COREFLEXIVE MODULES AND SEMIDUALIZING MODULES WITH
FINITE PROJECTIVE DIMENSION

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Abstract. Let $R$ and $S$ be rings and $\omega_R$ a semidualizing bimodule. For a subclass $T$ of the class of $\omega$-coreflexive modules and $n \geq 1$, we introduce and study modules of $\omega_T$-class $n$. By using the properties of such modules, we get some equivalent characterizations for $\omega_S$ having finite projective dimension. In particular, we prove that the projective dimension of $\omega_S$ is at most $n$ if and only if any module of $\omega_T$-class $n$ is $\omega$-coreflexive. Moreover, we get some equivalent characterizations for $\omega_S$ having finite projective dimension at most two or one in terms of the properties of (adjoint) $\omega$-coreflexive and $\omega$-cotorsionless modules. Finally, we give some partial answers to the Wakamatsu tilting conjecture.

1. Introduction

It is well known that the (Auslander) transpose is one of the most powerful tools in representation theory of artin algebras and Gorenstein homological algebra, see [AB, ARS, EJ], and references therein. However, this notion does not have its dual version as many notions in classical homological algebra do. So, a natural question is: How to dualize the (Auslander) transpose of modules appropriately? To this aim, we introduced in [TH1, TH3] the notions of the cotranspose and adjoint cotranspose of modules with respect to a semidualizing bimodule $\omega$. Then we showed in [TH1, TH2, TH3] that many interesting notions and results related to the (Auslander) transpose have counterparts related to the (adjoint) cotranspose. For example, the counterparts of torsionless, reflexive and $n$-torsionfree modules are $\omega$-cotorsionless, $\omega$-coreflexive and $n$-$\omega$-cotorsionfree modules, respectively. As a continue of these three papers, this paper is devoted to developing a further general theory introduced in them.

Wakamatsu in [W1] introduced and studied the so-called generalized tilting modules, which are usually called Wakamatsu tilting modules, see [BR, MR]. The Wakamatsu tilting conjecture is an important homological conjecture in representation theory of artin algebras, which states that for a Wakamatsu tilting module $\omega_R$ over an artin algebra $R$, the projective (or injective) dimensions of $\omega_R$ and $\omega_{\text{End}(R)}$ are identical ([BR, MR]). This conjecture situates between the famous finitistic dimension conjecture and the Gorenstein symmetry conjecture; in particular, the latter one is a special case of the Wakamatsu tilting conjecture. All these conjectures remain still open. By [W1, Theorem], the Wakamatsu tilting conjecture is equivalent to that for a Wakamatsu tilting module $\omega_R$ over an artin algebra $R$, the

2010 Mathematics Subject Classification. 18G25, 16E10, 16E30.
Key words and phrases: HT-projections, Modules of $\omega_T$-class $n$, (Adjont) $\omega$-corelexible modules, (Adjont) $\omega$-cotorsionless modules, (Adjont) $n$-$\omega$-cospherical modules, Projective dimension, Wakamatsu Tilting Conjecture.
projective (or injective) dimension of $R\omega$ is finite if and only if so is the projective (or injective) dimension of $\omega \End(R\omega)$. Huang in [Hu] generalized this equivalent version to left and right noetherian rings.

Observe that the Wakamatsu tilting conjecture makes sense for arbitrary rings. Let $R$ and $S$ be arbitrary rings. By [W2, Corollary 3.2], we have that a bimodule $R\omega S$ is semiendowing if and only if $R\omega$ is Wakamatsu tilting with $S = \End(R\omega)$, and if and only if $\omega S$ is Wakamatsu tilting with $R = \End(\omega S)$. We proved in [TH3, Proposition 5.1] that for a semiendowing bimodule $R\omega S$, the projective dimensions of $R\omega$ and $\omega S$ are identical provided that both of them are finite. So, over arbitrary rings $R$ and $S$, the Wakamatsu tilting conjecture is equivalent to that for a semiendowing bimodule $R\omega S$, the projective dimension of $R\omega$ is finite if and only if so is the projective dimension of $\omega S$. In this paper, we will study when the projective dimension of $\omega S$ is at most $n$ by using the properties of modules of $\omega$-$T$-class $n$, (adjoint) $\omega$-cotorsionless and $\omega$-coreflexive modules.

This paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

Let $R$ and $S$ be rings and $\omega R$ a semiendowing bimodule. In Section 3, we introduce and study Hom-Tensor projections and Tensor-Hom injections as duals of double dual embeddings in [J3]. Let $M$ be a left $R$-module and $F$ a left $S$-module. An epimorphism $\omega_S F \xrightarrow{1 \otimes \phi} \omega_S \Hom_R(\omega, M)$ of left $R$-modules is called a Hom-Tensor projection if it is obtained by applying the functor $\omega_S \otimes -$ to an epimorphism $F \xrightarrow{\phi} \Hom_R(\omega, M)$ of left $S$-modules. We prove that the kernel of a Hom-Tensor projection with $F$ adjoint $\omega$-coreflexive and $\omega \otimes S F$ 1-$\omega$-cospherical is the $\omega$-cotorsionless submodule of a 1-$\omega$-cospherical left $R$-module; conversely, the $\omega$-cotorsionless submodule of a 1-$\omega$-cospherical left $R$-module is the kernel of a special Hom-Tensor projection. We also get an adjoint version of this result about Tensor-Hom injections.

Jans introduced in [J3] the notion of modules of $D$-class $n$ in terms of the properties of double dual embeddings, and proved that for a left and right noetherian ring $R$ and $n \geq 1$, the right self-injective dimension of $R$ is at most $n$ if and only if any finitely generated left $R$-module of $D$-class $n$ is reflexive; and the global dimension of $R$ is at most $n + 1$ if and only if $\Hom_R(M, R)$ is projective for any finitely generated left $R$-module $M$ of $D$-class $n$. Motivated by Jans’s philosophy, in Section 4 we introduce and study modules of $\omega$-$T$-class $n$ in terms of the properties of $\Hom$-Tensor projections, where $T$ is a subclass of the class of adjoint $\omega$-coreflexive left $S$-modules and $n \geq 1$. We prove that if $U_n$ is a left $R$-module of $\omega$-$T$-class $n$, then there exists a collection of exact sequences $0 \to \Hom_R(\omega, U_i) \to F_{i-1} \to \Hom_R(\omega, U_{i-1}) \to 0$ $(2 \leq i \leq n)$ of left $S$-modules with all $F_i \in T$ and $U_i$ left $R$-modules; conversely, if there exists a collection of exact sequences as above, then $U_n$ can be selected of $\omega$-$T$-class $n$. Let $T$ be a subclass of the weak Auslander class with respect to $\omega$ containing all projective left $S$-modules. We prove that the projective dimension of $\omega S$ is at most $n$ if and only if any left $R$-module of $\omega$-$T$-class $n$ is $\omega$-coreflexive, and if and only if $\Tor^P_0(\omega, V) = 0$ for any adjoint $\omega$-cotorsionless left $S$-module $V$.

As a supplement to this result, we get that the projective dimension of $\omega S$ is at most $n + 1$ if and only if $\Tor^P_1(\omega, \Hom_R(\omega, U_n)) = 0$ for any left $R$-module $U_n$ of $\omega$-$T$-class $n$.

In Section 5, we first obtain some useful exact sequences to describe the kernel and cokernel of the canonical valuation homomorphism $\omega \otimes_S \Hom_R(\omega, M) \to M$.
with $M$ a left $R$-module; and then prove that any $n$-$\omega$-cospherical left $R$-module is $\omega$-coreflexive provided that either the projective dimension of $\omega_S$ is at most $n$ or $\omega_S$ admits a projective resolution ultimately closed at $n$.

In Section 6, we characterize when $\omega_S$ has small projective dimension in terms of the properties of (adjoint) $\omega$-coreflexive modules and $\omega$-cotorsionless modules. We prove that if the projective dimension of $R\omega$ is at most two, then the projective dimension of $\omega_S$ is at most two if and only if any $2$-$\omega$-cospherical left $R$-module is $\omega$-coreflexive, and if and only if any adjoint $\omega$-coreflexive left $S$-module is adjoint $2$-$\omega$-cospherical, if and only if any left $R$-module of $\omega$-$T$-class 2 is $\omega$-coreflexive, if and only if $\text{Tor}_{\omega}^2(S, V) = 0$ for any adjoint $\omega$-cotorsionless left $S$-module $V$, and if and only if $\text{Tor}_{\omega}^2(\omega, \text{Hom}_R(\omega, U)) = 0$ for any $\omega$-cotorsionless left $R$-module $U$. Moreover, we get that the projective dimension of $\omega_S$ is at most one if and only if any $1$-$\omega$-cospherical left $R$-module is $\omega$-cotorsionless (or $\omega$-coreflexive), if and only if any $\omega$-cotorsionless left $R$-module is $\omega$-coreflexive, and if and only if $\text{Tor}_{\omega}^2(\omega, V) = 0$ for any adjoint $\omega$-cotorsionless left $S$-module module $V$.

In Section 7, we study the Wakamatsu tilting conjecture in some special cases. Let $S$ be a left artinian ring, $R\omega_S = S\omega_S$ and $m,n \geq 1$. We prove that if the projective dimension of $S\omega$ is at most $n$ and the Ext-grade of $\text{Tor}_{m}^S(\omega, N)$ with respect to $\omega$ is at most $n - 1$ for any finitely presented left $S$-module $N$, then the projective dimensions of $S\omega$ and and $\omega_S$ are identical. Then we apply this result to get that if the projective dimension of $S\omega$ is at most $n$ and the projective dimension of $\text{Hom}_S(P_i(\omega), \omega)$ is finite for any $0 \leq i \leq n - 2$, where $P_i(\omega)$ is the $(i+1)$-st term in a minimal projective resolution of $S\omega$, then the projective dimensions of $S\omega$ and and $\omega_S$ are identical. As a consequence, we get that if the projective dimension of $S\omega$ is at most one, then the projective dimensions of $S\omega$ and and $\omega_S$ are identical. Finally, we get that for an artin algebra $S$, if the right self-injective dimension of $S$ is at most $n$ and the projective dimensions of the first $n - 1$ terms in a minimal injective resolution of $S_S$ are finite, then the left and right self-injective dimensions of $S$ are identical.

2. Preliminaries

Throughout this paper, all rings are associative rings with unites. For a ring $R$, we use $\text{Mod} R$ (resp. $\text{Mod} R^{op}$) to denote the class of left (resp. right) $R$-modules.

