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Gorenstein flatness and injectivity over Gorenstein rings

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Abstract Let R be a Gorenstein ring. We prove that if I is an ideal of R such that R/I is a semi-simple ring, then the Gorenstein flat dimension of R/I as a right R-module and the Gorenstein injective dimension of R/I as a left R-module are identical. In addition, we prove that if $R \to S$ is a homomorphism of rings and ${}_SE$ is an injective cogenerator for the category of left S-modules, then the Gorenstein flat dimension of S as a right R-module and the Gorenstein injective dimension of E as a left R-module are identical. We also give some applications of these results.

Keywords: Gorenstein flat, Gorenstein injective, Gorenstein rings

MSC(2000): 16E10, 16E30

1 Introduction

In classical homological algebra, the projective, injective and flat dimensions of modules are important and fundamental research objects. As a generalization of the notion of projective dimension of modules, Auslander and Bridger introduced in [1] the notion of G-dimension for finitely generated R-modules over a two-sided Noetherian ring R. For any module over a general ring, Enochs and Jenda introduced in [2] the notions of Gorenstein projective dimension and Gorenstein injective dimension; and furthermore, Enochs, Jenda and Torrecillas introduced in [3] the notion of Gorenstein flat dimension. In the recent years, these Gorenstein homological dimensions have become a vigorously active area of research. In particular, Holm in [4] gave some nice characterizations of them.

Note that Xu proved in [5] that for a commutative ring R, a simple R-module is flat if and only if it is injective. Furthermore, Ding and Chen proved in [6] that if R is a commutative coherent ring, then the flat dimension and injective dimension of any simple R-module are identical.

Throughout this paper, R is a Gorenstein ring (that is, R is a two-sided Noetherian ring with finite left and right self-injective dimensions) and all modules considered are unitary. We prove that if I is an ideal of R such that R/I is a semi-simple ring, then the Gorenstein flat dimension of R/I as a right R-module and the Gorenstein injective dimension of R/I as a left R-module are identical. As an immediate consequence of this result, we have that if R is commutative,

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then the Gorenstein flat dimension and Gorenstein injective dimension of any simple R-module T are identical. In particular, T is Gorenstein flat if and only if it is Gorenstein injective. In addition, we prove that if $R \to S$ is a homomorphism of rings and ${}_S E$ is an injective cogenerator for the category of left S-modules, then the Gorenstein flat dimension of S as a right R-module and the Gorenstein injective dimension of E as a left R-module are identical. As an application, we have that if S is an Artinian algebra with the center R, then, as R-modules, the Gorenstein flat dimension of S and the Gorenstein injective dimension of $\mathbb{D}(S)$ are identical, where $\mathbb{D}$ is the usual duality of S.

2 Preliminaries

In this section, we give some definitions and notations and collect some preliminary results which are often used in Section 3.

We use $\mathrm{Mod}R$ (resp. $\mathrm{Mod}R^{\mathrm{op}}$) to denote the category of left (resp. right) R-modules. A module $M \in \mathrm{Mod}R$ is called Gorenstein injective if there exists an exact sequence in $\mathrm{Mod}R$ with all terms injective: $\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$, such that $M = \mathrm{Ker}(I^0 \to I^1)$ and the sequence is still exact after applying the functor $\mathrm{Hom}_R(I, -)$ for any injective left R-module I. The (left) Gorenstein injective dimension of M, denoted by $\mathrm{l.Gid}_R(M)$, is defined as $\inf\{n|$ there exists an exact sequence $0 \to M \to Q^0 \to Q^1 \to \cdots \to Q^n \to 0$ with Q^i Gorenstein injective for any $0 \leqslant i \leqslant n\}$ (see [2, 4]).

Lemma 2.1. *Let M be a module in* $\mathrm{Mod}R$ *and n be a non-negative integer. Then the following statements are equivalent*:

(1) $\mathrm{l.Gid}_R(M) \leqslant n$;

(2) $\mathrm{Ext}^i_R(Q, M) = 0$ *for any $i \geqslant n+1$ and injective left R-module Q.*

Proof. Because R is Gorenstein, any module in $\mathrm{Mod}R$ has finite Gorenstein injective dimension by [7, Theorem 3.2]. Then by [4, Theorem 2.22], we get the assertion.

