Five notes on the algorithmic unified framework of splitting and contraction methods for linearly constrained convex optimization

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Preface

The beauty of mathematics is not exclusive to pure mathematics. In the research on optimization methods, we consistently strive for simplicity and unity. Simplicity encourages others to use the methods, while unity provides a sense of aesthetic satisfaction. Many convex optimization problems encountered in science and engineering involve linear constraints. By introducing the multipliers, these problems can be transformed into finding the saddle points of their Lagrange functions. A saddle point acts as a balance between two conflicting interests; solving it iteratively is akin to negotiation, requiring both sides to move toward each other. The equivalent mathematical expression of a saddle point is a variational inequality. Building on this idea, over the past decade, using basic university mathematics and general optimization principles, we have proposed a unified framework for a splitting contraction algorithm, with a particularly simple convergence proof. Some of the work based on this framework has been praised by renowned international scholars as “Very Simple Yet Powerful” and “Elegant”.

Our recent work shows that the framework not only has a simple convergence proof but also reveals a clear approach to constructing algorithms. For convex optimization problems with multi-block separable linear constraints, one only needs to understand the basic process of Gaussian elimination for solving the system of linear equations to design a series of splitting contraction methods. “Splitting” means that each iteration step is implemented by solving relatively simple subproblems, while “contraction” refers to the iterative points converging increasingly closer to the solution set. Each method we painstakingly put together in the past is a special case within this series of methods that can now be designed. Our experience confirms once again that the algorithms favored in engineering must be simple. I have written five short summaries, each two pages long, introducing the main results in convex optimization splitting contraction algorithms that were achieved since 2010. Some of these results are recent works in collaboration with my students, while others are my more mature perspectives after the age of seventy. For older scholars with some research foundation, continuing to collaborate with students after retirement has been proven to be the right choice and worthwhile. The specific contents of these five summary notes are:

1. **Introduction to Our Unified Algorithm Framework**. This note presents our unified framework for algorithms, providing a convergence proof, an equivalent representation of convergence conditions, and the resulting generalized prediction-correction Proximal Point Algorithm (PPA). The key identity ”product-to-sum and difference” that plays a critical role in the convergence proof could be contributed to the solid mathematical foundation I achieved from pre-Cultural Revolution secondary education in mathematics and physics.

2. ***H*-norm PPA under the Variational Inequality Perspective**. It explains that the H-norm PPA, as understood in the context of variational inequalities, is a specific method within our unified framework. Examples are provided to illustrate the construction of Customized PPA, which is tailored to specific needs, and Balanced PPA, which distributes computational difficulty in the primal and dual sub-problems of each iteration. These methods are also applicable to solving multi-block separable convex optimization problems.

3. **Deriving ADMM as a Specific Algorithm in the Unified Framework**. This summary shows that the Alternating Direction Method of Multipliers (ADMM) can be interpreted as a specific algorithm within the unified framework. While this interpretation may seem redundant, it provides a convenient approach for proving convergence rates. My two collaborative papers with Xiaoming Yuan on convergence rates of ADMM in both ergodic and point-wise senses rely on this prediction-correction decomposition. This method of interpretation has also been adopted by several renowned international scholars in their theorem proofs.

4. **Provide a class of solution methods for three-block separable convex optimization using a unified framework**. It was pointed out that directly extending ADMM to solve problems with three separable blocks does not guarantee convergence. However, our unified framework can guide the construction of algorithms for multi-block problems. Using convex optimization with three separable blocks as an example, we demonstrate how, by applying the same Gaussian-type prediction with different corrections, one can construct a family of prediction-correction splitting contraction algorithms.

5. **Generalized PPA for Large-Scale Linearly Constrained Separable Convex Optimization Problems**. This note presents a generalized PPA for large-scale, linearly constrained separable convex optimization problems, developed under the guidance of our unified framework. Each iteration of the algorithm follows a prediction-correction step that flows as naturally as the "elimination and back-substitution" process in Gaussian elimination for solving system of linear equations. The only difference is that here, "elimination" is achieved by solving relatively simple convex optimization subproblems.

For linearly constrained convex optimization, these short summaries illustrate that our unified framework not only covers foundational algorithms such as PPA, ALM, and ADMM, but also provides solution methods for multi-block separable convex optimization problems. I am deeply grateful for the support of my colleagues. These works contributed to my receiving the "Operations Research Society of China Scientific and Technological Award" in Operations Research in 2014, and elected as a Fellow of the "Operations Research Society of China" in 2024.

To fit everything into two-page format required considerable effort for me to edit and summarize the contents. Hopefully a graduate student with some background in optimization could likely browse through it in one day, and a week should be enough to fully understand the concepts. At the age of seventy-six, I often reflect on my academic legacy. I am fortunate that I still have time and energy to organize and present these materials. I welcome any corrections from readers should there be any errors.

Prof. Dr. Bingsheng He
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A. Splitting and contraction methods for convex optimization

Optimization, VI, Algorithmic framework and Generalized-PPA © Bingsheng He

1 Convex optimization and its related variational inequality

We consider the linearly constrained convex optimization

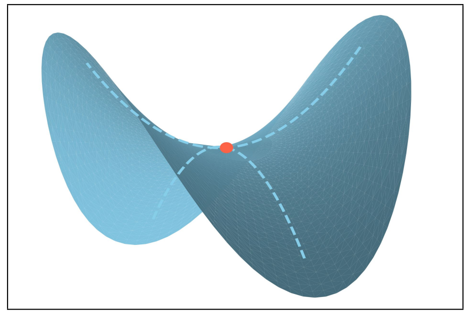
$$\min\{\theta(u) \mid \mathcal{A}u = b,\ u \in \mathcal{U}\}. \tag{1.1}$$

The Lagrangian function of the problem (1.1) is

$$L(u,\lambda) = \theta(u) - \lambda^T(\mathcal{A}u - b), \tag{1.2}$$

which is defined on $\mathcal{U} \times \Re^m$. A pair of $(u^*,\lambda^*) \in \mathcal{U} \times \Re^m$ is called a saddle point of the Lagrangian function (1.2), if

$$L_{u\in\mathcal{U}}(u,\lambda^*) \ge L(u^*,\lambda^*) \ge L_{\lambda\in\Re^m}(u^*,\lambda). \tag{1.3}$$



The two inequalities in (1.3) can be written as $(u^*,\lambda^*) \in \mathcal{U} \times \Re^m$,

$$\begin{cases} L(u,\lambda^*) - L(u^*,\lambda^*) \ge 0, & \forall u \in \mathcal{U}, \\ L(u^*,\lambda^*) - L(u^*,\lambda) \ge 0, & \forall \lambda \in \Re^m. \end{cases} \Longleftrightarrow \begin{cases} \theta(u) - \theta(u^*) + (u-u^*)^T(-\mathcal{A}^T\lambda^*) \ge 0, & \forall u \in \mathcal{U}, \\ (\lambda-\lambda^*)^T(\mathcal{A}u^* - b) \ge 0, & \forall \lambda \in \Re^m. \end{cases} \tag{1.4}$$

Combining the above two inequalities, the saddle point is described as the solution of the following VI :

$$\text{(VI)} \qquad w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w-w^*)^T F(w^*) \ge 0, \quad \forall\, w \in \Omega, \tag{1.5}$$

where
$$w = \begin{pmatrix} u \\ \lambda \end{pmatrix}, \qquad F(w) = \begin{pmatrix} -\mathcal{A}^T\lambda \\ \mathcal{A}u - b \end{pmatrix} \qquad \text{and} \qquad \Omega = \mathcal{U} \times \Re^m. \tag{1.6}$$

Please notice that $(w-\tilde{w})^T(F(w) - F(\tilde{w})) \equiv 0$. The problem (1.1) is translated to VI (1.5).

For the multi-block separable convex optimization, we take the three-block problem

$$\min\{\theta_1(x) + \theta_2(y) + \theta_3(z) \mid Ax + By + Cz = b, x \in \mathcal{X}, y \in \mathcal{Y}, z \in \mathcal{Z}\} \tag{1.7}$$

as an example. Its corresponding variational inequality has the form (1.5), where $\Omega = \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \times \Re^m$, and

$$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \theta(u) = \theta_1(x) + \theta_2(y) + \theta_3(z), \quad w = \begin{pmatrix} x \\ y \\ z \\ \lambda \end{pmatrix}, \quad F(w) = \begin{pmatrix} -A^T\lambda \\ -B^T\lambda \\ -C^T\lambda \\ Ax + By + Cz - b \end{pmatrix}. \tag{1.8}$$

For the $F(w)$ in (1.8), we still have $(w-\tilde{w})^T(F(w) - F(\tilde{w})) \equiv 0$, for any w and $\tilde{w}$ in the space containing Ω.

2 Algorithmic unified framework for monotone variational inequalities

We focus on how to solve the variational inequality (1.5), following is our algorithmic framework.