**Definition 2.1.** ([HW]). Let $R$ and $S$ be rings. An $(R,S)$-bimodule $R\omega_S$ is called semidualizing if

1. An $(R,S)$-bimodule $R\omega_S$ is called semidualizing if the following conditions are satisfied.

   a1) $R\omega$ admits a degreewise finite $R$-projective resolution.
   a2) $\omega_S$ admits a degreewise finite $S$-projective resolution.
   b1) The homothety map $\text{RR}_R \pi^{-1}_S \text{Hom}_{S^{op}}(\omega, \omega)$ is an isomorphism.
   b2) The homothety map $\text{SS}_S \psi^{-1}_S \text{Hom}_R(\omega, \omega)$ is an isomorphism.
   c1) $\text{Ext}_R^{\omega - 1}(\omega, \omega) = 0$.
   c2) $\text{Ext}_S^{\omega - 1}(\omega, \omega) = 0$.

2. A semidualizing bimodule $R\omega_S$ is called faithful if the following conditions are satisfied.
   e1) If $M \in \text{Mod} R$ and $\text{Hom}_R(\omega, M) = 0$, then $M = 0$.
   e2) If $N \in \text{Mod} S^{op}$ and $\text{Hom}_{S^{op}}(\omega, N) = 0$, then $N = 0$. 

3
Let $R$ be a ring. Recall from [W1, W2] that a module $\omega$ in \text{Mod}$R$ is called \textit{generalized tilting} (it is usually called \textit{Wakamatsu tilting}, see [BR, MR]) if it satisfies the conditions (a1) and (c1) in Definition 2.1, and there exists an exact sequence

$$0 \rightarrow R \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots \rightarrow W' \rightarrow \cdots$$

in \text{Mod}$R$ with all $W^i$ isomorphic to direct summands of finite sums of copies of $R\omega$, such that it remains still exact after applying the functor $\text{Hom}_{R}(\_, R\omega)$. A close relation between semidualizing bimodules and Wakamatsu tilting modules was established in [W2, Corollary 3.2] (see the introduction).

By [HW, Proposition 3.1], we have that any semidualizing bimodule over a commutative ring is faithful. The following example illustrates that there exist sufficiently many (faithful) semidualizing bimodules.

**Example 2.2.**

1. For any ring $R$, $R_R$ is semidualizing.
2. Let $R$ be an artin algebra, and let $\{T_1, \ldots, T_n\}$ be a complete set of non-isomorphic simple left $R$-module. Then $\omega := \oplus_{i=1}^{n} I^0(T_i)$ is Wakamatsu tilting, where $I^0(T_i)$ is the injective envelope of $T_i$ for any $1 \leq i \leq n$. By [W2, Corollary 3.2], we have that $R\omega$ is semidualizing, where $S = \text{End}(R\omega)$.
3. Let $k$ be a field. Then both $A = k[x, y]/(x, y)^2$ and $S = A[u, v]/(u, v)^2$ are commutative artinian non-Gorenstein local rings; and $\text{Hom}_A(S, A)$ and $S \otimes_A \text{Hom}_A(A, k)$ are mutually non-isomorphic semidualizing $(S, S)$-bimodules with infinite projective and injective dimensions ([S, Example 2.3.2]).
4. Let $R$ be a flat $S$-algebra over a commutative ring $S$. If $S_E$ is a semidualizing bimodule, then $E \otimes_S R$ is a faithfully semidualizing $(R, R)$-bimodule ([HW, Proposition 3.2]).

From now on, $R$ and $S$ are arbitrary associative rings with unit and $R\omega$ is a semidualizing bimodule. We write $(\_, \_):= \text{Hom}(\_, \_)$.

Let $M \in \text{Mod}$R. Then we have a canonical valuation homomorphism

$$\theta_M: \omega \otimes_S M \rightarrow M$$

defined by $\theta_M(x \otimes f) = f(x)$ for any $x \in \omega$ and $f \in M_{\omega}$. $M$ is called $\omega$-cotorsionless if $\theta_M$ is an epimorphism; and $M$ is called $\omega$-coreflexive if $\theta_M$ is an isomorphism (see [TH1]). We use $\text{Cot}_{\omega}(R)$ and $\text{Cor}_{\omega}(R)$ to denote the subclasses of $\text{Mod}$R consisting of $\omega$-cotorsionless modules and $\omega$-coreflexive modules, respectively.

Let $N \in \text{Mod}$S. Then we have a canonical valuation homomorphism

$$\mu_N: N \rightarrow (\omega \otimes_S N)_{\omega}$$

defined by $\mu_N(y)(x) = x \otimes y$ for any $y \in N$ and $x \in \omega$. $N$ is called adjoint $\omega$-cotorsionless if $\mu_N$ is a monomorphism; and $N$ is called adjoint $\omega$-coreflexive if $\mu_N$ is an isomorphism. We use $\text{Acot}_{\omega}(S)$ and $\text{Acor}_{\omega}(S)$ to denote the subclasses of $\text{Mod}$S consisting of adjoint $\omega$-cotorsionless modules and adjoint $\omega$-coreflexive modules, respectively.

**Definition 2.3.**

1. The weak Auslander class $wA_{\omega}(S)$ with respect to $\omega$ consists of all left $S$-modules $N$ satisfying
   
   \begin{itemize}
   \item[(A1)] $\text{Tor}^{S}_{i \geq 1}(\omega, N) = 0$, and
   \item[(A2)] $N \in \text{Acor}_{\omega}(S)$.
   \end{itemize}
The Auslander class $A_{\omega}(S)$ with respect to $\omega$ consists of all left $S$-modules $N$ satisfying (A1), (A2) and 

(A3) $\text{Ext}_R^1(\omega, \omega \otimes_S N) = 0$.

We will heavily use the following two lemmas in the sequel.

**Lemma 2.4.** Let $M \in \text{Mod } R$ and $N \in \text{Mod } S$. Then we have

1. $(\theta_M)_*: \mu_M = 1_{M_*}$.
2. $\theta_{\omega \otimes_S N} : (1_\omega \otimes \mu_N) = 1_{\omega \otimes_S N}$.
3. There exists an equivalence of categories

$$\text{Acor}_{\omega}(S) \overset{\sim}{\longrightarrow} \text{Cor}_{\omega}(R).$$

**Proof.** See [TH2, Lemma 6.1] for the assertions (1) and (2). The assertion (3) is a direct consequence of (1) and (2). \qed

Following [HW], set

$$F_\omega(R) := \{ \omega \otimes_S F \mid F \text{ is flat in } \text{Mod } S \},$$

$$P_\omega(R) := \{ \omega \otimes_S P \mid P \text{ is projective in } \text{Mod } S \},$$

$$I_\omega(S) := \{ I_* \mid I \text{ is injective in } \text{Mod } R \},$$

$$R^1(\omega) := \{ M \in \text{Mod } R \mid \text{Ext}_R^1(\omega, M) = 0 \}.$$ 

The modules in $F_\omega(R)$, $P_\omega(R)$ and $I_\omega(S)$ are called $\omega$-flat, $\omega$-projective and $\omega$-injective respectively. We use $I(R)$ to denote the subclass of $\text{Mod } R$ consisting of injective modules, and use $P(S)$ and $F(S)$ to denote the subclasses of $\text{Mod } S$ consisting of projective modules and flat modules, respectively. For a module $M \in \text{Mod } R$, we use $\text{Add}_R M$ to denote the subclass of $\text{Mod } R$ consisting of all direct summands of direct sums of copies of $M$.

**Lemma 2.5.** ([LHX, Proposition 2.4(1)] and [HW, Lemma 4.1 and Corollary 6.1]).

1. $\text{Add}_R \omega = P_\omega(R) \subseteq F_\omega(R) \cup I(R) \subseteq \text{Cor}_{\omega}(R) \cap R^1(\omega)$.
2. $P(S) \subseteq F(S) \cup I_\omega(S) \subseteq A_\omega(S) \subseteq wA_\omega(S) \subseteq \text{Acor}_\omega(S)$.

Motivated by the notion of $n$-spherical modules given in [AB], we introduce the following

**Definition 2.6.** Let $n \geq 1$.

1. ([TH1]) A module $M \in \text{Mod } R$ is called $n$-$\omega$-cospherical if $\text{Ext}_R^{1 \leq i \leq n}(\omega, M) = 0$.
2. A module $N \in \text{Mod } S$ is called adjoint $n$-$\omega$-cospherical if $\text{Tor}_R^S(1 \leq i \leq n)(\omega, N) = 0$.

We shall say that any module in $\text{Mod } R$ is $0$-$\omega$-cospherical, and any module in $\text{Mod } S$ is adjoint $0$-$\omega$-cospherical.

Let $M \in \text{Mod } R$. We use

$$0 \rightarrow M \overset{f^{-1}(M)}{\rightarrow} I^0(M) \overset{f^0(M)}{\rightarrow} I^1(M) \overset{f^1(M)}{\rightarrow} \cdots \overset{f^{i-1}(M)}{\rightarrow} I^i(M) \overset{f^i(M)}{\rightarrow} \cdots$$

to denote a minimal injective resolution of $M$ in $\text{Mod } R$.

**Definition 2.7.** ([TH1]). Let $M \in \text{Mod } R$ and $n \geq 1$. 


(1) $\text{cTr}_\omega M := \text{Coker}(\phi^0(M))$ is called the cotranspose of $M$ with respect to $\omega_*$.  
(2) $M$ is called $n$-$\omega$-cotorsionfree if and only if it is adjoint $n$-$\omega$-cospherical.

By [TH1, Proposition 3.2] (see Corollary 5.2(1) below), we have that for a module $M \in \text{Mod } R$, $M$ is 1-$\omega$-cotorsionfree if and only if it is 1-$\omega$-cospherical, and $M$ is 2-$\omega$-cotorsionfree if and only if it is $\omega$-cospherical. Note that the notion of $\omega$-cospherical modules has appeared in [Az].

Let $N \in \text{Mod } S$ and we use

$$\cdots \to f_i(N) \xrightarrow{f_{i-1}(N)} \cdots \to f_1(N) \xrightarrow{f_0(N)} f_0(N) \xrightarrow{f_{-1}(N)} N \to 0 \quad (2.1)$$

to denote a minimal flat resolution of $N$ in $\text{Mod } S$, where each $f_i(N) \to \text{Coker } f_i(N)$ is a flat cover of $\text{Coker } f_i(N)$. The existence of such a resolution is guaranteed by the fact that any module has a flat cover (see [BBE]). Based on the fact that $(\omega \otimes_S -)$ is an adjoint pair, the counterpart of Definition 2.7 was given in [TH3] as follows.

