HOMOLOGICAL DIMENSIONS OVER ALMOST GENTLE ALGEBRAS

DEMIN WANG, ZHAOYONG HUANG, AND YU-ZHE LIU*

ABSTRACT. We provide a method for computing the global dimension and self-injective dimension of almost gentle algebras, and prove that an almost gentle algebra is Gorenstein if it satisfies the Auslander condition.

CONTENTS

1. INTRODUCTION

In [2], Assem and Skowroński introduced gentle algebras and use them to study tilted algebras and derived equivalence. Afterwards, (skew-)gentle algebras play an important role in representation theory; and theirs module categories and derived categories were described by using string algebras, Auslander–Reiten theory, combinatorial methods and geometric models, see [1, 7–9, 20, 21, 23] and so on. In particular, the global dimension and self-injective dimension of (locally) gentle algebras have been studied, and (locally) gentle algebras were shown to be Gorenstein by using different methods [12, 13, 17].

As a generalization of gentle algebras, almost gentle algebras are important finite-dimensional algebras introduced by Green and Schroll [15], which are monomial special

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multiserial algebras, and are closely related to some hypergraphs and Brauer configuration algebras. In [11] and [16], complexes on almost gentle algebras and the trivial extension of almost gentle algebras were studied, respectively. Based on the above, we have the following questions.

Question 1.1.

(1) *How to characterize the global dimension and self-injective dimension of almost gentle algebras?*
(2) *Whether or when are almost gentle algebras Gorenstein?*

Let $\Bbbk$ be an algebraically closed field. In this paper, we provide a method for computing the global dimension and self-injective dimension of almost gentle algebras by using forbidden paths introduced by Avella-Alaminos and Geiß [5] (see Section 3.1). We define claw, written as κ, and anti-claw, written as ξ (Definition 3.1), to describe indecomposable projective modules and injective modules, and introduce tow sets $\mathcal{F}(v)$ and $\mathcal{F}(\xi)$ whose elements are some special forbidden paths on the bound quiver $(\mathcal{Q},\mathcal{I})$ of an almost gentle algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$, which are decided by some claws, where v is any vertex of the quiver $\mathcal{Q}$, see Notations 3.11, 4.15, and Definition 4.18. By [14, Corollary 2.4], we have that the finitistic dimension conjecture holds true for monomial algebras. Observe that an algebra is monomial if and only if so is its opposite algebra. It then follows from [3, Proposition 6.10] that the left and right self-injective dimensions of a monomial algebra, and hence, of an almost gentle algebra A, are identical. For simplicity, we call this common quantity the *self-injective dimension* of A. The global dimension $\mathrm{gl.dim}A$ and self-injective dimension $\mathrm{inj.dim}A$ of an almost gentle algebra A can be described by the following results.

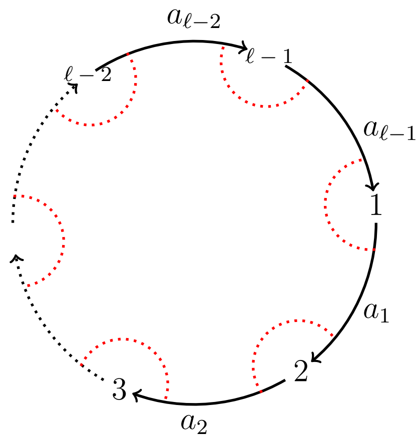
Theorem 1.2.

(1) (Theorem 3.13)
$$\mathrm{gl.dim}A = \sup_{F\in\mathcal{F}} \ell(F),$$
where $\mathcal{F} := \bigcup_{v\in\mathcal{Q}_0} \mathcal{F}(v)$ *and* $\ell(F)$ *is the length of* F*.*
(2) (Theorem 4.20)
$$\mathrm{inj.dim}A = \sup_{F\in\mathcal{F}_{\mathrm{a}}} \ell(F),$$
where $\mathcal{F}_{\mathrm{a}} := \bigcup_{v\in\mathcal{Q}_0} \mathcal{F}(\xi_v)$ *with* ξ_v *the anti-claw corresponding to the indecomposable injective module* $E(v)$*.*

Theorem 1.2(2) provides a method for judging the Gorensteiness of almost gentle algebras as follows.

Theorem 1.3. *For an almost gentle algebra* $A = \Bbbk\mathcal{Q}/\mathcal{I}$*, the following statements are equivalent.*

(1) $\mathrm{inj.dim}A = \infty$.
(2) (A direct consequence of Theorem 4.20) *There exists an anti-claw* ξ *corresponding to some indecomposable injective module such that* $\mathcal{F}(\xi)$ *contains a forbidden cycle whose length is infinite.*
(3) (Theorem 5.4) $(\mathcal{Q},\mathcal{I})$ *contains an oriented cycle* $\mathscr{C} = a_0a_1\cdots a_{\ell-1} =$



such that $a_{\overline{i}}a_{\overline{i+1}} \in \mathcal{I}$ (for any $x \in \mathbb{N}$, $\overline{x}$ is defined as x modulo ℓ), and there is a vertex v on $\mathscr{C}$ such that one of the following condition holds.
(A) *there is an arrow α ($\neq a_{\overline{v-1}}$) ending at v satisfying $\alpha a_v \in \mathcal{I}$;*
(B) *there is an arrow β ($\neq a_v$) starting at v satisfying $a_{\overline{v-1}}\beta \in \mathcal{I}$.*
The oriented cycle as above is said to be a forbidden cycle.

(4) (Theorem 5.1(1)) *There is a vertex v on some forbidden cycle $\mathscr{C}$ such that v is not invalid* (*see Definition 4.11 for the definition of invalid vertices*).

Recall that a left and right Noetherian ring R is called *Gorenstein* if its left and right self-injective dimensions are finite, and R is said to satisfy the *Auslander condition* if the flat dimension of the i-th term in a minimal injective coresolution of R as a left R-module is at most $i-1$ for any $i \geqslant 1$. The Auslander condition is left and right symmetric [10, Theorem 3.7]. Bass [6] proved that a commutative Noetherian ring R is Gorenstein if and only if it satisfied the Auslander condition. Based on it, Auslander and Reiten [4] conjectured that an Artin algebra is Gorenstein if it satisfies the Auslander condition. We call this conjecture **ARC** for short. It is situated between the well known Nakayama conjecture and the generalized Nakayama conjecture [4, p.2]. All these conjectures remain still open. As an application of the above results, we obtain the following result.

Theorem 1.4.

(1) (Corollary 5.6) *For an almost gentle algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$, the projective dimension of the injective envelope of A_A is infinite.*
(2) (Theorem 5.7) **ARC** *holds true for almost gentle algebras.*

2. Preliminaries

Throughout this paper, assume that $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, \mathfrak{s}, \mathfrak{t})$ is a finite connected quiver. Here, $\mathcal{Q}_0$ and $\mathcal{Q}_1$ respectively are the vertex and arrow sets of $\mathcal{Q}$, and $\mathfrak{s}$ and $\mathfrak{t}$ are functions from $\mathcal{Q}_1$ to $\mathcal{Q}_0$ respectively sending each arrow to its source and sink. For arbitrary two arrows α and β of the quiver $\mathcal{Q}$, if $\mathfrak{t}(\alpha) = \mathfrak{s}(\beta)$, then the composition of α and β is denoted by $\alpha\beta$. For any $v \in \mathcal{Q}_0$, we use e_v to denote the path of length zero corresponding to v. For a path on $\mathcal{Q}$, we use $\ell(p) = n$ to denote the length of p. Let A be a finite-dimensional algebra. We use $\mathsf{mod}(A)$ to denote the category of finitely generated right A-modules. For arbitrary two modules M and N, if M is a direct summand of N, then we write $M \leq_{\oplus} N$. Moreover, we use $S(v_0), P(v_0)$ and $E(v_0)$ to denote the simple, indecomposable projective and indecomposable injective modules corresponding to the vertex $v_0 \in \mathcal{Q}_0$, respectively.

2.1. **Almost gentle algebras.** Recall from [15] that a bound quiver $(\mathcal{Q},\mathcal{I})$ is called an *almost gentle pair* if it satisfies the following conditions:

(1) I, the admissible ideal of the path algebra $\Bbbk\mathcal{Q} := \mathrm{span}_{\Bbbk}(\mathcal{Q}_l \mid l \in \mathbb{N})$, is a subspace of the $\Bbbk$-linear space $\Bbbk\mathcal{Q}$ which is generated by some paths on $\mathcal{Q}$ of length two;

(2) for any arrow $a \in \mathcal{Q}_1$, there is at most one arrow $b \in \mathcal{Q}_1$ such that $ab \notin \mathcal{I}$ and at most one arrow $c \in \mathcal{Q}_1$ such that $ca \notin \mathcal{I}$.

Definition 2.1. [15] A finite-dimensional algebra A is called an *almost gentle algebra* if its bound quiver is an almost gentle pair.

Example 2.2. Let $\mathcal{Q}$ be the quiver shown in Figure 2.1, and let $\mathcal{I}$ be the admissible ideal of $\Bbbk\mathcal{Q}$ given by the paths $a_{1,2_{\mathrm{L}}}a_{2_{\mathrm{L}},3_{\mathrm{L}}}$, $a_{2,3}a_{3,4}$, $a_{1,2}a_{2,3}$, $a_{3,4}a_{4,5}$, $a_{2,3'}a_{3',4}$, $a_{2_{\mathrm{R}},3_{\mathrm{R}}}a_{3_{\mathrm{R}},5}$, $a_{1,2}a_{2,3'}$, $a_{3',4}a_{4,5}$, $a_{1,2}a_{2,4}$, $a_{3_{\mathrm{L}},4_{\mathrm{L}}}b_{4_{\mathrm{L}},5}$, $b_{1,2_{\mathrm{R}}}a_{2_{\mathrm{R}},3_{\mathrm{R}}}$. Then $(\mathcal{Q},\mathcal{I})$ is an almost gentle pair

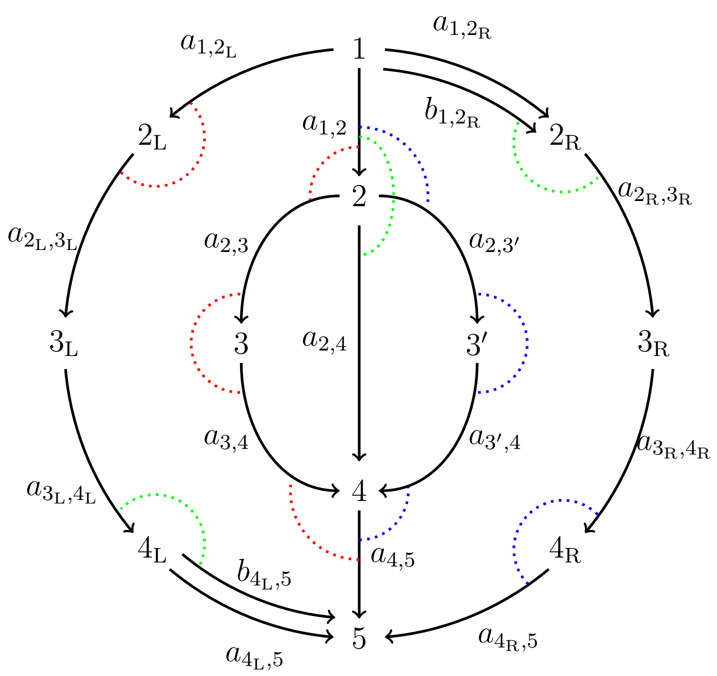


Figure 2.1. An almost gentle pair

and $A = \Bbbk\mathcal{Q}/\mathcal{I}$ is an almost gentle algebra.

2.2. **Strings and string modules.** For any arrow $a \in \mathcal{Q}_1$, Butler and Ringel [9] introduced the *formal inverse* a, written as a^{-1}, and, naturally, define $\mathfrak{t}(a^{-1}) = \mathfrak{s}(a)$ and $\mathfrak{s}(a^{-1}) = \mathfrak{t}(a)$. We denote by $\mathcal{Q}_1^{-1} = \{a^{-1} \mid a \in \mathcal{Q}_1\}$ the set of all formal inverses of arrows. Then any path $p = a_1a_2\cdots a_n$ in $(\mathcal{Q}, I)$ naturally provides a formal inverse path $p^{-1} = a_n^{-1}a_{n-1}^{-1}\cdots a_1^{-1}$ of p. For any path p, we define $(p^{-1})^{-1} = p$, and for any path e_v of length zero corresponding to $v \in \mathcal{Q}_0$, we define $e_v^{-1} = e_v$.

In a bound quiver $(\mathcal{Q},\mathcal{I})$ of a finite-dimensional algebra A, a *string* (of length m) on $(\mathcal{Q},\mathcal{I})$ is a sequence $s = (a_1, \ldots, a_m)$ such that

- each a_i $(1 \leqslant i \leqslant m)$ is either an arrow or a formal inverse of the arrow;
- if $a_i \in \mathcal{Q}_1$ and $a_{i+1} \in \mathcal{Q}_1^{-1}$, then $a_i \notin a_{i+1}^{-1}$;
- $\mathfrak{t}(a_i) = \mathfrak{s}(a_{i+1})$ holds for all $1 \leqslant i \leqslant n-1$.

In particular, a string of length zero is said to be a *simple string*; and a string which is one of the forms $\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$ and $\bullet \longleftarrow \bullet \longleftarrow \cdots \longleftarrow \bullet$ (length $\geqslant 0$) is said to be a *directed string*; and we define 0, the zero vector in $\Bbbk\mathcal{Q}$, is also a string which is said to be *a trivial string*.

Two strings s_1 and s_2 are called *equivalent* if $s_1 = s_2^{-1}$ or $s_1 = s_2$.

We denote by $\mathrm{Str}(A)$ the set of all equivalent classes of strings. Now, similar to [9], we define a string module with respect to a given string.

Definition 2.3 (String modules)**.** Suppose that $s = a_1a_2\cdots a_n$ is a string on $\mathcal{Q}$, where $a_1,\ldots,a_n$ are arrows such that $\mathfrak{s}(a_i) = v_i$ $(1 \leqslant i \leqslant n)$ and $\mathfrak{t}(a_n) = v_{n+1}$. It induces a *string module*, denote it by $\mathbb{M}(s)$, which satisfies the following conditions:

- for $v \in \mathcal{Q}_0$, we have that $\dim_k(\mathbb{M}(s)e_v)$ equals to the multiplicities of s transverses v;
- any $a_i \in \mathcal{Q}_1$ on s provides an identity from $\Bbbk_i$ to $\Bbbk_{i+1}$, where $\Bbbk_i$ and $\Bbbk_{i+1}$ are copies of $\Bbbk$ which, as two $\Bbbk$-vector spaces, are direct summands of $\mathbb{M}(s)e_{v_i}$ and $\mathbb{M}(s)e_{v_{i+1}}$, respectively;
- any arrow $a \in \mathcal{Q}_1$ which does not on s provides a zero action $\mathbb{M}(s)a = 0$.

For a directed string $s = a_1a_2\cdots a_n$ with $a_1, a_2, \ldots, a_n \in \mathcal{Q}_1$, if either $\mathfrak{t}(s)$ is a sink or for each arrow α with $\mathfrak{s}(\alpha) = \mathfrak{t}(s)$, and the concatenation $s\alpha = 0$, then s is said to be a *right maximal directed string*. Dually, a *left maximal directed string* is defined. Furthermore, the string module induced by a (right maximal) directed string s is said to be a (*right maximal*) *directed string module*.

Remark 2.4. (1) A *band* (of length m) is a string $b = (b_1, \ldots, b_m)$ with $\mathfrak{t}(b_m) = \mathfrak{s}(b_1)$ such that

- b^2 is a string of length $2m$;
- b is not a non-trivial power of some string, that is, there is no string s such that $b = s^n$ for some $n \geqslant 2$.

Two bands $b = b_1\cdots b_m$ and $b' = b'_1\cdots b'_m$ are called *equivalent* if either $b[t] = b'$ or $b[t]^{-1} = b'$, where $b[t] = b_{1+t}b_{2+t}\cdots b_{n-1}b_nb_1\cdots b_t$. We denote by $\mathrm{Ban}(A)$ the set of all equivalent classes of bands on the bound quiver of A.

Suppose that $b = b_1b_2\cdots b_n$ is a band on $\mathcal{Q}$, where $\mathfrak{s}(a_i) = v_i$ $(1 \leqslant i \leqslant n)$. For any $u \in \mathbb{N}^+$ and $0 \neq \lambda \in \Bbbk$, it induces a *band module*, denote it by $\mathbb{M}(s, J_n(\lambda))$, which satisfies the following conditions:

- for $v \in \mathcal{Q}_0$, we have $\dim_k(\mathbb{M}(s)e_v) = tu$, where u is the multiplicities of s transverses v;
- any $b_i \in \mathcal{Q}_1$ with $1 < i \leqslant n$ provides an identity from $\Bbbk_i^{\oplus u}$ to $\Bbbk_{i+1}^{\oplus u}$, where $\Bbbk_i$ and $\Bbbk_{i+1}$ are copies of $\Bbbk$ which, as two $\Bbbk$-vector spaces, are direct summands of $\mathbb{M}(s, J_n(\lambda))e_{v_i}$ and $\mathbb{M}(s, \boldsymbol{J}_n(\lambda))e_{v_{i+1}}$, respectively;
- the arrow b_1 provides a $\Bbbk$-linear map $\boldsymbol{J}_n(\lambda) := \begin{pmatrix} 1 & & & \\ \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{pmatrix}$ from $\Bbbk_1^{\oplus u}$ to $\Bbbk_2^{\oplus u}$;
- any right A-action $\mathbb{M}(s)a$ given by the arrow $a \in \mathcal{Q}_1$ does not on s is a zero action.

All indecomposable modules defined on a string algebra have been described in [9] by the following bijection:

$$\mathbb{M} : \mathrm{Str}(A) \cup (\mathrm{Ban}(A) \times \mathscr{J}) \to \mathrm{ind}(\mathsf{mod}(A)).$$

Here, $\mathscr{J}$ is the set of all Jordan blocks $J_n(\lambda)$ with $\lambda \neq 0$ and $\mathrm{ind}(\mathsf{mod}(A))$ is the set of all isoclasses of indecomposable modules over a string algebra A.

(2) Strings and bands originated from V-sequences and primitive V-sequences introduced by Wald and Waschbüsch [22]. (Primitive) V-sequences are used to describe the module categories of biserial algebras.

Example 2.5. Let $A = \Bbbk\mathcal{Q}/\mathcal{I}$ be the almost gentle algebra given in Example 2.2. Then $s_1 = a_{1,2_{\rm R}}b_{1,2_{\rm R}}^{-1}$ and $s_2 = a_{2,4}a_{4,5}a_{4_{\rm L},5}^{-1}b_{4_{\rm L},5}a_{4,5}^{-1}$ are strings, but they are not directed strings; and $s_3 = a_{2,4}a_{4,5}$ is a direct string. The string modules $\mathbb{M}(s_1)$, $\mathbb{M}(s_2)$ and $\mathbb{M}(s_3)$ are shown in Figure 2.2.