Algorithms in a unified framework **(Each iteration of the method consists of a prediction and a correction)**

[Prediction Step]. Start from a given v^k, find a predictor $\tilde{w}^k \in \Omega$, which satisfies

$$\tilde{w}^k \in \Omega, \quad \theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (v - \tilde{v}^k)^T Q(v^k - \tilde{v}^k), \quad \forall w \in \Omega, \tag{2.1a}$$

where the prediction matrix Q is not necessarily symmetric, but the kernel of $Q^T + Q$ is positive definite.

v is called the essential variable in the iteration which can be equal to w, or a part of the whole vector of w.

[Correction Step]. Find the correction matrix M which satisfied (2.2). The new iteration v^{k+1} is given by

$$v^{k+1} = v^k - M(v^k - \tilde{v}^k). \tag{2.1b}$$

Convergence Conditions **(It is easy to find the matrix M which satisfies the conditions, see the details in §4)**

[Convergence Conditions]. For the prediction matrix Q in (2.1a) and the correction matrix M in (2.1b), there is a positive definite matrix H, such that

$$HM = Q \quad \text{and} \quad G = Q^T + Q - M^T HM \succ 0. \tag{2.2}$$

3 Convergence proof of the methods in the algorithmic framework

Theorem A. For solving the VI (1.5), let $\{v^k\}$, $\{\tilde{w}^k\}$ be the sequences generated by (2.1). If the conditions (2.2) are satisfied, then we have

$$\tilde{w}^k \in \Omega, \;\; \theta(u)-\theta(\tilde{u}^k)+(w-\tilde{w}^k)^T F(\tilde{w}^k) \geq \tfrac{1}{2}\big(\|v-v^{k+1}\|_H^2-\|v-v^k\|_H^2\big)+\tfrac{1}{2}\|v^k-\tilde{v}^k\|_G^2, \;\; \forall w \in \Omega. \quad (3.1)$$

and

$$\|v^{k+1}-v^*\|_H^2 \leq \|v^k-v^*\|_H^2 - \|v^k-\tilde{v}^k\|_G^2, \quad \forall v^* \in \mathcal{V}^*. \quad (3.2)$$

Proof. Treating the term $Q(v^k-\tilde{v}^k)$ in the RHS of (2.1a) by using $Q = HM$ (see (2.2)) and the correction formula (2.1b), we obtain $Q(v^k-\tilde{v}^k) = HM(v^k-\tilde{v}^k) = H(v^k-v^{k+1})$. Thus we get

$$\tilde{w}^k \in \Omega, \quad \theta(u)-\theta(\tilde{u}^k)+(w-\tilde{w}^k)^T F(\tilde{w}^k) \geq (v-\tilde{v}^k)^T H(v^k-v^{k+1}), \quad \forall w \in \Omega. \quad (3.3)$$

Applying the identity

$$(a-b)^T H(c-d) = \tfrac{1}{2}\big(\|a-d\|_H^2-\|a-c\|_H^2\big)+\tfrac{1}{2}\big(\|b-c\|_H^2-\|b-d\|_H^2\big) \quad (3.4)$$

to the RHS of (3.3) with $a=v$, $b=\tilde{v}^k$, $c=v^k$ and $d=v^{k+1}$, we obtain

$$(v-\tilde{v}^k)^T H(v^k-v^{k+1}) = \tfrac{1}{2}\big(\|v-v^{k+1}\|_H^2-\|v-v^k\|_H^2\big)+\tfrac{1}{2}\big(\|v^k-\tilde{v}^k\|_H^2-\|v^{k+1}-\tilde{v}^k\|_H^2\big). \quad (3.5)$$

To the second part of the RHS of (3.5), by using $HM=Q$ and $2v^TQv = v^T(Q^T+Q)v$, it follows that

$$\begin{aligned}\|v^k-\tilde{v}^k\|_H^2-\|v^{k+1}-\tilde{v}^k\|_H^2 &\overset{(2.1b)}{=} \|v^k-\tilde{v}^k\|_H^2 - \|(v^k-\tilde{v}^k)-M(v^k-\tilde{v}^k)\|_H^2 \\ &= (v^k-\tilde{v}^k)^T(Q^T+Q-M^THM)(v^k-\tilde{v}^k) \overset{(2.2)}{=} \|v^k-\tilde{v}^k\|_G^2. \quad (3.6)\end{aligned}$$

Substituting (3.6) in (3.5), and then in (3.3), we get the assertion (3.1) directly. Setting the $w\in\Omega$ in (3.1) by any fixed w^*, then using $(\tilde{w}^k-w^*)^TF(\tilde{w}^k) = (\tilde{w}^k-w^*)^TF(w^*)$ and $\theta(\tilde{u}^k)-\theta(u^*)+(\tilde{w}^k-w^*)^TF(w^*) \geq 0$, we obtain (3.2) and the theorem is completely proved. □

This theorem is proved under weak conditions: $Q^T+Q \succeq 0, \; H \succeq 0, \; HM=Q, \; G=Q^T+Q-M^THM \succeq 0.$

Assertion (3.1) is useful for the convergence rate proof of ADMM, see SIAM Numer. Anal. 2012, 50:700-709.

4 The equivalent convergence conditions and the generalized PPA

Under the condition that the prediction matrix Q is nonsingular, it is easy to construct the correction matrix M which satisfies the convergence conditions (2.2). In fact, because $Q^T+Q \succ 0$, we can take

$$D \succ 0, \qquad G \succ 0 \qquad \text{and} \qquad D+G = Q^T+Q. \quad (4.1)$$

Afterwards, we let

$$HM=Q \qquad \text{and} \qquad M^THM = D. \quad (4.2)$$

Because $\left\{\begin{array}{l} HM=Q, \\ M^THM=D.\end{array}\right. \Leftrightarrow \left\{\begin{array}{l} Q^TM=D, \\ HM=Q.\end{array}\right. \Leftrightarrow \left\{\begin{array}{l} M=Q^{-T}D, \\ H=QD^{-1}Q^T,\end{array}\right.$ we get the matrices M, H and G, which satisfy the conditions (2.2). There are infinite combinations of D and G which satisfy conditions (4.1).

Choosing matrix D that satisfies condition (4.1), we get $M=Q^{-T}D$, and $H=QD^{-1}Q^T$ is positive definite.
The correction $v^{k+1}=v^k-M(v^k-\tilde{v}^k)$ (can be achieved by solving $Q^T(v^{k+1}-v^k)=D(\tilde{v}^k-v^k)$.

The generalized PPA by choosing a special D. We can take a special pair of D and G in (4.1) by

$$D = G = \tfrac{1}{2}(Q^T+Q). \quad (4.3)$$

In this case, $M=\frac{1}{2}Q^{-T}(Q^T+Q)$. Because $D=G$, the contractive inequality (3.2) becomes

$$\|v^{k+1}-v^*\|_H^2 \leq \|v^k-v^*\|_H^2 - \|v^k-\tilde{v}^k\|_D^2, \quad \forall v^* \in \mathcal{V}^*. \quad (4.4)$$

Moreover, since $D=M^THM$ (see (4.2)) and $M(v^k-\tilde{v}^k)=v^k-v^{k+1}$ (see (2.1b)), it follows that

$$\|v^{k+1}-v^*\|_H^2 \leq \|v^k-v^*\|_H^2 - \|v^k-v^{k+1}\|_H^2, \quad \forall v^* \in \mathcal{V}^*. \quad (4.5)$$

The inequality (4.5) is just the main convergence result of the classical PPA, a favorable formula ! Because the each iteration consists of a prediction and a correction, we call the related method as a generalized PPA.

In practice, in the generalized PPA, we suggest to take $D=\alpha(Q^T+Q)$ and $\alpha \in [0.5, 1)$.

B. PPA belongs to the algorithmic unified framework

Proximal Point Algorithm, Algorithmic unified framework © Bingsheng He

1 Preliminary theorem for convex optimization

Theorem B. Let $\mathcal{X} \subset \Re^n$ be a closed convex set, $\theta(x)$ and $f(x)$ be convex functions and $f(x)$ be differentiable. Assume that the solution set of the minimization problem $\min\{\theta(x) + f(x) \,|\, x \in \mathcal{X}\}$ is nonempty. Then,

$$x^* \in \arg\min\{\theta(x) + f(x) \,|\, x \in \mathcal{X}\} \tag{1.1}$$

if and only if

$$x^* \in \mathcal{X}, \;\; \theta(x) - \theta(x^*) + (x - x^*)^T \nabla f(x^*) \ge 0, \;\; \forall\, x \in \mathcal{X}. \tag{1.2}$$

2 Convex optimization and its related variational inequality

1). min-max problem. Let (x^*, y^*) be the solution of the min-max problem

$$\min_x \max_y \{\Phi(x,y) = \theta_1(x) - y^T Ax - \theta_2(y) \,|\, x \in \mathcal{X}, y \in \mathcal{Y}\}. \tag{2.1}$$

Using the notation of $\Phi(x,y)$, we have

$$\begin{cases} x^* \in \mathcal{X}, & \theta_1(x) - \theta_1(x^*) + (x - x^*)^T(-A^T y^*) \ge 0, \quad \forall\, x \in \mathcal{X}, \\ y^* \in \mathcal{Y}, & \theta_2(y) - \theta_2(y^*) \;+\; (y - y^*)^T(\;Ax^*) \;\;\ge 0, \quad \forall\, y \in \mathcal{Y}. \end{cases}$$

Furthermore, it can be written as a variational inequality in the compact form:

$$\text{(VI)} \qquad u^* \in \Omega, \quad \theta(u) - \theta(u^*) + (u - u^*)^T F(u^*) \ge 0, \quad \forall\, u \in \Omega, \tag{2.2}$$

where $u = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \theta(u) = \theta_1(x) + \theta_2(y), \qquad F(u) = \begin{pmatrix} -A^T y \\ Ax \end{pmatrix}$ and $\Omega = \mathcal{X} \times \mathcal{Y}$. (2.3)

Please notice that for the $F(u)$ in (2.3), we have $(u - \tilde{u})^T(F(u) - F(\tilde{u})) \equiv 0$.