**Definition 2.8.** ([TH3]). Let $N \in \text{Mod } S$ and $n \geq 1$.

(1) $\text{acTr}_\omega N := \text{Ker}(1_{\omega} \otimes f_0(N))$ is called the adjoint cotranspose of $N$ with respect to $\omega_*$. 
(2) $N$ is called adjoint $n$-$\omega$-cotorsionfree if $\text{acTr}_\omega N$ is $n$-$\omega$-cospherical.

By Corollary 5.2(2) below, we have that for a module $N \in \text{Mod } S$, $N$ is adjoint 1-$\omega$-cotorsionfree if and only if it is adjoint $\omega$-cospherical; and $N$ is adjoint 2-$\omega$-cotorsionfree if and only if it is adjoint $\omega$-cospherical.

The following result about the properties of (adjoint) $\omega$-cotorsionless and $\omega$-cospherical is useful.

**Proposition 2.9.**

(1) Let

$$0 \to K \xrightarrow{\lambda} F \xrightarrow{\phi} N \to 0$$

be an exact sequence in $\text{Mod } S$ with $F \in \text{Acot}_\omega(S)$ and $N \in \text{Acot}_\omega(S)$. Then $N \cong \text{Im}(1_{\omega} \otimes \phi)_*$ and $K \cong H_*$, where $H = \text{Ker}(1_{\omega} \otimes \phi)$.

(2) Let

$$0 \to M \xrightarrow{\psi} I \xrightarrow{\alpha} H \to 0$$

be an exact sequence in $\text{Mod } R$ with $I \in \text{Cor}_\omega(R)$ and $M \in \text{Cot}_\omega(R)$. Then $M \cong \text{Im}(1_{\omega} \otimes \psi)_*$ and $H \cong \omega \otimes_S K$, where $K = \text{Coker } \psi_*$.

**Proof.** (1) By assumption, we have the following exact sequence

$$0 \to H \xrightarrow{\delta} \omega \otimes_S F \xrightarrow{1_{\omega} \otimes \phi} \omega \otimes_S N \to 0$$

in $\text{Mod } R$ with $H = \text{Ker}(1_{\omega} \otimes \phi)$. Consider the following exact commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\lambda} & F & \xrightarrow{\phi} & N & \longrightarrow & 0 \\
& & 1 & \downarrow h \downarrow & \mu_F & \downarrow & \mu_N \\
0 & \longrightarrow & H_* & \xrightarrow{\delta_*} & (\omega \otimes_S F)_* & \xrightarrow{(1_{\omega} \otimes \phi)_*} & (\omega \otimes_S N)_* & \longrightarrow & 0
\end{array}
$$
where \( h \) is an induced homomorphism. Because \( \mu_F \) is an isomorphism and \( \mu_N \) is a monomorphism by assumption, we have that \( N \cong \text{Im} \mu_N \cong \text{Im}(1_\omega \otimes \phi)_* \) and \( h \) is an isomorphism by the snake lemma.

(2) By assumption, we have the following exact sequence
\[ 0 \to M \xrightarrow{\psi} I \xrightarrow{\pi} K \to 0 \]
in \( \text{Mod} \, S \) with \( K = \text{Coker} \psi \). Consider the following commutative diagram with exact rows
\[
\begin{array}{ccc}
\omega \otimes_S M & \xrightarrow{1_\omega \otimes \psi} & \omega \otimes_S I \\
\theta_M & & \theta_I \\
0 & \xrightarrow{\psi} & M & \xrightarrow{\alpha} & I & \xrightarrow{\gamma} & H & \to 0, \\
\end{array}
\]
where \( \gamma \) is an induced homomorphism. Because \( \theta_I \) is an isomorphism and \( \theta_M \) is an epimorphism by assumption, we have that \( M = \text{Im} \theta_M \cong \text{Im}(1_\omega \otimes \psi)_* \) and \( \gamma \) is an isomorphism by the snake lemma.

\[ \square \]

3. Hom-Tensor projections and Tensor-Hom injections

We begin with the following definition which will be convenient for our exposition.

**Definition 3.1.** Let \( M \in \text{Mod} \, R \) and \( F \in \text{Mod} \, S \). An epimorphism
\[ 1_\omega \otimes \phi : \omega \otimes_S F \to \omega \otimes_S M_* \]
in \( \text{Mod} \, R \) is called a Hom-Tensor projection (HT-projection for short) if it is obtained by applying the functor \( \omega \otimes_S - \) to an epimorphism \( \phi : F \to M \) in \( \text{Mod} \, S \).

To study the properties of HT-projections, we need the following

**Lemma 3.2.** Let \( A, B \in \text{Mod} \, R \). Then the following statements are equivalent.

1. \( A \cong \text{Im} \theta_B \).
2. \( A \in \text{Cot}_{\omega}(R) \) and there exists a monomorphism \( f : A \to B \) in \( \text{Mod} \, R \) such that \( f_* \) is an isomorphism.

**Proof.** (1) \( \Rightarrow \) (2) Let \( A \cong \text{Im} \theta_B \) and \( g : A \to \text{Im} \theta_B \) be an isomorphism in \( \text{Mod} \, R \). Since \( \theta_{\omega \otimes_S B_*} \) is an epimorphism and \( \omega \otimes_S B_* \in \text{Cot}_{\omega}(R) \) by Lemma 2.4(2), we have \( A \in \text{Cot}_{\omega}(R) \) by [TH1, Corollary 3.8]. Let \( \theta_B = i \cdot p \) be the natural epic-monic decomposition of \( \theta_B \) with \( p : \omega \otimes_S B_* \to \text{Im} \theta_B \) and \( i : \text{Im} \theta_B \to B \). Then \( f := i \cdot g \) is monic. Note that \( (\theta_B)_* = i_* \cdot p_* \) and \( (\theta_B)_* \) is a split epimorphism by Lemma 2.4(1). It yields that \( i_* \) is an epimorphism and hence an isomorphism. Thus \( f_* = i_* \cdot g_* \) is an isomorphism.

(2) \( \Rightarrow \) (1) Let \( A \) be \( \omega \)-cotorsionless and \( f : A \to B \) be a monomorphism in \( \text{Mod} \, R \) such that \( f_* \) is an isomorphism. Consider the following commutative diagram with the bottom row exact
\[
\begin{array}{ccc}
\omega \otimes_S A & \xrightarrow{1_\omega \otimes f_*} & \omega \otimes_S B_* \\
\theta_A & & \theta_B \\
0 & \xrightarrow{f} & A & \xrightarrow{g} & B.
\end{array}
\]
Since $A \in \text{Cot}_\omega(R)$ and $f_*$ is an isomorphism, $\theta_A$ is an epimorphism and $1_\omega \otimes f_*$ is also an isomorphism. So we have
\[
\text{Im } \theta_B = \text{Im}(\theta_B \cdot (1_\omega \otimes f_*)) = \text{Im}(f \cdot \theta_A) = \text{Im } f \cong A.
\]

For a module $M \in \text{Mod } R$, we call $\text{Im } \theta_M$ the $\omega$-cotorsionless submodule of $M$. The following addresses the relation between HT-projections and the $\omega$-cotorsionless submodules of $1$-$\omega$-cospherical modules.

**Theorem 3.3.** Let $M \in \text{Mod } R$ and $F \in \text{Mod } S$. If
\[
1_\omega \otimes \phi: \omega \otimes_S F \rightarrow \omega \otimes_S M
\]
is a HT-projection with $F \in \text{Acor}_\omega(S)$ and $\omega \otimes_S F$ $1$-$\omega$-cospherical in $\text{Mod } R$, then $H := \text{Ker}(1_\omega \otimes \phi)$ is the $\omega$-cotorsionless submodule of a $1$-$\omega$-cospherical module in $\text{Mod } R$.

Conversely, if $H$ is the $\omega$-cotorsionless submodule of a $1$-$\omega$-cospherical module in $\text{Mod } R$, then there exists an exact sequence
\[
0 \rightarrow H \rightarrow E \xrightarrow{\alpha} Y \rightarrow 0
\]
in $\text{Mod } R$ with $E$ injective and $\alpha: E \rightarrow Y$ a HT-projection.

**Proof.** Let
\[
0 \rightarrow H \rightarrow \omega \otimes_S F \xrightarrow{1_\omega \otimes \phi} \omega \otimes_S M \rightarrow 0
\]
be an exact sequence in $\text{Mod } R$ with $1_\omega \otimes \phi$ a HT-projection, $F \in \text{Acor}_\omega(S)$, $\omega \otimes_S F$ $1$-$\omega$-cospherical in $\text{Mod } R$ and $H = \text{Ker}(1_\omega \otimes \phi)$. Then we have the following exact sequence
\[
0 \rightarrow K \rightarrow F \xrightarrow{\phi} M_\ast \rightarrow 0\quad (3.1)
\]
in $\text{Mod } S$, where $K = \text{Ker } \phi$. Because $F \in \text{Acor}_\omega(S)$ and $M_\ast \in \text{Acot}_\omega(S)$ by assumption and Lemma 2.4(1) respectively, we have $K \cong H_\ast$ by Proposition 2.9(1).