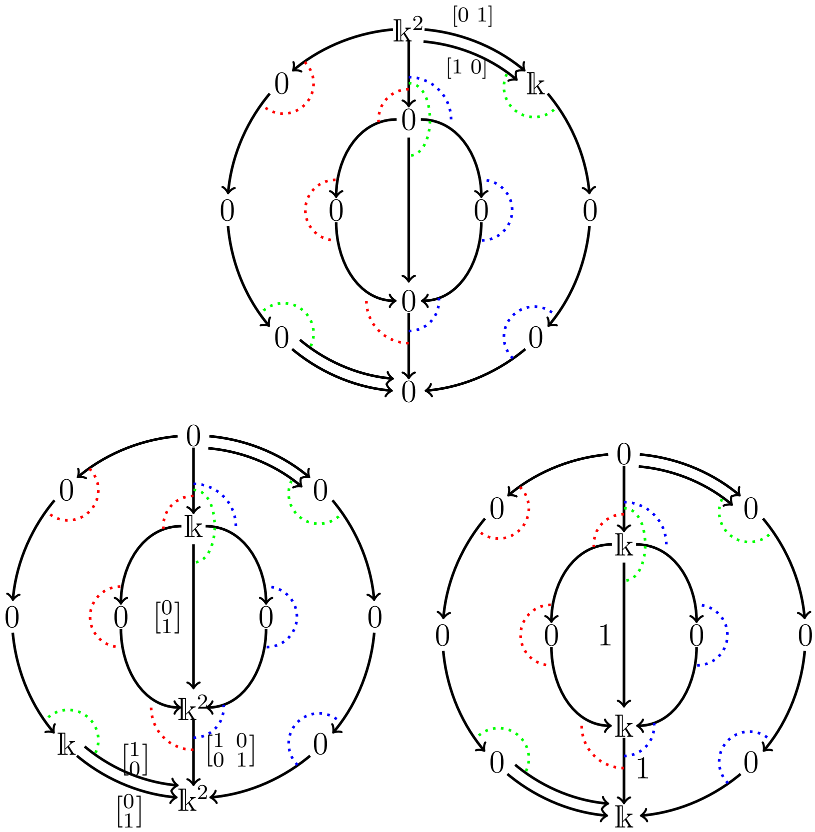


Figure 2.2. String modules

The string s_1 with a Jordan block $\boldsymbol{J}_n(\lambda)$ ($\lambda \neq 0$), i.e., the pair $(s_1, \boldsymbol{J}_n(\lambda)$, is a band, which describe the band module $\mathbb{M}(s_1, \boldsymbol{J}_n(\lambda))$ shown in Figure 2.3.

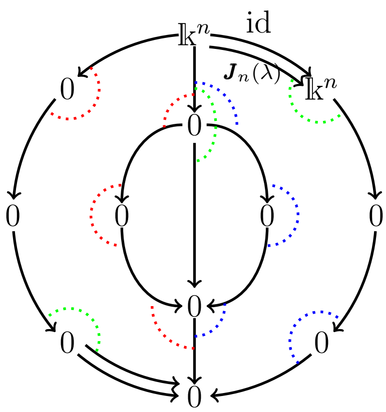


Figure 2.3. A band module

All indecomposable modules over a string algebra can be divided into two classes, one is a collection of all string modules and the other one is a collection of all band modules. But, it does not hold true over almost gentle algebras in general, see the indecomposable module shown in Figure 2.4.

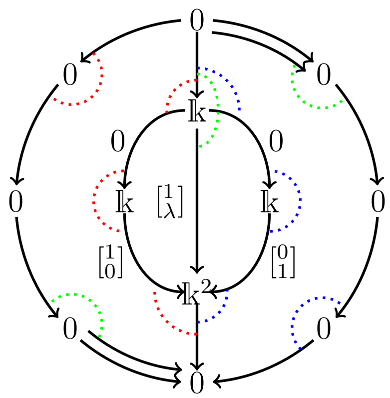


FIGURE 2.4. An indecomposable module which is neither string nor band ($\lambda \neq 0$)

3. GLOBAL DIMENSION OF ALMOST GENTLE ALGEBRAS

In this section, we provide a description of the global dimensions of almost gentle algebras. To do this, we need to compute the projective resolution of any simple module.

3.1. **Syzygies of directed string modules.**

Definition 3.1.

(1) A *claw* is a sequence of some directed strings $s_1, \cdots, s_n$, written as

$$s_1 \Wedge s_2 \Wedge \cdots \Wedge s_n,$$

whose sources coincide (cf. FIGURE 3.1). For simplicity, we use the symbol $s_1 \Wedge s_1 \Wedge \cdots \Wedge s_n$ to emphasize that $s_1, \ldots, s_n$ have the same sources.

(2) Dually, we can define that an *anti-claw* is a sequence of some directed strings $s_1, \cdots, s_n$, written as

$$s_1 \Vee s_2 \Vee \cdots \Vee s_n,$$

whose sinks coincide (cf. FIGURE 3.2). For simplicity, we use the symbol $s_1 \Vee s_2 \Vee \cdots \Vee s_n$ to emphasize that $s_1, \ldots, s_n$ have the same sink.

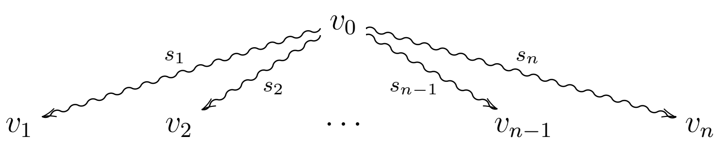


FIGURE 3.1. The claw $s_1 \Wedge s_2 \Wedge \cdots \Wedge s_n$ given by directed strings $s_1, s_2, \ldots, s_n$

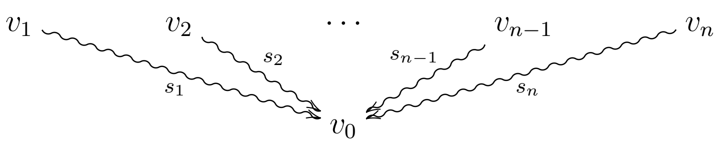


FIGURE 3.2. The anti-claw $s_1 \Vee s_2 \Vee \cdots \Vee s_n$ given by directed strings $s_1, s_2, \ldots, s_n$

From the definition of the almost gentle algebras, it is obvious that claws and anti-claws can be used to describe the indecomposable projective and injective modules, respectively. For any module M and $n \geqslant 1$, we use $\Omega_n(M)$ to denote the n-th syzygy of M, in particular, we write $\Omega_0(M) = M$.

Lemma 3.2. *Let A be an almost gentle algebra. Take M to be a directed string module given by some directed string δ. Then $\Omega_1(M)$ is a direct sum of some directed string modules, that is,*

$$\Omega_1(M) = \Omega_1(\mathbb{M}(\delta)) \cong \bigoplus_{i=1}^{r} \mathbb{M}(s_i) \tag{3.1}$$

for some directed strings $s_1, \ldots, s_r$.

Proof. First of all, assume that the source of δ is v_0. Then the top of $\mathbb{M}(\delta)$ is isomorphic to the simple module $S(v_0)$. It follows that the projective cover of $\mathbb{M}(\delta)$ is isomorphic to the indecomposable projective module $P(v_0)$. Now, let $s'_1 \wedge s'_2 \wedge \cdots \wedge s'_n$ be the claw corresponding to $P(v_0)$, and, for each $1 \leqslant t \leqslant n$, suppose $s'_t = a_{t,1}a_{t,2}\cdots a_{t,m_t}$ ($a_{t,1}$, $a_{t,2}$, $\ldots$, a_{t,m_t} are arrows). Then there are two integers $1 \leqslant j \leqslant n$ and $1 \leqslant l \leqslant m_j$ such that $\delta = a_{j,1}a_{j,2}\cdots a_{j,l}$. One can check that

$$\Omega_1(M) \cong \mathbb{M}(a_{j,l+1}\cdots a_{j,m_j}) \oplus \bigoplus_{\substack{1\leqslant t\leqslant n \\ t\neq j}} \mathbb{M}(a_{t,2}\cdots a_{t,m_t}).$$

Cf. FIGURE 3.3. In this case, it is easy to see that each indecomposable direct summand

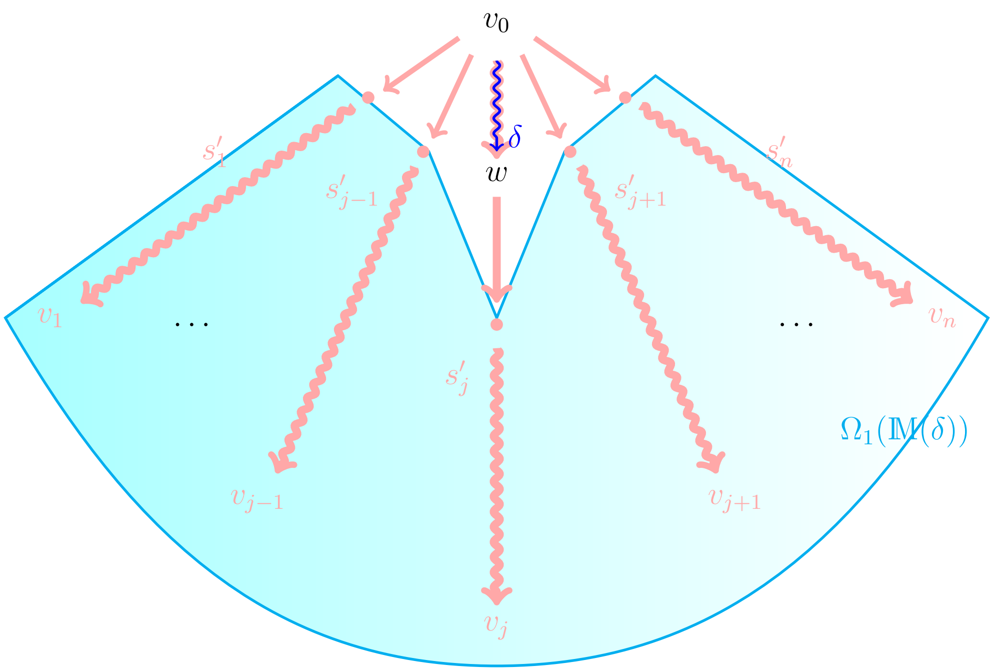


FIGURE 3.3. The projective cover $P(v_0)$ and 1-st syzygy $\Omega_1(\mathbb{M}(\delta))$ of $\mathbb{M}(\delta)$

of $\Omega_1(\mathbb{M}(\delta))$ is a directed string module. □

Lemma 3.3. *Keep the notations in Lemma 3.2. Then, for each $n \geqslant 1$, the n-th syzygy $\Omega_n(M)$ of M is a direct sum of some directed string modules.*

Proof. By Lemma 3.2, we have that $\Omega_1(M)$ is a direct sum of some directed string modules. It is clear that if $\Omega_n(M)$ is a direct sum of some directed string modules, then so is $\Omega_{n+1}(M) = \Omega_1(\Omega_n(M))$, thus the assertion follows. □

Recall that a *forbidden path* (of length $n \geqslant 1$) on a gentle pair $(\mathcal{Q}, \mathcal{I})$ is path $a_1 \cdots a_n$ such that $a_i a_{i+1} \in \mathcal{I}$ holds for all $1 \leqslant i \leqslant n-1$. Sometimes an oriented cycle $\mathscr{C} = a_1 a_2 \cdots a_n$ with $a_1 a_2, \ldots, a_{n-1}a_n, a_n a_1 \in \mathcal{I}$ is called a *forbidden cycle* or a *full relational* *orientated cycle* [5, Section 2.2]. Naturally, we can define the forbidden path on any almost gentle pair in the same way. A *forbidden path* of length zero is a path e_v corresponding to a vertex $v \in \mathcal{Q}_0$ such that one of the following conditions is satisfied:

(1) a path e_v of length zero corresponding to $v \in \mathcal{Q}_0$ such that
   - there is a unique arrow a with $\mathfrak{t}(a) = v$ and there is a unique arrow b with $\mathfrak{s}(a) = v$;
   - $ab \in \mathcal{I}$;

(2) a path e_v of length zero corresponding to a source $v \in \mathcal{Q}_0$ such that $\{a \in \mathcal{Q}_1 \mid \mathfrak{s}(a) = v\}$ contains only one element;

(3) a path e_v of length zero corresponding to a sink $v \in \mathcal{Q}_0$ such that $\{a \in \mathcal{Q}_1 \mid \mathfrak{t}(a) = v\}$ contains only one element.

In [5], forbidden paths were used to compute the so-called AG-invariants of gentle algebras which describe the derived equivalence of gentle one-cycle algebras. In [18,19], the standard forms of gentle one-cycle algebras and their geometric models by using forbidden paths and AG-invariants were provided.

Let α be an arrow and v the vertex $\mathfrak{t}(\alpha)$ (resp. $\mathfrak{s}(\alpha)$). A vertex $v \in \mathcal{Q}_0$ is called an $(\alpha, \downarrow)$-*relational vertex* (resp. $(\alpha, \uparrow)$-*relational vertex*) if there is an arrow β with $\mathfrak{s}(\beta) = v$ (resp. $\mathfrak{t}(\beta) = v$) such that $\alpha\beta \in \mathcal{I}$ (resp. $\beta\alpha \in \mathcal{I}$). A vertex $v \in \mathcal{Q}_0$ is called a *relational vertex* if it is either $(\alpha, \downarrow)$-relational or $(\alpha', \uparrow)$-relational for some arrows α and α', that is, there are two arrows a and b with $\mathfrak{t}(a) = v = \mathfrak{s}(b)$ such that $ab \in \mathcal{I}$.

Example 3.4. Consider the bound quiver $(\mathcal{Q}, \mathcal{I})$ given in Example 2.2. The vertex 2 is an $(a_{1,2}, \downarrow)$-relational vertex because $a_{2,3}$, $a_{2,4}$, and $a_{2,3'}$ are arrows starting at 2 such that $a_{1,2}a_{2,3}$, $a_{1,2}a_{2,4}$, and $a_{1,2}a_{2,3'}$ are belong to $\mathcal{I}$. Moreover, 2 is also an $(a_{2,4}, \uparrow)$-relational vertex by $a_{1,2}a_{2,4} \in \mathcal{I}$. Similarly, 2_{R} is both a $(b_{1,2_{\mathrm{R}}}, \downarrow)$-relational vertex and an $(a_{2_{\mathrm{R}},3_{\mathrm{R}}}, \uparrow)$-relational vertex, but it is not an $(a_{1,2_{\mathrm{R}}}, \downarrow)$-relational vertex since $a_{1,2_{\mathrm{R}}}a_{2_{\mathrm{R}},3\mathrm{R}}$ does not lie in $\mathcal{I}$.

Clearly, in the case for $(\mathcal{Q}, \mathcal{I})$ to be an almost gentle pair and $\mathcal{Q}$ to be a quiver of type $\mathbb{A}$, the path e_v corresponding to a vertex $v \in \mathcal{Q}_0$ is a forbidden path if and only if v is a relational vertex. However, in the case for $(\mathcal{Q}, \mathcal{I})$ to be an almost gentle pair, v to be relational does not admit that e_v is forbidden.

The following result can be shown by using Proposition 3.3.

Proposition 3.5. *Keep the notations in Lemma 3.2. Then, for each $n \geqslant 1$, every indecomposable direct summand of $\Omega_n(M)$ is one of the following*

(1) *a right maximal directed string module;*
(2) *a simple module corresponding to some relational vertex;*
(3) *a simple module corresponding to some sink.*

Proof. Assume that $D \leq_{\oplus} \Omega_n(M)$ is an indecomposable direct summand. Then, by Proposition 3.3, we have $D \cong \mathbb{M}(s)$ for some directed string. If $s = a_1a_2\cdots a_l$ is not right maximal ($a_1, \ldots, a_l$ are arrows, $l \geqslant 0$, and we have $s = e_v$ for some $v \in \mathcal{Q}_0$ and $D \cong \mathbb{M}(s) \cong S(v)$ in the case for $l = 0$), then there is an arrow α such that $s\alpha = a_1a_2\cdots a_l\alpha$, as a path on $\mathcal{Q}$, does not belong to $\mathcal{I}$. Now, assume that $s\alpha_1\cdots\alpha_m = a_1a_2\cdots a_l\alpha_1\alpha_2\cdots\alpha_m$ is right maximal. Then we can choose these arrows $\alpha_1 \ldots \alpha_m \in \mathcal{Q}_1$ such that $s\alpha_1 \ldots \alpha_m$ is a right maximal directed string since A is an almost gentle algebra, there is no oriented cycle $\alpha_1\alpha_2 \ldots \alpha_m$ over $\mathcal{Q}$. Then the projective cover

$$\tilde{p}_n : P(\Omega_{n-1}(M)) \to \Omega_{n-1}(M)$$

of $\Omega_{n-1}(M)$ satisfies the following conditions:

- $P(\Omega_{n-1}(M))$ has an indecomposable projective direct summand $P(v)$ such that v is a source of some right maximal directed string s' which is of the form

$$s' = b_1 b_2 \cdots b_k s \alpha_1 \alpha_2 \cdots \alpha_m = b_1 b_2 \cdots b_k a_1 a_2 \cdots a_l \alpha_1 \alpha_2 \cdots \alpha_m;$$

- $P(v)$ can be described as the claw $s'_1 \wedge s'_2 \wedge \cdots \wedge s'_r$ such that $s' = s'_j$ for some $1 \leqslant j \leqslant r$.

In this case, we have $D \cong \mathbb{M}(a_1 a_2 \cdots a_l \alpha_1 \alpha_2 \cdots \alpha_m) \leq_{\oplus} \mathrm{Ker}(\tilde{p}_n)$ $(= \Omega_n(M))$, which contradicts that $D \cong \mathbb{M}(s)$ in the case of $l \geqslant 1$.

Therefore, we have $l = 0$, i.e., $s = e_v$. In this case, $l = m = 0$ holds, and then $s' = b_1 b_2 \cdots b_k$ is right maximal. By the above argument, we have that $\mathfrak{t}(s') = \mathfrak{t}(b_k)$ and v coincide. Thus v is either a relational vertex or a sink. □

Example 3.6. Consider the almost gentle algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$ given in Example 2.2, the projective cover of the directed string module $\mathbb{M}(a_{1,2})$ is

$$p_0 : P_0 = P(1) \longrightarrow \mathbb{M}(a_{1,2}) = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$$

whose kernel is given by

$$\tilde{p}_0 : \mathrm{Ker}(p_0) = (2_{\mathrm{L}}) \oplus (2_{\mathrm{R}}) \oplus \left(\begin{smallmatrix} 2_{\mathrm{R}} \\ 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{smallmatrix}\right) \longrightarrow P(1),$$

where $P(1) = \left(\begin{smallmatrix} & & 1 & & \\ 2_{\mathrm{L}} & 2 & & 2_{\mathrm{R}} & 2_{\mathrm{R}} \\ & & & & 3_{\mathrm{R}} \\ & & & & 4_{\mathrm{R}} \end{smallmatrix}\right)$ is described by the claw

$$\kappa = a_{1,2_{\mathrm{L}}} \wedge a_{1,2} \wedge b_{1,2_{\mathrm{R}}} \wedge a_{1,2_{\mathrm{R}}} a_{2_{\mathrm{R}},3_{\mathrm{R}}} a_{3_{\mathrm{R}},4_{\mathrm{R}}},$$

and $\Omega_1(\mathbb{M}(a_{1,2})) = \left(\begin{smallmatrix} 2_{\mathrm{R}} \\ 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{smallmatrix}\right) \cong \mathbb{M}(e_{2_{\mathrm{R}}}) \oplus \mathbb{M}(e_2) \oplus \mathbb{M}(a_{2_{\mathrm{R}},3_{\mathrm{R}}} a_{3_{\mathrm{R}},4_{\mathrm{R}}})$. Each indecomposable direct summand of $\Omega_1(\mathbb{M}(a_{1,2}))$ is a directed string module. It is clear that $\mathbb{M}(e_{2_{\mathrm{R}}}) \cong S(2_{\mathrm{R}})$ and $\mathbb{M}(e_2) \cong S(2)$ are simple and $\mathbb{M}(a_{2_{\mathrm{R}},3_{\mathrm{R}}} a_{3_{\mathrm{R}},4_{\mathrm{R}}})$ is a right maximal directed string module.

