# Homological Invariants Related to Semidualizing Bimodules

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**Abstract**

Let $R$ and $S$ be rings and ${}_RC_S$ a semidualizing bimodule. We show that the supremum of the $C$-projective dimensions of $C$-flat left $R$-modules is less than or equal to that of left $R$-modules with finite $C$-projective dimension, and the latter one is less than or equal to the supremum of $C$-injective dimensions of projective (or flat) left $S$-modules. We also show that the supremum of the $C$-projective dimensions of injective left $R$-modules and that of the $C$-injective dimensions of projective left $S$-modules are identical provided that both of them are finite. Finally, we show that the supremum of the $C$-projective dimensions of $C$-flat left $R$-modules (a relative homological invariant) and that of the projective dimensions of flat left $S$-modules (an absolute homological invariant) coincide.

## 1 Introduction

The study of semidualizing modules in commutative rings was initiated by Foxby in [10] and by Golod in [12]. Then Holm and White extended it in [16]

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to arbitrary associative rings. Many authors have studied the properties of semidualizing modules and related modules, see for example, [10], [12], [15]–[16], [25], [28], [32]–[40] and the references therein. Among various research areas on semidualizing modules, one basic theme is to extend the "absolute" classical results in homological algebra to the "relative" setting with respect to semidualizing modules. One of the motivations of this paper comes from a classical result due to Jensen, which states that any flat left $R$-module has finite projective dimension over a ring $R$ with finite left finitistic dimension ([23, Proposition 6]). Simson extended this result to skeletally small additive categories ([29, Theorem 2.7]). Another comes from Emmanouil and Talelli's work [7], in which the relations among the supremum of the projective lengths of injective left $R$-modules, that of the injective lengths of projective left $R$-modules, the finitistic dimension and the left self-injective dimension of a ring $R$ were established. We are interested in whether these results have relative counterparts with respect to semidualizing modules. The paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

Let $R$ and $S$ be rings and ${}_RC_S$ a semidualizing bimodule. In Section 3, we show that the supremum of the $C$-projective dimensions of $C$-flat left $R$-modules is less than or equal to that of left $R$-modules with finite $C$-projective dimension, and the latter one is less than or equal to the supremum of $C$-injective dimensions of projective (or flat) left $S$-modules. The former part of this result is a $C$-version of the Jensen's result mentioned above.

In Section 4, we show that the supremum of the $C$-projective dimensions of injective left $R$-modules and that of the $C$-injective dimensions of projective left $S$-modules are identical provided that both of them are finite. If $S$ is a right coherent ring, then any $C$-Gorenstein projective left $R$-module is $C$-Gorenstein flat provided that the supremum of the $C$-projective dimensions of $C$-flat left $R$-modules is finite. In the final of this section, we give a negative answer to the following open question posed by White in [40]: for a commutative ring $R$, if $M$ is a left $R$-module with finite projective dimension, must the projective and $C$-Gorenstein projective dimensions of $M$ be identical?

In Section 5, we prove that if $R$ is a left noetherian ring, then the direct sum of the first $(n+1)$ terms in a minimal injective resolution of ${}_RC$ is a $\Sigma$-embedding cogenerator for the category of modules with $C$-projective dimension at most $n$; and if the supremum of the $C$-projective dimensions of $C$-flat left $R$-modules is at most $m$, then the direct sum of the first $(m+n+1)$ terms in a minimal injective resolution of ${}_RC$ is a $\Sigma$-embedding cogenerator for the category of modules with $C$-flat dimension at most $n$. Finally, we show that the supremum of the $C$-projective dimensions of $C$-flat left $R$-modules (a relative homological invariant) and that of the projective dimensions of flat left $S$-modules (an absolute homological invariant) coincide.

## 2 Preliminaries

Throughout this paper, all rings are associative rings with unit. Let $R$ be a ring. We use $\operatorname{Mod} R$ (resp. $\operatorname{Mod} R^{op}$) to denote the category of left (resp. right) $R$-modules, and use $\operatorname{mod} R$ (resp. $\operatorname{mod} R^{op}$) to denote the category of finitely presented left (resp. right) $R$-modules. Let $M \in \operatorname{Mod} R$. We use $\operatorname{Add}_R M$ (resp. $\operatorname{add}_R M$) to denote the subcategory of $\operatorname{Mod} R$ consisting of all direct summands of direct sums (resp. finite direct sums) of copies of $M$. We use

$$0 \to M \to I^0(M) \xrightarrow{f^0} I^1(M) \xrightarrow{f^1} \cdots \xrightarrow{f^{i-1}} I^i(M) \xrightarrow{f^i} \cdots$$

to denote a minimal injective resolution of $M$.

Let $\mathcal{X}$ be a full subcategory of $\operatorname{Mod} R$. We write

$$\mathcal{X}^{\perp} := \{M \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geqslant 1}(X, M) = 0\}, \text{ and}$$

$${}^{\perp}\mathcal{X} := \{M \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geqslant 1}(M, X) = 0\}.$$

A sequence

$$\mathbb{M} := \cdots \to M_1 \to M_2 \to M_3 \to \cdots$$

in $\operatorname{Mod} R$ is called $\operatorname{Hom}_R(\mathcal{X}, -)$-*exact* (resp. $\operatorname{Hom}_R(-, \mathcal{X})$-*exact*) if $\operatorname{Hom}_R(X, \mathbb{M})$ (resp. $\operatorname{Hom}_R(\mathbb{M}, X)$) is exact for any $X \in \mathcal{X}$. An exact sequence (of finite or infinite length):

$$\cdots \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

in $\operatorname{Mod} R$ is called an $\mathcal{X}$-*resolution* of $M$ if all $X_i$ are in $\mathcal{X}$. The $\mathcal{X}$-*projective dimension* $\mathcal{X}\text{-}\operatorname{pd}_R M$ of $M$ is defined as $\inf\{n \mid$ there exists an $\mathcal{X}$-resolution

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

of $M$ in $\operatorname{Mod} R\}$. Dually, the notions of an $\mathcal{X}$-*coresolution* and the $\mathcal{X}$-*injective dimension* $\mathcal{X}\text{-}\operatorname{id}_R M$ of $M$ are defined. In particular, we use $\operatorname{pd}_R M$,

$\mathrm{fd}_R M$ and $\mathrm{id}_R M$ to denote the projective, flat and injective dimensions of $M$ respectively.

We first give the following

**Lemma 2.1.** *Let $\mathcal{X}$ and $\mathcal{C}$ be full subcategories of* $\mathrm{Mod}\, R$ *with $\mathcal{C}$ additive.*

*(1) If $(\mathcal{X} \cup \mathcal{C}) \subseteq \mathcal{C}^{\perp}$ and $\mathcal{C}$-$\mathrm{pd}_R X \leqslant m(<\infty)$ for any $X \in \mathcal{X}$, then for a module $M \in \mathrm{Mod}\, R$ with $\mathcal{X}$-$\mathrm{pd}_R M \leqslant n(<\infty)$, we have $\mathcal{C}$-$\mathrm{pd}_R M \leqslant m+n$.*

*(2) If $(\mathcal{X} \cup \mathcal{C}) \subseteq {}^{\perp}\mathcal{C}$ and $\mathcal{C}$-$\mathrm{id}_R X \leqslant m(<\infty)$ for any $X \in \mathcal{X}$, then for a module $M \in \mathrm{Mod}\, R$ with $\mathcal{X}$-$\mathrm{id}_R M \leqslant n(<\infty)$, we have $\mathcal{C}$-$\mathrm{id}_R M \leqslant m+n$.*

*Proof.* (1) Let $M \in \mathrm{Mod}\, R$ with $\mathcal{X}$- $\mathrm{pd}_R M \leqslant n$ and

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0 \tag{2.1}$$

be an exact sequence in $\mathrm{Mod}\, R$ with all $X_i$ in $\mathcal{X}$. Because $\mathcal{X} \subseteq \mathcal{C}^{\perp}$ by assumption, the exact sequence (2.1) is $\mathrm{Hom}_R(\mathcal{C}, -)$-exact. Because $\mathcal{C}$-$\mathrm{pd}_R X \leqslant m$ and $\mathcal{C} \subseteq \mathcal{C}^{\perp}$ by assumption, for any $0 \leqslant i \leqslant n$ we have a $\mathrm{Hom}_R(\mathcal{C}, -)$-exact exact sequence

$$0 \to C_i^m \to \cdots \to C_i^1 \to C_i^0 \to X_i \to 0$$

in $\mathrm{Mod}\, R$ with all $C_i^j$ in $\mathcal{C}$. By [17, Corollary 3.7], we get an exact sequence

$$0 \to C_{m+n} \to C_{m+n-1} \to \cdots \to C_1 \to C_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ with all $C_t$ being direct sums of some modules in $\{C_i^j\}_{0 \leqslant i \leqslant n}^{0 \leqslant j \leqslant m}$. Because $\mathcal{C}$ is additive, we have that all $C_t$ are in $\mathcal{C}$ and $\mathcal{C}$-$\mathrm{pd}_R M \leqslant m+n$.