Applying the functor $\omega \otimes_S -$ to (3.1) yields that $H$ is isomorphic to a quotient module of $\omega \otimes_S K$. Using Lemma 2.4(2) and [TH1, Corollary 3.8], we get $H \in \text{Cot}_\omega(R)$. Let $L = \text{Im } \theta_M$ and let $\theta_M = i \cdot p$ be the natural epic-monic decomposition of $\theta_M$ with $p: \omega \otimes_S M_\ast \rightarrow L$ and $i: L \rightarrow M$. Then
\[
i_* \cdot p_* \cdot \mu_{M_\ast} = (\theta_M)_* \cdot \mu_{M_\ast} = 1_{M_\ast}
\]
by Lemma 2.4(1). It implies that $i_*$ is an epimorphism, and hence an isomorphism. So $p_* \cdot \mu_{M_\ast}$ is also an isomorphism. Set $H' = \text{Ker}(p \cdot (1_\omega \otimes \phi))$. Consider the following commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \rightarrow & H & \rightarrow & \omega \otimes_S F & \xrightarrow{1_\omega \otimes \phi} & \omega \otimes_S M_\ast & \rightarrow & 0 \\
& & \downarrow \lambda & & \downarrow & & \downarrow p & & \\
0 & \rightarrow & H' & \rightarrow & \omega \otimes_S F & \xrightarrow{p \cdot (1_\omega \otimes \phi)} & L & \rightarrow & 0,
\end{array}
\]
Diagram (3.1)
where $\lambda$ is an induced homomorphism which is monic. Because $(1_\omega \otimes \phi)_* \cdot \mu_F = 
abla_{M_{\omega}} : \phi$ and $\omega \otimes_S F$ is $1\omega$-cospHERical in $\text{Mod} R$, applying the functor $\text{Hom}_R(\omega, -)$ to Diagram (3.1) gives the following commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_* & \longrightarrow & (\omega \otimes_S F)_* & \phi \mu_{M_\omega}^{-1} & M_* & \longrightarrow & 0 \\
\downarrow \lambda_* & & \downarrow & & \downarrow \mu_* & & \downarrow p_* \mu_{M_\omega} & & \downarrow & \\
0 & \longrightarrow & H'_* & \longrightarrow & (\omega \otimes_S F)_{p*}(1_\omega \otimes \phi) & L_* & \longrightarrow & \text{Ext}_R^1(\omega, H') & \longrightarrow & 0.
\end{array}
$$

Because $p_* \cdot \mu_{M_\omega}$ is an isomorphism, we have that $\text{Ext}_R^1(\omega, H') = 0$ and $\lambda_*$ is also an isomorphism. Then it follows from Lemma 3.2 that $H$ is the $\omega$-cotorSionless submodule of a $1\omega$-cospHERical module $H'$.

Conversely, let $H$ be the $\omega$-cotorSionless submodule of a $1\omega$-cospHERical module $H'$ in $\text{Mod} R$. By Lemma 3.2, there exists a monomorphism $f : H \rightarrow H'$ such that $f_*$ is an isomorphism. Consider the following commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H & \xrightarrow{\psi} & E & \xrightarrow{\alpha} & Y & \longrightarrow & 0 \\
\downarrow f & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H' & \xrightarrow{e} & E & \xrightarrow{\beta} & Y' & \longrightarrow & 0,
\end{array}
$$

where $E$ is injective, $e$ is an embedding, $\psi = e \cdot f$, $Y = \text{Coker} \psi$ and $Y' = \text{Coker} e$.

We claim that $\alpha : E \rightarrow Y$ is a $\text{HT}$-projection. Since $H'$ is $1\omega$-cospHERical, we have the following commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_* & \xrightarrow{\psi_*} & E_* & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\
\downarrow f_* & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H'_* & \xrightarrow{e_*} & E_* & \xrightarrow{\beta_*} & Y'_* & \longrightarrow & 0,
\end{array}
$$

where $Z = \text{Coker} \psi_*$. Since $f_*$ is an isomorphism, we have $Z \cong Y'_*$. By Proposition 2.9(2) and its proof, we have that $Y \cong \omega \otimes_S Z$ and $\alpha : E \rightarrow Y$, up to isomorphism, is formed by tensoring $\pi : E_* \rightarrow Z(\cong Y'_*)$ with $\omega \otimes_S -$. The claim is proved.

As a consequence of Theorem 3.3, we have the following

**Corollary 3.4.** Let $M \in \text{Mod} R$ and $F \in \text{Mod} S$, and let $1_\omega \otimes \phi : \omega \otimes_S F \rightarrow \omega \otimes_S M_\omega$ be a $\text{HT}$-projection with $F \in \text{Cor}_{\omega}(S)$ and $\omega \otimes_S F$ $1\omega$-cospHERical in $\text{Mod} R$. Then $H := \text{Ker}(1_\omega \otimes \phi)$ is a $\omega$-cotorSionless and $1\omega$-cospHERical module in $\text{Mod} R$ provided that one of the following conditions is satisfied.

1. $M \in \text{Cor}_{\omega}(R)$.
2. $\omega \otimes_S M_\omega \in \text{Cor}_{\omega}(R)$ and $\rho_{\omega S}$ is faithful.

Conversely, if $H$ is a $\omega$-cotorSionless and $1\omega$-cospHERical module in $\text{Mod} R$ and

$$
0 \rightarrow H \rightarrow E \rightarrow Y \rightarrow 0
$$

is an exact sequence in $\text{Mod} R$ with $E$ injective, then $E \rightarrow Y$ is a $\text{HT}$-projection.
Proof. By Theorem 3.3, we have that \( H \in \text{Cot}_\omega(R) \). From the exact sequence
\[
0 \to H \to \omega \otimes_S F \xrightarrow{1_\omega \otimes \phi} \omega \otimes_S M_* \to 0
\]
in \( \text{Mod} R \), we get the following commutative diagram with exact rows:
\[
\begin{array}{ccc}
F & \xrightarrow{\phi} & M_* \\
\downarrow{\mu_F} & & \downarrow{\mu_{M_*}} \\
(\omega \otimes_S F)_* & \xrightarrow{(1_\omega \otimes \psi)_*} (\omega \otimes_S M_*)_* & \xrightarrow{\text{Ext}^1_R(\omega, H)} 0,
\end{array}
\]
where \( \mu_F \) is an isomorphism.

Case 1. Let \( M \in \text{Cor}_\omega(R) \). Then by Lemma 2.4(3), we have that \( M_* \in \text{Acot}_\omega(S) \) and \( \mu_{M_*} \) is an isomorphism.

Case 2. Let \( \omega \otimes S M_* \in \text{Cor}_\omega(R) \) and \( R\omega S \) be faithful. Then \( \theta_{\omega \otimes S M_*} \) is an isomorphism. Since \( \theta_{\omega \otimes S M_*} \cdot (1_\omega \otimes \mu_{M_*}) = 1_{\omega \otimes S M_*} \) by Lemma 2.4(2), we have that \( 1_\omega \otimes \mu_{M_*} \) is an epimorphism. Since \( \omega \) is faithful, we have that \( \mu_{M_*} \) is an epimorphism by [HW, Lemma 3.1], and hence an isomorphism by Lemma 2.4(1).

Consequently, in either case, \( (1_\omega \otimes \phi)_* \) is epic and \( \text{Ext}^1_R(\omega, H) = 0 \), that is, \( H \) is \( 1_\omega \)-cospherical.

The converse part of the corollary stems from the proof of the corresponding part of Theorem 3.3 using the fact that \( H \) is its own \( \omega \)-cotorsionless submodule. \( \square \)

In the rest of this section, we elaborate adjoint counterparts of the above notions and results about HT-projections.

**Definition 3.5.** Let \( N \in \text{Mod} S \) and \( I \in \text{Mod} R \). A monomorphism
\[
\psi_* : (\omega \otimes_S N)_* \hookrightarrow I_*
\]
in \( \text{Mod} S \) is called a **Tensor-Hom-injection** (TH-injection for short) if it is obtained by applying the functor \( \text{Hom}_R(\omega, -) \) to the monomorphism \( \psi : \omega \otimes S N \hookrightarrow I \) in \( \text{Mod} R \).

To study the properties of TH-injections, we need the following

**Lemma 3.6.** Let \( M, N \in \text{Mod} S \). Then the following statements are equivalent.

1. \( N \cong \text{Im} \mu_M \).
2. \( N \in \text{Acot}_\omega(S) \) and there exists an epimorphism \( g : M \twoheadrightarrow N \) in \( \text{Mod} S \) such that \( 1_\omega \otimes g \) is an isomorphism.

**Proof.** (1) \( \Rightarrow \) (2) Let \( N \cong \text{Im} \mu_M \) and \( t : \text{Im} \mu_M \twoheadrightarrow N \) be an isomorphism in \( \text{Mod} S \). Since \( \mu_{\omega \otimes_S M} \) is a monomorphism and \( (\omega \otimes_S M)_* \in \text{Acot}_\omega(S) \) by Lemma 2.4(1), we have \( N \in \text{Acot}_\omega(S) \). Let \( \mu_M = i \cdot p \) be the natural epic-monic decomposition of \( \mu_M \) with \( p : M \twoheadrightarrow \text{Im} \mu_M \) and \( i : \text{Im} \mu_M \hookrightarrow (\omega \otimes_S M)_* \). Then \( g := t \cdot p \) is epic. Note that
\[
\theta_{\omega \otimes_S M} \cdot (1_\omega \otimes i) \cdot (1_\omega \otimes p) = \theta_{\omega \otimes_S M} \cdot (1_\omega \otimes \mu_M) = 1_{\omega \otimes_S M}
\]
by Lemma 2.4(2). It implies that \( 1_\omega \otimes p \) is a monomorphism, and hence an isomorphism. Thus \( 1_\omega \otimes g = (1_\omega \otimes t) \cdot (1_\omega \otimes p) \) is an isomorphism.

(2) \( \Rightarrow \) (1) Let \( N \in \text{Acot}_\omega(S) \) and \( g : M \twoheadrightarrow N \) an epimorphism in \( \text{Mod} S \) such that \( 1_\omega \otimes g \) is an isomorphism. Consider the following exact commutative diagram...
with the upper row exact

\[
\begin{array}{c}
M \xrightarrow{\mu_M} \text{Coker} K \xrightarrow{\mu_N} N \xrightarrow{\omega N} 0\\
(\omega \otimes S M)_* \xrightarrow{(1_\omega \otimes g)_*} (\omega \otimes S M)_*.
\end{array}
\]

Since \( N \in \text{Cot}_\omega(S) \) and \( 1_\omega \otimes g \) is a monomorphism and \((1_\omega \otimes g)_*\) is also an isomorphism. So we have

\[
\text{Ker} \mu_M = \text{Ker}(1_\omega \otimes g)_* \cdot \mu_M = \text{Ker}(\mu_N \cdot g) = \text{Ker} g.
\]

It induces that \( N \cong \text{Im} \mu_M \).

For a module \( N \in \text{Mod} S \), we call \( \text{Im} \mu_N \) the adjoint \( 1_\omega \)-cotorsionfree quotient module of \( N \). The following addresses the relation between TH-injections and the adjoint \( \omega \)-cotorsionless quotient modules of adjoint \( 1_\omega \)-cospherical modules.

**Theorem 3.7.** Let \( N \in \text{Mod} S \) and \( I \in \text{Mod} R \). If

\[
\psi_* : (\omega \otimes S N)_* \rightarrow I_*
\]

is a TH-injection with \( I \in \text{Cor}_\omega(R) \) and \( I_* \) adjoint \( 1_\omega \)-cospherical in \( \text{Mod} S \), then \( K := \text{Coker} \psi_* \) is the adjoint \( \omega \)-cotorsionless quotient module of an adjoint \( 1_\omega \)-cospherical module in \( \text{Mod} S \).