Furthermore, we get the minimal projective resolution of $\mathbb{M}(a_{1,2})$ as follows:

$$0 \longrightarrow P_2 = P(3_{\mathrm{L}}) \oplus P(3_{\mathrm{R}}) \xrightarrow{p_2} P_1 = P(2_{\mathrm{L}}) \oplus P(2_{\mathrm{R}}) \oplus P(2_{\mathrm{R}})$$
$$\xrightarrow{p_1} P_0 = P(1) \xrightarrow{p_0} \mathbb{M}(a_{1,2}) = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) \longrightarrow 0,$$

where $\Omega_2(\mathbb{M}(a_{1,2})) \cong \left(\begin{smallmatrix} 3_{\mathrm{L}} \\ 4_{\mathrm{L}} \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{smallmatrix}\right) = P(3_{\mathrm{L}}) \oplus P(4_{\mathrm{R}}) \cong \mathbb{M}(a_{3_{\mathrm{L}},4_{\mathrm{L}}}) \oplus \mathbb{M}(a_{3_{\mathrm{R}},4_{\mathrm{R}}})$ is a direct sum of two right maximal directed string modules.

3.2. **Projective dimension of right maximal directed string modules.** For any vertex v of an almost gentle pair $(\mathcal{Q}, \mathcal{I})$, we define

$$v^{\downarrow} := \{w \in \mathcal{Q}_0 \mid \text{there is an arrow } a \in \mathcal{Q}_1 \text{ such that } \mathfrak{s}(a) = v \text{ and } \mathfrak{t}(a) = w\}.$$

Let $\kappa = s_1 \wedge s_2 \wedge \cdots \wedge s_n$ be a claw, where $s_i = a_{i,1} a_{i,2} \cdots a_{i,l_i}$ ($l_1 \geqslant 1$ and $a_{i,1}, \ldots, a_{i,l_i}$ are arrows). A vertex $\mathfrak{t}(a_{i,j})$ $(1 \leqslant j \leqslant l_i)$ is said to be a (κ, i)-*relational vertex* if there is an arrow α with $\mathfrak{s}(\alpha) = \mathfrak{t}(a_{i,j})$ such that $a_{i,j}\alpha \in \mathcal{I}$. All (κ, i)-relational vertices are called κ-*relational vertices* for simplicity.

Lemma 3.7. *Let M be an indecomposable module over an almost gentle algebra A corresponding to a right maximal directed string $\delta = a_1 a_2 \cdots a_l$ ($l \geqslant 1$, and $a_1, a_2, \ldots, a_l$ are arrows). Then $\Omega_1(M)$ is projective if and only if all vertices in $\mathfrak{s}(\delta)^{\downarrow} \backslash \{\mathfrak{t}(a_1)\}$ are not κ-relational, where κ is the claw corresponding to $P(\mathfrak{s}(\delta))$.*

Proof. By the definition of almost gentle algebras, the indecomposable projective module P_0 ($\cong P(\mathfrak{s}(\delta))$) given by the projective cover $p_0 : P_0 \to M$ of M can be described by the claw

$$\kappa = s_1 \wedge\!\!\!\wedge\, s_2 \wedge\!\!\!\wedge\, \cdots \wedge\!\!\!\wedge\, s_m,$$

where $s_j = a_1a_2\cdots a_l = \delta$ for some $1 \leqslant j \leqslant m$, and $s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_m$ are paths of length $\geqslant 1$. Notice that s_j is right maximal, then we have

$$\Omega_1(M) \cong \bigoplus_{\substack{1\le i\le m \\ i\ne j}} X_i$$

where each X_i is a directed string module, see Lemma 3.2. For simplicity, assume $s_i = a_{i,1}\cdots a_{i,t_i}$ for all $1 \leqslant i \leqslant m$, $i \neq j$, here $t_i \geqslant 1$. Then we obtain

$$X_i \cong \mathbb{M}(a_{i,2}\cdots a_{i,t_i})$$

(note that X_i is isomorphic to $S(\mathfrak{t}(a_{1,i}))$ in the case for $t_i = 1$).

Assume that $\Omega_1(M)$ is projective. If there exists a vertex $\mathfrak{t}(a_{i,1})$ in $\mathfrak{s}(\delta)^{\downarrow}$ which is κ-relational, then we have $t_i = 1$, $X_i \cong S(\mathfrak{t}(a_{1,i}))$, and there is a right maximal directed string s' of length $\geqslant 1$ with $\mathfrak{s}(s') = \mathfrak{t}(s_i) = \mathfrak{t}(a_{i,1})$. It is easy to see that $S(\mathfrak{t}(a_{1,i}))$ ($\leq_{\oplus} \Omega_1(M)$) is not projective by the existence of the directed string s' of length $\geqslant 1$. Thus, $\Omega_1(M)$ is not a projective module, a contradiction.

On the other hand, if all vertices in $\mathfrak{s}(\kappa)^{\downarrow}\backslash\{\mathfrak{t}(a_1)\}$ are not κ-relational, then for any $1 \leqslant i \leqslant m$, $i \neq j$, one of the following conditions is satisfied:

(1) $t_i = 1$, and $\mathfrak{t}(s_i)$ $(= \mathfrak{t}(a_{i,1}))$ is a sink of $(\mathcal{Q}, \mathcal{I})$;

(2) $t_i \geqslant 2$.

Thus, X_i is isomorphic to either the simple module $S(\mathfrak{t}(s_i))$ corresponding to $\mathfrak{t}(s_i) \in \mathcal{Q}_0$ or the right maximal directed string module $\mathbb{M}(s_i')$ with $\mathfrak{s}(s_i') = \mathfrak{t}(s_i)$.

In the case (1), since $\mathfrak{t}(s_i)$ is a sink of $(\mathcal{Q}, \mathcal{I})$, we obtain that $S(\mathfrak{t}(s_i))$ is a simple projective module.

In the case (2), assume $s_i' = b_{i,1}\cdots b_{i,\ell_i}$. Then there is no arrow $b \neq b_{i,1}$ with $\mathfrak{s}(b_{i,1}) = \mathfrak{s}(b) = \mathfrak{t}(s_i)$. Otherwise, by the definition of almost gentle algebras, we have $a_{i,1}b \in \mathcal{I}$, it follows that $\mathfrak{t}(s_i) = \mathfrak{t}(a_{i,1})$ is a (κ, i)-relational vertex, a contradiction.

Then, by the right maximality of s_i', we have that $\mathbb{M}(s_i')$ is projective. Therefore, X_i is a projective module, and so is $\Omega_1(M)$, as required. □

Similarly, one gets the following result.

Lemma 3.8. *Let $S(v)$ be a simple module over an almost gentle algebra A corresponding to the vertex $v \in \mathcal{Q}_0$. Then $\Omega_1(S(v))$ is projective if and only if all vertices in $v^{\downarrow}$ are not κ-relational, where κ is the claw corresponding to $P(v)$.*

Example 3.9. Let $A = \Bbbk\mathcal{Q}/\mathcal{I}$ be the almost gentle algebra given in Example 2.2. Keep the notations in Example 3.6. We claim that the 2-nd syzygy

$$\Omega_2(\mathbb{M}(a_{1,2})) \cong \Omega_1(S(2_{\mathrm{L}})) \oplus \Omega_1(S(2_{\mathrm{R}})) \oplus \Omega_1\left(\left(\begin{smallmatrix} 2_{\mathrm{R}} \\ 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{smallmatrix}\right)\right)$$

of $\mathbb{M}(a_{1,2})$ is projective. Let κ_{11}, κ_{12}, and κ_{13} be the claws corresponding to the indecomposable projective module $P(2_{\mathrm{L}})$, $P(2_{\mathrm{R}})$, and $P(2_{\mathrm{R}})$, respectively. For the vertex 2_{L}, we have $2_{\mathrm{L}}^{\downarrow} = \{3_{\mathrm{L}}\}$ which is a set containing only one element 3_{L}. Since 3_{L} is not a κ_{11}-relational vertex, we have that $\Omega_1(S_{2_{\mathrm{L}}})$ is projective by Lemma 3.8. Similarly, $\Omega_1(S(2_{\mathrm{R}}))$ is projective. On the other hand, we have $1^{\downarrow} = \{2_{\mathrm{L}}, 2, 2_{\mathrm{R}}\}$ where $2 = \mathfrak{t}(a_{1,2})$. Then $1^{\downarrow}\backslash\{\mathfrak{t}(a_{1,2})\} = \{2_{\mathrm{L}}, 2_{\mathrm{R}}\}$. Notice that 2_{L} is κ_0-relational (κ_0 is the claw κ given in Example 3.6) and 2_{R} is not κ_0-relational, thus $\Omega_1(\mathbb{M}(a_{1,2}))$ is not projective by Lemma

3.7. In this case, the indecomposable direct summand $\begin{pmatrix} 2_{\mathrm{R}} \\ 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{pmatrix} \le_{\oplus} \Omega_1(\mathbb{M}(a_{1,2}))$ is projective, it follows that $\Omega_1\left(\begin{pmatrix} 2_{\mathrm{R}} \\ 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{pmatrix}\right)$ is zero. Therefore, $\Omega_2(\mathbb{M}(a_{1,2}))$ is projective. The claim is proved.

3.3. **Global dimension.** In this subsection, we describe the global dimension of almost gentle algebras. To do this, we provide a description for the n-th syzygy of the simple module to be non-projective.

Proposition 3.10. *Keep the notations in Lemma 3.8. Then $\Omega_{n-1}(S(v))$ is not projective if and only if there is a forbidden path p of length n starting at v.*

Proof. Let $p = b_1 b_2 \cdots b_n$ be a forbidden path on $(\mathcal{Q}, \mathcal{I})$, where $v = v_0$, $\mathfrak{s}(b_i) = v_{i-1}$ $(1 \leqslant i \leqslant n)$ and $\mathfrak{t}(b_n) = v_n$, and let κ_0 be the claw corresponding to the indecomposable projective module $P(v_0) = P_0$ given by the projective cover $P_0 \to M$ of M. Then v_1 is a κ_0-relational vertex. By Lemma 3.8, we have that $\Omega_1(S(v))$ is not projective. To be more precisely, by Proposition 3.5, we have that $\Omega_1(S(v))$ has a non-projective indecomposable direct summand X_1 which is either a right maximal directed string module or a simple module such that $\mathrm{top}(X_1) \cong S(v_1)$.

Let κ_1 be the claw corresponding to the indecomposable projective module $P(v_1)$. Then $v_2 \in v_1^{\downarrow}$ is a κ_1-relational vertex, it follows that $\Omega_1(X_1)$ is not projective by using Lemma 3.7. In this case, $\Omega_2(S(v))$ is not projective since $\Omega_1(X_1) \le_{\oplus} \Omega_2(S(v))$, and $b_1 b_2$ is a forbidden path end with v_2 corresponding to it since v_2 is κ_1-relational. Then, we can fix a family of right maximal directed string modules $X_2, \ldots, X_n$ such that $X_t \le_{\oplus} \Omega_t(S(v))$ holds for all $1 \leqslant t \leqslant n$ and $X_1, X_2, \ldots, X_{n-1}$ are not projective by induction. Thus, $\Omega_{n-1}(S(v))$ is non-projective. Moreover, we obtain a correspondence

$$\Omega_t(S(v)) \mapsto v_t \tag{3.2}$$

such that $\mathrm{top}(X_t) \cong S(v_t)$ holds for all $1 \leqslant t \leqslant n$.

On the other hand, for the case for $\Omega_{n-1}(S(v))$ to be non-projective, we suppose that the lengths of all forbidden paths starting at v are less than or equal to $n-1$. Consider all right maximal forbidden paths $p_1, \ldots, p_m$ start at v, and assume

$$p_j = a_{j,1} a_{j,2} \cdots a_{j,l_j} = \quad v_{j,0} \xrightarrow{a_{j,1}} v_{j,1} \xrightarrow{a_{j,2}} \cdots \xrightarrow{a_{j,l_j}} v_{j,l_j}$$

$$(a_{j,k} a_{j,k+1} \in \mathcal{I} \text{ holds for all } 1 \leqslant k < l_j \leqslant n-1).$$

Here, a *right maximal forbidden path* is a forbidden path $p = a_1 \cdots a_l$ $(a_1, \ldots, a_l$ are arrows, and $a_k a_{k+1} \in \mathcal{I})$ such that $pb \notin \mathcal{I}$ holds for all arrows $b \in \mathcal{Q}_1$ with $\mathfrak{t}(p) = \mathfrak{s}(b)$. In this case, if $t \leqslant l_j$, then one can check that there is a module $X_{j,t} \le_{\oplus} \Omega_t(S(v))$ which is either a right maximal directed string or a simple module such that $\mathrm{top}(X_{j,t}) = v_{j,t}$ by Proposition 3.5. By the right maximality of p_j, all vertices in $(v_{j,l_j})^{\downarrow}$ are not κ_{j,l_j}-relational, where κ_{j,l_j} is the claw corresponding to $P(v_{j,l_j})$. Applying Lemma 3.7 to $\Omega_1(\Omega_{j-1}(S(v)))$, all X_{j,l_j} are projective since each indecomposable direct summand of $\Omega_{j-1}(S(v))$ is either a right maximal directed string module or a simple module by Proposition 3.5 again. Thus, for $L = \max\{l_j \mid 1 \leqslant j \leqslant m\}$, we obtain that $\bigoplus_{l_j = L} X_{j,l_j}$ is projective which is isomorphic to $\Omega_L(S(v))$. In the case for $L < n-1$, we have that $\Omega_n(S(v)) = 0$ is projective, a contradiction. In the case for $L = n-1$, we have that $\Omega_{n-1}(S(v)) = \Omega_L(S(v)) \cong \bigoplus_{l_j = L} X_{j,l_j}$ is projective, a contradiction. □

Notation 3.11. *For any vertex v of an almost gentle pair $(\mathcal{Q}, \mathcal{I})$, we use $\mathcal{F}(v)$ to denote the set of all forbidden paths starting at $v \in \mathcal{Q}_0$.*

For a module M, we use proj.dim$M$ to denote the projective dimension of M. We are now in a position to prove the following result.

Theorem 3.12. *For any simple module $S(v)$ over an almost gentle algebra A, we have*

$$\text{proj.dim}S(v) = \sup_{F\in\mathcal{F}(v)} \ell(F), \tag{3.3}$$

where $\ell(F)$ is the length of F.

Proof. We only prove it in the case for proj.dim$S(v) = n$. The proof of the case for proj.dim$S(v) = \infty$ is similar.

By Proposition 3.10, there exists a forbidden path $F_0 \in \mathcal{F}(v)$ such that $\ell(F_0) = n$ since $\Omega_{n-1}(S(v))$ is not projective. Notice that if there is a forbidden path F_0' in $\mathcal{F}(v)$ that satisfies $\ell(F_0') \geqslant n+1$, then $\Omega_n(S(v))$ is not projective by Proposition 3.10 again. It contradicts that proj.dim$S(v) = n$. Thus, F_0 is the forbidden path in $\mathcal{F}(v)$ such that

$$\text{proj.dim}S(v) = \ell(F_0) = \sup_{F\in\mathcal{F}(v)} \ell(F),$$

as required. □

The following result provides a description of the global dimension of almost gentle algebras.

Theorem 3.13. *Let A be an almost gentle algebra with the bound quiver $(\mathcal{Q}, \mathcal{I})$. Then*

$$\text{gl.dim}A = \sup_{F\in\mathcal{F}} \ell(F),$$

where $\mathcal{F} = \bigcup_{v\in\mathcal{Q}_0} \mathcal{F}(v)$ is the set of all forbidden paths.

Proof. Notice a forbidden path F starting at a vertex v yields that $\Omega_{\ell(F)}(S(v))$ is non-zero, then we have

$$\text{gl.dim}\Lambda = \sup_{v\in\mathcal{Q}_0} \text{proj.dim}S(v) = \sup_{v\in\mathcal{Q}_0}\sup_{F\in\mathcal{F}(v)} \ell(F) = \sup_{F\in \bigcup_{v\in\mathcal{Q}_0}\mathcal{F}(v)} \ell(F) = \sup_{F\in\mathcal{F}} \ell(F),$$

as required. □

Example 3.14. The global dimension of the almost gentle algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$ given in Example 2.2 is 4 which is decided by the forbidden paths

$$a_{1,2}a_{2,3'}a_{3',4}a_{4,5} \text{ and } a_{1,2}a_{2,3}a_{3,4}a_{4,5}.$$

To be more precisely, the above two forbidden paths show that the projective dimension proj.dim$S(1)$ of the simple module $S(1)$ is 4, and one can check that proj.dim$S(4) \geqslant$ proj.dim$S(v)$ holds for all $v \in \mathcal{Q}_0$.

4. Self-injective dimension of almost gentle algebras

If $\Lambda = \Bbbk\mathcal{Q}/\mathcal{I}$ is a finite-dimensional algebra, then

$$\text{inj.dim}\Lambda = \sup_{i\in\mathcal{Q}_0} \text{proj.dim}(D(\Lambda e_i)),$$

where $D(\Lambda e_i) := \text{Hom}_{\Bbbk}(\Lambda e_i, \Bbbk)$ is the indecomposable injective Λ-module, written as $E(i)$, corresponding to the vertex $i \in \mathcal{Q}_0$. Thus, we can compute the self-injective dimension inj.dimΛ of Λ by using minimal projective resolutions of injective modules.

4.1. **Syzygies of injective modules.** An indecomposable injective module of an algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$ corresponding to $v \in \mathcal{Q}_0$ is isomorphic to $D(Ae_v)$ which describes all paths on the bound quiver $(\mathcal{Q},\mathcal{I})$ end at v. Thus, each indecomposable injective module can be described by an anti-claw (see Definition 3.1). In this subsection, we compute the 1-st syzygy of an indecomposable injective module over an almost gentle algebra.

Lemma 4.1. *Let $(\mathcal{Q},\mathcal{I})$ be an almost gentle pair, and let $v \in \mathcal{Q}_0$ be a $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex, i.e., the vertex v satisfying $\mathfrak{t}^{-1}(v) = c$ and $\mathfrak{s}^{-1}(v) = d$. Then, in this bound quiver $(\mathcal{Q},\mathcal{I})$, there is an integer t with $0 \leqslant t \leqslant \min\{c, d\}$ such that the number of all non-zero paths crossing v of length two is t.*

(Here, we say a path $p = 1 \xrightarrow{a_1} 2 \longrightarrow \cdots \longrightarrow n \xrightarrow{a_n} n+1$ crosses a vertex v if v is a vertex lying in $\{2, 3, \ldots, n\}$.)