(2) It is dual to (1). □

**Definition 2.2.** ([16]). Let $R$ and $S$ be rings.

(1) An ($R$-$S$)-bimodule ${}_RC_S$ is called *semidualizing* if the following conditions are satisfied.

(a1) ${}_RC$ admits a degreewise finite $R$-projective resolution.

(a2) $C_S$ admits a degreewise finite $S$-projective resolution.

(b1) The homothety map ${}_RR_R \xrightarrow{{}_R\gamma} \mathrm{Hom}_{S^{op}}(C, C)$ is an isomorphism.

(b2) The homothety map ${}_SS_S \xrightarrow{\gamma_S} \mathrm{Hom}_R(C, C)$ is an isomorphism.

(c1) $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$, that is ${}_RC$ is *self-orthogonal.*

(c2) $\mathrm{Ext}_{S^{op}}^{\geqslant 1}(C, C) = 0$, that is $C_S$ is *self-orthogonal.*

(2) A semidualizing bimodule ${}_RC_S$ is called *faithful* if the following conditions are satisfied:

(f1)If $M \in \operatorname{Mod} R$ and $\operatorname{Hom}_R(C, M) = 0$, then $M = 0$.

(f2) If $N \in \operatorname{Mod} S^{op}$ and $\operatorname{Hom}_{S^{op}}(C, N) = 0$, then $N = 0$.

Typical examples of semidualizing bimodules include the free module of rank one, dualizing modules over a Cohen-Macaulay local ring and the ordinary Matlis dual bimodule ${}_\Lambda D(\Lambda)_\Lambda$ of ${}_\Lambda\Lambda_\Lambda$ over an artin algebra $\Lambda$. Over a commutative ring, all semidualizing modules are faithful ([16, Proposition 3.1]).

**From now on, $R$ and $S$ are arbitrary rings and we fix a semidualizing bimodule ${}_RC_S$.** For convenience, we write $(-)_* := \operatorname{Hom}(C, -)$, and

$$ {}_RC^{\perp} := \{M \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{i\geqslant 1}(C, M) = 0\}, $$
$$ {C_S}^{\top} := \{N \in \operatorname{Mod} S \mid \operatorname{Tor}_{i\geqslant 1}^S(C, N) = 0\}. $$

Following [16], set

$$ \mathcal{F}_C(R) := \{C \otimes_S F \mid F \text{ is flat in } \operatorname{Mod} S\}, $$

$$ \mathcal{P}_C(R) := \{C \otimes_S P \mid P \text{ is projective in } \operatorname{Mod} S\}, $$
$$ \mathcal{I}_C(S) := \{I_* \mid I \text{ is injective in } \operatorname{Mod} R\}. $$

The modules in $\mathcal{F}_C(R)$, $\mathcal{P}_C(R)$ and $\mathcal{I}_C(S)$ are called *$C$-flat*, *$C$-projective* and *$C$-injective* respectively. Symmetrically, the classes of $\mathcal{F}_C(S^{op})$, $\mathcal{P}_C(S^{op})$ and $\mathcal{I}_C(R^{op})$ are defined. Set $(-)^+ := \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive and $\mathbb{Q}$ is the additive group of rational numbers. We have the following

**Lemma 2.3.**

*(1) If $M \in \mathcal{F}_C(R)$, then $M^+ \in \mathcal{I}_C(R^{op})$.*

*(2) If $S$ is a right coherent ring and $N \in \mathcal{I}_C(R^{op})$, then $N^+ \in \mathcal{F}_C(R)$.*

*Proof.* (1) It follows directly from the adjoint isomorphism theorem.

(2) Let $S$ be a right coherent ring and $N \in \mathcal{I}_C(R^{op})$. Then there exists an injective module $I$ in $\operatorname{Mod} S^{op}$ such that $N = I_*$. By [11, Lemma 2.16(c)] we have

$$ C \otimes_S I^+ \cong {I_*}^+ (= N^+). $$

By [9, Theorem 2.2], we have that $I^+ \in \operatorname{Mod} S$ is flat. So $N^+ (\cong C \otimes_S I^+) \in \mathcal{F}_C(R)$. $\square$

Let $M \in \operatorname{Mod} R$ and $N \in \operatorname{Mod} S$. Then we have the following two canonical valuation homomorphisms:

$$\theta_M : C \otimes_S M_* \to M$$

defined by $\theta_M(c \otimes f) = f(c)$ for any $c \in C$ and $f \in M_*$; and

$$\mu_N : N \to (C \otimes_S N)_*$$

defined by $\mu_N(x)(c) = c \otimes x$ for any $x \in N$ and $c \in C$.

**Definition 2.4.** ([16])

(1) The *Auslander class* $\mathcal{A}_C(S)$ with respect to $C$ consists of all left $S$-modules $N$ satisfying the following conditions.

(A1) $N \in {C_S}^{\top}$.

(A2) $C \otimes_S N \in {}_RC^{\perp}$.

(A3) $\mu_N$ is an isomorphism in $\operatorname{Mod} S$.

(2) The *Bass class* $\mathcal{B}_C(R)$ with respect to $C$ consists of all left $R$-modules $M$ satisfying the following conditions.

(B1) $M \in {}_RC^{\perp}$.

(B2) $M_* \in {C_S}^{\top}$.

(B3) $\theta_M$ is an isomorphism in $\operatorname{Mod} R$.

For a subcategory $\mathcal{X}$ of $\operatorname{Mod} R$ and $n \geqslant 0$, we write

$$\mathcal{X}\text{-}\operatorname{pd}^{\leqslant n}(R) := \{M \in \operatorname{Mod} R \mid \mathcal{X}\text{-}\operatorname{pd}_R M \leqslant n\},$$

and

$$\mathcal{X}\text{-}\operatorname{pd}^{<\infty}(R) := \{M \in \operatorname{Mod} R \mid \mathcal{X}\text{-}\operatorname{pd}_R M < \infty\}.$$

We use $\mathcal{I}(R)$ to denote the subcategory of $\operatorname{Mod} R$ consisting of injective modules. The following two lemmas will be used frequently in the sequel.

**Lemma 2.5.**

*(1)* $\mathcal{I}(R) \cup \mathcal{F}_C(R)\text{-}\operatorname{pd}^{<\infty}(R) \subseteq \mathcal{B}_C(R) \subseteq {}_RC^{\perp} = \mathcal{P}_C(R)^{\perp}$.

*(2)* $\mathcal{I}_C(R^{op}) \subseteq {}^{\perp}\mathcal{I}_C(R^{op})$ *and* $\mathcal{I}_C(S) \subseteq {}^{\perp}\mathcal{I}_C(S)$.