Conversely, if \( K \) is the adjoint \( \omega \)-cotorsionless quotient of an adjoint \( 1_\omega \)-cospherical module in \( \text{Mod} S \), then there exists an exact sequence

\[
0 \rightarrow X \xrightarrow{\lambda} P \rightarrow K \rightarrow 0
\]

in \( \text{Mod} S \) with \( P \) projective and \( \lambda : X \rightarrow P \) is a TH-injection.

**Proof.** Let

\[
0 \rightarrow (\omega \otimes S N)_* \xrightarrow{\psi_*} I_* \rightarrow K \rightarrow 0
\]

be an exact sequence in \( \text{Mod} S \) with \( \psi_* \) a TH-injection, \( I \in \text{Cor}_\omega(R) \), \( I_* \) adjoint \( 1_\omega \)-cospherical in \( \text{Mod} S \) and \( K = \text{Coker} \psi_* \). Then we have the following exact sequence

\[
0 \rightarrow (\omega \otimes S N)_* \xrightarrow{\psi_*} I_* \rightarrow K \rightarrow 0
\]

in \( \text{Mod} R \), where \( H = \text{Coker} \psi \). Because \( I \in \text{Cor}_\omega(R) \) and \( \omega \otimes S N \in \text{Cot}_\omega(R) \) by assumption and Lemma 2.4(2) respectively, we have \( H \cong \omega \otimes S K \) by Proposition 2.9(2). Applying the functor \((-)_*\) to (3.2) yields that \( K \) is isomorphic to a submodule of \( H_* \). Using Lemma 2.4(1) and [TH3, Proposition 4.2], we get \( K \in \text{Cot}_\omega(S) \). Let \( L = \text{Im} \mu_N \) and let \( \mu_N = i : p \) be the natural epimonic decomposition of \( \mu_N \) with \( p : N \rightarrow L \) and \( i : L \rightarrow (\omega \otimes S N)_* \). Then \( 1_{\omega \otimes S N} \cdot \mu_N = \theta_{\omega \otimes S N} \cdot (1_\omega \otimes i) \cdot (1_\omega \otimes p) \) by Lemma 2.4(2). It implies that \( 1_\omega \otimes p \) is a monomorphism and hence an isomorphism. So \( \theta_{\omega \otimes S N} \cdot (1_\omega \otimes i) \) is also an isomorphism. Set \( K' = \text{Coker}(\psi_* \cdot i) \). Consider the following exact commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \rightarrow & L & \xrightarrow{\psi_* \cdot i} & I_* & \rightarrow & K' & \rightarrow & 0 \\
\downarrow \quad i & & \downarrow \quad 1_\mu & & \downarrow \quad 1_V & & & & \\
0 & \rightarrow & (\omega \otimes S N)_* & \xrightarrow{\psi_*} & I_* & \rightarrow & K & \rightarrow & 0,
\end{array}
\]
where and $\nu$ is an induced homomorphism which is epic. Because $\psi \cdot \theta_{\omega \otimes N} = \theta_I \cdot (1_{\omega} \otimes \psi_*)$ and $I_*$ is adjoint 1-$\omega$-cospherical in $\text{Mod} S$, applying the functor $\omega \otimes S$ to Diagram (3.2) gives the following commutative diagram with exact rows

$$
\begin{array}{cccccccc}
0 & \longrightarrow & \text{Tor}_1^S(\omega, K') & \longrightarrow & \omega \otimes S L & \longrightarrow & \omega \otimes S I_* & \longrightarrow & 0 \\
& & \downarrow \phi \circ \delta & & \downarrow \theta_{\omega \otimes N} \circ (1_{\omega} \otimes i) & & \downarrow 1_{\omega} \otimes \psi & & \\
0 & \longrightarrow & \omega \otimes S N & \longrightarrow & \omega \otimes S I_* & \longrightarrow & \omega \otimes S K & \longrightarrow & 0.
\end{array}
$$

Because $\theta_{\omega \otimes N} : (1_{\omega} \otimes i)$ is an isomorphism, we have that $\text{Tor}_1^S(\omega, K') = 0$ and $1_{\omega} \otimes \nu$ is also an isomorphism. Then it follows from Lemma 3.6 that $K$ is the adjoint $\omega$-cotorsionless quotient module of an adjoint 1-$\omega$-cospherical module $K'$.

Conversely, let $K$ be the adjoint $\omega$-cotorsionless quotient module of an adjoint 1-$\omega$-cospherical module $K'$. By Lemma 3.6, there exists an epimorphism $g : K' \rightarrow K$ such that $1_{\omega} \otimes g$ is an isomorphism. Consider the following commutative diagram with exact rows:

$$
\begin{array}{cccccccc}
0 & \longrightarrow & X' & \longrightarrow & P & \longrightarrow & K' & \longrightarrow & 0 \\
& & \downarrow f & & \downarrow \phi & & \downarrow g & & \\
0 & \longrightarrow & X & \longrightarrow & \lambda & \longrightarrow & K & \longrightarrow & 0,
\end{array}
$$

where $P$ is projective, $\phi = g \cdot \alpha$, $X' = \text{Ker} \alpha$ and $X = \text{Ker} \phi$.

We claim that $\lambda : X \hookrightarrow P$ is a TH-injection. Since $K'$ is adjoint 1-$\omega$-cospherical, we have the following commutative diagram with exact rows:

$$
\begin{array}{cccccccc}
0 & \longrightarrow & \omega \otimes S X' & \longrightarrow & \omega \otimes S P & \longrightarrow & \omega \otimes S K' & \longrightarrow & 0 \\
& & \downarrow \gamma & & \downarrow 1_{\omega} \otimes \alpha & & \downarrow 1_{\omega} \otimes \psi & & \\
0 & \longrightarrow & \omega \otimes S Z & \longrightarrow & \omega \otimes S P & \longrightarrow & \omega \otimes S K & \longrightarrow & 0,
\end{array}
$$

where $Z = \text{Ker}(1_{\omega} \otimes \phi)$. Since $1_{\omega} \otimes g$ is an isomorphism, we have $Z \cong \omega \otimes S X'$. By Proposition 2.9(1) and its proof, we have that $X \cong Z_*$ and $\lambda : X \hookrightarrow P$, up to isomorphism, is formed by applying $(-)_*$ to $\delta : Z(\cong \omega \otimes S X') \rightarrow \omega \otimes S P$. The claim is proved. \hfill $\square$

As a consequence of Theorem 3.7, we have the following

**Corollary 3.8.** Let $N \in \text{Mod} S$ and $I \in \text{Mod} R$, and let

$$
\psi_* : (\omega \otimes S M)_* \rightarrow I_*
$$

be a TH-injection with $I \in \text{Cor}_\omega(R)$ and $I_*$ adjoint 1-$\omega$-cospherical in $\text{Mod} S$. Then $K := \text{Coker} \psi_*$ is an adjoint $\omega$-cotorsionless and adjoint 1-$\omega$-cospherical module in $\text{Mod} S$ provided that one of the following conditions is satisfied.

1. $M \in \text{Acor}_\omega(S)$.
2. $(\omega \otimes S M)_* \in \text{Acor}_\omega(S)$ and $R \omega_S$ is faithful.

Conversely, if $K$ is an adjoint $\omega$-cotorsionless and adjoint 1-$\omega$-cospherical module in $\text{Mod} S$ and

$$
0 \rightarrow X \rightarrow F \rightarrow K \rightarrow 0
$$
is an exact sequence in Mod $S$ with $P$ projective, then $X \rightarrow F$ is a TH-injection.

**Proof.** By Theorem 3.7, we have that $K \in \text{Acot}_\omega(S)$. From the exact sequence

$$0 \rightarrow (\omega \otimes_S M)_s \xrightarrow{\psi_*} I_* \rightarrow K \rightarrow 0$$

in Mod $S$, we get the following commutative diagram with exact rows:

$$
\begin{array}{c}
0 \\
\xrightarrow{\text{Tor}_1^S(\omega, K)} \omega \otimes_S (\omega \otimes_S M)_s \xrightarrow{1_* \otimes \psi_*} \omega \otimes_S I_* \\
\xrightarrow{\theta_{\omega \otimes_S M}} \omega \otimes_S M \xrightarrow{\psi} I_* \\
0
\end{array}
$$

where $\theta_I$ is an isomorphism.

Case 1. Let $M \in \text{Acot}_\omega(S)$. Then by Lemma 2.4(3), we have that $\omega \otimes_S M \in \text{Cot}_\omega(R)$ and $\theta_{\omega \otimes_S M}$ is an isomorphism.

Case 2. Let $(\omega \otimes_S M)_s \in \text{Acot}_\omega(S)$ and $\mu_{\omega S}$ be faithful. Then $\mu_{(\omega \otimes_S M)_s}$ is an isomorphism. Since $(\theta_{\omega \otimes_S M}_s) \cdot \mu_{(\omega \otimes_S M)_s} = 1_{(\omega \otimes_S M)_s}$, by Lemma 2.4(1), we have that $(\theta_{\omega \otimes_S M}_s)$ is an isomorphism. Since $\omega$ is faithful, we have that $\theta_{\omega \otimes_S M}$ is a monomorphism, and hence an isomorphism by Lemma 2.4(2).

Consequently, in either case, $L_\omega \otimes \psi_*$ is monic and Tor$_1^S(\omega, K) = 0$, that is, $K$ is adjoint $1$-$\omega$-cospherical.

The converse part of the corollary stems from the proof of the corresponding part of Theorem 3.7 using the fact that $K$ is its own adjoint $\omega$-cotorsionless quotient module. \qed

### 4. Modules of $\omega$-$\mathcal{T}$-class $n$ and finite projective dimension

Motivated by the notion of modules of $D$-class $n$ introduced in [J3], in this section, we first introduce the notion of modules of $\omega$-$\mathcal{T}$-class $n$ as follows. Then we give some equivalent characterizations for $\omega_S$ having finite projective dimension in terms of the properties of modules of $\omega$-$\mathcal{T}$-class $n$.