2). Linearly constrained convex optimization The Lagrangian function of the linearly constrained convex optimization problem $\min\{\theta(u) \mid \mathcal{A}u = b, \; u \in \mathcal{U}\}$ is

$$L(u, \lambda) = \theta(u) - \lambda^T(\mathcal{A}u - b), \quad \text{which defined on } \mathcal{U} \times \Re^m.$$

The saddle point $(u^*, \lambda^*) \in \mathcal{U} \times \Re^m$ of the Lagrangian function can be characterized as

$$\begin{cases} u^* \in \mathcal{U}, \;\; L(u, \lambda^*) - L(u^*, \lambda^*) \ge 0, \;\; \forall\, u \in \mathcal{U}, & (2.4a) \\ \lambda^* \in \Re^m, \; L(u^*, \lambda^*) - L(u^*, \lambda) \ge 0, \;\; \forall\, \lambda \in \Re^m. & (2.4b) \end{cases}$$

Using a more compact form, the saddle-point can be characterized as the solution of the following VI:

$$\text{(VI)} \qquad w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \ge 0, \quad \forall\, w \in \Omega, \tag{2.5}$$

where $w = \begin{pmatrix} u \\ \lambda \end{pmatrix}, \qquad F(w) = \begin{pmatrix} -\mathcal{A}^T \lambda \\ \mathcal{A}u - b \end{pmatrix}$ and $\Omega = \mathcal{U} \times \Re^m$. (2.6)

Please notice that $(w - \tilde{w})^T(F(w) - F(\tilde{w})) \equiv 0$. We focus on solving the variational inequalities (2.2) and (2.5).

3 Algorithmic unified framework for monotone variational inequalities

Algorithms in a unified framework **(Each iteration of the method consists of a prediction and a correction)**

[Prediction Step]. Start from a given v^k, find a predictor $\tilde{w}^k \in \Omega$, which satisfies

$$\tilde{w}^k \in \Omega, \quad \theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (v - \tilde{v}^k)^T Q(v^k - \tilde{v}^k), \quad \forall w \in \Omega, \tag{3.1a}$$

where the prediction matrix Q is not necessarily symmetric, but the kernel of $Q^T + Q$ is positive definite.

v can be equal to w, or a part vector of the whole vector of w. In this note, v is the whole vector w.

[Correction Step]. Find the correction matrix M which satisfied (3.2). The new v^{k+1} is given by

$$v^{k+1} = v^k - M(v^k - \tilde{v}^k). \tag{3.1b}$$

Convergence Conditions

[Convergence Conditions]. For the prediction matrix Q in (3.1a) and the correction matrix M in (3.1b), there is a positive definite matrix H, such that

$$HM = Q \quad \text{and} \quad G = Q^T + Q - M^T HM \succ 0. \tag{3.2}$$

4 Proximal point algorithm for monotone variational inequalities

Definition (k-th iteration of the relaxed PPA) $\mathcal{H} \succ 0$. Start with a given w^k, find a $\tilde{w}^k$, such that

[Prediction] $\tilde{w}^k \in \Omega, \quad \theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (w - \tilde{w}^k)^T \mathcal{H}(w^k - \tilde{w}^k), \quad \forall\, w \in \Omega.$ (4.1a)

[Correction] $w^{k+1} = w^k - \alpha(w^k - \tilde{w}^k), \quad \alpha \in (0,2). \qquad \big(\alpha \in [1.2, 1.8] \text{ is suggested}\big).$ (4.1b)

The relaxed PPA is a special algorithm of (3.1) with $Q = \mathcal{H}$ and $M = \alpha I$. By setting $H = \frac{1}{\alpha}\mathcal{H}$, we have

$$HM = \mathcal{H} = Q \qquad \text{and} \qquad G = Q^T + Q - M^T H M = (2-\alpha)\mathcal{H} \succ 0. \tag{4.2}$$

Taking $\alpha = 1$, we get the classical PPA. PPA belongs to (3.1) and satisfies the convergence conditions (3.2).

5 PPA (Customized PPA and Balanced PPA) for VI

The main task of the each iteration of the relaxed PPA is to find a predictor $\tilde{w}^k$ which satisfies (4.1a).

1). PPA for VI (2.2) (Bingsheng He and Xiaoming Yuan, SIAM Imaging Science 5, (2012) 119-149)

Start with given u^k, find a $\tilde{u}^k$, such that (4.1a) is satisfied, where

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad w = u, \quad F(w) = \begin{pmatrix} -A^T y \\ Ax \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} rI_n & A^T \\ A & sI_m \end{pmatrix}, \quad rs > \|A^T A\|. \tag{5.1}$$

The concrete formula of (4.1a) with F and H given in (5.1) is $\boxed{\text{the uwave parts is } F(\tilde{w}^k)}$

$$\begin{cases} \tilde{x}^k \in \mathcal{X}, \ \ \theta_1(x) - \theta_1(\tilde{x}^k) + (x - \tilde{x}^k)^T\{\underline{-A^T\tilde{y}^k} + rI_n(\tilde{x}^k - x^k) + A^T(\tilde{y}^k - y^k)\} \ge 0, \quad \forall x \in \mathcal{X}, \\ \tilde{y}^k \in \mathcal{Y}, \ \ \theta_2(y) - \theta_2(\tilde{y}^k) + (y - \tilde{y}^k)^T\{\underline{A\tilde{x}^k} + A(\tilde{x}^k - x^k) + sI_m(\tilde{y}^k - y^k)\} \ge 0, \quad \forall y \in \mathcal{Y}. \end{cases}$$

According to Theorem B, to complete the PPA iteration, we need only to solve the following sub-problems:

$$\begin{cases} \tilde{x}^k = \text{argmin}\{\theta_1(x) - x^T A^T y^k + \frac{1}{2}r\|x - x^k\|^2 \,|\, x \in \mathcal{X}\}, & (5.2a) \\ \tilde{y}^k = \text{argmin}\{\theta_2(y) + y^T A(2\tilde{x}^k - x^k) + \frac{1}{2}s\|y - y^k\|^2 \,|\, y \in \mathcal{Y}\}. & (5.2b) \end{cases}$$

2). Customized PPA and Balanced PPA for VI (2.5)

Start with given w^k, find a $\tilde{w}^k$, such that (4.1a) is satisfied, where

$$F(w) = \begin{pmatrix} -\mathcal{A}^T\lambda \\ \mathcal{A}u - b \end{pmatrix}, \quad \mathcal{H} = \mathcal{H}_1 = \begin{pmatrix} \beta\mathcal{A}^T\mathcal{A} + \delta I_n & \mathcal{A}^T \\ \mathcal{A} & \frac{1}{\beta}I_m \end{pmatrix} \ \text{ or } \ \mathcal{H} = \mathcal{H}_2 = \begin{pmatrix} rI_n & \mathcal{A}^T \\ \mathcal{A} & \frac{1}{r}\mathcal{A}\mathcal{A}^T + \delta I_m \end{pmatrix}. \tag{5.3}$$

Customized PPA for VI (2.5) (For customized PPA, see He, Yuan, Zhang 2013 COA, Gu, He, Yuan 2014 COA)

The concrete formula of (4.1a) with $\mathcal{H} = \mathcal{H}_1$ given in (5.3) is $\boxed{\text{the uwave parts is } F(\tilde{w}^k)}$

$$\begin{cases} \tilde{u}^k \in \mathcal{U}, \ \ \theta(u) - \theta(\tilde{u}^k) + (u - \tilde{u}^k)^T\{\underline{-\mathcal{A}^T\tilde{\lambda}^k} + (\beta\mathcal{A}^T\mathcal{A} + \delta I_n)(\tilde{u}^k - u^k) + \mathcal{A}^T(\tilde{\lambda}^k - \lambda^k)\} \ge 0, \ \forall u \in \mathcal{U}, \\ \tilde{\lambda}^k \in \Re^m, \qquad (\lambda - \tilde{\lambda}^k)^T\{(\underline{\mathcal{A}\tilde{u}^k - b}) + \mathcal{A}(\tilde{u}^k - u^k) + (1/\beta)(\tilde{\lambda}^k - \lambda^k)\} \ge 0, \ \forall \lambda \in \Re^m. \end{cases}$$

