions of S_0 and S_0' we have

$$[u' - (Mu' + q)]_i > 0 \quad \text{and} \quad [u - (Mu + q)]_i > 0.$$

It follows that

$$\{P_\Omega[u' - (Mu' + q)] - P_\Omega[u^* - (Mu^* + q)]\}_i = [(I - M)(u' - u^*)]_i$$

and

$$\{P_\Omega[u - (Mu + q)] - P_\Omega[u^* - (Mu^* + q)]\}_i = [(I - M)(u - u^*)]_i.$$

(3) $u_i^* = 0$, $(Mu^* + q)_i \geqslant \delta$. In this case $[u^* - (Mu^* + q)]_i \leqslant -\delta$. Similary, according to the definitions of S_0 and S_0' we have

$$[u' - (Mu' + q)]_i \leqslant 0 \quad \text{and} \quad [u - (Mu + q)]_i \leqslant 0.$$

It follows that

$$\{P_\Omega[u' - (Mu' + q)] - P_\Omega[u^* - (Mu^* + q)]\}_i = 0$$

and

$$\{P_\Omega[u - (Mu + q)] - P_\Omega[u^* - (Mu^* + q)]\}_i = 0.$$

Therefore, for this u, from the above discussion we have

$$\{P_\Omega[u-(Mu+q)]-P_\Omega[u^*-(Mu^*+q)]\}$$
$$=r\cdot\{P_\Omega[u'-(Mu'+q)]-P_\Omega[u^*-(Mu^*+q)]\}$$

and thus

$$\begin{aligned} e(u) &= e(u)-e(u^*) \\ &= u-u^*-\{P_\Omega[u-(Mu+q)]-P_\Omega[u^*-(Mu^*+q)]\} \\ &= r(u'-u^*)-r\cdot\{P_\Omega[u'-(Mu'+q)]-P_\Omega[u^*-(Mu^*+q)]\} \\ &= r\cdot e(u'). \end{aligned}$$

It follows that $w(u)=w(u')$ and

$$\frac{\|e(u)\|}{\|u-u^*\|}\geqslant\eta, \quad \text{for all } u\in\{u^k\}.$$

Therefore the PC Algorithm is globally linearly convergent for all starting vector $u^0\in\mathbb{R}^n$. □

4. Extensions and conclusions

For $0<\gamma<2$, with the same direction $d(u)$ and the same steplength $\rho(u)$, the sequence $\{u^k\}$ generated by the iterative scheme

$$u^{k+1}=u^k-\gamma\rho(u^k)d(u^k)$$

or

$$u^{k+1}=P_\Omega[u^k-\gamma\rho(u^k)d(u^k)]$$

also converges to a solution point u^*. Since we have proved that $(u-u^*)^{\mathrm{T}}d(u)\geqslant\|e(u)\|^2$ for all $u\in\mathbb{R}^n$ in Theorem 1, a recommended choice of γ would be $\geqslant 1$.

The main advantages of our method are its simplicity and ability to handle the linear variational inequalities which might otherwise be excluded by some algorithms. Our method performs no transformation of the matrix elements. The method allows the optimal exploitation of the sparsity of the matrix M and may thus be efficient for large sparse problems. Since the method is easy to parallelise, it may be even more favorable for parallel computation.

However, in worse cases, the search direction (6) may lead to a slow convergence. Therefore, we suggest the following modified iterative scheme:

$$u^{k+1}=u^k-\gamma(I+M)^{-1}e(u^k). \tag{16}$$

Because $e(u)$ is the residue of the linear projection equation (3), the modified method can be viewed as an extension of the damped-Newton's method for unconstrained optimization. Let $G=(I+M)^{\mathrm{T}}(I+M)$. Using (6), (8) and (16), by a simple computation we get

$$\begin{aligned}
\|u^{k+1}-u^*\|_G^2 &= \|(I+M)[(u^k-u^*)-\gamma(I+M)^{-1}e(u^k)]\|^2 \\
&= \|(I+M)(u^k-u^*)-\gamma e(u^k)\|^2 \\
&= \|u^k-u^*\|_G^2 - 2\gamma(u^k-u^*)^{\mathrm{T}}(I+M^{\mathrm{T}})e(u^k) + \gamma^2\|e(u^k)\|^2 \\
&\leqslant \|u^k-u^*\|_G^2 - \gamma(2-\gamma)\|e(u^k)\|^2.
\end{aligned} \tag{17}$$

With the same approach, we can prove the convergence of this modified method.

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