*Proof.* (1) By [16, Lemma 4.1 and Corollary 6.1] and [35, Theorem 3.9], we have

$$\mathcal{I}(R) \cup \mathcal{F}_C(R)\text{-}\operatorname{pd}^{<\infty}(R) \subseteq \mathcal{B}_C(R) \subseteq {}_RC^{\perp}.$$

It is well known that $\operatorname{Ext}_R^n(\oplus_{i\in I}A_i, M) \cong \prod_{i\in I}\operatorname{Ext}_R^n(A_i, M)$ for any family of modules $\{A_i\}_{i\in I}$, $M \in \operatorname{Mod} R$ and $n \geqslant 1$. Because $\mathcal{P}_C(R) = \operatorname{Add}_R C$ by [36, Proposition 3.4(2)], it is easy to get $\mathcal{P}_C(R)^{\perp} = {}_RC^{\perp}$.

(2) It follows from [16, Lemma 4.1 and Theorem 6.4(b)]. □

The following result is used frequently in the sequel.

**Lemma 2.6.** ([35, Theorem 3.9]) and [36, Theorem 3.5])

*(1)* $\operatorname{fd}_S M_* \leqslant \mathcal{F}_C(R)\text{-}\operatorname{pd}_R M$ *for any* $M \in \operatorname{Mod} R$*, the equality holds if* $M \in \mathcal{B}_C(R)$.

*(2)* $\operatorname{pd}_S M_* \leqslant \mathcal{P}_C(R)\text{-}\operatorname{pd}_R M$ *for any* $M \in \operatorname{Mod} R$*, the equality holds if* $M \in \mathcal{B}_C(R)$.

*(3)* $\operatorname{id}_R C \otimes_S N \leqslant \mathcal{I}_C(S)\text{-}\operatorname{id}_S N$ *for any* $N \in \operatorname{Mod} S$*, the equality holds if* $N \in \mathcal{A}_C(S)$.

The following notions were introduced by Holm and J$\phi$gensen in [15] for commutative rings. We give the non-commutative versions of them.

**Definition 2.7.** Let $M$ be in $\operatorname{Mod} R$.

(1) $M$ is called *C-Gorenstein projective* if the following conditions are satisfied.

(i) $\operatorname{Ext}_R^{\geqslant 1}(M, G) = 0$ for any $G \in \mathcal{P}_C(R)$.

(ii) There exists a $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact exact sequence

$$\mathbb{G} := 0 \to M \to G^0 \to G^1 \to \cdots$$

in $\operatorname{Mod} R$ with all $G^i$ in $\mathcal{P}_C(R)$.

(2) $M$ is called *C-Gorenstein flat* if the following conditions are satisfied.

(i) $\operatorname{Tor}^R_{\geqslant 1}(E, M) = 0$ for any $E \in \mathcal{I}_C(R^{op})$.

(ii) There exists an exact sequence

$$\mathbb{Q} := 0 \to M \to Q^0 \to Q^1 \to \cdots$$

in $\operatorname{Mod} R$ with all $Q^i$ in $\mathcal{F}_C(R)$, such that $E \otimes_R \mathbb{Q}$ is exact for any module $E \in \mathcal{I}_C(R^{op})$.

We use $\mathcal{GP}_C(R)$ to denote the subcategory of $\operatorname{Mod} R$ consisting of $C$-Gorenstein projective modules. Putting ${}_RC_S = {}_RR_R$, then $C$-Gorenstein projective modules and $C$-Gorenstein flat modules are the classical *Gorenstein projective modules* and *Gorenstein flat modules* respectively ([4, 5, 8, 14]).

**Lemma 2.8.** *For a module $M \in \operatorname{Mod} R$, if $\mathcal{P}_C(R)\text{-pd}_R M < \infty$, then $\mathcal{P}_C(R)\text{-pd}_R M = \mathcal{GP}_C(R)\text{-pd}_R M$.*

*Proof.* The case for commutative rings has been proved in [40, Proposition 2.16]. The argument there is valid in our setting, so we omit it. □

# 3 The $C$-Version of a Result of Jensen

In this section, we investigate the relationship among some homological invariants related to ${}_RC_S$. We first define *the finitistic $C$-projective dimension* $\text{F}\mathcal{P}_C\text{-dim}\, R$ of $R$ as

$$\text{F}\mathcal{P}_C\text{-dim}\, R := \sup\{\mathcal{P}_C(R)\text{- pd}_R M \mid M \in \operatorname{Mod} R \text{ with } \mathcal{P}_C(R)\text{- pd}_R M < \infty\},$$

and define *the finitistic $C$-Gorenstein projective dimension* $\text{F}\mathcal{GP}_C\text{-dim}\, R$ of $R$ as

$$\text{F}\mathcal{GP}_C\text{-dim}\, R := \sup\{\mathcal{GP}_C(R)\text{- pd}_R M \mid M \in \operatorname{Mod} R \text{ with } \mathcal{GP}_C(R)\text{- pd}_R M < \infty\}.$$

We write the supremum of the $C$-projective dimensions of $C$-flat left $R$-modules as

$$\text{spclfc}\, R := \sup\{\mathcal{P}_C(R)\text{- pd}_R M \mid M \in \mathcal{F}_C(R)\}.$$

The following result is a $C$-version of [23, Proposition 6]. It plays a key role in the sequel.

**Proposition 3.1.** $\text{spclfc}\, R \leqslant \text{F}\mathcal{P}_C\text{-dim}\, R$.

*Proof.* The proof is modified from [23, Proposition 6]. Let $\text{F}\mathcal{P}_C\text{-dim}\, R < \infty$ and $M \in \mathcal{F}_C(R)$. Then $M \cong C \otimes_S F$ for some flat module $F$ in $\operatorname{Mod} S$. Now take an exact sequence

$$0 \to B \to F_0 \to F \to 0 \tag{3.1}$$

in $\operatorname{Mod} S$ with $F_0$ free and $B$ flat. Assume that $B$ is generated by $\aleph$ elements, where $\aleph$ is a finite or an infinite cardinal number. We claim that $\text{pd}_S B \leqslant \text{F}\mathcal{P}_C\text{-dim}\, R$.

We proceed by using transfinite induction on $\aleph$. If $\aleph \leqslant \aleph_0$, then there exists a pure exact sequence

$$0 \to B \to F' \to F'/B \to 0$$

in $\operatorname{Mod} S$ such that $F'$ is a free submodule of $F_0$ and $F'$ is generated by at most $\aleph_0$ elements. Hence $F'/B$ is a countably related flat module by [22]. Now it follows from [21, Lemma 2] (see also [27, Lamma 1.2]) that $\operatorname{pd}_S F'/B \leqslant 1$. So $B$ is projective and $\operatorname{pd}_S B \leqslant \mathrm{F}\mathcal{P}_C\text{-}\dim R$. Next from the proof of [23, Proposition 6], we know that there exists a transfinite sequence $(C_\beta)_{\beta<\Omega}$ of pure submodules $C_\beta$ such that $B = \bigcup_{\beta<\Omega} C_\beta$ with $C_{\beta_1} \subseteq C_{\beta_2}$ for $\beta_1 \leqslant \beta_2$, and each $C_\beta$ is generated by less than $\aleph$ elements. Then by the induction hypothesis, we have $\operatorname{pd}_S C_\beta \leqslant \mathrm{F}\mathcal{P}_C\text{-}\dim R$. So $\operatorname{pd}_S B \leqslant \mathrm{F}\mathcal{P}_C\text{-}\dim R$ by [2, Proposition 3]. The claim is proved.

By the claim and the exact sequence (3.1), we have $\operatorname{pd}_S F < \infty$. Notice that $F \in \mathcal{A}_S(C)$ by [16, Lemma 4.1], so $\mu_F : F \to (C \otimes_S F)_*$ is an isomorphism, and hence $\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M = \mathcal{P}_C(R)\text{-}\operatorname{pd}_R(C \otimes_S F) = \operatorname{pd}_S(C \otimes_S F)_* = \operatorname{pd}_S F < \infty$ by Lemma 2.6(2). It follows that $\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M \leqslant \mathrm{F}\mathcal{P}_C\text{-}\dim R$. □

For a subcategory $\mathcal{X}$ of $\operatorname{Mod} R$, following [36] we write

$$\operatorname{id}_R \mathcal{X} := \sup\{\operatorname{id}_R X \mid X \in \mathcal{X}\}.$$

The following result improves [36, Proposition 3.6].