A module $N \in \mathrm{Mod}R^{\mathrm{op}}$ is called Gorenstein flat if there exists an exact sequence in $\mathrm{Mod}R^{\mathrm{op}}$ with all terms flat: $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$, such that $N = \mathrm{Im}(F_0 \to F^0)$ and the sequence is still exact after applying the functor $- \bigotimes_R I$ for any injective left R-module I. The (right) Gorenstein flat dimension of N, denoted by $\mathrm{r.Gfd}_R(N)$, is defined as $\inf\{n|$there exists an exact sequence $0 \to P_n \to \cdots \to P_1 \to P_0 \to N \to 0$ with P_i Gorenstein flat for any $0 \leqslant i \leqslant n\}$ (see [3, 4]).

Lemma 2.2. *Let N be a module in* $\mathrm{Mod}R^{\mathrm{op}}$ *and n be a non-negative integer. Then the following statements are equivalent*:

(1) $\mathrm{r.Gfd}_R(N) \leqslant n$;

(2) $\mathrm{Tor}^R_i(N, Q) = 0$ *for any $i \geqslant n+1$ and injective left R-module Q.*

Proof. Because R is Gorenstein, any module in $\mathrm{Mod}R^{\mathrm{op}}$ has finite Gorenstein flat dimension by [8, Proposition 1.6]. Then by [4, Theorem 3.14], we get the assertion.

Lemma 2.3 ([9, ChapterVI, Proposition 5.1])**.** *Let R and S be rings (not necessarily Gorenstein) with the condition $({}_R A, {}_S B_R, {}_S C)$. If ${}_S C$ is injective, then we have the isomorphism of Abelian groups for any $n \geqslant 1$:* $\mathrm{Ext}^n_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(\mathrm{Tor}^R_n(B, A), C)$.

3 Main results

For any $M \in \mathrm{Mod}R$ (resp. $\mathrm{Mod}R^{\mathrm{op}}$), $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers.

The following lemma plays a crucial role in the proof of the first main result in this paper.

Lemma 3.1. *Let R be a ring (not necessarily Gorenstein) and I be an ideal of R such that R/I is a semi-simple ring. Then for any $M \in \mathrm{Mod}R$ and a positive integer n, $\mathrm{Ext}_R^n(M, R/I) = 0$ if and only if $\mathrm{Tor}_n^R(R/I, M) = 0$.*

Proof. Assume that $\mathrm{Ext}_R^n(M, R/I) = 0$. By Lemma 2.3, we have an isomorphism:

$$\mathrm{Ext}_R^n(M, (R/I)^+) \cong (\mathrm{Tor}_n^R(R/I, M))^+, \tag{1}$$

where R/I is regarded as a right R-module. Since $I(R/I)^+ = 0$, $(R/I)^+$ is a semi-simple left R-module and $(R/I)^+ \cong \bigoplus_{i\in\Gamma} S_i$, where each S_i is a simple left R-module and Γ is an index set. Since R/I is semi-simple, each S_i is a direct summand of R/I.

Since $\mathrm{Ext}_R^n(M, R/I) = 0$, $\mathrm{Ext}_R^n(M, S_i) = 0$ for any $i \in \Gamma$. So $\mathrm{Ext}_R^n(M, \prod_{i\in\Gamma} S_i) \cong \prod_{i\in\Gamma} \mathrm{Ext}_R^n(M, S_i) = 0$. But $\prod_{i\in\Gamma} S_i$ is also a semi-simple module, so $(R/I)^+ \cong \bigoplus_{i\in\Gamma} S_i$ is isomorphic to a direct summand of $\prod_{i\in\Gamma} S_i$. Thus $\mathrm{Ext}_R^n(M, (R/I)^+) = 0$ and therefore $(\mathrm{Tor}_n^R(R/I, M))^+ = 0$ by (1). Since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator for $\mathrm{Mod}\mathbb{Z}$, $\mathrm{Tor}_n^R(R/I, M) = 0$.