According to Theorem B, to complete PPA iteration, we need only to solve the following sub-problems:

$$\begin{cases} \tilde{u}^k = \text{argmin}\big\{\theta(u) - u^T\mathcal{A}^T\lambda^k + \frac{1}{2}(u - u^k)^T(\beta\mathcal{A}^T\mathcal{A} + \delta I_n)(u - u^k) \,|\, u \in \mathcal{U}\big\}, & (5.4a) \\ \tilde{\lambda}^k = \lambda^k - \beta\big(\mathcal{A}[2\tilde{u}^k - u^k] - b\big). & (5.4b) \end{cases}$$

Balanced PPA for VI (2.5) (For balanced PPA, see He and Yuan, arXiv:2108.08554; S.J. Xu, 2023, JAMC)

The concrete formula of (4.1a) with $\mathcal{H} = \mathcal{H}_2$ given in (5.3) is $\boxed{\text{the uwave parts is } F(\tilde{w}^k)}$

$$\begin{cases} \tilde{u}^k \in \mathcal{U}, \ \ \theta(u) - \theta(\tilde{u}^k) + (u - \tilde{u}^k)^T\{\underline{-\mathcal{A}^T\tilde{\lambda}^k} + rI_n(\tilde{u}^k - u^k) + \mathcal{A}^T(\tilde{\lambda}^k - \lambda^k)\} \ge 0, \ \forall u \in \mathcal{U}, \\ \tilde{\lambda}^k \in \Re^m, \qquad (\lambda - \tilde{\lambda}^k)^T\{(\underline{\mathcal{A}\tilde{u}^k - b}) + \mathcal{A}(\tilde{u}^k - u^k) + (\frac{1}{r}\mathcal{A}\mathcal{A}^T + \delta I_m)(\tilde{\lambda}^k - \lambda^k)\} \ge 0, \ \forall \lambda \in \Re^m. \end{cases}$$

According to Theorem B, to complete PPA iteration, we need only to solve the following sub-problems:

$$\begin{cases} \tilde{u}^k = \text{argmin}\big\{\theta(u) - u^T\mathcal{A}^T\lambda^k + \frac{r}{2}\|u - u^k\|^2 \,|\, u \in \mathcal{U}\big\}, \quad \boxed{\text{Primal problem is easier than(5.4a)}} & (5.5a) \\ \tilde{\lambda}^k = \lambda^k - (\frac{1}{r}\mathcal{A}\mathcal{A}^T + \delta I_m)^{-1}\big(\mathcal{A}[2\tilde{u}^k - u^k] - b\big). \quad \boxed{\text{Only once Cholesky-Decomposition}} & (5.5b) \end{cases}$$

Algorithm (5.5) is called the balanced algorithm, because its two subproblems share the difficulties.

C. Explaining ADMM as an algorithm within the framework

The convergence rate proof benefit from this explaination © Bingsheng He

1 Two block convex optimization and its related VI and ADMM

We consider the two block separable convex optimization problem

$$\min\{\theta_1(x) + \theta_2(y) \mid Ax + By = b,\ x \in \mathcal{X}, y \in \mathcal{Y}\}, \tag{1.1}$$

where $\theta_1(x), \theta_2(y)$ are convex function, $A,\ B,\ b$ are the corresponding matrices and vector. $\mathcal{X}$ and $\mathcal{Y}$ are closed convex set. The saddle point of the Lagrangian function of the problem (1.1), say $w^* = (x^*, y^*, \lambda^*)$, can be characterized as the solution of the following variational inequality:

$$w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \ge 0, \quad \forall\, w \in \Omega, \tag{1.2a}$$

where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \quad u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \theta(u) = \theta_1(x) + \theta_2(y), \quad F(w) = \begin{pmatrix} -A^T\lambda \\ -B^T\lambda \\ Ax + By - b \end{pmatrix}. \tag{1.2b}$$

The k-th iteration of ADMM begins with a given $v^k = (y^k, \lambda^k)$, produces $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ via

$$\left\{\begin{array}{ll} x^{k+1} \in \arg\min\big\{\theta_1(x) - x^T A^T \lambda^k + \frac{1}{2}\beta\|Ax + By^k - b\|^2 \;\big|\; x \in \mathcal{X}\big\}, & (1.3a) \\ y^{k+1} \in \arg\min\big\{\theta_2(y) - y^T B^T \lambda^k + \frac{1}{2}\beta\|Ax^{k+1} + By - b\|^2 \;\big|\; y \in \mathcal{Y}\big\}, & (1.3b) \\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b). & (1.3c) \end{array}\right.$$

2 Split ADMM iteration into prediction and correction of the unified framework

We intentionally split ADMM (1.3) into a prediction-correction method.

The prediction of the k-th iteration begins with given $v^k = (y^k, \lambda^k)$, produces $(\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ via

$$\textbf{[Prediction]} \quad \left\{\begin{array}{ll} \tilde{x}^k \in \arg\min\big\{\theta_1(x) - x^T A^T \lambda^k + \frac{1}{2}\beta\|Ax + By^k - b\|^2 \;\big|\; x \in \mathcal{X}\big\}, & (2.1a) \\ \tilde{y}^k \in \arg\min\big\{\theta_2(y) - y^T B^T \lambda^k + \frac{1}{2}\beta\|A\tilde{x}^k + By - b\|^2 \;\big|\; y \in \mathcal{Y}\big\}, & (2.1b) \\ \tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b). & (2.1c) \end{array}\right.$$

According to the optimality of the convex optimization, the VI forms of three subproblems of (2.1) are

$$\left\{\begin{array}{ll} \tilde{x}^k \in \mathcal{X}, & \theta_1(x) - \theta_1(\tilde{x}^k) + (x - \tilde{x}^k)^T\big\{-A^T\lambda^k + \beta A^T(A\tilde{x}^k + By^k - b)\big\} \ge 0, \ \ \forall x \in \mathcal{X}, \\ \tilde{y}^k \in \mathcal{Y}, & \theta_2(y) - \theta_2(\tilde{y}^k) + (y - \tilde{y}^k)^T\big\{-B^T\lambda^k + \beta B^T(A\tilde{x}^k + B\tilde{y}^k - b)\big\} \ge 0, \ \ \forall y \in \mathcal{Y}, \\ \tilde{\lambda}^k \in \Re^m, & (\lambda - \tilde{\lambda}^k)^T\{(A\tilde{x}^k + By^k - b) + \frac{1}{\beta}(\tilde{\lambda}^k - \lambda^k)\} \ge 0, \ \ \forall \lambda \in \Re^m. \end{array}\right.$$

By using $\tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + By^k - b)$, the above inequalities can be written as

$$\left\{\begin{array}{lll} \tilde{x}^k \in \mathcal{X}, & \underline{\theta_1(x) - \theta_1(\tilde{x}^k)} + (x - \tilde{x}^k)^T\big\{\underline{-A^T\tilde{\lambda}^k} \big\} \ge 0, \ \ \forall x \in \mathcal{X}, & (2.2a) \\ \tilde{y}^k \in \mathcal{Y}, & \underline{\theta_2(y) - \theta_2(\tilde{y}^k)} + (y - \tilde{y}^k)^T\big\{\underline{-B^T\tilde{\lambda}^k} + \beta B^T B(\tilde{y}^k - y^k) \big\} \ge 0, \ \ \forall y \in \mathcal{Y}, & (2.2b) \\ \tilde{\lambda}^k \in \Re^m, & (\lambda - \tilde{\lambda}^k)^T\{\underline{(A\tilde{x}^k + B\tilde{y}^k - b)} - B(\tilde{y}^k - y^k) + \frac{1}{\beta}(\tilde{\lambda}^k - \lambda^k)\} \ge 0, \ \ \forall \lambda \in \Re^m. & (2.2c) \end{array}\right.$$

Furthermore, by using the notations of VI (1.2), we get the compact **VI form of the prediction (2.1)** :

$$\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (v - \tilde{v}^k)^T Q(v^k - \tilde{v}^k), \ \ \forall w \in \Omega, \tag{2.3a}$$

where

$$Q = \begin{pmatrix} \beta B^T B & 0 \\ -B & \frac{1}{\beta} I_m \end{pmatrix}. \tag{2.3b}$$

Let $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ be the output of the classical ADMM (1.3), then we have

$$x^{k+1} = \tilde{x}^k, \quad y^{k+1} = \tilde{y}^k \quad \text{and} \quad \lambda^{k+1} = \tilde{\lambda} + \beta B(y^k - \tilde{y}^k). \tag{2.4}$$

where $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ are generated by the prediction (2.1). According to (2.4)

$$\textbf{[Correction]} \quad v^{k+1} = v^k - M(v^k - \tilde{v}^k), \quad \text{where} \quad M = \begin{pmatrix} I & 0 \\ -\beta B & I_m \end{pmatrix}. \tag{2.5}$$

By setting $H = \begin{pmatrix} \beta B^T B & 0 \\ 0 & \frac{1}{\beta} I_m \end{pmatrix}$, we have $HM = Q$ and $\underline{G = Q^T + Q - M^T HM} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\beta} I_m \end{pmatrix}$. (2.6)