**Corollary 3.2.** $\sup\{\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M \mid M \in \operatorname{Mod} R$ *with* $\mathcal{F}_C(R)\text{-}\operatorname{pd}_R M < \infty\} \leqslant \mathrm{F}\mathcal{P}_C\text{-}\dim R \leqslant \operatorname{id}_R \mathcal{P}_C(R)$.

*Proof.* If $\mathrm{F}\mathcal{P}_C\text{-}\dim R < \infty$, then $\operatorname{spclfc} R \leqslant \mathrm{F}\mathcal{P}_C\text{-}\dim R$ by Proposition 3.1. It follows from Lemmas 2.5 and 2.1(1) that $\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M < \infty$ for any $M \in \operatorname{Mod} R$ with $\mathcal{F}_C(R)\text{-}\operatorname{pd}_R M < \infty$. Then the first inequality follows.

Let $M \in \operatorname{Mod} R$ with $\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M = n(< \infty)$ and

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to M \to 0$$

be an exact sequence in $\operatorname{Mod} R$ with all $C_i$ in $\mathcal{P}_C(R)(= \operatorname{Add}_R C$ by [36, Proposition 3.4(2)]). Then $\operatorname{Ext}^n_R(M, C_n) \neq 0$ and $\operatorname{id}_R C_n \geqslant n$. So $\operatorname{id}_R \mathcal{P}_C(R) \geqslant n$ and the second inequality follows. □

Motivated by [7, Section 2], we write the supremum of the $C$-injective dimensions of projective left $S$-modules as

$$\operatorname{siclp} S := \sup\{\mathcal{I}_C(S)\text{-}\operatorname{id}_S P \mid P \in \operatorname{Mod} S \text{ is projective}\},$$

and write the supremum of the $C$-injective dimensions of flat left $S$-modules as

$$\operatorname{siclf} S := \sup\{\mathcal{I}_C(S)\text{-}\operatorname{id}_S F \mid F \in \operatorname{Mod} S \text{ is flat}\}.$$

A special case of those commutative Noetherian rings $S$ with $\operatorname{siclp} S \leqslant n$, it was proved in [33, Theorem 2.6] that these are precisely the rings over which every finitely generated module can be embedded into a module with $C$-projective dimension at most $n$.

**Theorem 3.3.**

*(1)* $\operatorname{spclfc} R \leqslant \mathrm{F}\mathcal{GP}_C\text{-}\dim R = \mathrm{F}\mathcal{P}_C\text{-}\dim R \leqslant \operatorname{id}_R \mathcal{P}_C(R) = \operatorname{siclp} S = \operatorname{siclf} S$.

*(2) If $R$ is a left noetherian ring, then* $\mathrm{F}\mathcal{P}_C\text{-}\dim R \leqslant \operatorname{id}_R C = \operatorname{siclp} S$.

*Proof.* (1) By Proposition 3.1, Lemma 2.8 and Corollary 3.2, we have $\operatorname{spclfc} R \leqslant \mathrm{F}\mathcal{P}_C\text{-}\dim R \leqslant \mathrm{F}\mathcal{GP}_C\text{-}\dim R \leqslant \operatorname{id}_R \mathcal{P}_C(R)$.

Now suppose that $\mathrm{F}\mathcal{P}_C\text{-}\dim R = n(< \infty)$ and $M \in \operatorname{Mod} R$ with $\mathcal{GP}_C(R)\text{-}\operatorname{pd}_R M < \infty$. By [25, Corollary 3.4], there exists $M' \in \operatorname{Mod} R$ such that $\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M' = \mathcal{GP}_C(R)\text{-}\operatorname{pd}_R M$. So $\mathcal{GP}_C(R)\text{-}\operatorname{pd}_R M \leqslant n$. It follows that $\mathrm{F}\mathcal{GP}_C\text{-}\dim R \leqslant \mathrm{F}\mathcal{P}_C\text{-}\dim R$. The first equality follows.

Assume that $\operatorname{siclp} S = n(< \infty)$ and $M(\cong C \otimes_S P) \in \mathcal{P}_C(R)$ with projective in $\operatorname{Mod} S$. Then there exists an exact sequence:

$$0 \to P \to {I^0}_* \to {I^1}_* \to \cdots \to {I^n}_* \to 0 \tag{3.2}$$

in $\operatorname{Mod} S$ with all $I^i$ injective in $\operatorname{Mod} R$. By Lemma 2.5(1), all $I^i$ are in $\mathcal{B}_C(R)$. So ${I^i}_* \in {C_S}^{\top}$ and $C \otimes_S {I^i}_* \cong I^i$ for any $0 \leqslant i \leqslant n$. Then applying the functor $C \otimes_S -$ to (3.2) yields the following exact sequence

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^n \to 0$$

in $\operatorname{Mod} R$. It follows that $\operatorname{id}_R M \leqslant n$ and $\operatorname{id}_R \mathcal{P}_C(R) \leqslant \operatorname{siclp} S$. By using a dual argument, we get $\operatorname{siclp} S \leqslant \operatorname{id}_R \mathcal{P}_C(R)$. The second equality follows.

Obviously $\operatorname{siclp} S \leqslant \operatorname{siclf} S$. Now let $\operatorname{siclp} S = n(< \infty)$ and $F \in \operatorname{Mod} S$ be flat. Then $\mathrm{F}\mathcal{P}_C\text{-}\dim R \leqslant n$ and $\mathcal{P}_C(R)\text{-}\operatorname{pd}_R(C \otimes_S F) < \infty$ by the former argument. Let

$$0 \to C_m \to \cdots \to C_1 \to C_0 \to C \otimes_S F \to 0$$

be an exact sequence in $\operatorname{Mod} R$ with all $C_i$ in $\mathcal{P}_C(R)$. By Lemma 2.6(3), we have $\operatorname{id}_R C_i \leqslant \operatorname{siclp} S \leqslant n$. Thus $\operatorname{id}_R(C \otimes_S F) \leqslant n$. Note that $F \in \mathcal{A}_C(R)$ by

[16, Lemma 4.1]. Then $\mathcal{I}_C(S)\text{-id}_S F \leqslant n$ by Lemma 2.6(3) again. It yields that $\operatorname{siclf} S \leqslant n$. Therefore we conclude that $\operatorname{siclp} S = \operatorname{siclf} S$.

(2) Let $R$ be a left noetherian ring. Then $\operatorname{id}_R \mathcal{P}_C(R) \leqslant \operatorname{id}_R C$ by [3, Theorem 1.1]. Now the first inequality follows from Corollary 3.2.

Since $\operatorname{id}_R C = \mathcal{I}_C(S)\text{-id}_S S$ by Lemma 2.6(3), we have $\operatorname{id}_R C \leqslant \operatorname{siclp} S$. Now let $\operatorname{id}_R C = n(< \infty)$. Since $R$ is left noetherian, by [3, Theorem 1.1] we have $\operatorname{id}_R G \leqslant n$ for any $G \in \mathcal{P}_C(R)$. It follows from Lemma 2.6(3) that $\mathcal{I}_C(S)\text{-id}_S P \leqslant n$ for any projective module $P$ in $\operatorname{Mod} S$. Thus $\operatorname{siclp} S \leqslant n$ and $\operatorname{siclp} S \leqslant \operatorname{id}_R C$. □

Note that Theorem 3.3(1) extends [14, Theorem 2.28] and [7, Proposition 2.1]. The inequality in Theorem 3.3(2) can be strict, as illustrated in the following example. We refer to [1] for the notions about quivers and their representations.