Conversely, assume that $\mathrm{Tor}_n^R(R/I, M) = 0$. By Lemma 2.3, we have an isomorphism:

$$\mathrm{Ext}_R^n(M, (R/I)^{++}) \cong (\mathrm{Tor}_n^R((R/I)^+, M))^+, \tag{2}$$

where R/I is regarded as a left R-module. Since $(R/I)^+ I = 0$, $(R/I)^+$ is a semi-simple right R-module and $(R/I)^+ \cong \bigoplus_{j\in\Gamma'} S'_j$, where each S'_j is a simple right R-module and Γ' is an index set. Since R/I is semi-simple, each S'_j is a direct summand of R/I.

Since $\mathrm{Tor}_n^R(R/I, M) = 0$, $\mathrm{Tor}_n^R(S'_j, M) = 0$ for any $j \in \Gamma'$ and $\mathrm{Tor}_n^R((R/I)^+, M) = 0$. So $\mathrm{Ext}_R^n(M, (R/I)^{++}) = 0$ by (2). On the other hand, by [10, p. 48], the canonical evaluation homomorphism $R/I \to (R/I)^{++}$ is a monomorphism. Since $I(R/I)^{++} = 0$, $(R/I)^{++}$ is also a semi-simple left R-module and R/I is isomorphic to a direct summand of $(R/I)^{++}$. Thus $\mathrm{Ext}_R^n(M, R/I) = 0$.

The following is one of main results in this paper.

Theorem 3.2. *If I is an ideal of R such that R/I is a semi-simple ring, then* $\mathrm{r.Gfd}_R(R/I) = \mathrm{l.Gid}_R(R/I)$.

Proof. We first prove $\mathrm{l.Gid}_R(R/I) \leqslant \mathrm{r.Gfd}_R(R/I)$. Without loss of generality, suppose $\mathrm{r.Gfd}_R(R/I) = m < \infty$. Then, for any $i \geqslant m+1$ and injective left R-module Q, we have that $\mathrm{Tor}_i^R(R/I, Q) = 0$ by Lemma 2.2. So $\mathrm{Ext}_R^i(Q, R/I) = 0$ by Lemma 3.1. It follows from Lemma 2.1 that $\mathrm{l.Gid}_R(R/I) \leqslant m$. This proves $\mathrm{l.Gid}_R(R/I) \leqslant \mathrm{r.Gfd}_R(R/I)$.

We next prove the converse inequality. Without loss of generality, suppose $\mathrm{l.Gid}_R(R/I) = n < \infty$. Then, for any $i \geqslant n+1$ and injective left R-module Q, we have that $\mathrm{Ext}_R^i(Q, R/I) = 0$ by Lemma 2.1. So $\mathrm{Tor}_i^R(R/I, Q) = 0$ by Lemma 3.1. It follows from Lemma 2.2 that $\mathrm{r.Gfd}_R(R/I) \leqslant n$. This proves $\mathrm{r.Gfd}_R(R/I) \leqslant \mathrm{l.Gid}_R(R/I)$.

Recall that R is called a semi-local ring if $R/J(R)$ is a semi-simple ring, where $J(R)$ is the Jacobson radical of R. By Theorem 3.2, we immediately have the following

Corollary 3.3. *If R is a semi-local ring, then* $\mathrm{r.Gfd}_R(R/J(R)) = \mathrm{l.Gid}_R(R/J(R))$.

Corollary 3.4. *If R is a commutative ring, then for any simple R-module T,* $\mathrm{Gfd}_R(T) = \mathrm{Gid}_R(T)$; *in particular, T is Gorenstein flat if and only if it is Gorenstein injective.*

Proof. Let T be a simple R-module. Then there exists a maximal ideal $\mathfrak{m}$ of R such that $T \cong R/\mathfrak{m}$. So T is a simple ring and hence the assertion follows from Theorem 3.2.

The following is another main result in this paper.

Theorem 3.5. *Let $R \to S$ be a homomorphism of rings. If ${}_SE$ is an injective cogenerator for* Mod$S$, *then* $\mathrm{r.Gfd}_R(S) = \mathrm{l.Gid}_R(E)$.