**Definition 4.1.** Let $\mathcal{T}$ be a subclass of $\text{Acot}_\omega(S)$. An $\omega$-cotorsionless module $U_n$ in Mod $R$ is said to be of $C$-$\mathcal{T}$-class $n$ if there exist $F_1, \ldots, F_{n-1} \in \mathcal{T}$ and $U_2, \ldots, U_{n-1} \in \text{Cot}_\omega(R)$ such that

$$
\begin{align*}
0 & \rightarrow U_n \rightarrow \omega \otimes_S F_{n-1} \rightarrow \omega \otimes_S U_{n-1} \rightarrow 0, \\
0 & \rightarrow U_{n-1} \rightarrow \omega \otimes_S F_{n-2} \rightarrow \omega \otimes_S U_{n-2} \rightarrow 0, \\
& \vdots \\
0 & \rightarrow U_2 \rightarrow \omega \otimes_S F_1 \rightarrow \omega \otimes_S U_1 \rightarrow 0
\end{align*}
$$

are exact with all the above epimorphisms HT-projections. We shall say that any $\omega$-cotorsionless module is of $\omega$-$\mathcal{T}$-class 1.

It seems that it is not easy to grasp the definition of modules of $\omega$-$\mathcal{T}$-class $n$. The following result is helpful to comprehend it, which will be used frequently in the sequel.

**Theorem 4.2.** Let $\mathcal{T}$ be a subclass of $\text{Acot}_\omega(S)$. If a module $U_n \in \text{Mod } R$ is of $\omega$-$\mathcal{T}$-class $n$, then there exists a collection of exact sequences

$$
0 \rightarrow U_i \rightarrow F_i \rightarrow U_{i-1} \rightarrow 0 \quad (2 \leq i \leq n) \tag{4.1}
$$

in Mod $S$ with all $F_i \in \mathcal{T}$ and $U_i \in \text{Mod } R$. 

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Conversely, if there exists a collection of exact sequences as in (4.1), then $U_n$ can be selected of $\omega$-class $n$.

**Proof.** Let $U_n \in \text{Mod}\, R$ be of $\omega$-class $n$. Consider the exact sequences in Definition 4.1. For any $2 \leq i \leq n$, since $\omega \otimes S F_i \rightarrow \omega \otimes S U_{i-1}$ is a HT-projection, we have the following commutative diagram with exact rows

$$
\begin{array}{ccc}
F_i & \rightarrow & U_{i-1} \\
\downarrow \mu_{F_i} & & \downarrow \mu_{U_{i-1}} \\
0 & \rightarrow & (\omega \otimes S F_i) \\
\end{array}
$$

Note that $\mu_{F_i}$ is an isomorphism by assumption and that $\mu_{U_{i-1}}$ is a monomorphism by Lemma 2.4(1). Then we get an exact sequence

$$0 \rightarrow U_{i-1} \rightarrow F_i \rightarrow U_{i} \rightarrow 0 \quad (2 \leq i \leq n).$$

Conversely, assume that there exists a collection of exact sequences as in (4.1). First, consider the following exact sequence

$$0 \rightarrow H_1 \rightarrow F_1 \xrightarrow{\phi_1} U_1 \rightarrow 0$$
in $\text{Mod}\, S$ with $H_1 = \text{Ker} \phi_1$. Set $U_2 = \text{Ker}(1 \otimes \phi_1)$. Then we have an exact sequence

$$0 \rightarrow U_2 \rightarrow \omega \otimes S F_1 \xrightarrow{1 \otimes \phi_1} \omega \otimes S U_1 \rightarrow 0$$
in $\text{Mod}\, S$. Then $1 \otimes \phi_1$ is a HT-projection and $U_2$ is of $\omega$-class 2. Notice that $\omega \otimes S H_1 \in \text{Cot}_\omega(R)$ by Lemma 2.4(2), so $U_2 \in \text{Cot}_\omega(R)$ since it is isomorphic to a quotient module of $\omega \otimes S H_1$. Because $F_1 \in \text{Acot}_\omega(S)$ and $U_{1*} \in \text{Acot}_\omega(S)$ by assumption and Lemma 2.4(1) respectively, it follows from Proposition 2.9(1) and its proof that $H_1 \cong U_2$, and $U_{1*} \cong \text{Im}(1 \otimes \phi_1)$. So we get an exact sequence

$$0 \rightarrow U_2 \rightarrow F_1 \xrightarrow{(1 \otimes \phi_1) \mu_{F_1}} U_1 \rightarrow 0$$
in $\text{Mod}\, S$.

Next, consider the following exact sequence

$$0 \rightarrow H_2 \rightarrow F_2 \xrightarrow{\phi_2} U_2 \rightarrow 0$$
in $\text{Mod}\, S$ with $H_2 = \text{Ker} \phi_2$. Set $U_3 = \text{Ker}(1 \otimes \phi_2)$. By using an argument similar to above, we get an exact sequence

$$0 \rightarrow U_3 \rightarrow F_2 \xrightarrow{(1 \otimes \phi_2) \mu_{F_2}} U_2 \rightarrow 0$$
in $\text{Mod}\, S$ with $U_3$ of $\omega$-class 3.

Continuing this process, we get the desired assertion. \qed

The following two lemmas are useful for proving the next theorem.

**Lemma 4.3.** Let $N \in \text{Acot}_\omega(S)$ and $L \in \text{Cot}_\omega(R)$. If either $N$ or $L$ is given, then the other exists such that these two modules are connected by the following exact sequences

$$0 \rightarrow N \xrightarrow{\mu_N} (\omega \otimes S N) \rightarrow \text{Ext}_R^1(\omega, L) \rightarrow 0,$$

$$0 \rightarrow \text{Tor}_S^1(\omega, N) \rightarrow \omega \otimes S L \xrightarrow{\theta_L} L \rightarrow 0.$$
Proof. Given $N \in \text{Acot}_\omega(S)$, consider the following exact sequence 
\[ 0 \to N_1 \to P \to N \to 0 \]
in $\text{Mod } S$ with $P$ projective. Then we get the following exact sequence 
\[ 0 \to L \to \omega \otimes_S P \to \omega \otimes_S N \to 0 \]
in $\text{Mod } R$ with $L = \text{Ker}(\omega \otimes_S P \to \omega \otimes_S N)$. Notice that $\omega \otimes_S N_1 \in \text{Cot}_\omega(R)$ by Lemma 2.4(2) and that $L$ is isomorphic to a quotient module of $\omega \otimes_S N_1$, so $L \in \text{Cot}_\omega(R)$.

Now consider the following commutative diagram with exact rows 
\[
\begin{array}{cccccc}
0 & \to & N_1 & \to & P & \to & N & \to & 0 \\
& & \downarrow{\mu_P} & & \downarrow{\mu_N} & & \\
0 & \to & L_\ast & \to & (\omega \otimes_S P)_\ast & \to & (\omega \otimes_S N)_\ast & \to & \text{Ext}^1_R(\omega, L) & \to & 0.
\end{array}
\]

Since $\mu_P$ is an isomorphism and $\mu_N$ is a monomorphism by Lemma 2.5(2) and assumption respectively, we have the following two exact sequences 
\[ 0 \to N \xrightarrow{\mu_N} (\omega \otimes_S N)_\ast \to \text{Ext}^1_R(\omega, L) \to 0, \]
\[ 0 \to L_\ast \to (\omega \otimes_S P)_\ast (\cong P) \to N \to 0. \]

Then we get the following commutative diagram with exact rows 
\[
\begin{array}{cccccc}
0 & \to & \text{Tor}^S_1(\omega, N) & \to & \omega \otimes_S L_\ast & \to & \omega \otimes_S (\omega \otimes_S P)_\ast & \to & \omega \otimes_S N & \to & 0 \\
& & & \downarrow{\theta_L} & & \downarrow{\theta_{\omega \otimes_S P}} & & & & \\
0 & \to & L_\ast & \to & \omega \otimes_S P & \to & \omega \otimes_S N & \to & 0,
\end{array}
\]

where $\theta_{\omega \otimes_S P}$ is an isomorphism by Lemma 2.5(1). It yields the following exact sequence 
\[ 0 \to \text{Tor}^S_1(\omega, N) \to \omega \otimes_S L_\ast \xrightarrow{\theta_L} L \to 0. \]

If $L$ is given, then we get the assertion dually. \hfill \Box

Lemma 4.4. Let $\phi : F \to N$ be an epimorphism in $\text{Mod } S$ with $F \in \text{Acor}_\omega(S)$ and $N \in \text{Acot}_\omega(S)$. Then we have the following exact sequence 
\[ \text{Tor}^S_1(\omega, F) \to \text{Tor}^S_1(\omega, N) \to \omega \otimes_S H_\ast \xrightarrow{\theta_H} H \]
in $\text{Mod } R$, where $H = \text{Ker}(1_\omega \otimes \phi)$.

Proof. By assumption, we have the following exact sequence 
\[ 0 \to H \xrightarrow{\alpha} \omega \otimes_S F \xrightarrow{1_\omega \otimes \phi} \omega \otimes_S N \to 0 \]
in $\text{Mod } R$. Then we get the following commutative diagram with exact rows 
\[
\begin{array}{cccccc}
F & \to & N & \to & 0 \\
\downarrow{\phi} & & \downarrow{\mu_N} & & \\
0 & \to & H_\ast & \to & (\omega \otimes_S F)_\ast (\cong F) & \xrightarrow{(1_\omega \otimes \phi)} (\omega \otimes_S N)_\ast.
\end{array}
\]

Because $F \in \text{Acor}_\omega(S)$ and $N \in \text{Acot}_\omega(S)$ by assumption, $\mu_F$ is an isomorphism and $\mu_N$ is a monomorphism. So we get the following exact sequence 
\[ 0 \to H_\ast \xrightarrow{\alpha} (\omega \otimes_S F)_\ast (\cong F) \xrightarrow{\phi \mu_F^{-1}} N \to 0 \]
in Mod $S$ and the following commutative diagram with exact rows

$$
\begin{array}{cccccc}
\text{Tor}^S_1(\omega, F) & \rightarrow & \text{Tor}^S_1(\omega, N) & \rightarrow & \omega \otimes_S H & \rightarrow & \omega \otimes_S (\omega \otimes_S F) \\
& & & & \downarrow{\theta_H} & \downarrow{\theta_{\omega \otimes_S F}} & \\
0 & \rightarrow & H & \rightarrow & \omega \otimes_S F.
\end{array}
$$

Also because $F \in \text{Acot}_\omega(S)$, we have $\omega \otimes_S F \in \text{Cor}_\omega_R(R)$ by Lemma 2.4(3). So $\theta_{\omega \otimes_S F}$ is an isomorphism and we get the desired exact sequence. \hfill \Box

From now on, we fix $T$ a subclass of $w\mathcal{A}_\omega(S)$ containing all projective left $S$-modules, that is, $\mathcal{P}(S) \subseteq T \subseteq w\mathcal{A}_\omega(S)$. We use $\text{pd}_{S^\omega} \omega$ and $\text{fd}_{S^\omega} \omega$ to denote the projective and flat dimensions of $\omega_S$, respectively. The following result establishes a relationship between the finiteness of $\text{pd}_{S^\omega} \omega$ and the properties of modules of $\omega$-$T$-class $n$, $\omega$-coreflexive modules and adjoint $\omega$-cotorsionless modules.