Proof. Assume v is shown as in Figure 4.1. Then, by the definition of almost gentle

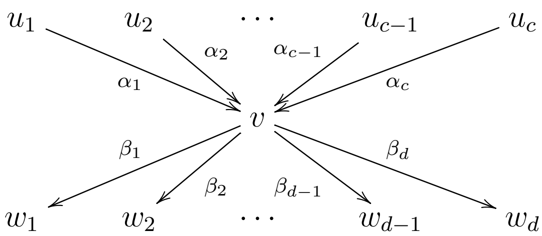


Figure 4.1. $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex

pairs, for any α_i ($1 \leqslant i \leqslant c$), there is at most one arrow β_j ($1 \leqslant j \leqslant d$) such that $\alpha_i\beta_j \notin \mathcal{I}$. In this case, if β_j exists, then for any $\imath \neq i$, we have $\alpha_\imath\beta_j \in \mathcal{I}$, and for any $\jmath \neq j$, we have $\alpha_i\beta_\jmath \in \mathcal{I}$. Thus, without loss of generality, we assume

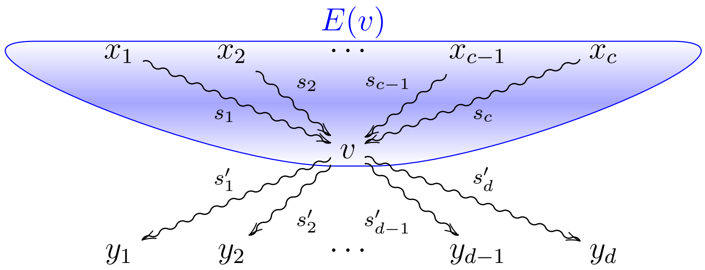
$$\alpha_i\beta_j \begin{cases} \notin \mathcal{I} & \text{if } 1 \leqslant i = j \leqslant t; \\ \in \mathcal{I} & \text{if } t < i \leqslant c. \end{cases} \tag{4.1}$$

Suppose that the number of all paths crossing v of length at least two is T. We claim that $T = t$. By the first condition of (4.1), we have t paths $\alpha_1\beta_1, \ldots, \alpha_t\beta_t \notin \mathcal{I}$ crossing v, then $T \geqslant t$. In the following, we prove $T \leqslant t$. If $T > t$, then there is a path $\alpha_k\beta_l \notin \mathcal{I}$ crossing v such that $\alpha_k\beta_l \neq \alpha_i\beta_i$ for any $1 \leqslant i \leqslant t$. In this case, by the second condition of (4.1), we have $k \leqslant t$. Then $\alpha_k\beta_j \notin \mathcal{I}$ for some j with $j \neq k$ and $1 \leqslant j \leqslant d$. It follows that both β_k and β_j are arrows starting at v such that $\alpha_k\beta_k$ and $\alpha_k\beta_j$ are paths not lying in $\mathcal{I}$, this contradicts the definition of almost gentle pairs. The claim is proved. □

If there are t non-zero paths of length two on an almost gentle pair $(\mathcal{Q},\mathcal{I})$ crossing a $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex v, then, using the notation from Lemma 4.1 and Figure 4.1, the anti-claw corresponding to $E(v)$ is of the form

$$s_1 \curlyvee s_2 \curlyvee \cdots \curlyvee s_c,$$

where each s_i is a left maximal directed string of the form $\cdots\alpha_i$, and there are d right maximal directed string $s'_1, \ldots, s'_d$ starting at v, where each s'_j is a path of the form $\beta_j\cdots$, see Figure 4.2.



$s_i = \cdots \alpha_i \ (1 \leqslant i \leqslant c)$

$s'_j = \beta_j \cdots \ (1 \leqslant j \leqslant d)$

FIGURE 4.2. $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex v on an almost gentle pair $(\mathcal{Q}, \mathcal{I})$ ($s'_1, \ldots, s'_d$ are right maximal directed strings)

Lemma 4.2. *Let v be a $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex on an almost gentle pair $(\mathcal{Q}, \mathcal{I})$ is shown in FIGURE 4.2. Then $\Omega_1(E(v))$ has a decomposition as follows:*

$$\Omega_1(E(v)) \cong \varPsi \oplus \bigoplus_{\jmath \in J} S_\jmath \oplus \bigoplus_{\imath \in I} M_\imath,$$

where I and J are index sets, $\varPsi$ is either zero or a module whose top is a direct sum of some copies of $S(v)$, all $S_\jmath$ are copies of $S(v)$, and all $M_\imath$ are either simple modules or right maximal directed string modules.

(Note that $\varPsi$ may be not indecomposable, and, in this case, $\varPsi$ may contain a direct summand which is indecomposable projective.)

Proof. By Lemma 4.1, we may assume $\alpha_1\beta_1, \ldots, \alpha_t\beta_t \notin \mathcal{I}$. Then we have $s_1s'_1, \ldots, s_ts'_t \notin \mathcal{I}$ since $\mathcal{I}$ is an admissible ideal generated by some paths of length two, and so, for any $1 \leqslant i \leqslant c$, the claw corresponding to the indecomposable projective module $P(x_i)$ is of the form

$$\kappa = s_{i1} \curlywedge s_{i2} \curlywedge \cdots \curlywedge s_{im_i},$$

where

- $s_{i1} = \begin{cases} s_is'_i, & i \leqslant t; \\ s_i, & i \geqslant t+1, \end{cases}$ here, the path s_i is of the form

$$v_{i,1,1} \xrightarrow{a_{i,1,1}} v_{i,1,2} \xrightarrow{a_{i,1,2}} \cdots \xrightarrow{a_{i,1,\ell_{i1}}} v_{i,1,\ell_{i1}+1} \quad (v_{i,1,1} = x_i, v_{i,1,\ell_{i1}} + 1 = v), \tag{4.2}$$

  and the path s'_i is of the form

$$v_{i,1,\ell_{i1}+1} \xrightarrow{a_{i,1,\ell_{i1}+1}} v_{i,1,\ell_{i1}+2} \xrightarrow{a_{i,1,l_{i1}+2}} \cdots \xrightarrow{a_{i,1,l_{i1}}} v_{i,1,l_{i1}+1} \quad (v_{i,1,\ell_{i1}} = v, v_{i,1,l_{i1}} = y_i); \tag{4.3}$$

- s_{ij} is a path

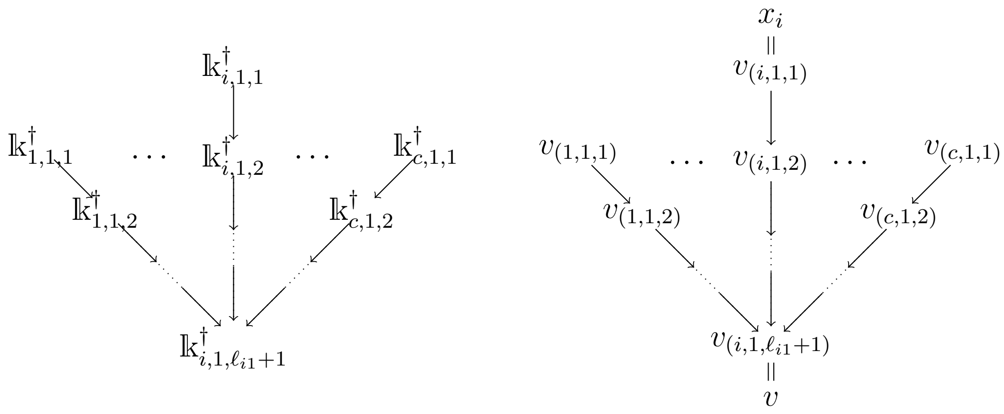
$$v_{i,j,1} \xrightarrow{a_{i,j,1}} v_{i,j,2} \xrightarrow{a_{i,j,2}} \cdots \xrightarrow{a_{i,j,l_{ij}}} v_{i,j,l_{ij}+1} \quad (x_i = v_{i,j,1})$$

  of length l_{ij} $(1 < j \leqslant m_i)$.

Then the projective cover of $E(v)$ is the homomorphism

$$p_0 : P_0 = \bigoplus_{i=1}^{c} P(x_i) \to E(v)$$

(the left picture of FIGURE 4.3 shows the quiver representation of $E(v)$, and the right picture of FIGURE 4.3 shows the anti-claw of $E(v)$),



Indecomposable injective module $E(v)$ (each $\Bbbk^{\dagger}_{i,j,k}$ is isomorphic to $\Bbbk$)

The anti-claw of $E(v)$

FIGURE 4.3. The indecomposable injective module $E(v)$ and the anti-claw of it

such that for each triple (i,j,k) with $1 \leqslant k \leqslant \ell_{ij}+1$, we have the following two facts (see FIGURE 4.4, the right picture):

- if $j=1$ and $1 \leqslant k \leqslant \ell_{ij}+1$, that is, $v_{i,j,k} \in \{x_i = v_{i,1,1}, v_{i,1,2}, \dots, v_{i,1,\ell_{ij}+1}\}$ is a vertex on the path s_i, then p_0 sends each one-dimensional $\Bbbk$-linear space $\Bbbk_{i,1,k}$, the direct summand of the $\Bbbk$-linear space $P_0 e_{v_{i,1,k}}$ corresponding to the vertex $v_{i,1,k}$, to the one-dimensional direct summand, written as $\Bbbk^{\dagger}_{i,j,k}$, of the $\Bbbk$-linear space $E(v)e_{v_{i,j,k}}$ corresponding to the vertex $v_{i,j,k}$ by using the identity
$$(\mathrm{id} : x \mapsto x) \in \mathrm{Hom}_{\Bbbk}(\Bbbk,\Bbbk) \cong \mathrm{Hom}_{\Bbbk}(\Bbbk_{i,1,k}, \Bbbk^{\dagger}_{i,j,k});$$
- if (i,j,k) does not lie in the above two cases, then p_0 sends each $\Bbbk_{i,j,k}$ to zero.

Then the 1-st syzygy $\Omega^1(E(v))$ is a direct sum of some modules of the following three classes:

- the directed string module M_{ij} corresponding to the string $\hat{s}_{ij} := a_{i,j,2} \cdots a_{i,j,l_{ij}}$ which is obtained from s_{ij} $(1 < j \leqslant m_i)$; if $m_i = 1$, then $\hat{s}_{ij} = e_{\mathfrak{t}(a_{i,j,1})}$ is a simple string, that is, the above directed string module is simple;
- the simple module isomorphic to $S(v)$ which is given by the indecomposable projective module $P(x_i)$ with $i \geqslant t+1$ because $s_i s'_j = 0$ holds for all $1 \leqslant j \leqslant d$;
- the module $\mathit{\Psi}$ given by the $\Bbbk$-linear maps $(\varphi_{i,1,1}, \varphi_{i,1,2}, \cdots, \varphi_{i,1,\ell_{i1}})_{1\leqslant i\leqslant t}$ whose top is $S(v)^{\oplus(t-1)}$ (for $0 \leqslant t \leqslant 1$, we put $\mathit{\Psi} = 0$), and in this case, if $c=2$, then $\mathit{\Psi}$ corresponds to a claw whose top is $S(v)$.

Therefore, we obtain
$$\Omega_1(E(v)) \cong \mathit{\Psi} \oplus S(v)^{\oplus(c-t)} \oplus \Bigg(\bigoplus_{(i,j)\in I} M_{ij} \Bigg), \tag{4.4}$$
where I is some index set. □

Corollary 4.3. *The top of the direct summand*
$$\mathit{\Psi}_0 := \mathit{\Psi} \oplus \bigoplus_{\jmath \in J} S_\jmath \ \leq_{\oplus} \Omega_1(E(v))$$

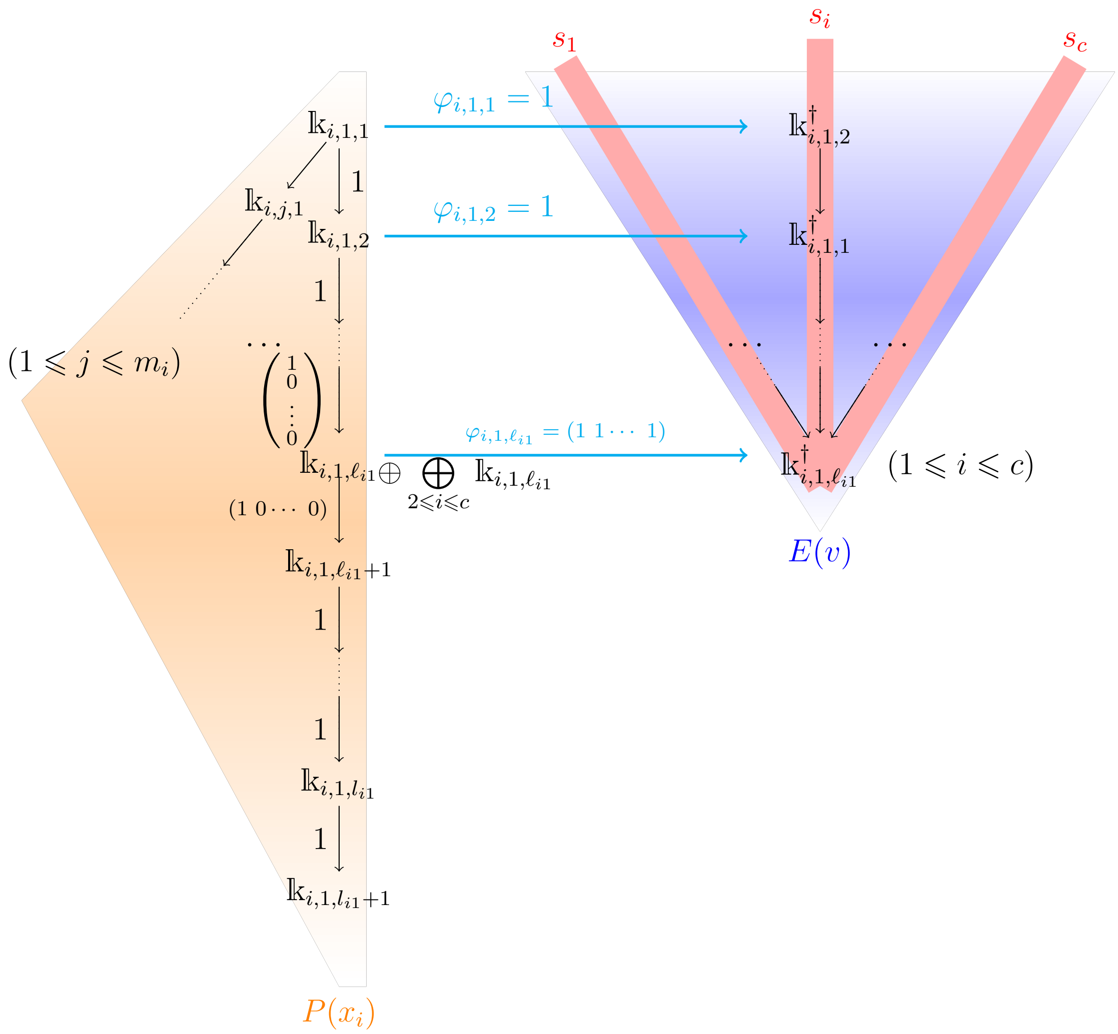


FIGURE 4.4. The direct summand $P(x_i)$ of the projective module $P_0 = \bigoplus_{1\leqslant i\leqslant c} P(x_i)$ given by the projective cover of $E(v)$ (where P_0e_v, as a $\Bbbk$-linear space, has a direct summand $\Bbbk^{\oplus c} = \bigoplus_{1\leqslant i\leqslant c} \Bbbk_{i,1,\ell_{i1}}$)

given in Lemma 4.2 is isomorphic to $S(v)^{\oplus(c-1)}$.

Proof. It is a direct corollary of the formula (4.4) in the proof of Lemma 4.2. □

Example 4.4. Let $A = \Bbbk\mathcal{Q}/\mathcal{I}$ be the almost gentle algebra given by Example 2.2.

(1) Consider the indecomposable injective module $E(2_{\mathrm{R}}) = ({}^{1}{}_{2_{\mathrm{R}}}{}^{1})$ and its projective cover $P(1)^{\oplus 2} \to E(2_{\mathrm{R}})$. The anti-claw corresponding to $E(2_{\mathrm{R}})$ is

$$a_{1,2_{\mathrm{R}}} \curlyvee b_{1,2_{\mathrm{R}}}.$$

Here, $v = 2_{\mathrm{R}}$, $s_1 = a_{1,2_{\mathrm{R}}}$, and $s_2 = b_{1,2_{\mathrm{R}}}$. The claw corresponding to $P(1)$ is

$$a_{1,2_{\mathrm{L}}} \Cap a_{1,2} \Cap a_{1,2_{\mathrm{R}}} a_{2_{\mathrm{R}},3_{\mathrm{R}}} a_{3_{\mathrm{R}},4_{\mathrm{R}}} \Cap b_{1,2_{\mathrm{R}}}.$$

Here, $s_1' = a_{2_{\mathrm{R}},3_{\mathrm{R}}} a_{3_{\mathrm{R}},4_{\mathrm{R}}}$. We have

$$\Omega_1(E(2_{\mathrm{R}})) \cong \begin{pmatrix} 2_{\mathrm{R}} \\ 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{pmatrix} \oplus \Big((2_{\mathrm{L}}) \oplus (2) \oplus (2_{\mathrm{R}}) \oplus \begin{pmatrix} 2_{\mathrm{R}} \\ 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{pmatrix} \oplus (2_{\mathrm{L}}) \oplus (2) \Big), \tag{4.5}$$

see FIGURE 4.5, where $\varPsi \cong \begin{pmatrix} 2_{\mathrm{R}} \\ 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{pmatrix}$ is the first direct summand shown in (4.5), $\bigoplus_{\jmath\in J} S_\jmath =$

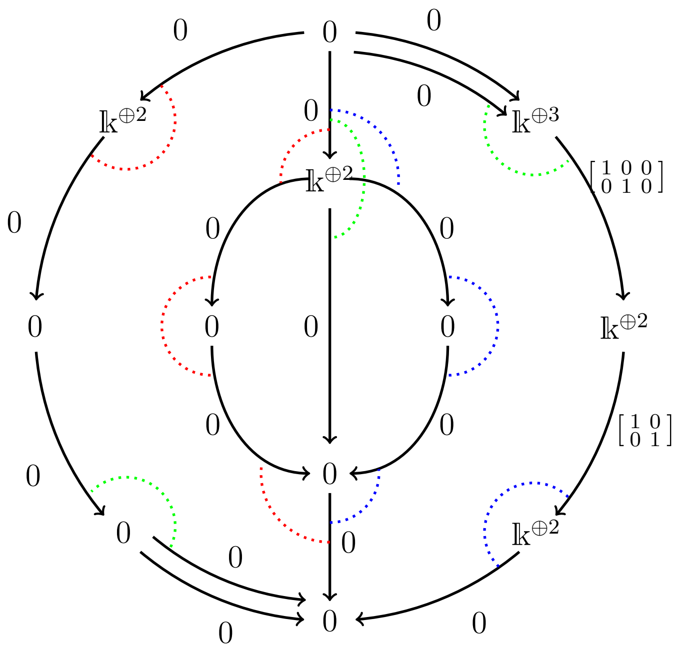
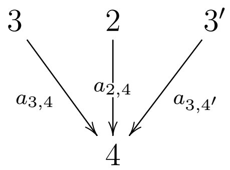


FIGURE 4.5. The 1st-syzygy $\Omega_1(E(2_{\mathrm{R}}))$ of the injective module $E(2_{\mathrm{R}})$

