A Counterexample to Left-Right Symmetry for Diagonal Auslander-Regularity

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Abstract: We construct a finite-dimensional Auslander-regular algebra R with global dimension three, which is right diagonal but not left diagonal; moreover, any simple right R-module with grade i has projective dimension i, but there is a simple left R-module with grade one and projective dimension two. It answers a question raised by Iyama negatively.

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1 Introduction

For a ring R and a right R-module M, we use

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^i(M) \to \cdots$$

to denote a minimal injective coresolution of M, and use $\mathrm{pd}_R M$, $\mathrm{fd}_R M$ and $\mathrm{id}_R M$ to denote the projective, flat and injective dimensions of M, respectively. The number $\operatorname{grade} M := \inf\{i \geqslant 0 \mid \mathrm{Ext}^i_R(M, R) \neq 0\}$ is the *grade* of M. We use $\mathrm{gl.\,dim}\, R$ to denote the global dimension of R.

In [1] Bass proved a fundamental theorem which states that a commutative noetherian ring R is a Gorenstein ring (that is, $\mathrm{id}_R R < \infty$) if and only if $\mathrm{fd}_R E^i(R) \leqslant i$ for any $i \geqslant 0$. Let R be a left and right noetherian ring. Auslander proved that the above condition is left-right symmetric [4, Theorem 3.7]; in this case, R is said to satisfy the *Auslander condition*. In addition, the ring R is called *Auslander n-Gorenstein* with $n \geqslant 1$ if $\mathrm{fd}_R E^i(R_R) \leqslant i$ for any $0 \leqslant i \leqslant n-1$. Thus R satisfies the Auslander condition if and only if it is Auslander n-Gorenstein for all n. A left and right noetherian ring R is called *Auslander-regular* if it satisfies the Auslander condition and $\mathrm{gl.\,dim}\, R < \infty$ [2]. Since the Auslander condition is left-right symmetric, so is the notion of

Auslander-regular rings. In [3] Iyama called an Auslander-regular ring R *right diagonal* if $\mathrm{fd}_R I = i$ for any non-zero direct summand I of $E^i(R_R)$ and $i \geqslant 0$. Symmetrically, the notion of *left diagonal Auslander-regular rings* is defined. He proved that the notion of diagonal Auslander-regular artin algebras is left-right symmetric in the case for the global dimension of the algebra being at most two [3, Theorem 3.4], and that for an Auslander 1-Gorenstein artin algebra R, the condition that a simple R-module S with $\operatorname{grade} S = 0$ is projective is left-right symmetric [3, 3.4.1]. Then he asks whether these two results hold true in general.

Question 1.1. ([3, Question 3.6.2]) Let R be an artin algebra. Is the condition that R is diagonal Auslander-regular left-right symmetric? More generally, is the following condition $(*)$ left-right symmetric for an Auslander n-Gorenstein ring R?

$(*)$ A simple R-module S with $\operatorname{grade} S = i$ satisfies $\mathrm{pd}_R S = i$ for any $0 \leqslant i \leqslant n-1$.

In this paper, we construct a finite-dimensional Auslander-regular algebra R with $\mathrm{gl.dim}\, R = 3$, which is right diagonal but not left diagonal; moreover, any simple right R-module S with $\operatorname{grade} S = i$ satisfies $\mathrm{pd}_R S = i$, but there is a simple left R-module S' with $\operatorname{grade} S' = 1$ and $\mathrm{pd}_{R^{\mathrm{op}}} S' = 2$. This answers Question 1.1 negatively.

2 A counterexample

Let $\Bbbk$ be an algebraically closed field and let $R := \Bbbk\mathcal{Q}/\mathcal{I}$ be a finite-dimensional $\Bbbk$-algebra given by the quiver $\mathcal{Q}$:

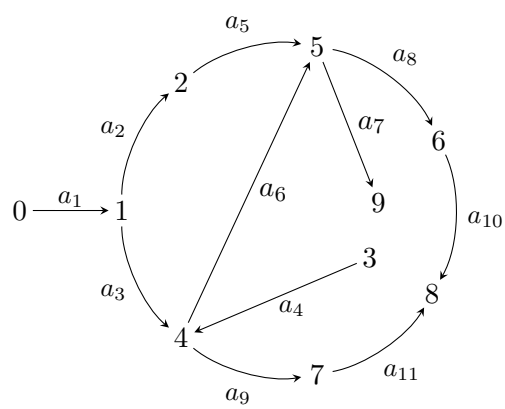

Figure 2.1: The quiver $\mathcal{Q}$ of R.

with the ideal $\mathcal{I}$ given by the relations

$$a_2a_5 = a_3a_6, a_6a_8a_{10} = a_9a_{11} \text{ and } a_5a_7 = a_4a_9 = a_4a_6a_8 = 0.$$

For any $i \in \mathcal{Q}_0 = \{0, 1, \ldots, 9\}$, associated with the vertex i, we write

$$P(i) := \text{the indecomposable projective right } R\text{-module},$$

$$I(i) := \text{the indecomposable injective right } R\text{-module},$$
$$S(i) := \text{the indecomposable simple right } R\text{-module},$$
$$P'(i) := \text{the indecomposable projective left } R\text{-module},$$
$$I'(i) := \text{the indecomposable injective left } R\text{-module},$$
$$S'(i) := \text{the indecomposable simple left } R\text{-module}.$$

Lemma 2.1. *It holds that*

(1)

$$(\mathrm{pd}_R I(i))_{i=0}^{9} = (1,2,2,3,2,1,1,1,0,0), \tag{2.1}$$
$$(\mathrm{id}_R P(i))_{i=0}^{9} = (0,1,1,0,1,2,2,1,3,2), \tag{2.2}$$
$$(\mathrm{pd}_R S(i))_{i=0}^{9} = (1,2,2,3,2,1,1,1,0,0), \tag{2.3}$$
$$(\mathrm{id}_R S(i))_{i=0}^{9} = (0,1,1,0,1,2,2,2,3,2). \tag{2.4}$$