**Theorem 4.5.** For any $n \geq 1$, the following statements are equivalent.

1. $\text{pd}_{S^\omega} \omega \leq n$.
2. Any module of $\omega$-$\mathcal{P}(S)$-class $n$ in $\text{Mod} R$ is $\omega$-coreflexive.
3. Any module of $\omega$-$T$-class $n$ in $\text{Mod} R$ is $\omega$-coreflexive.
4. $\text{Tor}^S_n(\omega, V) = 0$ for any $V \in \text{Acot}_\omega(S)$.
5. $\text{Tor}^n_{n+1}(\omega, N) = 0$ for any $N \in \text{Mod} S$.

**Proof.** (1) $\Leftrightarrow$ (5) It is trivial since $\text{pd}_{S^\omega} \omega = \text{fd}_{S^\omega} \omega$. The implication (3) $\Rightarrow$ (2) is also trivial.

(2) $\Rightarrow$ (4) If $n = 1$, then the assertion follows from Lemma 4.3. Now let $V \in \text{Acot}_\omega(S)$ and $n \geq 2$. By the proof of Lemma 4.3, there exists an exact sequence

$$
0 \rightarrow U_{n+1} \rightarrow P \rightarrow V \rightarrow 0
$$

in $\text{Mod} S$ with $P$ projective. By Theorem 4.2 and its proof, we have the following two exact sequences

$$
\begin{array}{cccccc}
0 & \rightarrow & U_{n+1} & \rightarrow & P_{n-1} & \rightarrow & \cdots & \rightarrow & P_1 & \rightarrow & U_1 & \rightarrow & 0, \\
& & & & & & & & & & & & 0 \rightarrow U_n \rightarrow \omega \otimes_S P_{n-1} \rightarrow \cdots \rightarrow \omega \otimes_S U_{n-1} & \rightarrow & 0
\end{array}
$$

with all $P_i \in \text{Mod} S$ projective, $U_n \in \text{Mod} R$ of $\omega$-$\mathcal{P}(S)$-class $n$ and $U_{n-1} = \text{Im} f_{n-1}$, such that $1_\omega \otimes f_{n-1}$ is a HT-projection. Then by Lemma 4.4, we have the following exact sequence

$$
0 \rightarrow \text{Tor}^S_n(\omega, U_{n-1}) \rightarrow \omega \otimes_S U_n \rightarrow \theta_{U_n} U_n \rightarrow 0.
$$

By (2), $U_n \in \text{Cor}_\omega(R)$ and $\theta_{U_n}$ is an isomorphism. So $\text{Tor}^S_n(\omega, U_{n-1}) = 0$, and hence

$$
\text{Tor}^n_{n-1}(\omega, U_{n-1}) \cong \text{Tor}^S_n(\omega, U_{n-1}) \cong \text{Tor}^S_1(\omega, U_{n-1}) = 0.
$$

(4) $\Rightarrow$ (3) Let $U_n \in \text{Mod} R$ be of $\omega$-$T$-class $n$. Then by Theorem 4.2, there exists an exact sequence

$$
0 \rightarrow U_{n+1} \rightarrow T_{n-1} \rightarrow \cdots \rightarrow T_1 \rightarrow U_1 \rightarrow 0
$$

in $\text{Mod} S$ with all $T_i \in T$ such that $U_n \cong \text{Ker}(1_\omega \otimes f_{n-1})$. By Lemma 4.4, we have the following exact sequence

$$
0 \rightarrow \text{Tor}^S_1(\omega, U_{n-1}) \rightarrow \omega \otimes_S U_n \rightarrow \theta_{U_n} U_n \rightarrow 0.
$$

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where $U_{n-1*} = \text{Im } f_{n-1}$. In addition, we have the following exact sequence

$$0 \to U_{1*} \to I^0(U_{1*}) \xrightarrow{f^0(U_{1*})} I^1(U_{1*}) \to \text{cTr}_U U_1 \to 0$$

in $\text{Mod } S$. By Lemma 2.5(2), we have

$$\text{Tor}_2^S(\omega, I^0(U_{1*})) = 0 = \text{Tor}_2^S(\omega, I^1(U_{1*})).$$

Put $V = \text{Im } f^0(U_{1*})$. Then $V \in \text{Acot}_\omega(S)$. So by (4), we have

$$\text{Tor}_1^S(\omega, U_{n-1*}) \cong \text{Tor}_{n-1}^S(\omega, U_{1*}) \cong \text{Tor}_n^S(\omega, V) = 0.$$  

It follows from (4.2) that $\theta_{U_{n*}}$ is an isomorphism and $U_n \in \text{Cor}_\omega(R)$.

(4) $\Rightarrow$ (5) Let $N \in \text{Mod } S$ and

$$0 \to V \to P \to N \to 0$$

be an exact sequence in $\text{Mod } S$ with $P$ projective. Then $V \in \text{Acot}_\omega(S)$. Conversely, let $V \in \text{Acot}_\omega(S)$. Then by [TH3, Lemma 4.7(1)], there exists an exact sequence

$$0 \to V \to E \to N \to 0$$

in $\text{Mod } S$ with $E$ $\omega$-injective. Note that $\text{Tor}_{\geq 1}^S(\omega, E) = 0$ by Lemma 2.5(2). Now the assertion follows easily from the dimension shifting.

As a consequence of Theorem 4.5, we have the following

**Corollary 4.6.** For any $n \geq 1$, the following statements are equivalent.

1. $U_{n*} \in \text{Acor}_\omega(S)$ for any $U_n$ of $\omega$-$\mathcal{P}(S)$-class $n$ in $\text{Mod } R$.
2. $U_{n*} \in \text{Acor}_\omega(S)$ for any $U_n$ of $\omega$-$\mathcal{T}$-class $n$ in $\text{Mod } R$.
3. $[\text{Tor}_n^S(\omega, V)]_* = 0$ for any $V \in \text{Acot}_\omega(S)$.

If $\text{pd}_{S,\omega} \omega \leq n$, then these equivalent conditions are satisfied.

**Proof.** Let $U_n \in \text{Mod } R$ be of $\omega$-$\mathcal{P}(S)$-class $n$ and $V \in \text{Acot}_\omega(S)$. From the proof of the implications $(2) \Rightarrow (4) \Rightarrow (3)$ in Theorem 4.5, we know that if either $U_n$ or $V$ is given, then the other exists, such that $\text{Ker } \theta_{U_{n*}} \cong \text{Tor}_n^S(\omega, V)$. It implies

$$\text{Ker } \theta_{U_{n*}} \cong \text{Ker } \theta_{U_{n*}} \cong [\text{Tor}_n^S(\omega, V)]_*.$$  

Then by Lemma 2.4(1), we have that $U_{n*} \in \text{Acor}_\omega(S)$ if and only if $\mu_{U_{n*}}$ is an isomorphism, if and only if $\theta_{U_{n*}}$ is an isomorphism, and if and only if $[\text{Tor}_n^S(\omega, V)]_* = 0$. This proves $(1) \Leftrightarrow (3)$. Similarly, we have $(2) \Leftrightarrow (3)$.

The last assertion follows immediately from Theorem 4.5.

The following result is a supplement to Theorem 4.5.

**Theorem 4.7.** For any $n \geq 1$, the following statements are equivalent.

1. $\text{pd}_{S,\omega} \omega \leq n + 1$.
2. $\text{Tor}_1^S(\omega, U_{n*}) = 0$ for any module $U_n$ of $\omega$-$\mathcal{P}(S)$-class $n$ in $\text{Mod } R$.
3. $\text{Tor}_1^S(\omega, U_{n*}) = 0$ for any module $U_n$ of $\omega$-$\mathcal{T}$-class $n$ in $\text{Mod } R$.

**Proof.** $(1) \Rightarrow (3)$ Let $U_n \in \text{Mod } R$ be of $\omega$-$\mathcal{T}$-class $n$. Then by Theorem 4.2, there exists an exact sequence

$$0 \to U_{n*} \to T_{n-1} \to \cdots \to T_1 \to U_{1*} \to 0$$
in \text{Mod} \, S with all \( T_i \in \mathcal{T} \). Then \( \text{Tor}^S_{\geq 1}(\omega, T_i) = 0 \) for any \( 1 \leq i \leq n - 1 \). On the other hand, we have the following exact sequence

\[ 0 \to U_1 \to I^0(U_1)_* \xrightarrow{f_0(U_1)_*} I^1(U_1)_* \to \text{cTr}_\omega U_1 \to 0 \]

in \text{Mod} \, S. Note that \( \text{Tor}^S_{\geq 1}(\omega, I^0(U_1)_*) = 0 = \text{Tor}^S_{\geq 1}(\omega, I^1(U_1)_*) \) by Lemma 2.5(2). So by (1), we have

\[ \text{Tor}^S_1(\omega, U_{n*}) \cong \text{Tor}^S_{n+2}(\omega, \text{cTr}_\omega U_1) = 0. \]

(3) \Rightarrow (2) It is trivial.