ADMM (1.3) is split into the prediction-correction method (2.3)-(2.5). According to the framework, we have

$$\|v^{k+1}-v^*\|_H^2 \le \|v^k-v^*\|_H^2 - \|v^k-\tilde{v}^k\|_G^2, \quad \forall v^* \in \mathcal{V}^*. \tag{2.7}$$

Theorem C1. Let the sequences $\{v^k\}$ and $\{\tilde{v}^k\}$ be generated by the prediction-correction method (2.3)-(2.5), then we have

$$\|v^k-\tilde{v}^k\|_G^2 = \frac{1}{\beta}\|\lambda^k-\tilde{\lambda}^k\|^2 \ge \|v^k-v^{k+1}\|_H^2, \tag{2.8}$$

and consequently,

$$\|v^{k+1}-v^*\|_H^2 \le \|v^k-v^*\|_H^2 - \|v^k-v^{k+1}\|_H^2, \quad \forall v^* \in \mathcal{V}^*. \tag{2.9}$$

Proof. From (2.2b), we know that the optimal condition of the y-subproblem is

$$\tilde{y}^k \in \mathcal{Y}, \quad \theta_2(y)-\theta_2(\tilde{y}^k) + (y-\tilde{y}^k)^T\{-B^T\tilde{\lambda}^k + \beta B^T B(\tilde{y}^k - y^k)\} \ge 0, \quad \forall\, y \in \mathcal{Y}. \tag{2.10}$$

According to (2.4), $\lambda^{k+1} = \tilde{\lambda}^k + \beta B(y^k-\tilde{y}^k)$ and $\tilde{y}^k = y^{k+1}$, the inequality (2.10) can be written as

$$y^{k+1} \in \mathcal{Y}, \quad \theta_2(y)-\theta_2(y^{k+1}) + (y-y^{k+1})^T\{-B^T\lambda^{k+1}\} \ge 0, \quad \forall\, y \in \mathcal{Y}. \tag{2.11}$$

The above inequality is hold also for the last iteration, *i. e.*, we have

$$y^k \in \mathcal{Y}, \quad \theta_2(y)-\theta_2(y^k) + (y-y^k)^T\{-B^T\lambda^k\} \ge 0, \quad \forall\, y \in \mathcal{Y}. \tag{2.12}$$

Setting $y=y^k$ and $y=y^{k+1}$ in in (2.11) and (2.12), respectively, and then adding them, we get

$$(\lambda^k-\lambda^{k+1})^T B(y^k-y^{k+1}) \ge 0. \tag{2.13}$$

Using $\lambda^k - \tilde{\lambda}^k = (\lambda^k-\lambda^{k+1}) + \beta B(y^k-y^{k+1})$ (see (2.4)) and the inequality (2.13), we obtain

$$\begin{aligned} \frac{1}{\beta}\|\lambda^k-\tilde{\lambda}^k\|^2 &= \frac{1}{\beta}\|(\lambda^k-\lambda^{k+1}) + \beta B(y^k-y^{k+1})\|^2 \\ &\ge \beta\|B(y^k-y^{k+1})\|^2 + \frac{1}{\beta}\|\lambda^k-\lambda^{k+1}\|^2 = \|v^k-v^{k+1}\|_H^2. \end{aligned} \tag{2.14}$$

The assertion (2.8) follows from (2.14) directly. Consequently, the assertion (2.9) follows from (2.7) and (2.8). The theorem is proved. $\square$

Note that from (2.9) we have $\quad \sum_{k=1}^{\infty}\|v^k-v^{k+1}\|_H \le \|v^0-v^*\|_H^2, \ \forall v^* \in \mathcal{V}^*. \qquad (2.15)$

3 Important property of ADMM by using the algorithmic framework

Theorem C2. Let the sequences $\{v^k\}$ and $\{\tilde{v}^k\}$ be generated by the prediction (2.3) and correction (2.5), then we have

$$\|v^{k+1}-v^{k+2}\|_H \le \|v^k-v^{k+1}\|_H^2, \quad \forall v^* \in \mathcal{V}^*. \tag{3.1}$$

Proof. According to (2.3), we have

$$\theta(u)-\theta(\tilde{u}^k) + (w-\tilde{w}^k)^T F(\tilde{w}^k) \ge (v-\tilde{v}^k)^T Q(v^k-\tilde{v}^k), \ \ \forall w \in \Omega \tag{3.2}$$

and

$$\theta(u)-\theta(\tilde{u}^{k+1}) + (w-\tilde{w}^{k+1})^T F(\tilde{w}^{k+1}) \ge (v-\tilde{v}^{k+1})^T Q(v^{k+1}-\tilde{v}^{k+1}), \ \ \forall w \in \Omega. \tag{3.3}$$

Setting the w in (3.2) and (3.3) by $\tilde{w}^{k+1}$ and $\tilde{w}^k$, respectively, and then adding the resulting inequalities together and using $(\tilde{w}^k-\tilde{w}^{k+1})^T\big(F(\tilde{w}^k)-F(\tilde{w}^{k+1})\big)=0$, we obtain

$$(\tilde{v}^k-\tilde{v}^{k+1})^T Q\{(v^k-\tilde{v}^k)-(v^{k+1}-\tilde{v}^{k+1})\} \ge 0. \tag{3.4}$$

Adding $\{(v^k-\tilde{v}^k)-(v^{k+1}-\tilde{v}^{k+1})\}^T Q\{(v^k-\tilde{v}^k)-(v^{k+1}-\tilde{v}^{k+1})\}$ to the both sides of (3.4), we get

$$(v^k-v^{k+1})^T Q\{(v^k-\tilde{v}^k)-(v^{k+1}-\tilde{v}^{k+1})\} \ge \frac{1}{2}\|(v^k-\tilde{v}^k)-(v^{k+1}-\tilde{v}^{k+1})\|_{(Q^T+Q)}^2.$$

Because $HM=Q$ and $M(v^k-\tilde{v}^k)=(v^k-v^{k+1})$ (see (2.5)), the above inequality becomes

$$(v^k-v^{k+1})^T H\{(v^k-v^{k+1})-(v^{k+1}-v^{k+2})\} \ge \frac{1}{2}\|(v^k-\tilde{v}^k)-(v^{k+1}-\tilde{v}^{k+1})\|_{(Q^T+Q)}^2. \tag{3.5}$$

Using the identity $\|a\|_H^2 - \|b\|_H^2 = 2a^T H(a-b) - \|a-b\|_H^2$ and (3.5), we get

$$\begin{aligned} &\|v^k-v^{k+1}\|_H^2 - \|v^{k+1}-v^{k+2}\|_H^2 \\ &\quad = 2(v^k-v^{k+1})^T H\{(v^k-v^{k+1})-(v^{k+1}-v^{k+2})\} - \|(v^k-v^{k+1})-(v^{k+1}-v^{k+2})\|_H^2 \\ &\quad \ge \|(v^k-\tilde{v}^k)-(v^{k+1}-\tilde{v}^{k+1})\|_{(Q^T+Q)}^2 - \|(v^k-v^{k+1})-(v^{k+1}-v^{k+2})\|_H^2. \end{aligned} \tag{3.6}$$

Since $(v^k-v^{k+1}) = M(v^k-\tilde{v}^k)$ and $Q^T+Q-M^T HM = G$, the RHS of (3.6) is $\|(v^k-\tilde{v}^k)-(v^{k+1}-\tilde{v}^{k+1})\|_G^2$. The assertion (3.1) follows immediately. $\square$

Theorem C3. Let $\{v^k\}$ be the sequence generated by ADMM, then for any integer $t>0$, we have

$$\|v^t-v^{t+1}\|_H^2 \le \frac{1}{t+1}\sum_{k=0}^{t}\|v^k-v^{k+1}\|_H \le \frac{1}{t+1}\sum_{k=0}^{\infty}\|v^k-v^{k+1}\|_H \le \frac{1}{t+1}\|v^0-v^*\|_H^2, \ \forall v^* \in \mathcal{V}^*.$$

Proof. The assertion follows from (3.1) and (2.15) directly. $\square$ See H and Yuan, Numer. Math. 130 (2015) 567-577.

D. Constructing SC methods for three block convex optimization

Different corrections based on the same prediction © Bingsheng He

1 The variational inequality and the algorithmic framework

Finding the saddle point of the Lagrangian function is reduced to solving the variational inequality

$$\text{(VI)} \qquad w^* \in \Omega, \quad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \ge 0, \quad \forall\, w \in \Omega. \tag{1.1}$$

For solving VI (1.1), we have proposed the following algorithmic unified framework:

[Prediction Step]. Start from a given v^k, find a predictor $\tilde{w}^k \in \Omega$, which satisfies

$$\tilde{w}^k \in \Omega, \quad \theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (v - \tilde{v}^k)^T Q(v^k - \tilde{v}^k), \quad \forall w \in \Omega, \tag{1.2a}$$

where the matrix Q is not necessarily symmetric, but the kernel of $Q^T + Q$ is positive definite.