**Example 3.4.** Let $R$ be the bound quiver algebra $kQ/J^2$, where $k$ is a field, $Q$ is the quiver

$$\circlearrowleft \circ 1 \longrightarrow \circ 2 \longrightarrow \circ 3 \longrightarrow \circ 4,$$

$kQ$ is the path $k$-algebra of $Q$, and $J$ is the two-sided ideal of $kQ$ generated by the arrows. If $C$ is the $(R, R)$-bimodule $R$, then $\mathrm{F}\mathcal{P}_C\text{-dim}\, R = 0$ ([13, Example 1.2]), but $\operatorname{id}_R C = \infty$.

# 4 Some Relative Homological Invariants

In classical homological algebra, it is known that for any module $M \in \operatorname{Mod} R$ with $\operatorname{fd}_R M \leqslant n$, the $n$th yoke in every flat resolution of $M$ is flat. As described in the following result, an analogous result holds for $C$-flat dimension of modules.

**Lemma 4.1.** *Let ${}_RC_S$ be faithful and $M \in \operatorname{Mod} R$. If there exist two exact sequences:*

$$0 \to M_n \to \cdots \to C_1 \to C_0 \to M \to 0, \text{ and}$$

$$0 \to D_n \to \cdots \to D_1 \to D_0 \to M \to 0$$

*in* $\operatorname{Mod} R$ *with all* $C_i$ *and* $D_i$ *in* $\mathcal{F}_C(R)$*, then* $M_n \in \mathcal{F}_C(R)$*.*

*Proof.* Applying the functor $(-)^+$ to both of the given exact sequences, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & M^+ & \longrightarrow & D_0^+ & \longrightarrow & \cdots & \longrightarrow & D_{n-1}^+ & \longrightarrow & D_n^+ & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_0} & & & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \\ 0 & \longrightarrow & M^+ & \longrightarrow & C_0^+ & \longrightarrow & \cdots & \longrightarrow & C_{n-1}^+ & \longrightarrow & M_n^+ & \longrightarrow & 0. \end{array}$$

By Lemma 2.3(1), all modules $C_i^+$ and $D_i^+$ lie in $\mathcal{I}_C(R^{op})$. Then the existence of all $f_i$ follows from Lemma 2.5(2). Now we may view the sequence $(f_0, \cdots, f_{n-1}, f_n)$ as a quasi-isomorphism between the following two complexes:

$$0 \to D_0^+ \to \cdots \to D_{n-1}^+ \to D_n^+ \to 0, \text{ and}$$

$$0 \to C_0^+ \to \cdots \to C_{n-1}^+ \to M_n^+ \to 0.$$

We therefore obtain an exact sequence:

$$0 \to D_0^+ \to D_1^+ \oplus C_0^+ \to \cdots \to D_n^+ \oplus C_{n-1}^+ \to M_n^+ \to 0.$$

Then $M_n^+ \in \mathcal{I}_C(R^{op}) (\subseteq \mathcal{A}_C(R^{op}))$ and $M_n^+ \otimes_R C \in \operatorname{Mod} S^{op}$ is injective by Lemma 2.3(1) and [16, Lemma 5.1(c)]. Note that $M_n^+ \otimes_R C \cong \operatorname{Hom}_R(C, M_n)^+$ by [11, Lemma 2.16(c)]. So $\operatorname{Hom}_R(C, M_n) \in \operatorname{Mod} S$ is flat by [11, Corollary 2.18(b)], and hence it is in $\mathcal{A}_C(S)$ by [16, Lemma 4.1]. Then $M_n \in \mathcal{B}_C(R)$ by [34, Lemma 1.7]. It follows from [16, Lemma 5.1(a)] that $M_n \in \mathcal{F}_C(R)$. □

The following example shows that the assumption about the faithfulness of ${}_RC_S$ in the above lemma is necessary.

**Example 4.2.** Let $k$ be an algebraically closed field and let $R = kQ$ be the path $k$-algebra of dimension 3 of the quiver

$$1\circ \longrightarrow \circ 2.$$

Put $C = I(1) \oplus I(2)$. Then ${}_RC_R$ is a non-faithful semidualizing bimodule and there exist two exact sequences:

$$0 \to S(2) \to P(1) \to S(1) \to 0, \text{ and}$$

$$0 \to I(1) \to I(1)^2 \to S(1) \to 0,$$

where $I(1)$ and $P(1)$ are in $\mathcal{F}_C(R)$, but $S(2)$ is not in $\mathcal{F}_C(R)$.

Motivated by the corresponding notions introduced in [7], in an analogous way we write the supremum of the $C$-projective dimensions of injective left $R$-modules as

$$\operatorname{spcli} R := \sup\{\mathcal{P}_C(R)\text{-}\operatorname{pd}_R I \mid I \in \operatorname{Mod} R \text{ is injective}\},$$

and write the supremum of the $C$-flat dimensions of injective left $R$-modules as

$$\operatorname{sfcli} R := \sup\{\mathcal{F}_C(R)\text{-}\operatorname{pd}_R I \mid I \in \operatorname{Mod} R \text{ is injective}\}.$$

Next we turn to further investigate the relationship of aforementioned relative invariants. The following two results extend [7, Proposition 2.2 and Corollary 2.3] respectively.

**Theorem 4.3.**

*(1) If* $\operatorname{spcli} R < \infty$ *and* $\operatorname{siclp} S < \infty$*, then* $\operatorname{spcli} R = \operatorname{siclp} S$*.*

*(2) If* ${}_RC_S$ *is faithful, then* $\operatorname{spcli} R \leqslant \operatorname{sfcli} R + \operatorname{spclfc} R$*.*

*Proof.* (1) Let $\operatorname{spcli} R = n$, and let $I \in \operatorname{Mod} R$ be injective with $\mathcal{P}_C(R)$-$\operatorname{pd}_R I = n$. Thus there exists an exact sequence

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to I \to 0$$

in $\operatorname{Mod} R$ with all $C_i \in \mathcal{P}_C(R)$. Then $\operatorname{Ext}_R^n(I, C_n) \neq 0$, which implies that $\operatorname{id}_R C_n \geqslant n$. We may assume that $C_n \cong C \otimes_S P$ for some projective module $P$ in $\operatorname{Mod} S$. Then $\mathcal{I}_C(S)$-$\operatorname{id}_S P = \operatorname{id}_R C_n \geqslant n$ by Lemma 2.6(3), implying that $\operatorname{siclp} S \geqslant n$. With the aid of Lemma 2.6(2), a similar argument gives the converse inequality.

(2) Let $\operatorname{sfcli} R = n(< \infty)$ and $\operatorname{spclfc} R = m(< \infty)$, and let $I \in \operatorname{Mod} R$ be injective. Since $I \in \mathcal{B}_C(R)$, by [35, Theorem 3.9 and Proposition 3.7] there exists an exact sequence

$$0 \to K_n \to C_{n-1} \to \cdots \to C_0 \to I \to 0$$

in $\operatorname{Mod} R$ with all $C_i$ in $\mathcal{P}_C(R)$. Since $\mathcal{F}_C(R)$-$\operatorname{pd}_R I \leqslant \operatorname{sfcli} R = n$, it follows from Lemma 4.1 that $K_n \in \mathcal{F}_C(R)$. Since $\operatorname{spclfc} R = m$, we have $\mathcal{P}(R)_C$-$\operatorname{pd}_R K_n \leqslant m$ and $\mathcal{P}(R)_C$-$\operatorname{pd}_R I \leqslant m + n$. □

**Corollary 4.4.** *Let* ${}_RC_S$ *be faithful. Then the following statements are equivalent.*

*(1)* $\operatorname{spcli} R = \operatorname{siclp} S < \infty$*.*

*(2)* $\operatorname{sfcli} R < \infty$ *and* $\operatorname{siclp} S < \infty$.

*Proof.* The implication (1) ⇒ (2) is trivial.