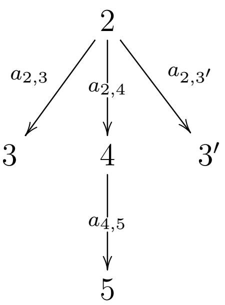
0 (the index set J is an empty set), and $\bigoplus_{\imath\in I} M_\imath \cong (2_{\mathrm{L}})\oplus(2)\oplus(2_{\mathrm{R}})\oplus\begin{pmatrix} 2_{\mathrm{R}} \\ 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{pmatrix}\oplus(2_{\mathrm{L}})\oplus(2)$ which is a direct sum of some right maximal directed string modules. Thus, we have

$$\Psi_0 = \Psi \cong \begin{pmatrix} 2_{\mathrm{R}} \\ 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{pmatrix}.$$

(2) Consider the indecomposable injective module $E(4) = \left(\begin{smallmatrix} 3 & 2 & 3' \\ & 4 & \end{smallmatrix}\right)$ and its projective cover $P(3)\oplus P(2)\oplus P(3') \to E(4)$. The anti-claw corresponding to $E(4)$ is

$a_{3,4} \vee a_{2,4} \vee a_{3,4'} =$  ,

and $P(3)$, $P(2)$ and $P(3)$ are indecomposable projective modules corresponding to the claws

$a_{3,4}$, $a_{2,3} \wedge a_{2,4} \wedge a_{2,3'} =$  , and $a_{3',4}$,

respectively. Then we have $c = 3$, $d = 1$, $s_1 = a_{3,4}$, $s_2 = a_{2,4}$, $s_3 = a_{3,4'}$, and $s_1' = a_{2,4}a_{4,5}$. Thus,

$$\Omega_1(E(4)) \cong \left(\begin{smallmatrix} 4 \\ 5 \end{smallmatrix}\right) \oplus (4) \oplus ((3)\oplus(3')),$$

where $\Psi \cong \left(\begin{smallmatrix} 4 \\ 5 \end{smallmatrix}\right)$, $\bigoplus_{\jmath\in J} S_\jmath \cong (4)$ (the index set J contains only one element), and $\bigoplus_{\imath\in I} M_\imath \cong (3)\oplus(3')$ (the index set I is $\{1,2\}$, $M_1 \cong (3)$, and $M_2 \cong (3')$), see FIGURE

4.6. Thus, we have

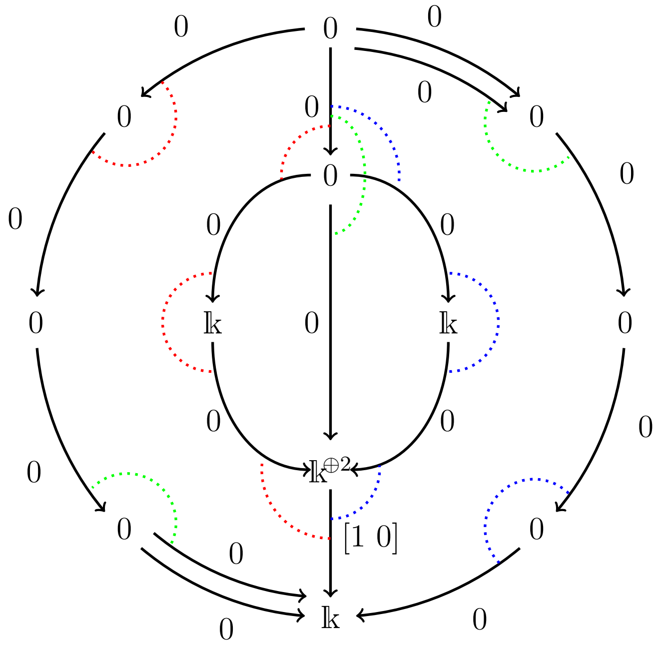


FIGURE 4.6. The 1st-syzygy $\Omega_1(E(4))$ of the injective module $E(4)$

$$\varPsi_0 \cong \binom{4}{5} \oplus (4).$$

4.2. **The projectivity of the direct summand $\varPsi_0$.** To compute proj.dim$D(A)$, we need a method to describe the projectivity of $\varPsi_0$. We say that a $(c^{\text{in}}, d^{\text{out}})$-type vertex v of $\mathcal{Q}$ is a *gentle vertex* if the following conditions hold:

- $c \leqslant 2$ and $d \leqslant 2$, in this case, we say that v is a $((\leqslant 2)^{\text{in}}, (\leqslant 2)^{\text{out}})$-type vertex for simplicity;
- the number of arrows ending at v is less than or equal to 2, and that of arrows starting at v is less than or equal to 2;
- there is at most one path of length two crossing v such that it lies in $\mathcal{I}$, and there at most one path of length two crossing v such that it does not lie in $\mathcal{I}$.

A bound quiver $(\mathcal{Q}, \mathcal{I})$ is a gentle pair if $\mathcal{I}$ is generated by some paths of length two and all vertices of $\mathcal{Q}$ are gentle.

Lemma 4.5. *Keep the notation in Lemma 4.2 and let $c = 2$. If v is a gentle vertex, then the direct summand $\varPsi_0 \leq_{\oplus} \Omega_1(E(v))$ is projective.*

(Note that if $c \leqslant 1$, then $E(v)$ is a directed string module whose projective dimension can be computed by using Lemma 3.7.)

Proof. Since v is a $(c^{\text{in}}, d^{\text{out}})$-type vertex, if v is a gentle vertex, then we have $c \leqslant 2$ and $d \leqslant 2$. In the following, we show that $\varPsi_0$ is projective.

(1) In the case for $d = 0$, we have that v is a sink of $\mathcal{Q}$, and $E(v)$ corresponds to a string which is of the following form

$$v_{1,1,1} \xrightarrow{a_{1,1,1}} v_{1,1,2} \xrightarrow{a_{1,1,2}} \cdots \xrightarrow{a_{1,1,\ell_{11}}} v \xleftarrow{a_{2,1,\ell_{21}}} \cdots \xleftarrow{a_{2,1,2}} v_{2,1,2} \xleftarrow{a_{2,1,1}} v_{2,1,1}$$
$$(v_{1,1,\ell_{11}+1} = v = v_{2,1,\ell_{21}+1}),$$

then, by Lemma 4.2, we have

$$\Omega_1(E(v)) \cong P(v) \oplus \bigoplus_{\imath\in I} M_\imath,$$

where I is some index set, all $M_\imath$ are directed string modules, and $\Psi_0 \cong S(v) \cong P(v)$ is both simple and projective.

(2) In the case for $d = 1$, there is a unique arrow α such that $\mathfrak{s}(\alpha) = v$. Since v is a gentle vertex, then either $a_{1,1,\ell_{11}}\alpha \in \mathcal{I}$ or $a_{2,1,\ell_{21}}\alpha \in \mathcal{I}$ holds. Assume $a_{1,1,\ell_{11}}\alpha \in \mathcal{I}$ and let s' be the right maximal directed string given by the path

$$v_{2,1,\ell_{21}+1} \xrightarrow{a_{2,1,\ell_{21}+1}} v_{2,1,\ell_{21}+2} \xrightarrow{v_{2,1,\ell_{21}+2}} \cdots \xrightarrow{a_{2,1,l_{21}}} v_{2,1,l_{21}+1}$$

on $(\mathcal{Q},\mathcal{I})$ satisfying $\alpha = a_{2,1,\ell_{21}+1}$. Then $a_{2,1,\ell_{21}}\alpha \notin \mathcal{I}$, and we have

$$\Omega_1(E(v)) \cong \mathbb{M}(s') \oplus \bigoplus_{\imath\in I} M_\imath$$

by Lemma 4.2, where I is some index set, all $M_\imath$ are directed string modules, and

$$\Psi_0 \cong \mathbb{M}(s') \cong P(v)$$

is both a directed string module and a projective module since $\mathcal{I}$ is generated by some paths of length two.

(3) In the case for $d = 2$, there are two arrow α and β such that $\mathfrak{s}(\alpha) = v = \mathfrak{s}(\beta)$, and, without loss of generality, we have $a_{1,1,\ell_{11}}\alpha \in \mathcal{I}$, $a_{1,1,\ell_{11}}\beta \notin \mathcal{I}$, $a_{2,1,\ell_{21}}\alpha \notin \mathcal{I}$, $a_{2,1,\ell_{21}}\beta \in \mathcal{I}$. Then we have two right maximal directed strings s'_1 and s'_2 on $(\mathcal{Q},\mathcal{I})$ which are given by the paths

$$v_{1,1,\ell_{11}+1} \xrightarrow{a_{1,1,\ell_{11}+1}} v_{1,1,\ell_{11}+2} \xrightarrow{a_{1,1,\ell_{11}+2}} \cdots \xrightarrow{a_{1,1,l_{11}}} v_{1,1,l_{11}+1}$$

$$\text{and } v_{2,1,l_{21}+1} \xleftarrow{a_{2,1,l_{21}}} \cdots \xleftarrow{a_{2,1,\ell_{21}+2}} v_{2,1,\ell_{21}+2} \xleftarrow{a_{2,1,\ell_{21}+1}} v_{2,1,\ell_{21}+1},$$

respectively. Here, $a_{1,1,\ell_{11}+1} = \beta$, $a_{2,1,\ell_{21}+1} = \alpha$. Thus

$$\Omega_1(E(v)) \cong \mathbb{M}(s'_1(s'_2)^{-1}) \oplus \bigoplus_{\imath\in I} M_\imath$$

by Lemma 4.2, where I is some index set, all $M_\imath$ are directed string modules, and

$$\Psi_0 \cong \mathbb{M}(s'_1(s'_2)^{-1}) \cong P(v)$$

is projective.

Therefore, we conclude that Ψ_0 is projective. □

Example 4.6. Consider the 1-st syzygy $\Omega_1(E(2_\mathrm{R}))$ of the indecomposable injective module $E(2_\mathrm{R})$ given in Example 4.4, it follows from Lemma 4.5 that Ψ_0 is projective since 2_R is a gentle vertex. Of course, we can directly check that $\Psi_0 \cong P(2_\mathrm{R})$ is projective by using the definition of projection modules.

Lemma 4.7. *Keep the notation in Lemma 4.2 and Corollary 4.3, and let $c > 2$. Then the direct summand $\Psi_0 \leq_\oplus \Omega_1(E(v))$ is projective if and only if $d = 0$ (where t is the integer with $0 \leqslant t \leqslant \min\{c, d\}$ given in the proof of Lemma 4.2).*

Proof. By Lemma 4.2 and Corollary 4.3, if $d > 0$, then $\Psi_0 \leq_\oplus \Omega_1(E(v))$ is a module such that the following conditions hold:

- $\mathrm{top}(\Psi_0) \cong S(v)^{\oplus(c-1)}$;

- $\mathrm{rad}(\varPsi_0) \cong \bigoplus_{1\leqslant i\leqslant t} \mathbb{M}(s''_i)$, where s''_i is the path

$$v_{i,1,\ell_{i1}+2} \xrightarrow{a_{i,1,l_{i1}+2}} \cdots \xrightarrow{a_{i,1,l_{i1}}} v_{i,1,l_{i1}+1}$$

  satisfying $s' = a_{i,1,l_{i1}+1}s''_i$, see (4.3).

Then the projective cover of $\varPsi_0$ is

$$P(v)^{\oplus(c-1)} \to \varPsi_0$$

satisfying

$$\begin{aligned}&\dim_{\mathbb{k}}(P(v)^{\oplus(c-1)}) - \dim_{\mathbb{k}} \varPsi_0\\ &= (c-1)\Big(1+\sum_{i=1}^{d}\ell(s'_i)\Big) - \Big((c-1)+\sum_{i=1}^{t}\ell(s'_i)\Big)\\ &= (c-2)\sum_{i=1}^{t}\ell(s'_i) + (c-1)\sum_{i=t+1}^{d}\ell(s'_i) > 0.\end{aligned}$$

Here, $t=0$ admits $\sum_{i=1}^{t}\ell(s'_i)=0$, and $d=t=1$ admits $\sum_{i=t+1}^{d}\ell(s'_i)=0$. Thus, if $d>0$, then $\varPsi_0$ is non-projective.

If $d=0$, then v is a sink which admits that $\varPsi$ is isomorphic to a direct sum of some copies of $S(v)\cong P(v)$. □

Example 4.8. Consider the 1-st syzygy $\Omega_1(E(4))$ of the indecomposable injective module $E(4)$ given in Example 4.4. We have that $\varPsi_0 \cong \binom{4}{5}\oplus(4)$ is non-projective. Here, we have $c=3>2$, $d=1\neq 0$, $\mathbb{M}(s'_1)\cong P(4)$, $\ell(s'_1)=1$, and

$$\dim_{\mathbb{k}}(P(4)^{\oplus 2}) - \dim_{\mathbb{k}} \varPsi_0 = (3-1)(1+\ell(s'_1)) - 3 = 1 > 0.$$

Lemma 4.9. *Keep the notation in Lemma 4.2 and Corollary 4.3, and let $d>2$. Then, when $c\geqslant 1$, the direct summand $\varPsi_0 \leq_{\oplus} \Omega_1(E(v))$ is projective if and only if one of the following conditions holds.*

(1) *$c=1$ (in this case, the path s_1, see Figure 4.2 or Figure 4.4, is a unique path ending at v), and there is a unique right maximal path s'_j ($1\leqslant j\leqslant d$) such that $s_1s'_j\notin\mathcal{I}$, $\ell(s'_j)=1$, and $\mathfrak{t}(s'_j)$ is a sink of the quiver $\mathcal{Q}$.*

(2) *$c=1$, and there is a unique right maximal path $s'_j=$*

$$v = v_{1,1,\ell_{11}+1} \xrightarrow{a_{1,1,\ell_{11}+1}} v_{1,1,\ell_{11}+2} \xrightarrow{a_{1,1,\ell_{11}+2}} \cdots \xrightarrow{a_{1,1,l_{11}}} v_{1,1,l_{11}+1} \quad (\textit{cf. } (4.3))$$

($1\leqslant j\leqslant d$) such that $s_1s'_j\notin\mathcal{I}$, $\ell(s'_j)\geqslant 2$, and the vertex $v_{1,1,\ell_{11}+2}$ ($\in v^{\downarrow}$) on the path s'_j is not an $(a_{1,1,\ell_{11}+1},\downarrow)$-relational vertex.

(3) *$c=1$, and $s_1s'_j\in\mathcal{I}$ holds for all $1\leqslant j\leqslant d$.*

Proof. First of all, if $c>2$, then $\varPsi_0$ is non-projective by Lemma 4.7. Thus, we only need to consider the case for $1\leqslant c\leqslant 2$.

If $c=2$, we have two subcases as follows:

(a) $t=1$. In this subcase, cf. Figure 4.2, there is a unique path lying in $\{s'_1, s'_2, \ldots, s'_d\}$, assuming s'_1 without loss of generality, such that $s_1s'_1\notin\mathcal{I}$ holds. Then, by Lemma 4.2, $\varPsi_0$ is a string module corresponding to the string s'_1, then the projective cover is of the following form

$$P(v)\to\varPsi_0$$

which admits

$$\begin{aligned}
&\dim_{\Bbbk} P(v) - \dim_{\Bbbk} \varPsi_0 \\
&= \left(1 + \sum_{i=1}^{d} \ell(s'_i)\right) - \left(1 + \ell(s'_1)\right) \\
&= \sum_{i=2}^{d} \ell(s'_i) > 0. \ \ (\text{for } d > 2)
\end{aligned}$$

It follows that the kernel of $P(v) \to \varPsi_0$ is not zero. Thus, $\varPsi_0$ is non-projective as required.

(b) $t = 2$. In this subcase, cf. Figure 4.2, there are two paths lying in $\{s'_1, s'_2, \dots, s'_d\}$, assuming s'_1 and s'_2 without loss of generality, such that $s_1 s'_1 \notin \mathcal{I}$ and $s_2 s'_2 \notin \mathcal{I}$ hold. Then, by Lemma 4.2, $\varPsi_0$ is a string module corresponding to the string $(s'_1)^{-1} s'_2$, and we obtain that the projective cover is of the following form

$$P(v) \to \varPsi_0$$

which admits

$$\begin{aligned}
&\dim_{\Bbbk} P(v) - \dim_{\Bbbk} \varPsi_0 \\
&= \left(1 + \sum_{i=1}^{d} \ell(s'_i)\right) - \left(1 + \ell(s'_1) + \ell(s'_2)\right) \\
&= \sum_{i=3}^{d} \ell(s'_i) > 0. \ \ (\text{by using } d > 2)
\end{aligned}$$

It follows that $\varPsi_0$ is non-projective by an argument similar to that in (a).

Thus, $\varPsi_0$ is non-projective in the case of $c = 2$.

If $c = 1$, then $E(v)$ is both an injective module and a directed string module. We have the following two cases.

(c) there is a unique path lying in $\{s'_1, s'_2, \dots, s'_d\}$, assuming s'_1 without loss of generality, such that $s_1 s'_1 \notin \mathcal{I}$ holds. In this case, $\varPsi_0$ is a string module corresponding to the string s''_1 which is obtained by deleting the first arrow $a_{1,1,\ell_{11}+1}$ of s' where $\ell_{11} + 1 \geqslant 1$, $v_{1,1,\ell_{11}+1} = v$, see (4.3). Then the following two subcases need to be considered.

(c-1) $\ell(s'_1) = 1$. In this subcase, s''_1 is a path of length zero given by the vertex $v_{1,1,\ell_{11}+2}$. It follows that $\varPsi_0 = S(v_{1,1,\ell_{11}+2})$ is projective if and only if $v_{1,1,\ell_{11}+2} = \mathfrak{t}(s''_1) = \mathfrak{t}(s'_1)$ is a sink. We obtain (1).

(c-2) $\ell(s'_1) \geqslant 2$. In this subcase,

$$s''_1 = \ \ v_{1,1,\ell_{11}+2} \xrightarrow{a_{1,1,\ell_{11}+2}} \cdots \xrightarrow{a_{1,1,l_{11}}} v_{1,1,l_{11}+1}$$

is a right maximal directed string of length $\geqslant 1$. We have $\varPsi_0 \cong \mathbb{M}(s''_1)$. Thus, $\varPsi_0$ is non-projective if and only if there is at least one arrow α starting at $v_{1,1,\ell_{i1+2}}$, and by the definition of almost gentle algebra, we have $a_{1,1,\ell_{11}+1}\alpha \in \mathcal{I}$. That is, $\varPsi_0$ is non-projective if and only if $v_{1,1,\ell_{i1+2}}$ is $(a_{1,1,\ell_{11}+1}, \downarrow)$-relational. We obtain (2).

(d) For all path $s'_j \in \{s'_1, s'_2, \dots, s'_d\}$, we have $s_1 s'_j \in \mathcal{I}$. In this case, $\varPsi_0 = 0$ is projective. We obtain (3).