[Correction Step]. The new iteration (corrector) v^{k+1} is given by

$$v^{k+1} = v^k - M(v^k - \tilde{v}^k), \tag{1.2b}$$

where $M = Q^{-T}D$, D is chosen which is satisfied $D \succ 0,\ G \succ 0,\ D + G = Q^T + Q$.

Because $M = Q^{-T}D$, the correction (1.2b) can be achieved by $\quad Q^T(v^{k+1} - v^k) = D(\tilde{v}^k - v^k)$.

2 Prediction

We take the three block separable convex optimization as an example for explanation. The problem and the VI are described in **A**. We take the direct extension of ADMM for producing its prediction.

[Prediction]

$$\begin{cases} \tilde{x}^k \in \arg\min\big\{\theta_1(x) - x^T A^T \lambda^k + \frac{1}{2}\beta\|Ax + By^k + Cz^k - b\|^2 \mid x \in \mathcal{X}\big\}, & \text{(2.1a)}\\ \tilde{y}^k \in \arg\min\big\{\theta_2(y) - y^T B^T \lambda^k + \frac{1}{2}\beta\|A\tilde{x}^k + By + Cz^k - b\|^2 \mid y \in \mathcal{Y}\big\}, & \text{(2.1b)}\\ \tilde{z}^k \in \arg\min\big\{\theta_3(z) - z^T C^T \lambda^k + \frac{1}{2}\beta\|A\tilde{x}^k + B\tilde{y}^k + Cz - b\|^2 \mid z \in \mathcal{Z}\big\}, & \text{(2.1c)}\\ \tilde{\lambda}^k = \lambda^k - \beta\big(A\tilde{x}^k + By^k + Cz^k - b\big). & \text{(2.1d)} \end{cases}$$

According to the basic theorem of optimization, it follows from (2.1) that

$$\begin{cases} \tilde{x}^k \in \mathcal{X},\ \underline{\theta_1(x) - \theta_1(\tilde{x}^k)} + (x - \tilde{x}^k)^T\{\underline{-A^T\tilde{\lambda}^k} \} \ge 0,\ \forall x \in \mathcal{X}, & \text{(2.2a)}\\ \tilde{y}^k \in \mathcal{Y},\ \underline{\theta_2(y) - \theta_2(\tilde{y}^k)} + (y - \tilde{y}^k)^T\{\underline{-B^T\tilde{\lambda}^k} + \beta B^T B(\tilde{y}^k - y^k) \} \ge 0,\ \forall y \in \mathcal{Y}, & \text{(2.2b)}\\ \tilde{z}^k \in \mathcal{Z},\ \underline{\theta_3(z) - \theta_3(\tilde{z}^k)} + (z - \tilde{z}^k)^T\{\underline{-C^T\tilde{\lambda}^k} + \beta C^T B(\tilde{y}^k - y^k) + \beta C^T C(\tilde{z}^k - z^k)\} \ge 0,\ \forall z \in \mathcal{Z}, & \text{(2.2c)}\\ \qquad \underline{(A\tilde{x}^k + B\tilde{y}^k + C\tilde{z}^k - b)} - B(\tilde{y}^k - y^k) - C(\tilde{z}^k - z^k) + \frac{1}{\beta}(\tilde{\lambda}^k - \lambda^k) = 0. & \text{(2.2d)} \end{cases}$$

[The inequalities (2.2) can be written together as the prediction (1.2a)] where

$$w = \begin{pmatrix} x \\ y \\ z \\ \lambda \end{pmatrix}, \quad v = \begin{pmatrix} y \\ z \\ \lambda \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} \beta B^T B & 0 & 0 \\ \beta C^T B & \beta C^T C & 0 \\ -B & -C & \frac{1}{\beta}I \end{pmatrix}. \tag{2.3}$$

3 Correction by using the kernel matrix

Note that the matrix

$$Q^T + Q = \begin{pmatrix} B^T & 0 & 0 \\ 0 & C^T & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 2\beta I & \beta I & -I \\ \beta I & 2\beta I & -I \\ -I & -I & \frac{2}{\beta}I \end{pmatrix} \begin{pmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & I \end{pmatrix}$$

is positive definite (whenever B and C are both full column rank) and

$$Q^T = \begin{pmatrix} B^T & 0 & 0 \\ 0 & C^T & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \beta I & \beta I & -I \\ 0 & \beta I & -I \\ 0 & 0 & \frac{1}{\beta}I \end{pmatrix} \begin{pmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & I \end{pmatrix}.$$

The center part of the matrices $Q^T + Q$ and Q^T are called the kernel matrices, and denoted respectively by

$$\mathcal{Q}^T + \mathcal{Q} = \begin{pmatrix} 2\beta I & \beta I & -I \\ \beta I & 2\beta I & -I \\ -I & -I & \frac{2}{\beta}I \end{pmatrix} \text{ and } \mathcal{Q}^T = \begin{pmatrix} \beta I & \beta I & -I \\ 0 & \beta I & -I \\ 0 & 0 & \frac{1}{\beta}I \end{pmatrix}. \text{ Note that } \mathcal{Q}^{-T} = \begin{pmatrix} \frac{1}{\beta}I & -\frac{1}{\beta}I & 0 \\ 0 & \frac{1}{\beta}I & I \\ 0 & 0 & \beta I \end{pmatrix}. \tag{3.1}$$

$\mathcal{Q}^{-T}$ has the simple form because the kernel matrix $\mathcal{Q}^T$ is a upper triangular matrix ! Since the k-th iteration begins with a given (By^k, Cz^k, λ^k), for starting the next iteration, the correction of the k-th iteration needs only to offer $(By^{k+1}, Cz^{k+1}, \lambda^{k+1})$. This is very simple ! See some examples !

3.1 Constructing method I D has a simple form

$$D=\begin{pmatrix} B^T & 0 & 0\\ 0 & C^T & 0\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix} \nu\beta I & 0 & 0\\ 0 & \nu\beta I & 0\\ 0 & 0 & \frac{1}{\beta}I\end{pmatrix}\begin{pmatrix} B & 0 & 0\\ 0 & C & 0\\ 0 & 0 & I\end{pmatrix}, \quad \nu\in(0,1). \tag{3.2}$$

Since the kernel matrix of Q^T+Q can be decomposed as

$$\begin{pmatrix} 2\beta I & \beta I & -I\\ \beta I & 2\beta I & -I\\ -I & -I & \frac{2}{\beta}I\end{pmatrix}=\begin{pmatrix} \nu\beta I & 0 & 0\\ 0 & \nu\beta I & 0\\ 0 & 0 & \frac{1}{\beta}I\end{pmatrix}+\begin{pmatrix} (2-\nu)\beta I & \beta I & -I\\ \beta I & (2-\nu)\beta I & -I\\ -I & -I & \frac{1}{\beta}I\end{pmatrix},$$

the kernel matrices of D and G in the right hand side of the above equation are positive definite. The solution of the system of equation $Q^T(v^{k+1}-v^k)=D(\tilde{v}^k-v^k)$ can be obtained by solving

$$\begin{pmatrix} \beta I & \beta I & -I\\ 0 & \beta I & -I\\ 0 & 0 & \frac{1}{\beta}I\end{pmatrix}\begin{pmatrix} By^{k+1}-By^k\\ Cz^{k+1}-Cz^k\\ \lambda^{k+1}-\lambda^k\end{pmatrix}=\begin{pmatrix} \nu\beta I & 0 & 0\\ 0 & \nu\beta I & 0\\ 0 & 0 & \frac{1}{\beta}I\end{pmatrix}\begin{pmatrix} B\tilde{y}^k-By^k\\ C\tilde{z}^k-Cz^k\\ \tilde{\lambda}^k-\lambda^k\end{pmatrix}.$$

By using the inverse of the kernel matrix of Q^T in (3.1), the correction form can be simplified to

$$\begin{pmatrix} By^{k+1}\\ Cz^{k+1}\\ \lambda^{k+1}\end{pmatrix}=\begin{pmatrix} By^k\\ Cz^k\\ \lambda^k\end{pmatrix}-\begin{pmatrix} \nu I & -\nu I & 0\\ 0 & \nu I & \frac{1}{\beta}I\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix} B(y^k-\tilde{y}^k)\\ C(z^k-\tilde{z}^k)\\ \lambda^k-\tilde{\lambda}^k\end{pmatrix}. \tag{3.3}$$

3.2 Constructing method II G has a simple form

$$D=\begin{pmatrix} B^T & 0 & 0\\ 0 & C^T & 0\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix} (2-\nu)\beta I & \beta I & -I\\ \beta I & (2-\nu)\beta I & -I\\ -I & -I & \frac{1}{\beta}I\end{pmatrix}\begin{pmatrix} B & 0 & 0\\ 0 & C & 0\\ 0 & 0 & I\end{pmatrix}, \quad \nu\in(0,1). \tag{3.4}$$