Let $\operatorname{sfcli} R < \infty$ and $\operatorname{siclp} S < \infty$. Then $\operatorname{spclfc} R < \infty$ by Theorem 3.3(1). So $\operatorname{spcli} R < \infty$ by Theorem 4.3(2). Now the implication (2) ⇒ (1) follows from Theorem 4.3(1). □

In the following result, we give a sufficient condition when a $C$-Gorenstein projective module is $C$-Gorenstein flat.

**Proposition 4.5.** *Let $S$ be a right coherent ring. If $\operatorname{spclfc} R < \infty$ (in particular, if $\mathrm{F}\mathcal{P}_C\text{-}\dim R < \infty$), then any $C$-Gorenstein projective module in $\operatorname{Mod} R$ is $C$-Gorenstein flat.*

*Proof.* By Proposition 3.1, we have $\operatorname{spclfc} R \leqslant \mathrm{F}\mathcal{P}_C\text{-}\dim R$. Now let $S$ be a right coherent ring and $\operatorname{spclfc} R < \infty$. If $M \in \operatorname{Mod} R$ is $C$-Gorenstein projective module, then by definition there exists a $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact exact sequence

$$\mathbb{G} := \cdots \to P_1 \to P_0 \to G^0 \to G^1 \to \cdots$$

in $\operatorname{Mod} R$ with all $G^i$ in $\mathcal{P}_C(R)$, $P_i$ projective and $M \cong \operatorname{Im}(P_0 \to G^0)$, such that $\operatorname{Hom}_R(\mathbb{G}, H)$ is exact for any module $H \in \mathcal{P}_C(R)$. By using induction on the dimension, it is not difficult to get that $\operatorname{Hom}_R(\mathbb{G}, H')$ is exact for any $H' \in \operatorname{Mod} R$ with $\mathcal{P}_C(R)\text{-}\operatorname{pd}_R H' < \infty$.

Now let $E \in \mathcal{I}_C(R^{op})$. Then $E^+ \in \mathcal{F}_C(R)$ by Lemma 2.3(2), and so $\mathcal{P}_C(R)\text{-}\operatorname{pd}_R E^+ < \infty$ by assumption. It yields that $\operatorname{Hom}_R(\mathbb{G}, E^+)$ is exact. Thus $E \otimes_R \mathbb{G}$ is exact by the adjoint isomorphism theorem. It follows that $M$ is $C$-Gorenstein flat. □

Recall that the *big finitistic dimension* $\operatorname{FPD} R$ of $R$ is defined as

$$\operatorname{FPD} R := \sup\{\operatorname{pd}_R M \mid M \in \operatorname{Mod} R \text{ with } \operatorname{pd}_R M < \infty\}.$$

Following [7], we write the supremum of the projective dimensions of flat left $R$-modules as

$$\operatorname{splf} R := \sup\{\operatorname{pd}_R M \mid M \in \operatorname{Mod} R \text{ is flat}\}.$$

Putting ${}_RC_S = {}_RR_R$ in Proposition 4.5, we get immediately the following result, which is a slight generalization of [14, Proposition 3.4].

**Corollary 4.6.** *Let $R$ be a right coherent ring. If $\operatorname{splf} R < \infty$ (in particular, if $\operatorname{FPD} R < \infty$), then any Gorenstein projective module in $\operatorname{Mod} R$ is Gorenstein flat.*

We use $\mathcal{P}(R)$ to denote the subcategory of $\operatorname{Mod} R$ consisting of projective modules. Recall from [14] that a subcategory $\mathcal{X}$ of $\operatorname{Mod} R$ is called *projectively resolving* if $\mathcal{P}(R) \subseteq \mathcal{X}$ and $\mathcal{X}$ is closed under extensions and kernels of epimorphisms.

White posed in [40, Question 2.15] an open question: for a commutative ring $R$, if $M \in \operatorname{Mod} R$ with $\operatorname{pd}_R M < \infty$, must $\operatorname{pd}_R M = \mathcal{GP}_C(R)\text{-}\operatorname{pd}_R M$? The following example illustrates that the answer to each of the question and its non-commutative version is negative in general. In addition, Holm and White proved in [16, Corollary 6.4] that $\mathcal{P}_C(R)$ and $\mathcal{F}_C(R)$ are projectively resolving if ${}_RC_S$ is faithful. The following example also illustrates that this result is not true.

**Example 4.7.**

(1) Let $R$ be a non-self-injective commutative artinian local ring with maximal ideal $m$. For example we can take for $R$ the ring $k[[X, Y]]/(X^2, XY, Y^2)$ with $k$ a field (see [6, p.15]). Then $C := I^0(R/m)$ is a faithfully semidualizing module and $C$ is $C$-(Gorenstein) projective. But $C$ is an injective cogenerator for $\operatorname{Mod} R$, so we have $\operatorname{pd}_R C = \operatorname{id}_R R \neq 0$. We also claim that $R \notin \mathcal{P}_C(R)$. Otherwise, there exists a projective module $P$ in $\operatorname{Mod} R$ such that $R \cong C \otimes_R P$. It follows that $R$ is injective, a contradiction. Consequently, $\mathcal{P}_C(R)$ is not projectively resolving.

(2) Let $R$ be a Gorenstein artin algebra with $\operatorname{id}_R R = \operatorname{id}_{R^{op}} R = n \geqslant 1$. For example we can take for $R$ the bound quiver algebra $kQ/J^2$, where $k$ is an algebraically field, $Q$ is the quiver

$$\circ 1 \xrightarrow{\alpha_1} \circ 2 \xrightarrow{\alpha_2} \circ 3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_n} \circ n+1,$$

$kQ$ is the path $k$-algebra of $Q$, and $J$ is the two-sided ideal of $kQ$ generated by the arrows. Put $C := \oplus_{i=0}^{n} I^i(R)$. Then by [39, Corollary 3.2], it is easy to see that $C$ is a semidualizing $(R, S)$-bimodule, where $S = \operatorname{End}_R C$. Because $C$ is an injective cogenerator for $\operatorname{Mod} R$ by [19, Theorem 2], we have $\operatorname{pd}_R C = \operatorname{fd}_R C = \operatorname{id}_{R^{op}} R = n (\geqslant 1)$ by [20, Proposition 1]. But $C$ is $C$-(Gorenstein) projective.

The following result shows that the answer to the White's question mentioned above is positive under some condition.

**Proposition 4.8.** *Assume that $\mathcal{P}_C(R)$ is projectively resolving and $M \in \operatorname{Mod} R$. If $\operatorname{pd}_R M < \infty$, then we have*

$$\operatorname{pd}_R M = \mathcal{P}_C(R)\text{-}\operatorname{pd}_R M = \mathcal{GP}_C(R)\text{-}\operatorname{pd}_R M.$$

*Proof.* By assumption, $\mathcal{P}_C(R)$ is projectively resolving; in particular, $\mathcal{P}(R) \subseteq \mathcal{P}_C(R)$. So we have $\operatorname{pd}_R M \geqslant \mathcal{P}_C(R)\text{-}\operatorname{pd}_R M$. On the other hand, by Lemma 2.5(1), we have $\mathcal{P}_C(R) \subseteq \mathcal{P}(R)^{\perp} \cap {}^{\perp}\mathcal{P}(R)$. So, if $\operatorname{pd}_R M < \infty$, then $\operatorname{pd}_R M = \mathcal{P}_C(R)\text{-}\operatorname{pd}_R M$ by [18, Theorem 3.10]. It follows from Lemma 2.8 that $\mathcal{P}_C(R)\text{-}\operatorname{pd}_R M = \mathcal{GP}_C(R)\text{-}\operatorname{pd}_R M$. □

# 5 Σ-Embedding Cogenerators

Recall from [26] that a module $A \in \operatorname{Mod} R$ is called a *Σ-embedding cogenerator* for a subcategory $\mathcal{B}$ of $\operatorname{Mod} R$, if every module in $B$ admits an injection to a direct sum of copies of $A$.