Therefore, $\varPsi_0$ is projective if and only if the $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex v lies in one of the cases (c-1), (c-2) and (d). □

Example 4.10. (1) Consider the almost gentle pair $(\mathcal{Q},\mathcal{I}')$ given by the quiver $\mathcal{Q}$ provided in Example 2.2 and the admissible ideal $\mathcal{I}' = \mathcal{I}\backslash\{a_{1,2}a_{2,4}\} \cap \{a_{2,4}a_{2,5}\}$ see FIGURE 4.7. Then the vertex 2 is a $(1^{\mathrm{in}}, 3^{\mathrm{out}})$-type vertex such that the conditions given

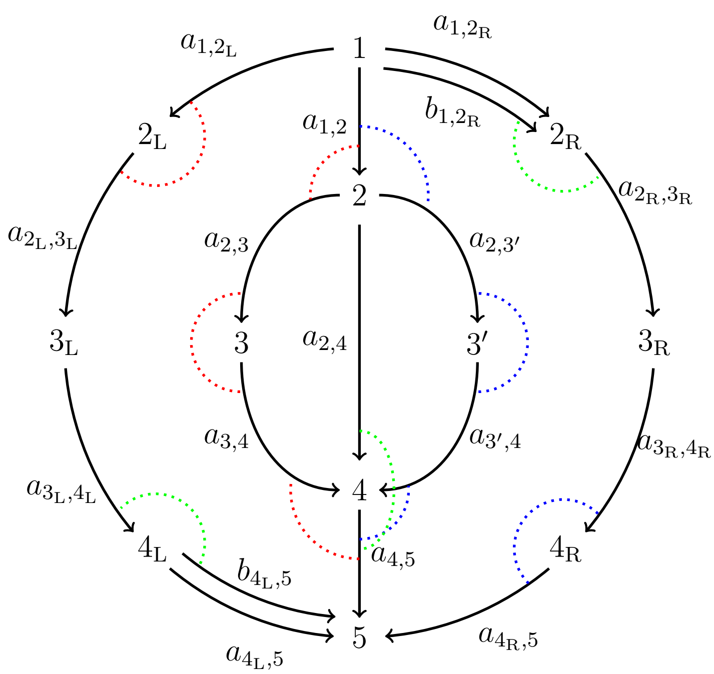


FIGURE 4.7. An almost gentle pair $(\mathcal{Q},\mathcal{I}')$ $(\mathcal{I}' = \mathcal{I}\backslash\{a_{1,2}a_{2,4}\} \cap \{a_{2,4}a_{2,5}\})$

in Lemma 4.9(1) are satisfied, except the condition $\mathfrak{t}(s'_j)$ to be a sink. That is, $c = 1$, $d = 3(> 2)$, $s_1 = a_{1,2}$, $s'_1 = a_{2,3}$, $s'_2 = a_{2,4}(= s'_j)$, $s'_3 = a_{2,3'}$, $\ell(s'_2) = 1$, and $s_1s'_2 \notin \mathcal{I}$ hold. In this instance, $\mathit{\Psi}_0$ is non-projective. Indeed, one can check that $\mathit{\Psi}_0 \cong S(4)$ is non-projective simple.

(2) Consider the almost gentle pair $(\mathcal{Q},\mathcal{I}'')$ given by the quiver $\mathcal{Q}$ provided in Example 2.2 and the admissible ideal $\mathcal{I}'' = \mathcal{I}\backslash\{a_{1,2}a_{2,4}\}$, Then the vertex 2 is a $(1^{\mathrm{in}}, 3^{\mathrm{out}})$-type vertex such that the conditions given in Lemma 4.9(2) are satisfied. Thus, for $E(2) = \binom{1}{2}$, we have that $\mathit{\Psi}_0$ is projective. Indeed, one can check that $\mathit{\Psi}_0 \cong \binom{4}{5} \cong P(4)$ since $P(1)$ is the indecomposable projective module corresponding to the claw

$$a_{1,2_L} \curlywedge a_{1,2}a_{2,4}a_{4,5} \curlywedge a_{1,2_R}a_{2_R,3_R}a_{3_R,4_R} \curlywedge b_{1,2_R}.$$

(3) The indecomposable injective module $E(2)$ over the almost gentle algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$ shown in Example 2.2 is the one corresponding to the $(1^{\mathrm{in}}, 3^{\mathrm{out}})$-type vertex 2. We have $c = 1$, $d = 3$ ($\geqslant 2$), $s_1 = a_{1,2}$, $s'_1 = a_{2,3}$, $s'_2 = a_{2,4}a_{4,5}$, and $s'_3 = a_{2,3'}$. Obviously, $4 \in 2^{\downarrow} = \{3, 4, 3'\}$ is a vertex on the path s'_2, and the length of s'_2 is two. Notice that $s_1s'_j \in \mathcal{I}$ holds for all $j \in \{1, 2, 3\}$, then $\mathit{\Psi}_0$ is projective by Lemma 4.9 (3). In fact, we can check that

$$\Omega_1(E(2)) \cong (2_L) \oplus (2_R) \oplus \begin{pmatrix} 2_R \\ 3_R \\ 4_R \end{pmatrix} = \bigoplus_{\imath\in I=\{1,2,3\}} M_\imath$$

$$\mathit{\Psi}_0 = 0, \quad \text{and} \bigoplus_{\jmath\in J=\emptyset} S_\jmath = 0$$

by using Lemma 4.2. Thus $\mathit{\Psi}_0 = 0$ is projective.

Definition 4.11. For a vertex v of almost gentle pair $(\mathcal{Q},\mathcal{I})$, assume that v is a $(c^{\mathrm{in}},d^{\mathrm{out}})$-type vertex. We call it an *invalid vertex* if one of the following conditions holds.

(1) v is a gentle vertex with $c=2$;
(2) c is arbitrary and $d=0$;
(3) $c=1$, there is a unique right maximal path s'_j $(1\leqslant j\leqslant d)$ such that $s_1s'_j\notin\mathcal{I}$, $\ell(s'_j)=1$, and $\mathfrak{t}(s'_j)$ is a sink of the quiver $\mathcal{Q}$;
(4) $c=1$, there is a unique right maximal path $s'_j=$

$$v=v_{1,1,\ell_{11}+1}\xrightarrow{a_{1,1,\ell_{11}+1}}v_{1,1,\ell_{11}+2}\xrightarrow{a_{1,1,\ell_{11}+2}}\cdots\xrightarrow{a_{1,1,l_{11}}}v_{1,1,l_{11}+1}$$

$(1\leqslant j\leqslant d)$ such that $s_1s'_j\notin\mathcal{I}$, $\ell(s'_j)\geqslant 2$, and the vertex $v_{1,1,\ell_{11}+2}$ is not an $(a_{1,1,\ell_{11}+1},\downarrow)$-relational vertex;
(5) $c=1$ and $s_1s'_j\in\mathcal{I}$ for any $1\leqslant j\leqslant d$;

Proposition 4.12. *Keep the notation in Lemma 4.2 and Corollary 4.3. For the indecomposable injective module $E(v)$ corresponding to the $(c^{\mathrm{in}},d^{\mathrm{out}})$-type vertex v, if $c\geqslant 1$, then the direct summand Ψ_0 $(\leq_{\oplus}\Omega_1(E(v)))$ is projective if and only if v is invalid .*

Proof. In the case for $1\leqslant c\leqslant 2$ and $d=0$, we have that $\Psi_0\cong S(v)^{\oplus(c-1)}$ is projective since v is a sink. Thus, in the condition (2), we assume $c\geqslant 3$.

The conditions (1)–(5) given in Definition 4.11 are given by Lemmata 4.5, 4.7 and 4.9, respectively. To be more precisely, if $c=2$, $d\leqslant 2$ and v is a gentle vertex, then Lemma 4.5 provides the condition (1); if $c\geqslant 3$, then Lemma 4.7 provides the condition (2); if $c=1$ and $d\geqslant 3$, then v satisfying the conditions given in Lemma 4.9 is a vertex such that the conditions (3)–(5) hold. Thus, we only need to consider the following two cases:

(A) $1\leqslant c\leqslant 2$, $d\leqslant 2$ and v is not a gentle vertex;
(B) $c=1$, $d\leqslant 2$, and v is a gentle vertex.

For the case (A), by the definition of gentle vertices, we have that one of the following cases occurs:

(a) $c=1$, $d=2$, $s_1s'_1\in\mathcal{I}$, and $s_1s'_2\in\mathcal{I}$;
(b) $c=2$, $d=1$, $s_1s'_1\in\mathcal{I}$, and $s_2s'_1\in\mathcal{I}$;
(c) $c=d=2$ such that $s_1s'_2$ and $s_1s'_2$ lie in $\mathcal{I}$, and at least one of $s_1s'_1$ and $s_2s'_2$ lies in $\mathcal{I}$.

In the case (a), we have $\Psi_0=0$ which is a trivial projective module, we obtain the condition (5).

In the case (b), we have $\Psi_0\cong S(v)$ which is non-projective since v does not be a sink of $\mathcal{Q}$.

In the case (c), if $s_1s'_1\in\mathcal{I}$ and $s_2s'_2\notin\mathcal{I}$, then Ψ_0 is a directed string module corresponding to the string s'_2. In this case, the kernel of the projective cover of Ψ_0, which is of the form $P_0\to\Psi_0$, is non-zero since the following formula

$$\dim_{\Bbbk}P_0-\dim_{\Bbbk}\Psi_0=(\ell(s'_1)+\ell(s'_2)+1)-(\ell(s'_2)+1)=\ell(s'_1)>0$$

holds. Thus, Ψ_0 is non-projective. Similarly, We have that Ψ_0 is non-projective in other cases.

For the case (B), if $d=0$ then it is trivial that $\Psi_0=0$. Thus, we have $1\leqslant d\leqslant 2$, and obtain the following two cases.

(d) $c=1$ and $d=1$.
(e) $c=1$ and $d=2$.

In the case (d), we have that v is a $(1^{\rm in}, 1^{\rm out})$-type vertex and s_1s_1' equals to

$$\overbrace{v_{1,1,1} \xrightarrow{a_{1,1,1}} \cdots \xrightarrow{a_{1,1,\ell_{11}}} v}^{s_1} = \overbrace{v_{1,1,\ell_{11}+1} \xrightarrow{a_{1,1,\ell_{11}+1}} \underbrace{v_{1,1,\ell_{11}+2} \xrightarrow{a_{1,1,\ell_{11}+2}} \cdots \xrightarrow{a_{1,1,l_{11}}} v_{1,1,l_{11}+1}}_{s_1''}}^{s_1'}.$$

Then $\Psi_0 \cong \mathbb{M}(s'')$ is projective if and only if one of the following conditions holds:

(d-1) $s_1s_1' \notin \mathcal{I}$, $\ell(s_1') = 1$, and $v_{1,1,\ell_{11}+2} = \mathfrak{t}(s_1')$ is a sink, that is, $v_{1,1,\ell_{11}+2}$ is a vertex lying in $v^{\downarrow}$ satisfying the condition (3);

(d-2) $s_1s_1' \notin \mathcal{I}$, $\ell(s_1') \geqslant 2$, and $v_{1,1,\ell_{11}+2}$ is not a $(a_{1,1,\ell_{11}+2}, \downarrow)$-relational vertex, that is, the condition (4) holds;

(d-3) $s_1s_1' \in \mathcal{I}$. In this case, the vertex v satisfies the condition (5).

If v is a $(1^{\rm in}, 2^{\rm out})$-type vertex, then at least one of s_1s_1' and s_1s_2' lies in $\mathcal{I}$. If $s_1s_1' \notin \mathcal{I}$, then $s_1s_2' \in \mathcal{I}$. Thus, similar to the case (d), Ψ_0 is projective if and only if one of the following conditions holds:

(e-1) $s_1s_1' \notin \mathcal{I}$, $\ell(s_1') = 1$, and $v_{1,1,\ell_{11}+2} = \mathfrak{t}(s_1')$ is a sink, this satisfies the condition (3);

(e-2) $s_1s_1' \notin \mathcal{I}$, $\ell(s_1') \geqslant 2$, and $v_{1,1,\ell_{11}+2}$ is not a $(a_{1,1,\ell_{11}+2}, \downarrow)$-relational vertex, this satisfies the condition (4);

The cases for $s_1s_1' \in \mathcal{I}$ and $s_1s_2' \notin \mathcal{I}$ are dual.

(e-3) s_1s_1' and s_1s_2' lie in $\mathcal{I}$. Then we have $\Psi_0 = 0$ is projective, this satisfies the condition (5).

□

4.3. **Self-injective dimension.** We divide all vertices of an almost gentle pair $(\mathcal{Q}, \mathcal{I})$ to three parts: sources; $(1^{\rm in}, d^{\rm out})$-vertices, and $((\geqslant 2)^{\rm in}, d^{\rm out})$-vertices. If a vertex v is a source, then $E(v)$ is a simple module corresponding to v, and in this situation we have computed the projective dimension proj.dim$E(v)$ in Proposition 3.10, i.e.,

$$\text{proj.dim}E(v) = \text{proj.dim}S(v) = \ell(F_v),$$

where F_v is the right maximal forbidden path starting at the sink v.

Next, we compute proj.dim$E(v)$ in the case for v being a vertex lies in the second and third situations. By Lemma 4.2, we have that $\Omega_1(E(v))$ is a direct sum of some right maximal directed string modules and the module Ψ_0. The projective resolution of any right maximal directed string module can be computed by using Lemma 3.3. Therefore, we need to construct the projective resolution of Ψ_0.

Lemma 4.13. *For any $n \geqslant 1$, the n-th syzygy $\Omega_n(\Psi_0)$ of Ψ_0 is a direct sum of some right maximal directed string modules and some simple module.*

Proof. Let v be a $(c^{\rm in}, d^{\rm out})$-type vertex shown in Figure 4.2, and assume that $s_is_j' \notin \mathcal{I}$ if and only if $0 \leqslant i = j \leqslant t$ $(< \min\{c, d\})$. Recall that any A-module M over the almost gentle algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$ can be described by the bound quiver $(\mathcal{Q}, \mathcal{I})$ as the quiver representation $(V_w, \varphi_a)_{w\in\mathcal{Q}_0, a\in\mathcal{Q}_1}$, where V_w is the $\Bbbk$-linear space which is $\Bbbk$-linear isomorphic to Me_w and φ_a is a $\Bbbk$-linear space $V_{\mathfrak{s}(a)} \to V_{\mathfrak{t}(a)}$ such that $\varphi_a\varphi_b = 0$ $(\mathfrak{t}(a) = \mathfrak{s}(b))$ if and only if $ab \in \mathcal{I}$.

Computing the kernel of the projective cover of $E(v)$, we obtain that the quiver representation of Ψ is of the form

$$(\Psi e_w, \varphi_a)_{w\in\mathcal{Q}_0, a\in\mathcal{Q}_1},$$

where for any vertex w on the path s''_j except v (s''_j a string obtained by deleting the first arrow of s'_j), and

$$\Psi e_w \cong \mathbb{M}(s''_j)e_w,$$

to be precise, we have $\mathrm{rad}(\Psi) \cong \bigoplus_{j=1}^{d} \mathbb{M}(s''_j)$. FIGURE 4.8 provides an example for the above, and for simplicity we assume that each path s'_j has no self-injection in this figure. Furthermore, $\mathrm{top}(\Psi) \cong S(v)^{\oplus(t-1)}$, and then the projective cover of Ψ is of the

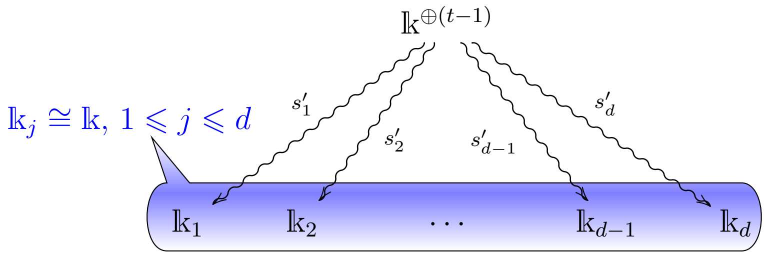


FIGURE 4.8. The quiver representation of Ψ

(all $\Bbbk$-linear spaces corresponding to vertices on each s'_j is one-dimensional, except the $\Bbbk$-linear space $\Bbbk^{\oplus(t-1)}$ corresponding to the vertex v)

form $P(v)^{\oplus(t-1)} \to \Psi$. Now, we have $\Omega_1(\Psi) \cong \bigoplus_{j=1}^{d} \mathbb{M}(s''_j)$, then it is clear that each direct summand of $\Omega_1(\Psi)$ is either a right maximal directed string module or a simple module, and so is $\Omega_n(\Psi_1)$ by using Proposition 3.5 and induction.

On the other hand, by Lemma 4.2, we have

$$\Psi_0 = \Psi \oplus \bigoplus_{\jmath \in J} S_\jmath,$$

it follows that $\Omega_n(\Psi_0) = \Omega_n(\Psi) \oplus \bigoplus_{\jmath \in J} \Omega_n(S_\jmath)$. Notice that $S_\jmath$ is a direct string module corresponding to a simple string, so each direct summand of $\Omega_n(S_\jmath)$ is either a right maximal directed string module or a simple module by using Proposition 3.5 and induction. □

For a directed string $\delta = a_1a_2\cdots a_l$, a forbidden path $F = b_1b_2\cdots b_l$ is said to be a *δ-forbidden* if one of the following conditions holds:

- $\mathfrak{s}(F) = \mathfrak{t}(\delta)$ and $\delta b_1 \notin \mathcal{I}$;
- $\mathfrak{s}(F) = \mathfrak{s}(\delta)$ and $a_1 \neq b_1$.

Example 4.14. The indecomposable injective module $E(2) = \binom{1}{2}$ over the almost gentle algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$ given in Example 2.2 is a directed string module corresponding to the directed string $a_{1,2}$. Then the forbidden paths $b_{1,2_\mathrm{R}}a_{2_\mathrm{R},3_\mathrm{R}}$, $a_{1,2_\mathrm{R}}$ and $b_{1,2_\mathrm{L}}a_{2_\mathrm{L},3_\mathrm{L}}$ are $a_{1,2}$-forbidden paths. But $a_{2,4}$ is not an $a_{1,2}$-forbidden path because $a_{1,2}a_{2,4} \in \mathcal{I}$.

Notation 4.15. Let $\mathcal{F}(\delta)$ be the set of all δ-forbidden paths. If $\ell(\delta) = 0$, then $\mathcal{F}(\delta) = \mathcal{F}(v)$, where $v = \mathfrak{s}(\delta) = \mathfrak{t}(\delta)$.

Now we show the following proposition.

Proposition 4.16. *Let δ be either a right maximal directed string or a simple string. Then $\Omega_{n-1}(\mathbb{M}(\delta))$ is non-projective if and only if there is a forbidden path $F \in \mathcal{F}(\delta)$ whose length is n.*

Proof. The case for δ to be simple is shown in Proposition 3.10. Now, we assume that δ is the right maximal directed string

$$v_{1,1} \xrightarrow{a_{1,1}} v_{1,2} \xrightarrow{a_{1,2}} \cdots \xrightarrow{a_{1,l_1}} v_{1,l_1+1}$$

with length $l_1 \geqslant 1$ and $v_{1,1}$ a $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex ($d \geqslant 1$), and the claw corresponding to $P(v_{1,1})$ is

$$\kappa = \delta_1 \wedge \delta_2 \wedge \cdots \wedge \delta_d,$$

where $\delta = \delta_1$, $\delta_i = v_{i,1} \xrightarrow{a_{i,1}} v_{i,2} \xrightarrow{a_{i,2}} \cdots \xrightarrow{a_{i,l_i}} v_{i,l_i+1}$ ($l_i \geqslant 1$, $1 \leqslant i \leqslant d$) is right maximal, and $v_{1,1} = v_{2,1} = \cdots = v_{d,1}$. Then we have

$$\Omega_1(\mathbb{M}(\delta)) \cong \bigoplus_{i=1}^{d} \mathbb{M}(\delta_i^*),$$

where

$$\delta_i^* = \begin{cases} 0, & \text{if } i = 1; \\ v_{i,2} \xrightarrow{a_{i,2}} \cdots \xrightarrow{a_{i,l_i}} v_{i,l_i+1}, & \text{if } i \geqslant 2. \end{cases}$$

In the case for $d = 1$, $\mathbb{M}(\delta)$ is projective, and so $\Omega_1(\mathbb{M}(\delta)) = 0$ is projective.