Proof. It is trivial that S can be regarded as a bimodule ${}_SS_R$, and ${}_SE$ can be regarded as a left R-module.

Let Q be any injective left R-module. Then by Lemma 2.3, for any $n \geqslant 1$, we have that $\mathrm{Ext}^n_R(Q, \mathrm{Hom}_S(S, E)) \cong \mathrm{Hom}_S(\mathrm{Tor}^R_n(S, Q), E)$. So $\mathrm{Ext}^n_R(Q, E) \cong \mathrm{Hom}_S(\mathrm{Tor}^R_n(S, Q), E)$. Because ${}_SE$ is an injective cogenerator for Mod$S$, $\mathrm{Ext}^n_R(Q, E) = 0$ if and only if $\mathrm{Tor}^R_n(S, Q) = 0$. It follows easily from Lemmas 2.1 and 2.2 that $\mathrm{r.Gfd}_R(S) = \mathrm{l.Gid}_R(E)$.

Finally, we give some applications of Theorem 3.5.

Corollary 3.6. *Let $R \to S$ be a homomorphism of rings. Then we have*

(1) $\mathrm{r.Gfd}_R(S) = \mathrm{l.Gid}_R(S^+)$.

(2) *If S is a QF ring* (*that is, S is a two-sided Noetherian ring and S is self-injective*), *then* $\mathrm{r.Gfd}_R(S) = \mathrm{l.Gid}_R(S)$.

Proof. By [10, Chapter I, Proposition 9.3], S^+ is an injective cogenerator for Mod$S$. On the other hand, it is well known that ${}_SS$ is an injective cogenerator for Mod$S$ if S is a QF ring. Thus both assertions follow immediately from Theorem 3.5.

Recall from [11] that an algebra S is called an Artinian algebra if S is a two-sided Artinian ring and S is finitely generated as an R-module, where R is the center of S. It is well known that if S is an Artinian algebra, then its center R is a commutative Artinian ring. We use $\mathbb{D}$ to denote the usual duality of S, that is, $\mathbb{D} = \mathrm{Hom}_R(\ , R/J(R))$.

Corollary 3.7. *If S is an Artinian algebra with the center R, then* $\mathrm{Gfd}_R(S) = \mathrm{Gid}_R(\mathbb{D}(S))$.

Proof. Because S is an Artinian algebra, $\mathbb{D}(S)$ is an injective cogenerator for Mod$S$. So we get the assertion from Theorem 3.5.

References

1 Auslander M, Bridger M. Stable Module Theory. Memoirs Amer Math Soc, Vol 94. Providence: Amer Math Soc, 1969

2 Enochs E E, Jenda O M G. Gorenstein injective and projective modules. *Math Z*, **220**(4): 611–633 (1995)

3 Enochs E E, Jenda O M G, Torrecillas B. Gorenstein flat modules. *J Nanjing Univ Math Biquarterly*, **10**(1): 1–9 (1993)

4 Holm H. Gorenstein homological dimensions. *J Pure Appl Algebra*, **189**(1-3): 167–193 (2004)

5 Xu J Z. Flatness and injectivity of simple modules over a commutative ring. *Comm Algebra*, **19**(2): 535–537 (1991)

6 Ding N Q, Chen J L. The homological dimensions of simple modules. *Bull Austral Math Soc*, **48**(2): 265–274 (1993)

7 Enochs E E, Jenda O M G. Gorenstein balance of Hom and tensor. *Tsukuba J Math*, **19**(1): 1–13 (1995)

8 Enochs E E, Jenda O M G. Gorenstein flat preenvelopes and resolvents. *J Nanjing Univ Math Biquarterly*, **12**(1): 1–9 (1995)

9 Cartan H, Eilenberg S. Homological Algebra. Reprint of the 1956 original, Princeton Landmarks in Math. Princeton: Princeton University Press, 1999

10 Stenström B. Rings of Quotients. Die Grund der Math Wissen in Einz, Vol. 217. Berlin: Springer-Verlag, 1975

11 Auslander M, Reiten I. Representation theory of Artin algebras III. Almost split sequences. *Comm Algebra*, **3**(3): 239–294 (1975)