**Theorem 5.1.** *Let $R$ be a left noetherian ring and $\mathcal{X}$ a subcategory of* $\operatorname{Mod} R$ *with $\mathcal{X} \subseteq C^{\perp}$, and let $m, n \geqslant 0$. Then we have*

*(1) If $\sup\{\mathcal{P}_C(R)\text{-}\operatorname{pd}_R X \mid X \in \mathcal{X}\} \leqslant m$, then $\oplus_{t=0}^{m+n} I^t(C)$ is a Σ-embedding cogenerator for $\mathcal{X}\text{-}\operatorname{pd}^{\leqslant n}(R)$.*

*(2) If $\sup\{\mathcal{P}_C(R)\text{-}\operatorname{pd}_R X \mid X \in \mathcal{X}\} < \infty$ then $\oplus_{t\geqslant 0} I^t(C)$ is a Σ-embedding cogenerator for $\mathcal{X}\text{-}\operatorname{pd}^{<\infty}(R)$.*

*Proof.* (1) Let $M \in \operatorname{Mod} R$ with $\mathcal{X}\text{-}\operatorname{pd}_R M \leqslant n$ and $\sup\{\mathcal{P}_C(R)\text{-}\operatorname{pd}_R X \mid X \in \mathcal{X}\} \leqslant m$. Then by Lemma 2.1(1), we have an exact sequence

$$0 \to C_{m+n} \to C_{m+n-1} \to \cdots \to C_1 \to C_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all $C_t$ in $\mathcal{P}_C(R)(= \operatorname{Add}_R C)$. Because $R$ is a left noetherian ring, all $I^j(C_t)$ are in $\operatorname{Add}_R I^j(C)$ for any $j \geqslant 0$. By [26, Corollary 1.3] (cf. [17, Corollary 3.5]), $M$ can be embedded into (a direct summand of) $\oplus_{t=0}^{m+n} I^t(C_t)$. So $M$ can be embedded into a direct sum of copies of $\oplus_{t=0}^{m+n} I^t(C)$. It follows that $\oplus_{t=0}^{m+n} I^t(C)$ is a $\sum$-embedding cogenerator for $\mathcal{X}\text{-}\operatorname{pd}^{\leqslant n}(R)$.

(2) It is a direct consequence of (1). □

Putting $\mathcal{X} = \mathcal{P}_C(R)$ in Theorem 5.1, we have the following result in which the first assertion is a $C$-version of [26, Theorem 2.2].

**Corollary 5.2.** *Let $R$ be a left noetherian ring, and let $n \geqslant 0$. Then we have*

*(1) $\oplus_{t=0}^{n} I^t(C)$ is a Σ-embedding cogenerator for $\mathcal{P}_C(R)\text{-}\operatorname{pd}^{\leqslant n}(R)$.*

*(2) $\oplus_{t\geqslant 0} I^t(C)$ is a Σ-embedding cogenerator for $\mathcal{P}_C(R)\text{-}\operatorname{pd}^{<\infty}(R)$.*

As another application of Theorem 5.1, we have the following

**Corollary 5.3.** *Let $R$ be a left noetherian ring, and let $m, n \geqslant 0$. Then we have*

*(1) If* $\operatorname{spclfc} R \leqslant m$ *(in particular, if* $\mathrm{F}\mathcal{P}_C\text{-}\dim R \leqslant m$ *or* $\mathrm{id}_R C \leqslant m$*), then* $\oplus_{t=0}^{m+n} I^t(C)$ *is a $\Sigma$-embedding cogenerator for* $\mathcal{F}_C(R)\text{-}\mathrm{pd}^{\leqslant n}(R)$.

*(2) If* $\operatorname{spclfc} R < \infty$ *(in particular, if* $\mathrm{F}\mathcal{P}_C\text{-}\dim R < \infty$ *or* $\mathrm{id}_R C < \infty$*), then* $\oplus_{t\geqslant 0} I^t(C)$ *is a $\Sigma$-embedding cogenerator for* $\mathcal{F}_C(R)\text{-}\mathrm{pd}^{<\infty}(R)$.

*Proof.* By Theorem 3.3, we have

$$\operatorname{spclfc} R \leqslant \mathrm{F}\mathcal{P}_C\text{-}\dim R \leqslant \mathrm{id}_R C.$$

Note that $\mathcal{F}_C(R) \subseteq C^{\perp}$ by Lemma 2.5(1). So, Putting $\mathcal{X} = \mathcal{F}_C(R)$ in Theorem 5.1, then the assertions follow. □

Putting $C = R$ in Corollary 5.3, we have the following result in which the second assertion generalizes [26, Corollary 2.3].

**Corollary 5.4.** *Let $R$ be a left noetherian ring, and let $m, n \geqslant 0$. Then we have*

*(1) If* $\operatorname{splf} R \leqslant m$ *(in particular, if* $\mathrm{FPD}\, R \leqslant m$ *or* $\mathrm{id}_R R \leqslant m$*), then* $\oplus_{t=0}^{m+n} I^t(R)$ *is a $\Sigma$-embedding cogenerator for the subcategory of* $\operatorname{Mod} R$ *consisting of modules with flat dimension at most $n$.*

*(2) If* $\operatorname{splf} R < \infty$ *(in particular, if* $\mathrm{FPD}\, R < \infty$ *or* $\mathrm{id}_R R < \infty$*), then* $\oplus_{t\geqslant 0} I^t(C)$ *is a $\Sigma$-embedding cogenerator for the subcategory of* $\operatorname{Mod} R$ *consisting of modules with finite flat dimension.*

In view of Proposition 4.5 and Corollary 5.3, it is necessary for us to get more information about (the finiteness of) $\operatorname{spclfc} R$.

**Lemma 5.5.** *For any $m, n \geqslant 0$, we have*

*(1) If* $\mathrm{pd}_S N \leqslant n$ *for any flat module $N$ in* $\operatorname{Mod} S$*, then* $\mathcal{P}_C(R)\text{-}\mathrm{pd}_R M \leqslant m + n$ *for any module* $M \in \operatorname{Mod} R$ *with* $\mathcal{F}_C(R)\text{-}\mathrm{pd}_R M \leqslant m$.

*(2) If* $\mathrm{pd}_S N \leqslant n$ *(resp.* $< \infty$*) for any module* $N \in \operatorname{Mod} S$ *with* $\mathrm{fd}_S N < \infty$*, then* $\mathcal{P}_C(R)\text{-}\mathrm{pd}_R M \leqslant n$ *(resp.* $< \infty$*) for any module* $M \in \operatorname{Mod} R$ *with* $\mathcal{F}_C(R)\text{-}\mathrm{pd}_R M < \infty$.

*Proof.* (1) Let $M \in \operatorname{Mod} R$ with $\mathcal{F}_C(R)$-$\operatorname{pd}_R M \leqslant m$. By Lemmas 2.5(1) and 2.6(1), we have $M \in \mathcal{B}_C(R)$ and $\operatorname{fd}_S M_* \leqslant m$. Then by assumption and the dimension shifting, we have $\operatorname{pd}_S M_* \leqslant m+n$. So there exists an exact sequence

$$0 \to P_{m+n} \to \cdots \to P_1 \to P_0 \to M_* \to 0 \tag{5.1}$$

in $\operatorname{Mod} S$ with all $P_i$ projective. Applying the functor $C \otimes_S -$ to (5.1) yields the following exact sequence

$$0 \to C \otimes_S P_{m+n} \to \cdots \to C \otimes_S P_1 \to C \otimes_S P_0 \to C \otimes_S M_* (\cong M) \to 0$$

in $\operatorname{Mod} R$ with all $C \otimes_S P_i$ in $\mathcal{P}_C(R)$. It follows that $\mathcal{P}_C(R)$-$\operatorname{pd}_R M \leqslant m+n$.