In the case for $d \geqslant 2$, for any $i \geqslant 2$, we have $\mathbb{M}(\delta_i^*) \neq 0$. Furthermore, $\mathbb{M}(\delta_i^*)$ is non-projective if and only if $v_{i,2}$ is $(a_{i,1}, \downarrow)$-relational. It follows that

- there is an arrow $a_{i,2}^{(1)}$ such that $a_{i,1}a_{i,2}^{(1)}$ is a forbidden path lying in $\mathcal{F}(\delta)$;
- the 1-st syzygy $\Omega_1(\mathbb{M}(\delta))$ is non-projective;
- there is a directed string $\delta_i^{(1)}$ which is of the form $a_{i,2}^{(1)} \cdots$ such that

$$\mathbb{M}(\delta_i^{(1)*}) \leq_{\oplus} \Omega_1(\mathbb{M}(\delta_i^*)) \leq_{\oplus} \Omega_2(\mathbb{M}(\delta)),$$

  where $\delta_i^{(1)*}$ is the string obtained by deleting the first arrow $a_{i,2}^{(1)}$ of $\delta_i^{(1)}$, see Figure 4.9.

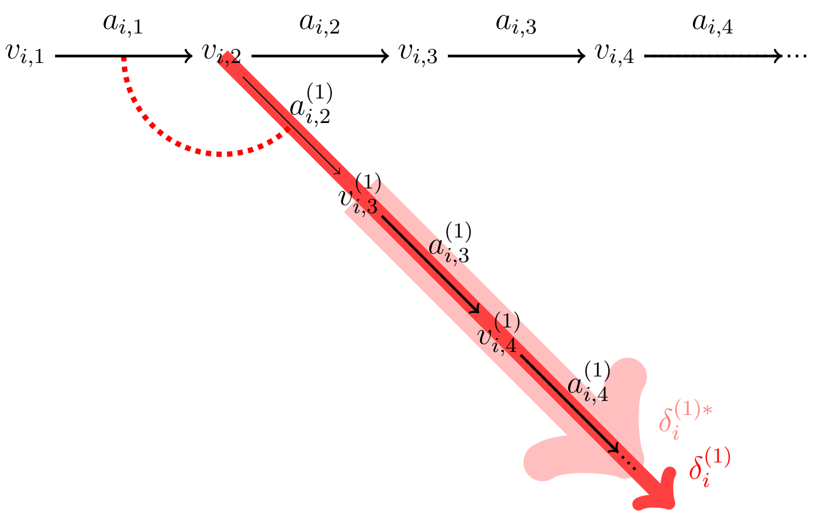


Figure 4.9. $\Omega_1(\mathbb{M}(\delta))$ is non-projective

Therefore, if $\mathbb{M}(\delta_i)$ is non-projective, then so is $\Omega_1(\mathbb{M}(\delta))$, and in this case, we get a forbidden path $a_{i,1}a_{i,2}^{(1)}$ lying in $\mathcal{F}(\delta)$; if $\mathbb{M}(\delta_i^{(1)*})$ is non-projective, then so is $\Omega_2(\mathbb{M}(\delta))$, and in this case, by the method similar to above, there is a directed string $\delta_i^{(2)}$ which is of the form $a^{(2)}_{i,3}\cdots$ such that $a^{(1)}_{i,2}a^{(2)}_{i,3}$ is a forbidden path, then we get a forbidden path $a_{i,1}a^{(1)}_{i,2}a^{(2)}_{i,3}$ lying in $\mathcal{F}(\delta)$, see FIGURE 4.10. Repeating the above step, we can find

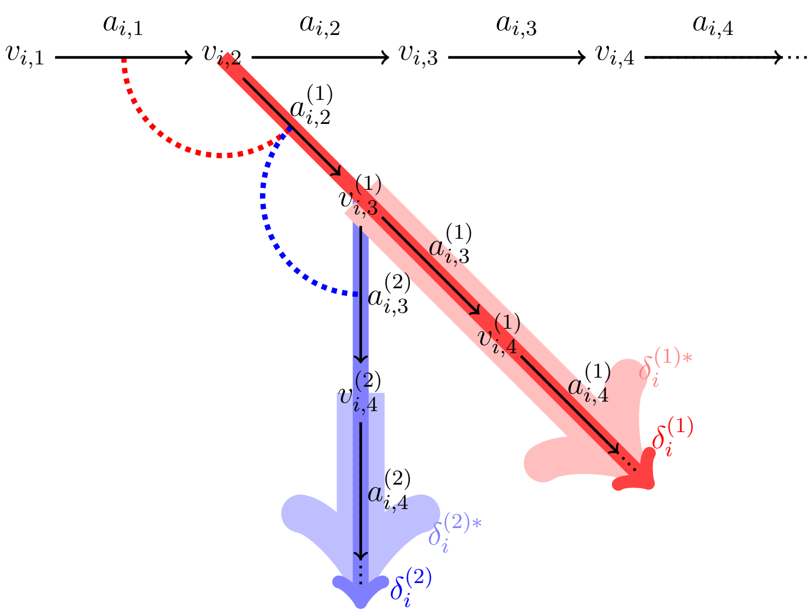


FIGURE 4.10. $\Omega_2(\mathbb{M}(\delta))$ is non-projective (note that $a^{(1)}_{i,3}$ and $a^{(2)}_{i,3}$ may be the same)

a forbidden path of length n which lies in $\mathcal{F}(\delta)$ if $\Omega_{n-1}(\mathbb{M}(\delta))$ is non-projective.

Conversely, if δ is a simple string corresponding to v_1, and by assumption there is a forbidden path of length n lying in $\mathcal{F}(\delta)=\mathcal{F}(v_1)$. Then $\Omega_{n-1}(\mathbb{M}(\delta))$ is non-projective by Proposition 3.10. Next, consider the case for δ to be a right maximal directed string

$$v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} \cdots \xrightarrow{a_l} v_{l+1}$$

with length $l \geqslant 1$. Since $\mathcal{F}(\delta)$ contains a forbidden path of length n and δ is a right maximal directed string, all forbidden paths lying in $\mathcal{F}(\delta)$ share the same source $v_{1,1}$ by the definition of $\mathcal{F}(\delta)$. Suppose that there is a forbidden path $F=b_1b_2\cdots b_n$ which is of the form

$$w_1 \xrightarrow{b_1} w_2 \xrightarrow{b_2} \cdots \xrightarrow{b_n} w_{n+1}$$

such that $w_1=v_1$ and $a_1\neq b_1$. The 1-st syzygy $\Omega_1(\mathbb{M}(\delta))$ contains a direct summand M_1 which is a directed string module (see Lemma 3.2) such that $M_1b_2=0$ and $\mathrm{top}(M_1)\cong S(w_2)$ by computing the kernel of the projective cover of $\mathbb{M}(\delta)$. Note that M_1 is non-projective. Otherwise, $M_1\cong e_{w_2}A$. It follows that $M_1b_2=e_{w_2}Ab_2\neq 0$ since $e_{w_2}Ab_2$ contains at least one element $b_2\neq 0$. This is a contradiction. Then $\Omega_1(\mathbb{M}(\delta))$ is non-projective.

Since w_2 is $(b_1,\downarrow)$-relational, the projective cover of M_1 is $P(w_2)\to M_1$, then $\Omega_1(M_1)$ has a direct summand M_2 which is a directed string module (see Lemma 3.2) such that $M_2b_3=0$ and $\mathrm{top}(M_2)\cong S(w_3)$ by computing the kernel of the projective cover of M_1. Similarly, we have $M_2\leq_\oplus \Omega_1(M_1)$ $(\leq_\oplus \Omega_2(\mathbb{M}(\delta)))$ is non-projective, and so $\Omega_2(\mathbb{M}(\delta))$ is non-projective.

Therefore $\Omega_{n-1}(\mathbb{M}(\delta))$ is non-projective by induction. □

Example 4.17. Recall that the minimal projective resolution of the indecomposable injective module $E(2)$ over the almost gentle algebra $A = \Bbbk\mathcal{Q}/\mathcal{I}$ given in Example 2.2 is

$$0 \longrightarrow P_2 = P(3_{\mathrm{L}}) \oplus P(3_{\mathrm{R}}) \xrightarrow[p_2]{\begin{pmatrix} a_{2_{\mathrm{L}},3_{\mathrm{L}}} & 0 \\ 0 & a_{2_{\mathrm{R}},3_{\mathrm{R}}} \\ 0 & 0 \end{pmatrix}} P_1 = P(2_{\mathrm{L}}) \oplus P(2_{\mathrm{R}}) \oplus P(2_{\mathrm{R}})$$

$$\xrightarrow[p_1]{(a_{1,2_{\mathrm{L}}}\ b_{1,2_{\mathrm{R}}}\ a_{1,2_{\mathrm{R}}})} P_0 = P(1) \xrightarrow[p_0]{} \mathbb{M}(a_{1,2}) \cong E(2) \longrightarrow 0,$$

see Example 3.6. Then we have

$$\Omega_1(E(2)) \cong (2_{\mathrm{L}}) \oplus (2_{\mathrm{R}}) \oplus \begin{pmatrix} 2_{\mathrm{R}} \\ 3_{\mathrm{R}} \\ 4_{\mathrm{R}} \end{pmatrix} = S(2_{\mathrm{L}}) \oplus S(2_{\mathrm{R}}) \oplus P(2_{\mathrm{R}})$$

is non-projective since $S(2_{\mathrm{L}})$ and $S(2_{\mathrm{R}})$ are non-projective. Here, $E(2)$ is also a right maximal directed string module, and the directed string corresponding to it is $a_{1,2}$. We have three $a_{1,2}$-forbidden paths of length $\geqslant 1$ as follows:

$$a_{1,2_{\mathrm{L}}}a_{2_{\mathrm{L}},3_{\mathrm{L}}},\ b_{1,2_{\mathrm{R}}}a_{2_{\mathrm{R}},3_{\mathrm{R}}},\ \text{and}\ a_{1,2_{\mathrm{R}}}.$$

We have that $S(2_{\mathrm{L}})$ and $S(2_{\mathrm{R}})$ correspond to the forbidden paths $a_{1,2_{\mathrm{L}}}a_{2_{\mathrm{L}},3_{\mathrm{L}}}$ and $b_{1,2_{\mathrm{R}}}a_{2_{\mathrm{R}},3_{\mathrm{R}}}$ of length two, respectively. The fact that $P(2_{\mathrm{R}}) \leq_{\oplus} \Omega_1(E(2))$ yields $\Omega_1(P(2_{\mathrm{R}})) = 0$. Proposition 4.16 shows that the forbidden paths $a_{1,2_{\mathrm{L}}}a_{2_{\mathrm{L}},3_{\mathrm{L}}}$ and $b_{1,2_{\mathrm{R}}}a_{2_{\mathrm{R}},3_{\mathrm{R}}}$, as two elements in $\mathcal{F}(a_{1,2})$, exist.

In Lemma 4.2, we show that the 1-st syzygy of an indecomposable injective $E(v)$ (v is a $(c^{\mathrm{in}}, d^{\mathrm{out}})$)-type vertex module over an almost gentle algebra can be described by three parts:

- a module $\varPsi$ whose top is a direct sum of some copies of $S(v)$;
- a semi-simple module which is of the form $S(v)^{\oplus(c-t)}$, where t is an integer satisfying $0 \leqslant t \leqslant c$;
- a direct sum whose direct summand is either a right maximal directed module or a simple module;

Lemma 4.13 shows that any direct summand of the n-th ($n \geqslant 1$) syzygy $\Omega_n(\varPsi_0)$ of $\varPsi_0 = \varPsi \oplus S(v)^{\oplus(c-d)}$ is either a right maximal directed string module or a simple module. Propositions 3.10 and 4.16 provide a correspondence from any right maximal directed string module or simple module to a forbidden path on the bound quiver of an almost gentle algebra. Thus, for any $n \geqslant 1$, each non-projective direct summand of $\Omega_n(E(v))$ can be described by using a forbidden path of length $n+1$, except the direct summand $\varPsi_0$ of $\Omega_1(E(v))$. The projectivity of $\varPsi_0$ is described in Proposition 4.12, thus we can compute the projective resolution of any indecomposable injective module over almost gentle algebra.

Definition 4.18. For an indecomposable injective module corresponding a $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex v, assume that the anti-claw of $E(v)$, written as ξ in this section, is of the form shown in Figure 4.2. A forbidden path F is said to be a *ξ-forbidden path* if it satisfies one of the following conditions:

(1) there is an integer $1 \leqslant i \leqslant c$ such that $F \in \mathcal{F}(\mathfrak{s}(s_i))$;
(2) F is a forbidden path starting at v such that v is not an invalid vertex.

The set of all ξ-forbidden paths is written as $\mathcal{F}(\xi)$. In particular, if $c = 0$, then $E(v) = S(v)$ is simple, and we have $\mathcal{F}(\xi) = \mathcal{F}(v)$.

Theorem 4.19. *Let $A = (\mathcal{Q}, \mathcal{I})$ be an almost gentle algebra. Then for any $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex v (keep the notation in Lemma 4.2 and Figure 4.2), we have*

$$\mathrm{proj.dim}E(v) = \sup_{F\in\mathcal{F}(\xi)} \ell(F),$$

where ξ is the anti-claw corresponding to $E(v)$.

(Note that if $\mathcal{F}(\xi) = \emptyset$, then take $\sup_{F\in\mathcal{F}(\xi)} \ell(F) = 0$.)

Proof. If $c = 0$, then $E(v) = S(v)$. Thus, we have

$$\mathrm{proj.dim}E(v) = \mathrm{proj.dim}S(v) = \sup_{F\in\mathcal{F}(v)} \ell(F).$$

When $c \geqslant 1$, we have two cases as follows:

(1) v is invalid;

(2) v is not invalid.

Assume $\xi = s_1 \vee s_2 \vee \cdots \vee s_c$. In the case (1), $\mathcal{F}(\xi)$ contains c ($\geqslant 1$) right maximal forbidden paths $F_1, \ldots, F_c$ which respectively correspond to $s_1, \ldots, s_c$ such that $s_i b_{i1} \notin \mathcal{I}$, where b_{i1} is the first arrow of $F_i = b_{i1}b_{i2}\cdots b_{il_i}$ ($l_i \geqslant 1$), see Definition 4.18(1). By Proposition 4.16, we have that $\Omega_t(E(v))$ is non-projective if and only if there is an integer i with $1 \leqslant i \leqslant c$ such that $\ell(F_i) = t + 1$. Without loss of generality, suppose

$$\ell(F_1) = \sup_{1\leqslant i\leqslant c} \ell(F_i). \tag{4.6}$$

Since $\Omega_{\ell(F_1)-1}(E(v))$ is non-projective, we have $\mathrm{proj.dim}E(v) \geqslant \ell(F_1)$. If $\mathrm{proj.dim}E(v) > \ell(F_1)$, then $\Omega_{\ell(F_1)}(E(v))$ is non-projective. In this case, by Proposition 4.16, there exists a forbidden path $F_j \in \{F_1, \ldots, F_c\}$ such that $\ell(F_j) \geqslant \ell(F_1) + 1$. It contradicts the maximality of $\ell(F_1)$. Thus, we have

$$\mathrm{proj.dim}E(v) = \ell(F_1), \tag{4.7}$$

and therefore

$$\mathrm{proj.dim}E(v) = \sup_{1\leqslant i\leqslant c} \ell(F_i), \tag{4.8}$$

as required.

In the case (2), we obtain that $\mathcal{F}(\xi)$ contains c ($\geqslant 1$) right maximal forbidden paths $F_1, \ldots, F_c$ and F_{c+1}, where $F_1, \ldots, F_c$ respectively correspond to $s_1, \ldots, s_c$ such that $s_i b_{i1} \notin \mathcal{I}$ (b_{i1} is the first arrow of F_i); and F_{c+1} corresponds to v, that is, F_{c+1} is a forbidden path starting at v (see Definition 4.18(2)).

Next, we compute $\mathrm{proj.dim}\,\Psi_0$. By Proposition 4.12, v is not a sink since $d = 0$ contradicts Definition 4.11(2). We have the following subcases:

(2.1) $c = 1$ and $d \geqslant 1$;

(2.2) $c = 2$, then v is not a gentle vertex;

(2.3) $c \geqslant 3$, then Ψ_0 is non-projective by Lemma 4.7.

We only prove the case (2.1). The proofs of (2.2) and (2.3) are similar.

In the subcase (2.1), there is a unique s'_j ($1 \leqslant j \leqslant d$) such that $s_1 s'_j \notin \mathcal{I}$ by Definition 4.11(5). Without loss of generality, let $s_1 =$

$$v_{1,1,1} \xrightarrow{a_{1,1,1}} v_{1,1,2} \xrightarrow{a_{1,1,2}} \cdots \xrightarrow{a_{1,1,\ell_{11}}} v_{1,1,\ell_{11}+1} = v$$

and assume $j = 1$ and $s'_j = s'_1 =$

$$v = v_{1,1,\ell_{11}+1} \xrightarrow{a_{1,1,\ell_{11}+1}} v_{1,1,\ell_{11}+2} \xrightarrow{a_{1,1,\ell_{11}+2}} \cdots \xrightarrow{a_{1,1,l_{11}}} v_{1,1,l_{11}+1}.$$

By Definition 4.11(3)(4), we have that $v_{1,1,\ell_{11}+2}$ is $(a_{1,1,\ell_{11}+1},\downarrow)$-relational. Otherwise,
(2.1.1) $v_{1,1,\ell_{11}+2}$ is a sink, and so $\ell(s'_1)=1$, it contradicts Definition 4.11(3);
(2.1.2) $v_{1,1,\ell_{11}+2}$ is not a sink, and so $\ell(s'_1)\geqslant 1$, it contradicts Definition 4.11(2).
Thus, there is an arrow α_1 such that $\alpha_1\neq a_{1,1,\ell_{11}+1}$, $\mathfrak{s}(\alpha_1)=v$, and $a_{1,1,\ell_{11}}\alpha_1\in\mathcal{I}$ is a forbidden path lying in $\mathcal{F}(\xi)$. Thus, if $\mathfrak{t}(\alpha_1)$ is $(\alpha_1,\downarrow)$-relational, that is, there is an arrow α_2 such that $\mathfrak{t}(\alpha_1)=\mathfrak{s}(\alpha_2)$ and $\alpha_1\alpha_2\in\mathcal{I}$. then $\Omega_1(\varPsi)$ $(\leq_{\oplus}\Omega_2(E(v))$ has a direct summand M_1 such that $M_1\alpha_2=0$ and $\mathrm{top}(M_1)\cong S(\mathfrak{t}(\alpha_1))$, The direct summand M_1 is non-projective. Otherwise, $M_1\cong e_{\mathfrak{t}(\alpha_1)}A$, and so $M_1\alpha_2\cong e_{\mathfrak{t}(\alpha_1)}A\alpha_2\neq 0$. We find a forbidden path $a_{1,1,\ell_{11}}\alpha_1\alpha_2\in\mathcal{F}(\xi)$ whose length equals to 3. Next, any direct summand of $\Omega_{\geqslant 2}(\varPsi)$ is right maximal directed string module. We get

$$\mathrm{proj.dim}E(v)=\sup_{1\leqslant i\leqslant c+1}\ell(F_i) \tag{4.9}$$

by using an argument similar to the proof of (4.8). □

Now, we can prove the following result.