(2)

$$(\mathrm{pd}_{R^{\mathrm{op}}} I'(i))_{i=0}^{9} = (0,1,1,0,1,2,2,1,3,2), \tag{2.5}$$
$$(\mathrm{id}_{R^{\mathrm{op}}} P'(i))_{i=0}^{9} = (1,2,2,3,2,1,1,1,0,0), \tag{2.6}$$
$$(\mathrm{pd}_{R^{\mathrm{op}}} S'(i))_{i=0}^{9} = (0,1,1,0,1,2,2,2,3,2), \tag{2.7}$$
$$(\mathrm{id}_{R^{\mathrm{op}}} S'(i))_{i=0}^{9} = (1,2,2,3,2,1,1,1,0,0). \tag{2.8}$$

(3) $\mathrm{gl.dim}\, R = 3$.

(4) *The minimal injective coresolutions of* R_R *and* ${}_RR$ *are as follows.*

$$0 \to R_R \to I(8)^{\oplus 8} \oplus I(9)^{\oplus 4} \to I(0) \oplus I(5)^{\oplus 4} \oplus I(6)^{\oplus 2} \oplus I(7)^{\oplus 4}$$
$$\to I(1) \oplus I(2) \oplus I(4)^{\oplus 2} \to I(3) \to 0,$$

and

$$0 \to {}_RR \to I'(0)^{\oplus 8} \oplus I'(3)^{\oplus 4} \to I'(1) \oplus I'(2)^{\oplus 3} \oplus I'(4)^{\oplus 5} \oplus I'(7)$$
$$\to I'(5) \oplus I'(6)^{\oplus 2} \oplus I'(7) \oplus I'(9) \to I'(8) \to 0.$$

Proof. (1) It is straightforward to verify these equations by constructing the minimal projective resolution or injective coresolution of each corresponding module. For example, for the indecomposable injective right R-module $I(0)$, its minimal projective resolution is

$$0 \to P(1) \to P(0) \to I(0) \to 0,$$

and so $\mathrm{pd}_R I(0) = 1$.

(2) According to (1), we get (2) immediately by using the usual duality $\mathrm{Hom}_{\Bbbk}(-, \Bbbk)$.

Assertion (3) follows directly from (1) or (2). Assertion (4) follows by a direct calculation. □

Lemma 2.2. *It holds that*

$$(\operatorname{grade} S(i))_{i=0}^{9} = (1,2,2,3,2,1,1,1,0,0), \tag{2.9}$$

$$(\operatorname{grade} S'(i))_{i=0}^{9} = (0,1,1,0,1,2,2,1,3,2). \tag{2.10}$$

Proof. By [5, Lemma 1], we have

$$\operatorname{Ext}_R^t(S(i), R) \cong \operatorname{Hom}_R(S(i), E^t(R_R))$$

for any $0 \leqslant i \leqslant 9$ and $t \geqslant 0$. Thus, by Lemma 2.1(4), for $S(0)$, we have

$$\operatorname{Hom}_R(S(0), R) \cong \operatorname{Hom}_R(S(0), E^0(R_R)) \cong \operatorname{Hom}_R(S(0), I(8)^{\oplus 8} \oplus I(9)^{\oplus 4}),$$

and

$$\operatorname{Ext}_R^1(S(0), R) \cong \operatorname{Hom}_R(S(0), E^1(R_R)) \cong \operatorname{Hom}_R(S(0), I(0) \oplus I(5)^{\oplus 4} \oplus I(6)^{\oplus 2} \oplus I(7)^{\oplus 4}).$$

Since $\operatorname{Hom}_R(S(0), I(8)) = 0$ and $\operatorname{Hom}_R(S(0), I(9)) = 0$, we have $\operatorname{Hom}_R(S(0), R) = 0$. Since $\operatorname{Hom}_R(S(0), I(0)) \neq 0$, we have $\operatorname{Ext}_R^1(S(0), R) \neq 0$. Thus, $\operatorname{grade} S(0) = 1$. Similarly, we may compute the grades of other simple right R-modules and simple left R-modules, and obtain (2.9) and (2.10). □

By Lemma 2.1, we have that R is Auslander-regular with $\operatorname{gl.dim} R = 3$. Moreover, for any $0 \leqslant i \leqslant 3$ and any non-zero direct summand I of $E^i(R_R)$, one can check that $\operatorname{pd}_R I = i$. It follows that R is right diagonal Auslander-regular. On the other hand, by Lemma 2.1(4)(2), the indecomposable injective left R-module $I'(7)$ occurs as a direct summand of $E^2({}_R R)$, but $\operatorname{pd}_{R^{\mathrm{op}}} I'(7) = 1$. It means that R is not left diagonal Auslander-regular. Consequently, diagonal Auslander-regularity is not left-right symmetric.

By Lemmas 2.1 and 2.2, we have (2.3) = (2.9), that is,

$$\operatorname{pd}_R S(i) = \operatorname{grade} S(i) \text{ for any } 0 \leqslant i \leqslant 9.$$

By Lemmas 2.1 and 2.2 again, we have

$$(2.7) = (0,1,1,0,1,2,2,2,3,2) \neq (0,1,1,0,1,2,2,1,3,2) = (2.10),$$

To be precise, we have

$$\operatorname{pd}_{R^{\mathrm{op}}} S'(7) = 2 \neq 1 = \operatorname{grade} S'(7).$$

Therefore, the algebra R shows that condition $(*)$ is not left-right symmetric when $n \geqslant 2$.

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