(2) It has been essentially proved in (1). □

Let $k$ be a field, and let $S$ be a right-Noetherian $k$-algebra for which there exists a left-Noetherian $k$-algebra $R$ and a dualizing complex ${}_R D_S$. Then $\operatorname{pd}_S N < \infty$ for any module $N \in \operatorname{Mod} S$ with $\operatorname{fd}_S N < \infty$ ([24, Theorem]). So $\mathcal{P}_C(R)$-$\operatorname{pd}_R M < \infty$ for any module $M \in \operatorname{Mod} R$ with $\mathcal{F}_C(R)$-$\operatorname{pd}_R M < \infty$ by Lemma 5.5(2).

As a consequence of Lemma 5.5(1), we have the following result, which shows that the relative homological invariant $\operatorname{spclfc} R$ coincides with the absolute homological invariant $\operatorname{splf} S$. Compare it with Proposition 3.1.

**Theorem 5.6.** $\operatorname{spclfc} R = \operatorname{splf} S$.

*Proof.* Putting $m = 0$ in Lemma 5.5(1), it is easy to get $\operatorname{spclfc} R \leqslant \operatorname{splf} S$. Now let $\operatorname{spclfc} R = n (< \infty)$ and $N \in \operatorname{Mod} S$ be flat. Then $C \otimes_S N \in \mathcal{F}_C(R)$ and there exists an exact sequence

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to C \otimes_S N \to 0$$

in $\operatorname{Mod} R$ with all $C_i$ in $\mathcal{P}_C(R) (= \operatorname{Add}_R C)$. Applying the functor $\operatorname{Hom}_R(C, -)$ to it yields an exact sequence

$$0 \to C_{n*} \to \cdots \to C_{1*} \to C_{0*} \to (C \otimes_S N)_* \to 0$$

in $\operatorname{Mod} S$ with all $C_{i*}$ projective. Note that $N \in \mathcal{A}_C(S)$ by [16, Lemma 4.1]. Hence $N \cong (C \otimes_S N)_*$ and $\operatorname{pd}_S N \leqslant n$. Thus $\operatorname{splf} S \leqslant \operatorname{spclfc} R$ and the proof is complete. □

We finish the paper by the following interesting open problem suggested by a referee.

**Open Problem 5.7.** Let $\mathcal{C}$ be a preadditive category and $\mathrm{Mod}\,\mathcal{C}$ the category of additive contravariant functors from $\mathcal{C}$ to abelian groups. Similar problems on pure projective resolutions, pure projective and pure injective dimensions have been studied in $\mathrm{Mod}\,\mathcal{C}$ ([29]–[31]). It is interesting to study how to define suitably a semidualizing bimodule $T$ in $\mathrm{Mod}\,\mathcal{C}$ such that the results in this paper still hold true by replacing $C$ (in $\mathrm{Mod}\,R$) by $T$ (in $\mathrm{Mod}\,\mathcal{C}$).

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# References

[1] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras*, Vol. 1, Techniques of Representation Theory, London Mathematical Society Student Texts 65, Cambridge University Press, Cambridge, 2006.

[2] M. Auslander, *On the dimension of modules and algebras (III)*, Nagoya. Math. J. 9 (1955), 67–77.

[3] H. Bass, *Injective dimension in Noetherian rings*, Trans. Amer. Math. Sco. 102 (1962), 18–29.

[4] S. Bouchiba, *A variant theory for the Gorenstein flat dimension*, Colloq. Math. 140 (2015), 183–204.

[5] S. Bouchiba, *When flat and Gorenstein flat dimensions coincide?* Colloq. Math. 147 (2017), 77–85.

[6] L. W. Christensen, *Gorenstein dimensions*, Lecture Notes in Math. 1747, Springer-Verlag, Berlin-Heidelberg-New York, 2000.

[7] I. Emmanouil and O. Talelli, *On the flat length of injective modules,* J. London Math. Soc. 84 (2011), 408–432.

[8] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, de Gruyter Exp. in Math. 30, Walter de Gruyter, Berlin-New York, 2000.

[9] D. J. Fieldhouse, *Character modules*, Comment. Math. Helv. 46 (1971), 274–276.

[10] H.-B. Foxby, *Gorenstein modules and related modules*, Math. Scand. 31 (1972), 267–284.

[11] R. Göbel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, de Gruyter Expositions in Math. 41, 2nd revised and extended edition, Berlin-Boston 2012.

[12] E. S. Golod, *G-dimension and generalized perfect ideals*, Trudy Mat. Inst. Steklov. 165 (1984), 62–66.

[13] E. L. Green, *Finitistic dimensions of finite dimensional monomial algebras*, J. Algebra 136 (1991), 37–50.

[14] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra 189 (2004), 167–193.

[15] H. Holm and P. Jøgensen, *Semidualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Algebra 205 (2006), 423–445.

[16] H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. 47 (2007), 781–808.

[17] Z. Y. Huang, *Proper resolutions and Gorenstein categories*, J. Algebra 393 (2013), 142–169.

[18] Z. Y. Huang, *Homological dimensions relative to preresolving subcategories*, Kyoto J. Math. 54 (2014), 727–757.

[19] Y. Iwanaga, *On rings with finite self-injective dimension*, Comm. Algebra 7 (1979), 393–414.

[20] Y. Iwanaga, *On rings with finite self-injective dimension II*, Tsukuba J. Math. 4 (1980), 107–113.

[21] C. U. Jensen, *On homological dimensions of rings with countably generated ideals*, Math. Scand. 18 (1966), 97–105.

[22] C. U. Jensen, *Homological dimension of $\aleph_0$-coherent rings*, Math. Scand. 20 (1967), 55–66.

[23] C. U. Jensen, *On the vanishing of* $\varprojlim^{(i)}$, J. Algebra 15 (1970), 151–166.

[24] P. Jøgensen, *Finite flat and projective dimension*, Comm. Algebra 33 (2005), 2275–2279.

[25] Z. F. Liu, Z. Y. Huang and A. M. Xu, *Gorenstein projective dimension relative to a semidualizing bimodule*, Comm. Algebra 41 (2013), 1–18.

[26] J. Miyachi, *Injective resolutions of Noetherian rings and cogenerators*, Proc. Amer. Math. Soc. 128 (2000), 2233–2242.

[27] B. L. Osofsky, *Upper bounds of homological dimensions,* Nagoya. Math. J. 32 (1968), 315–322.

[28] M. Salimi, E. Tavasoli, P. Moradifar and S. Yassemi, *Syzygy and torsionless modules with respect to a semidualizing module*, Algebr. Represent. Theory 17 (2014), 1217–1234.

[29] D. Simson, *On pure global dimension of locally finitely presented Grothendieck categories*, Fund. Math. 96 (1977), 91–116.

[30] D. Simson, *Pure-periodic modules and a structure of pure-projective resolutions*, Pacific J. Math. 207 (2002), 235–256.

[31] D. Simson, *Flat complexes, pure periodicity and pure acyclic complexes*, J. Algebra 480 (2017), 298–308.

[32] R. Takahashi, *A new approximation theory which unifies spherical and Cohen-Macaulay approximations*, J. Pure Appl. Algebra 208 (2007), 617–634.

[33] X. Tang, *New characterizations of dualizing modules*, Comm. Algebra 40 (2012), 845–861.

[34] X. Tang, *FP-injectivity relative to a semidualizing bimodule*, Publ. Math. Debrecen 80 (2012), 311–326.

[35] X. Tang and Z. Y. Huang, *Homological aspects of the dual Auslander transpose*, Forum Math. 27 (2015), 3717–3743.

[36] X. Tang and Z. Y. Huang, *Homological aspects of the dual Auslander transpose II*, Kyoto J. Math. 57 (2017), 17–53.

[37] X. Tang and Z. Y. Huang, *Homological aspects of the adjoint cotranspose*, Colloq. Math. 150 (2017), 293–311.

[38] T. Wakamatsu, *On modules with trivial self-extensions*, J. Algebra 114 (1988), 106–114.

[39] T. Wakamatsu, *Tilting modules and Auslander's Gorenstein property*, J. Algebra 275 (2004), 3–39.

[40] D. White, *Gorenstein projective dimension with respect to a semidualizing module*, J. Commutative Algebra 2 (2010), 111–137.