Theorem 4.20. *Let $A=\Bbbk\mathcal{Q}/\mathcal{I}$ be an almost gentle algebra. Then*

$$\mathrm{inj.dim}A=\mathrm{proj.dim}D(A)=\sup_{F\in\mathcal{F}}\ell(F),$$

where $\mathcal{F}:=\bigcup_{v\in\mathcal{Q}_0}\mathcal{F}(\xi_v)$ and ξ_v is the anti-claw corresponding to the indecomposable injective module $E(v)$.

Proof. By Theorem 4.19, we have

$$\mathrm{proj.dim}E(v)=\sup_{F\in\mathcal{F}(\xi_v)}\ell(F)$$

for each v. Thus,

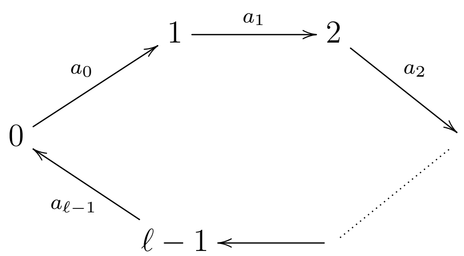
$$\mathrm{inj.dim}A=\mathrm{proj.dim}D(A)=\mathrm{proj.dim}\bigoplus_{v\in\mathcal{Q}_0}E(v)=\sup_{F\in\mathcal{F}}\ell(F).$$

□

5. Auslander–Reiten Conjecture

In this section, we prove that the Auslander–Reiten conjecture (**ARC** for short), mentioned in Introduction, holds true for almost gentle algebras.

5.1. **The second description of self-injective dimension.** Recall that an oriented cycle $\mathscr{C}=a_1a_2\cdots a_n$ is said to be a *forbidden cycle* if $a_1a_2,\ldots,a_{n-1}a_n$ and a_na_1 are generators of $\mathcal{I}$. For simplicity, assume that a forbidden cycle $\mathscr{C}$ of length ℓ is of the following form:



Theorem 5.1. *For an almost gentle algebra $A=\Bbbk\mathcal{Q}/\mathcal{I}$ whose bound quiver $(\mathcal{Q},\mathcal{I})$ has at least one forbidden cycle, then $\mathrm{inj.dim}A=\infty$ if and only if there is a vertex v on some forbidden cycle such that v is not invalid.*

Proof. If $\mathrm{inj.dim}A = \infty$, then, by Theorem 4.20, there is an anti-claw ξ such that $\mathcal{F}(\xi)$ contains a forbidden path F with $\ell(F) = \infty$ and F being of the form:

$$F = (\alpha_1 \cdots \alpha_{n-1})(\overbrace{\alpha_n\alpha_{n+1}\cdots\alpha_{n+\ell-1}}^{\mathscr{C}=a_0a_1\cdots a_{\ell-1}})(\overbrace{\alpha_n\alpha_{n+1}\cdots\alpha_{n+\ell-1}}^{\mathscr{C}=a_0a_1\cdots a_{\ell-1}})\cdots,\ (n \geqslant 2).$$

We get $\mathfrak{t}(\alpha_{n-1}) = \mathfrak{t}(\alpha_{n+\ell-1}) = \mathfrak{s}(\alpha_n)$ $(= \mathfrak{s}(a_v) = v$ for some $0 \leqslant v \leqslant \ell - 1)$. Assume that v is a $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex. Obviously, $c \geqslant 2$ and $d \geqslant 1$.

(a) When $c > 2$, v is invalid if and only if $d = 0$, see Definition 4.11. Thus, v is not invalid, a contradiction.

(b) When $c = 2$, we have

$$\mathrm{top}(\Omega_1(E(v))) \geq_{\oplus} \mathrm{top}(\Psi_0) \cong S(v)^{\oplus c-1} = S(v).$$

Since v is a vertex on a forbidden cycle, one can check that $\mathrm{proj.dim}(\mathrm{top}(\Psi_0)) = \infty$. It follows that Ψ_0 is not projective, then v is not invalid by Proposition 4.12.

Conversely, assume that every vertex on forbidden cycle is invalid. Then for any vertex v on each forbidden cycle, the direct summand $\Psi_0 = \Psi \oplus \bigoplus_{j\in J} S_j$ of $\Omega_1(E(v))$ is projective by Lemma 4.2 and Proposition 4.12. In this case, if $\mathrm{inj.dim}A = \infty = \mathrm{proj.dim}D(A)$, then there is a right maximal directed string module $M \cong \mathbb{M}(\delta)$ such that $\mathrm{proj.dim}M = \infty$, where $\delta = b_1b_2\cdots b_m$ is a right maximal directed string. It follows that $\mathcal{F}(\delta)$ contains a forbidden path F such that $\ell(F) = \infty$.

If $\ell(\delta) \geqslant 1$, then, by the definition of $\mathcal{F}(\delta)$, we have that $\mathfrak{s}(F)$ is a vertex on the claw

$$\kappa = r_1 \Lsh r_2 \Lsh \cdots \Lsh r_n$$
$$(\text{where } r_j = \ w_{j,1} \xrightarrow{\beta_{j,1}} w_{j,2} \xrightarrow{\beta_{j,2}} \cdots \xrightarrow{\beta_{j,l_j}} w_{j,l_j+1},\ 1 \leqslant j \leqslant n,$$
$$r_1 = \delta, \text{ and } \mathfrak{s}(b_1) = w_{1,1} = w_{2,1} = \ldots = w_{n,1})$$

corresponding to $P(\mathfrak{s}(\delta))$, the indecomposable projective module given by the projective cover of $\mathbb{M}(\delta)$, such that $\mathfrak{s}(F)$ is a κ-relational vertex lying in $\mathfrak{s}(\delta)^{\downarrow}\backslash\{\mathfrak{t}(b_1)\} = \{w_{j,2} \mid 2 \leqslant j \leqslant n\}$. Without loss of generality, assume $\mathfrak{s}(F) = w_{2,2}$. Then $l_2 = 1$ (i.e., $r_2 = \beta_{2,1}$) and F is of the following form:

$$F = (\alpha_1 \cdots \alpha_{n-1})(\overbrace{\alpha_n\alpha_{n+1}\cdots\alpha_{n+\ell-1}}^{\mathscr{C}=a_0a_1\cdots a_{\ell-1}})(\overbrace{\alpha_n\alpha_{n+1}\cdots\alpha_{n+\ell-1}}^{\mathscr{C}=a_0a_1\cdots a_{\ell-1}})\cdots \tag{5.1}$$
$$(\text{we take } \alpha_1\cdots\alpha_{n-1} = e_{\mathfrak{s}(\alpha_n)} \text{ in the case for } n = 0)$$

such that $\beta_{2,1}\alpha_1 \in \mathcal{I}$ (we take $\beta_{2,1}\alpha_0 \in \mathcal{I}$ in the case for $n = 0$). By using an argument similar to the proof of the necessity, we get that

(A) If $n = 0$, then $\beta_{2,1}F$ is a forbidden path which admits the vertex $\mathfrak{s}(\alpha_0) = \mathfrak{t}(\beta_{2,1}) = \mathfrak{t}(\alpha_{\ell-1})$ on the forbidden cycle $\mathscr{C}$ which is not invalid

(B) If $n \geqslant 1$, then $\mathfrak{s}(\alpha_n) = \mathfrak{t}(\alpha_{n-1}) = \mathfrak{t}(\alpha_{n+\ell-1})$ is a vertex on the forbidden cycle $\mathscr{C}$ which is not invalid

If $\ell(\delta) = 0$, then $\delta = e_v$ is a path of length zero corresponding to some vertex v, and $\mathcal{F}(\delta)$ contains a forbidden path F (which is of the form as shown in (5.1)) with $\ell(F) = \infty$ such that there is an arrow α starting at v and ending at $\mathfrak{s}(F)$ such that αF is a forbidden path. It follows the vertex $\mathfrak{s}(\alpha_n)$ is not invalid by using an argument similar to the proof of the necessity. □

Corollary 5.2. *Let $A = \Bbbk\mathcal{Q}/\mathcal{I}$ be an almost gentle algebra. If* $\mathrm{inj.dim}A = \infty$*, then the following statements hold.*

(1) *The bound quiver $(\mathcal{Q}, \mathcal{I})$ has at least one forbidden cycle.*

(2) *There is a vertex v on some forbidden cycle $\mathscr{C} = a_0a_1\cdots a_{\ell-1}$ such that one of the following statements holds:*
   (A) *there is an arrow α ending at v satisfying $\alpha a_v \in \mathcal{I}$;*
   (B) *there is an arrow β starting at v satisfying $a_{\overline{v-1}}\beta \in \mathcal{I}$, where $\overline{v-1}$ is $v-1$ modulo ℓ.*

Proof. The statement (1) holds by Theorem 5.1. Now we prove (2).

By Theorem 5.1, we can find a vertex v on some forbidden cycle $\mathscr{C} = a_0a_1\cdots a_{\ell-1}$ such that v is not invalid. Assume that v is a $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex ($c \geqslant 1$, $d \geqslant 1$), and for all arrows $\beta_1, \ldots, \beta_d$ starting at v (see Figure 4.1), we have $a_{\overline{v-1}}\beta_j \notin \mathcal{I}$ holds for any $1 \leqslant j \leqslant d$. Then, by the definition of almost gentle algebras, we have $d = 1$.

(a) When $c \geqslant 2$, if there are two integers i_1 and i_2 with $1 \leqslant i_1, i_2 \leqslant c$ such that $\alpha_{i_1}a_v$ and $\alpha_{i_2}a_v$ are paths of length two lying in $\mathcal{I}$, then it contradicts the definition of almost gentle pairs.
(b) When $c = 1$, there is no arrow α starting at v such that αa_v lies in $\mathcal{I}$. Then v is invalid by Definition 4.11(5), a contradiction.

(Notice that if all vertices on forbidden cycle satisfy the case (b), then $(\mathcal{Q}, \mathcal{I})$ is a forbidden cycle in the case for $\mathcal{Q}$ to be connected. Furthermore, A is self-injective in this case.) □

Lemma 5.3. *Let $A = \Bbbk\mathcal{Q}/\mathcal{I}$ be an almost gentle algebra such that the bound quiver $(\mathcal{Q}, \mathcal{I})$ contains at least one forbidden cycle. Assume that there is a vertex v on some forbidden cycle $\mathscr{C} = a_0a_1\cdots a_{\ell-1}$ such that one of the following conditions holds:*
(A) *there is an arrow α ending at v satisfying $\alpha a_v \in \mathcal{I}$;*
(B) *there is an arrow β starting at v satisfying $a_{\overline{v-1}}\beta \in \mathcal{I}$.*

Then $\mathrm{inj.dim} A = \infty$.

Proof. Assume that v is a $(c^{\mathrm{in}}, d^{\mathrm{out}})$-type vertex as shown in Figure 4.2. In Case (A), we have $c \geqslant 2$ and $d \geqslant 1$, see Figure 5.1. If $c > 2$, then v is invalid if and only if $d = 0$,

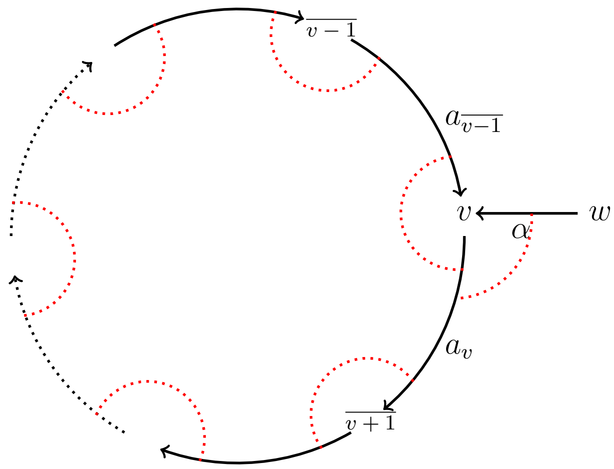


Figure 5.1. There exists an arrow α starting with v such that $\alpha a_v \in \mathcal{I}$

see Definition 4.11(2). If $c = 2$, then v is invalid if and only if v is gentle, see Definition 4.11(1). Thus, v is not invalid in Case (A). Let ξ be the anti-claw corresponding to $E(v)$. Then $\mathcal{F}(\xi)$ contains the forbidden path

$$F = (a_v a_{\overline{v+1}} \cdots a_{\ell-1})(\overbrace{a_0a_1\cdots a_{\ell-1}}^{\mathscr{C}})(\overbrace{a_0a_1\cdots a_{\ell-1}}^{\mathscr{C}})\cdots$$

whose length is infinite, and thus proj.dim$E(v) = \infty$ by Theorem 4.19.

In Case (B), we have $c \geqslant 1$ and $d \geqslant 2$. If $c \geqslant 2$, then one can check that v is not invalid by Definition 4.11(1)(2). In this case, we obtain proj.dim$E(v)$ is infinite by an argument similar to the proof of Case (A), and so is proj.dim$D(A)$ (= inj.dim$A$), as required. Now, consider the case for $c = 1$ in Case (B). Assume $\mathfrak{t}(\beta) = w$. Then we have $S(v) \leq_{\oplus} \mathrm{top}(E(w))$. It follows that $\Omega_1(E(w))$ has a direct summand, written as M, such that

- M is a right maximal directed string module;
- top$M = S(\overline{v+1})$.

Then proj.dim$M = \infty$ since $\overline{v+1}$ is a vertex on $\mathscr{C}$, and so is proj.dim$E(w)$. Thus proj.dim$D(A) = \infty$, as required. □

The following result provides a description of the infiniteness of inj.dim$A$.

Theorem 5.4. *Let $A = \Bbbk\mathcal{Q}/\mathcal{I}$ be an almost gentle algebra such that the bound quiver $(\mathcal{Q}, \mathcal{I})$ contains at least one forbidden cycle. $\mathscr{C} = a_0a_1\cdots a_{\ell-1}$. Then* inj.dim$A = \infty$ *if and only if there is a vertex v on $\mathscr{C}$ such that one of the conditions (A) and (B) given in Lemma 5.3 holds.*

Proof. It follows from Lemma 5.3 and Corollary 5.2. □

5.2. **Auslander–Reiten Conjecture.**

Lemma 5.5. *Let $A = \Bbbk\mathcal{Q}/\mathcal{I}$ be an almost gentle algebra whose bound quiver $(\mathcal{Q}, \mathcal{I})$ has at least one forbidden cycle. If* inj.dim$A = \infty$*, then the injective envelope E^0 of A has an infinite projective dimension.*

Proof. By Theorem 5.4, there exists a vertex v on some forbidden cycle $\mathscr{C} = a_0a_1\cdots a_{\ell-1}$ such that one of the conditions (A) and (B) given in Lemma 5.3 holds. If A satisfies the condition (A), that is, there is an arrow α ending at v satisfying $\alpha a_v \in \mathcal{I}$, then there are two cases as follows:

(1) For any arrow β starting at v, we have $\alpha\beta \in \mathcal{I}$.
(2) There exists a unique arrow β with $\mathfrak{s}(\beta) = v$, we have $\alpha\beta \notin \mathcal{I}$.

In Case (1), assume $\mathfrak{s}(\alpha) = w$ (cf. Figure 5.1). Then the injective envelope $E^0_{P(w)}$ of $P(w)$ has a direct summand which is isomorphic to $E(v)$. We have that the projective dimension proj.dim$E(v)$ of $E(v)$ is infinite, see the proof of Lemma 5.3(A), thus proj.dim$E^0_{P(w)} = \infty$, it follows that proj.dim$E^0 = \infty$.

In Case (2), there is a right maximal directed string $p = \beta_1\beta_2\cdots\beta_l$ ($\beta = \beta_1$) such that $\alpha p \notin \mathcal{I}$, cf. Figure 5.2. In this case, the injective envelope $E^0_{P(v)}$ of $P(v)$ has a direct summand $E(\mathfrak{t}(\beta))$ (i.e., $E(u_1)$, where u_1 is shown in Figure 5.2). By the definition of almost gentle algebras, we have $a_{\overline{v-1}}\beta \in \mathcal{I}$, and then the anti-claw ξ corresponding to $E(u_1)$ is of the form

$$\xi = \ r_1 \vee r_2 \vee \ \cdots \vee r_m$$

such that $r_1 = p$. Then

$$F = (a_{\overline{v+1}}\cdots a_{\ell-1})(\overbrace{a_0a_1\cdots a_{\ell-1}}^{\mathscr{C}})(\overbrace{a_0a_1\cdots a_{\ell-1}}^{\mathscr{C}})\cdots$$

is a forbidden path lying in $\mathcal{F}(\xi)$ whose length is infinite. By Theorem 4.19, we have proj.dim$E(u_1) = \infty$, and so proj.dim$E^0 = \infty$. □

Corollary 5.6. *Let A be an almost gentle algebra with* inj.dim$A = \infty$*, then* proj.dim$E^0 = \infty$*, where E^0 is the injective envelope of A_A.*

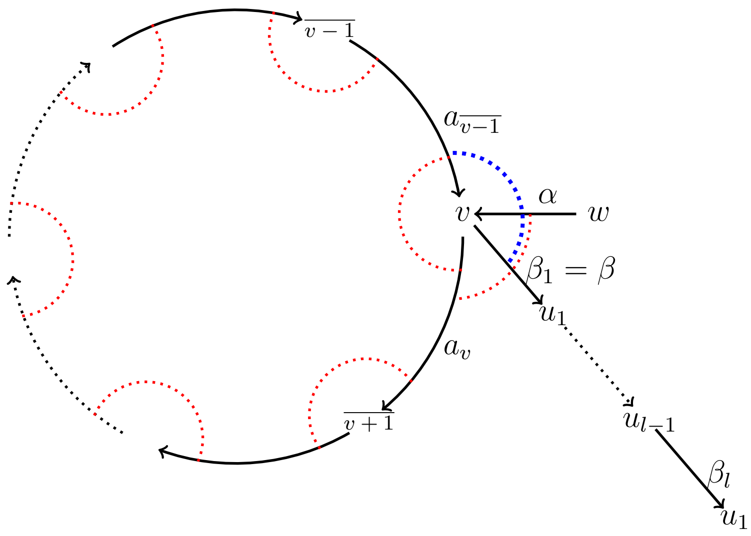


FIGURE 5.2. There exists an path $p = \beta_1 \cdots \beta_l$ starting with v such that $\alpha p \notin \mathcal{I}$ (i.e., such that $\alpha\beta_1 \notin \mathcal{I}$)

Proof. Since $\text{inj.dim}A = \infty$, there is an indecomposable injective module $E(v)$ such that $\text{proj.dim}E(v) = \infty$. Then by Theorem 4.19, the set $\mathcal{F}(\xi)$ given by the anti-claw ξ corresponding to $E(v)$ contains a forbidden path F whose length is infinite. Thus F provides a forbidden cycle $\mathscr{C}$, that is, $(\mathcal{Q}, \mathcal{I})$ contains a forbidden cycle. Then $\text{proj.dim}E^0 = \infty$ by Lemma 5.5, □

As a consequence, we obtain the following result.

Theorem 5.7. **ARC** *holds true for almost gentle algebras.*

Proof. Let A be an almost gentle algebra. If $\text{inj.dim}A < \infty$, then A is Gorenstein. If $\text{inj.dim}A = \infty$, then A does not satisfy the Auslander condition by Corollary 5.6. The proof is finished. □

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