Since the kernel matrix of Q^T+Q can be decomposed as

$$\begin{pmatrix} 2\beta I & \beta I & -I\\ \beta I & 2\beta I & -I\\ -I & -I & \frac{2}{\beta}I\end{pmatrix}=\begin{pmatrix} (2-\nu)\beta I & \beta I & -I\\ \beta I & (2-\nu)\beta I & -I\\ -I & -I & \frac{1}{\beta}I\end{pmatrix}+\begin{pmatrix} \nu\beta I & 0 & 0\\ 0 & \nu\beta I & 0\\ 0 & 0 & \frac{1}{\beta}I\end{pmatrix},$$

the kernel matrices of D and G in the right hand side of the above equation are positive definite. The solution of the system of equation $Q^T(v^{k+1}-v^k)=D(\tilde{v}^k-v^k)$ can be obtained by solving

$$\begin{pmatrix} \beta I & \beta I & -I\\ 0 & \beta I & -I\\ 0 & 0 & \frac{1}{\beta}I\end{pmatrix}\begin{pmatrix} By^{k+1}-By^k\\ Cz^{k+1}-Cz^k\\ \lambda^{k+1}-\lambda^k\end{pmatrix}=\begin{pmatrix} (2-\nu)\beta I & \beta I & -I\\ \beta I & (2-\nu)\beta I & -I\\ -I & -I & \frac{1}{\beta}I\end{pmatrix}\begin{pmatrix} B\tilde{y}^k-By^k\\ C\tilde{z}^k-Cz^k\\ \tilde{\lambda}^k-\lambda^k\end{pmatrix}.$$

By using the inverse of the kernel matrix of Q^T in (3.1), the correction form can be simplified to

$$\begin{pmatrix} By^{k+1}\\ Cz^{k+1}\\ \lambda^{k+1}\end{pmatrix}=\begin{pmatrix} By^k\\ Cz^k\\ \lambda^k\end{pmatrix}-\begin{pmatrix} (1-\nu)I & -(1-\nu)I & 0\\ 0 & (1-\nu)I & 0\\ -\beta I & -\beta I & I\end{pmatrix}\begin{pmatrix} B(y^k-\tilde{y}^k)\\ C(z^k-\tilde{z}^k)\\ \lambda^k-\tilde{\lambda}^k\end{pmatrix}. \tag{3.5}$$

3.3 Constructing method III D is proportional to (Q^T+Q)

$$D=\alpha\big[Q^T+Q\big]=\alpha\begin{pmatrix} B^T & 0 & 0\\ 0 & C^T & 0\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix} 2\beta I & \beta I & -I\\ \beta I & 2\beta I & -I\\ -I & -I & \frac{1}{\beta}I\end{pmatrix}\begin{pmatrix} B & 0 & 0\\ 0 & C & 0\\ 0 & 0 & I\end{pmatrix}, \quad \alpha\in(0,1). \tag{3.6}$$

The solution of the system of equations $Q^T(v^{k+1}-v^k)=D(\tilde{v}^k-v^k)$ can be obtained by solving

$$\begin{pmatrix} \beta I & \beta I & -I\\ 0 & \beta I & -I\\ 0 & 0 & \frac{1}{\beta}I\end{pmatrix}\begin{pmatrix} By^{k+1}-By^k\\ Cz^{k+1}-Cz^k\\ \lambda^{k+1}-\lambda^k\end{pmatrix}=\alpha\begin{pmatrix} 2\beta I & \beta I & -I\\ \beta I & 2\beta I & -I\\ -I & -I & \frac{2}{\beta}I\end{pmatrix}\begin{pmatrix} B(y^k-\tilde{y}^k)\\ C(z^k-\tilde{z}^k)\\ \lambda^k-\tilde{\lambda}^k\end{pmatrix}, \quad \alpha\in(0,1).$$

By using the inverse of the kernel matrix of Q^T in (3.1), the correction form can be simplified to

$$\begin{pmatrix} By^{k+1}\\ Cz^{k+1}\\ \lambda^{k+1}\end{pmatrix}=\begin{pmatrix} By^k\\ Cz^k\\ \lambda^k\end{pmatrix}-\alpha\begin{pmatrix} I & -I & 0\\ 0 & I & \frac{1}{\beta}I\\ -\beta I & -\beta I & 2I\end{pmatrix}\begin{pmatrix} B(y^k-\tilde{y}^k)\\ C(z^k-\tilde{z}^k)\\ \lambda^k-\tilde{\lambda}^k\end{pmatrix}, \quad \alpha\in(0,1). \tag{3.7}$$

E. Generalized PPA for multi-block convex optimization

Application of the generalized Proximal Point Algorithm © Bingsheng He

1 The variational inequality and the algorithmic framework

Consider the multi-block convex optimization problem

$$\min\{\sum_{i=1}^p \theta_i(x_i) \mid \sum_{i=1}^p A_i x_i = b,\ x_i \in \mathcal{X}_i\}. \tag{1.1}$$

The Lagrangian function of the problem (1.1) is $L(x_1,\ldots,x_p,\lambda) = \sum_{i=1}^p \theta_i(x_i) - \lambda^T(\sum_{i=1}^p A_i x_i - b)$, which is defined on $\Omega = \prod_{i=1}^p \mathcal{X}_i \times \Re^m$. Note tha the saddle point of the Lagrangian function, say $(x_1^*,\ldots,x_p^*,\lambda^*) \in \Omega$, can be described the solution of the following variational inequality:

$$\text{(VI)} \qquad w^* \in \Omega, \quad \theta(x) - \theta(x^*) + (w - w^*)^T F(w^*) \ge 0, \quad \forall\, w \in \Omega, \tag{1.2}$$

where $\Omega = \prod_{i=1}^p \mathcal{X}_i \times \Lambda$, $\theta(x) = \sum_{i=1}^p \theta_i(x_i)$,

$$w = \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ \lambda \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}, \quad F(w) = \begin{pmatrix} -A_1^T\lambda \\ \vdots \\ -A_p^T\lambda \\ \sum_{i=1}^p A_i x_i - b \end{pmatrix}. \tag{1.3}$$

We still use Ω^* to represent the solution set of the variational inequality (1.2). For solving VI (1.2), we have proposed the following algorithmic unified framework which consists a prediction and a correction.

[Prediction]. Start from a given w^k, find a predictor $\tilde{w}^k \in \Omega$, which satisfies

$$\tilde{w}^k \in \Omega, \quad \theta(x) - \theta(\tilde{x}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (w - \tilde{w}^k)^T Q(w^k - \tilde{w}^k), \quad \forall w \in \Omega, \tag{1.4a}$$

where the matrix Q is not necessarily symmetric, but the kernel of $Q^T + Q$ is positive definite.

[Correction]. The new iterate (corrector) w^{k+1} is given by

$$w^{k+1} = w^k - M(w^k - \tilde{w}^k). \tag{1.4b}$$

where $M = Q^{-T}D$, D is chosen which is satisfied $D \succ 0,\ G \succ 0,\ D + G = Q^T + Q$.

Because $M = Q^{-T}D$, the correction can be achieved by $\quad Q^T(w^{k+1} - w^k) = D(\tilde{w}^k - w^k). \quad$ (1.5)

2 Prediction

[24] B.S.He, S.J.Xu, X.M.Yuan, Handbook of Numerical Analysis, 24 (2023) 511-557.

Start from a given $(A_1x_1^k, A_2x_2^k, \cdots, A_px_p^k, \lambda^k)$, obtain $\tilde{w}^k = (\tilde{x}_1^k, \tilde{x}_2^k, \cdots, \tilde{x}_p^k, \tilde{\lambda}^k)$ via:

$$\left\{\begin{array}{l}
\tilde{x}_1^k \in \arg\min\{\theta_1(x_1) - x_1^T A_1^T \lambda^k + \frac{1}{2}\beta\|A_1(x_1 - x_1^k)\|^2 \mid x_1 \in \mathcal{X}_1\}; \\
\tilde{x}_2^k \in \arg\min\{\theta_2(x_2) - x_2^T A_2^T \lambda^k + \frac{1}{2}\beta\|A_1(\tilde{x}_1^k - x_1^k) + A_2(x_2 - x_2^k)\|^2 \mid x_2 \in \mathcal{X}_2\}; \\
\qquad \vdots \\
\tilde{x}_p^k \in \arg\min\{\theta_p(x_p) - x_p^T A_p^T \lambda^k + \frac{1}{2}\beta\|\sum_{j=1}^{p-1} A_j(\tilde{x}_j^k - x_j^k) + A_p(x_p - x_p^k)\|^2 \mid x_p \in \mathcal{X}_p\}; \\
\tilde{\lambda}^k = \lambda^k - \beta\big(\sum_{j=1}^p A_j\tilde{x}_j^k - b\big).
\end{array}\right. \tag{2.1}$$

The optimal condition of the x_i subproblem is

$$\tilde{x}_i^k \in \mathcal{X}_i,\ \ \theta_i(x_i) - \theta_i(\tilde{x}_i^k) + (x_i - \tilde{x}_i^k)^T\{-A_i^T\tilde{\lambda}^k + \beta\sum_{j=1}^i A_j(\tilde{x}_j^k - x_j^k) + A_i^T(\tilde{\lambda}^k - \lambda^k)\} \ge 0,\ \ \forall x_i \in \mathcal{X}_i.$$

The formula which yields $\tilde{\lambda}^k$ can be rewritten as

$$\tilde{\lambda}^k \in \Re^m, \quad (\lambda - \tilde{\lambda}^k)^T\{\big(\sum_{j=1}^p A_j\tilde{x}_j^k - b\big) + \tfrac{1}{\beta}(\tilde{\lambda}^k - \lambda^k)\} \ge 0, \quad \forall \lambda \in \Re^m.$$

By using the notations $\theta(x)$ and $F(w)$, we obtain the following variational inequality for the prediction.

Lemma 2.1 *Let $\tilde{w}^k \in \Omega$ be generated by* (2.1) *with given $(A_1x_1^k, A_2x_2^k, \cdots, A_px_p^k, \lambda^k)$. Then we have*

$$\tilde{w}^k \in \Omega,\ \ \theta(x) - \theta(\tilde{x}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \ge (w - \tilde{w}^k)^T Q(w^k - \tilde{w}^k),\ \ \forall\, w \in \Omega, \tag{2.2a}$$

where

$$Q = \begin{pmatrix}
\beta A_1^T A_1 & 0 & \cdots & 0 & A_1^T \\
\beta A_2^T A_1 & \beta A_2^T A_2 & \ddots & \vdots & A_2^T \\
\vdots & & \ddots & 0 & \vdots \\
\beta A_p^T A_1 & \beta A_p^T A_2 & \cdots & \beta A_p^T A_p & A_p^T \\
0 & 0 & \cdots & 0 & \frac{1}{\beta} I_m
\end{pmatrix}. \tag{2.2b}$$

3 Correction (1.5) by using the kernel matrix

also see arXiv:2107.01897v2[math.OC].

$$Q^T+Q=\begin{pmatrix} A_1^T & 0 & \cdots & 0 & 0\\ 0 & A_2^T & \ddots & \vdots & 0\\ \vdots & & \ddots & 0 & \vdots\\ 0 & 0 & \cdots & A_p^T & 0\\ 0 & 0 & \cdots & 0 & I_m \end{pmatrix}\begin{pmatrix} 2\beta I & \beta I & \cdots & \beta I & I\\ \beta I & 2\beta I & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & \beta I & I\\ \beta I & \cdots & \beta I & 2\beta I & I\\ I & \cdots & I & I & \frac{2}{\beta}I \end{pmatrix}\begin{pmatrix} A_1 & 0 & \cdots & 0 & 0\\ 0 & A_2 & \ddots & \vdots & 0\\ \vdots & & \ddots & 0 & \vdots\\ 0 & 0 & \cdots & A_p & 0\\ 0 & 0 & \cdots & 0 & I_m \end{pmatrix}.$$

is positive definite (whenever each $A_i, i=1,\dots,p$, is full column rank matrix) and

$$Q^T=\begin{pmatrix} A_1^T & 0 & \cdots & 0 & 0\\ 0 & A_2^T & \ddots & \vdots & 0\\ \vdots & & \ddots & 0 & \vdots\\ 0 & 0 & \cdots & A_p^T & 0\\ 0 & 0 & \cdots & 0 & I_m \end{pmatrix}\begin{pmatrix} \beta I & \beta I & \cdots & \beta I & 0\\ 0 & \beta I & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & \beta I & 0\\ 0 & \cdots & 0 & \beta I & 0\\ I & \cdots & I & I & \frac{1}{\beta}I \end{pmatrix}\begin{pmatrix} A_1 & 0 & \cdots & 0 & 0\\ 0 & A_2 & \ddots & \vdots & 0\\ \vdots & & \ddots & 0 & \vdots\\ 0 & 0 & \cdots & A_p & 0\\ 0 & 0 & \cdots & 0 & I_m \end{pmatrix}$$

The center part of the matrices Q^T+Q and Q^T are called their kernel matrix and denoted

$$\mathcal{Q}^T+\mathcal{Q}=\begin{pmatrix} 2\beta I & \beta I & \cdots & \beta I & I\\ \beta I & 2\beta I & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & \beta I & I\\ \beta I & \cdots & \beta I & 2\beta I & I\\ I & \cdots & I & I & \frac{2}{\beta}I \end{pmatrix} \quad\text{and}\quad \mathcal{Q}^T=\begin{pmatrix} \beta I & \beta I & \cdots & \beta I & 0\\ 0 & \beta I & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & \beta I & 0\\ 0 & \cdots & 0 & \beta I & 0\\ I & \cdots & I & I & \frac{1}{\beta}I \end{pmatrix}, \tag{3.1}$$

respectively. Note that $\mathcal{Q}^{-T}$ has the simple form because $\mathcal{Q}^T$ is a nonsingular upper triangular matrix !

Since the k-th iteration begins with a given $(A_1x_1^k, A_2x_2^k,\dots,A_px_p^k,\lambda^k)$, for starting the next iteration, the correction this iteration needs only to offer $(A_1x_1^{k+1}, A_2x_2^{k+1},\dots,A_px_p^{k+1},\lambda^{k+1})$. There are infinite combinations of D and G that meet the conditions, we only take the following example for illustration.

$\boldsymbol{D}$ is proportional to $\boldsymbol{G}$

$D=\alpha\big(Q^T+Q\big)$ and $G=(1-\alpha)\big(Q^T+Q\big)$, $\alpha\in(0,1)$

$$D=\alpha\begin{pmatrix} A_1^T & 0 & \cdots & 0 & 0\\ 0 & A_2^T & \ddots & \vdots & 0\\ \vdots & & \ddots & 0 & \vdots\\ 0 & 0 & \cdots & A_p^T & 0\\ 0 & 0 & \cdots & 0 & I_m \end{pmatrix}\begin{pmatrix} 2\beta I & \beta I & \cdots & \beta I & I\\ \beta I & 2\beta I & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & \beta I & I\\ \beta I & \cdots & \beta I & 2\beta I & I\\ I & \cdots & I & I & \frac{2}{\beta}I \end{pmatrix}\begin{pmatrix} A_1 & 0 & \cdots & 0 & 0\\ 0 & A_2 & \ddots & \vdots & 0\\ \vdots & & \ddots & 0 & \vdots\\ 0 & 0 & \cdots & A_p & 0\\ 0 & 0 & \cdots & 0 & I_m \end{pmatrix}. \tag{3.2}$$

The solution of the system of equations $Q^T(v^{k+1}-v^k)=D(\tilde v^k-v^k)$ can be obtained by solving

$$\begin{pmatrix} \beta I & \beta I & \cdots & \beta I & 0\\ 0 & \beta I & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & \beta I & 0\\ 0 & \cdots & 0 & \beta I & 0\\ I & \cdots & I & I & \frac{1}{\beta}I \end{pmatrix}\begin{pmatrix} A_1x_1^{k+1}-A_1x_1^k\\ A_2x_2^{k+1}-A_2x_2^k\\ \vdots\\ A_px_p^{k+1}-A_px_p^k\\ \lambda^{k+1}-\lambda^k \end{pmatrix}=\alpha\begin{pmatrix} 2\beta I & \beta I & \cdots & \beta I & I\\ \beta I & 2\beta I & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & \beta I & I\\ \beta I & \cdots & \beta I & 2\beta I & I\\ I & \cdots & I & I & \frac{2}{\beta}I \end{pmatrix}\begin{pmatrix} A_1\tilde x_1^k-A_1x_1^k\\ A_2\tilde x_2^k-A_2x_2^k\\ \vdots\\ A_p\tilde x_p^k-A_px_p^k\\ \tilde\lambda^k-\lambda^k \end{pmatrix}.$$

Because
$$\begin{pmatrix} \beta I & \beta I & \cdots & \beta I & 0\\ 0 & \beta I & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & \beta I & 0\\ 0 & \cdots & 0 & \beta I & 0\\ I & \cdots & I & I & \frac{1}{\beta}I \end{pmatrix}^{-1}=\begin{pmatrix} \frac{1}{\beta}I & -\frac{1}{\beta}I & 0 & \cdots & 0\\ 0 & \frac{1}{\beta}I & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & -\frac{1}{\beta}I & 0\\ 0 & \cdots & 0 & \frac{1}{\beta}I & 0\\ -I & 0 & \cdots & 0 & \beta I \end{pmatrix},$$
finally, we get

$$\begin{pmatrix} A_1x_1^{k+1}\\ A_2x_2^{k+1}\\ \vdots\\ A_px_p^{k+1}\\ \lambda^{k+1} \end{pmatrix}=\begin{pmatrix} A_1x_1^k\\ A_2x_2^k\\ \vdots\\ A_px_p^k\\ \lambda^k \end{pmatrix}-\alpha\begin{pmatrix} I & -I & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & I & -I & 0\\ I & \cdots & I & 2I & \frac{1}{\beta}I\\ -\beta I & 0 & \cdots & 0 & I \end{pmatrix}\begin{pmatrix} A_1x_1^k-A_1\tilde x_1^k\\ A_2x_2^k-A_2\tilde x_2^k\\ \vdots\\ A_px_p^k-A_p\tilde x_p^k\\ \lambda^k-\tilde\lambda^k \end{pmatrix},\quad \alpha\in(0,1). \tag{3.3}$$

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