(2) \Rightarrow (1) Let \( N \in \text{Mod} \, S \). Then we have the following commutative diagram with exact rows

\[
\begin{array}{cccc}
F_1(N) & \xrightarrow{f_0(N)} & F_0(N) & \to N & \to 0 \\
\mu_{F_1(N)} & & \mu_{F_0(N)} & & \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & \to (\alpha \text{Tr}_\omega N)_* & \xrightarrow{\omega \otimes_S F_1(N)_*} & (\alpha \otimes_S F_0(N))_* \\
& & \mu_{F_0(N)} & & \\
& & & & \\
\end{array}
\]

where \( \mu_{F_0(N)} \) and \( \mu_{F_1(N)} \) are isomorphisms by Lemma 2.5(2). So we get the following exact sequence

\[ 0 \to (\alpha \text{Tr}_\omega N)_* \xrightarrow{\mu_{F_1(N)}^{-1} \alpha} F_1(N) \xrightarrow{f_0(N)} F_0(N) \to N \to 0 \]

in \text{Mod} \, S. By Theorem 4.2, we have the following exact sequence

\[ 0 \to U_{n*} \to P_{n-1} \to \cdots \to P_1 \to (\alpha \text{Tr}_\omega N)_* \to 0 \]

in \text{Mod} \, S with all \( P_i \) projective such that \( U_n \) is of \( \omega\text{-}\mathcal{P}(S)\)-class \( n \). Then by (2), we have

\[ \text{Tor}^S_{n+2}(\omega, N) \cong \text{Tor}^S_1(\omega, U_{n*}) = 0. \]

It implies that \( \text{pd}_S^{\omega} \omega = \text{id}_S^{\omega} \omega \leq n + 1 \). \( \Box \)

For a module \( N \in \text{Mod} \, S \), the \( A_\omega(S)\)-projective dimension \( A_\omega(S)\)-pd \( R \) \( N \) of \( N \) is defined as \( \inf \{ n \mid \text{there exists an exact sequence} \}

\[ 0 \to A_n \to \cdots \to A_1 \to A_0 \to N \to 0 \]

in \text{Mod} \, S with all \( A_i \in A_\omega(S) \}. \) If no such \( n \) exists, then \( A_\omega(S)\)-pd \( R \) \( N \) = \( \infty \). As a byproduct of Theorem 4.2, we get the following

\textbf{Proposition 4.8.} For any \( n \geq 1 \), the following statements are equivalent.

\begin{enumerate}
  \item \( A_\omega(S)\)-pd \( S \) \( N \leq n + 1 \) for any \( N \in \text{Mod} \, S \).
  \item \( U_{n*} \in A_\omega(S) \) for any \( U_n \) of \( \omega\text{-}A_\omega(S)\)-class \( n \) in \text{Mod} \, R.
  \item \( U_{n*} \in A_\omega(S) \) for any \( U_n \) of \( \omega\text{-}\mathcal{P}(S)\)-class \( n \) in \text{Mod} \, R.
\end{enumerate}

\textbf{Proof.} (1) \Rightarrow (2) Let \( U_n \in \text{Mod} \, R \) be of \( \omega\text{-}A_\omega(S)\)-class \( n \) in \text{Mod} \, R. Then by Theorem 4.2, there exists an exact sequence

\[ 0 \to U_{n*} \to A_{n-1} \to \cdots \to A_1 \to U_{1*} \to 0 \]

in \text{Mod} \, S with all \( A_i \in A_\omega(S) \) and \( U_1 \in \text{Mod} \, R \). On the other hand, we have the following exact sequence

\[ 0 \to U_{1*} \to I^0(U_1)_* \xrightarrow{f_0(U_1)_*} I^1(U_1)_* \to \text{cTr}_\omega U_1 \to 0 \]

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in $\text{Mod } S$. So we get the following exact sequence

$$0 \rightarrow U_{n} \rightarrow A_{n-1} \rightarrow \cdots \rightarrow A_{1} \rightarrow I^{0}(U_{1})_{\ast} \overset{f^{0}(U_{1})}{\rightarrow} I^{1}(U_{1})_{\ast} \rightarrow \text{cTr}_{\omega} U_{1} \rightarrow 0$$

in $\text{Mod } S$, where $f^{0}(U_{1})_{\ast}, I^{1}(U_{1})_{\ast} \in \mathcal{A}_{\omega}(S)$ by Lemma 2.5(2). Because $\mathcal{A}_{\omega}(S)$ is projectively resolving and closed under direct summands by [HW, Theorem 6.2 and Proposition 4.2], we have $U_{n} \in \mathcal{A}_{\omega}(S)$ by [AB, Lemma 3.12].

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (1) Let $N \in \text{Mod } S$ and

$$0 \rightarrow K_{n} \rightarrow P_{n} \overset{f_{n-1}}{\rightarrow} \cdots \overset{f_{1}}{\rightarrow} P_{1} \overset{f_{0}}{\rightarrow} P_{0} \rightarrow N \rightarrow 0$$

be an exact sequence in $\text{Mod } S$ with all $P_{i}$ projective. Then for any $1 \leq i \leq n$, we have the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & K_{i} & \overset{1}{\rightarrow} & P_{1} & \overset{f_{1}}{\rightarrow} \cdots \overset{f_{i}}{\rightarrow} P_{i-1} & \\
\downarrow & & \downarrow & & & \\
0 & \rightarrow & U_{i_{\ast}} & \overset{1}{\rightarrow} & \omega \otimes_{S} P_{i_{\ast}} & \overset{1}{\rightarrow} \cdots \overset{1}{\rightarrow} \omega \otimes_{S} P_{i-1_{\ast}} & \\
\end{array}
\]

$K_{i} = \text{Ker } f_{i-1}$ and $U_{i} = \text{Ker } (1 \otimes f_{i-1})$. By Lemma 2.5(2), we have that all $\mu_{P_{i}}$ are isomorphisms. So $K_{i} \cong U_{i_{\ast}}$ for any $1 \leq i \leq n$. Then by Theorem 4.2, $U_{n}$ can be selected of $\omega\mathcal{P}(S)$-class $n$. So $K_{n}(\cong U_{n_{\ast}}) \in \mathcal{A}_{\omega}(S)$ by (3), and hence $\mathcal{A}_{\omega}(S)$-pd$_{S} N \leq n + 1$.

5. Some useful exact sequences

In this section, we give some exact sequences, which will be used frequently in the sequel. The following result is fundamental.

**Proposition 5.1.** Let

$$0 \rightarrow M \rightarrow U^{0} \overset{f}{\rightarrow} U^{1}$$

be an exact sequence in $\text{Mod } R$ satisfying the following conditions:

(1) Both $U^{0}$ and $U^{1}$ are in Cor$_{\omega}(R)$.

(2) $U^{0}_{\ast}$ is adjoint $1$-$\omega$-cospherical and $U^{1}_{\ast}$ is adjoint $2$-$\omega$-cospherical.

Then there exists an exact sequence

$$0 \rightarrow \text{Tor}^{S}_{2}(\omega, H) \rightarrow \omega \otimes_{S} M_{\ast} \overset{g_{\ast}}{\rightarrow} M \rightarrow \text{Tor}^{S}_{1}(\omega, H) \rightarrow 0$$

in $\text{Mod } R$, where $H = \text{Coker } f_{\ast}$.

**Proof.** By applying the functor $(-)_{\ast}$ to (5.1), we get an exact sequence

$$0 \rightarrow M_{\ast} \rightarrow U^{0}_{\ast} \overset{f_{\ast}}{\rightarrow} U^{1}_{\ast} \rightarrow H \rightarrow 0$$

in $\text{Mod } S$. Let

$$f = i \cdot p$$

with $p : U^{0} \rightarrow \text{Im } f$ and $i : \text{Im } f \hookrightarrow U^{1}$ and

$$f_{\ast} = i' \cdot p'$$

\[19\]
with $p': U^0_\ast \to \Im f_\ast$ and $i' : \Im f_\ast \to U^1_\ast$ be the natural epic-monic decompositions of $f$ and $f_\ast$, respectively. Since $\Tor^S_1(\omega, U^0_\ast) = 0$ and $\theta_U$ is an isomorphism by assumption, we have the following commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \Tor^S_1(\omega, \Im f_\ast) & \longrightarrow & \omega \otimes_R M_\ast & \longrightarrow & \omega \otimes_S U^0_\ast \\
& & \downarrow\theta_M & & \downarrow\theta_{U^0} & & \downarrow h \\
0 & \longrightarrow & M & \longrightarrow & U^0 & \longrightarrow & \Im f \\
& & & & \downarrow p & & \downarrow 0,
\end{array}
$$

where $h$ is an induced homomorphism. Then

$$
p \cdot \theta_U = h \cdot (1 \otimes p').
$$

In addition, by the snake lemma, we have

$$
\Ker \theta_M \cong \Tor^S_1(\omega, \Im f_\ast) \quad \text{and} \quad \Coker \theta_M \cong \Ker h.
$$

On the other hand, since $\Tor^S_1(\omega, U^1_\ast) = 0 = \Tor^S_2(\omega, U^1_\ast)$ by assumption, by applying the functor $\omega \otimes_S -$ to the exact sequence

$$
0 \to \Im f_\ast \longrightarrow U^1_\ast \longrightarrow H \to 0,
$$

we get the following exact sequence:

$$
0 \to \Tor^S_1(\omega, H) \to \omega \otimes_S \Im f_\ast \longrightarrow \omega \otimes_S U^1_\ast \longrightarrow \omega \otimes_S H \to 0
$$

and the isomorphism

$$
\Tor^S_1(\omega, \Im f_\ast) \cong \Tor^S_2(\omega, H).
$$

Because

$$
\begin{array}{ccc}
\omega \otimes_S U^0_\ast & \longrightarrow & \omega \otimes_S U^1_\ast \\
\downarrow \theta_{U^0} & & \downarrow \theta_{U^1} \\
U^0 & \longrightarrow & U^1
\end{array}
$$

is a commutative diagram, we have

$$
f \cdot \theta_{U^0} = \theta_{U^1} \cdot (1 \otimes f_\ast).
$$

Because $f_\ast = i' \cdot p'$, we get

$$
1 \otimes f_\ast = 1 \otimes (i' \cdot p') = (1 \otimes i') \cdot (1 \otimes p').
$$

Thus we have

$$
i \cdot h \cdot (1 \otimes p') = i \cdot p \cdot \theta_U = f \cdot \theta_{U^0} = \theta_{U^1} \cdot (1 \otimes f_\ast) = \theta_{U^1} \cdot (1 \otimes i') \cdot (1 \otimes p').
$$

Because $1 \otimes p'$ is epic, we get $i \cdot h = \theta_{U^1} \cdot (1 \otimes i')$. Notice that $i$ is monic and $\theta_{U^1}$ is an isomorphism, so

$$
\Coker \theta_M \cong \Ker h \cong \Ker(1 \otimes i') \cong \Tor^S_1(\omega, H).
$$

Consequently we obtain the desired exact sequence.

In the following, we give some applications of Proposition 5.1.

**Corollary 5.2.**
Let there exists an exact sequence
\[ 0 \to \Tor^S_2(\omega, \Coker \omega, M) \to \omega \otimes S M_{\ast} \stackrel{\theta_{\omega \otimes S}}{\longrightarrow} M \to \Tor^S_1(\omega, \Coker \omega, M) \to 0 \]
in \text{Mod } R.

(2) Let \( N \in \text{Mod } S \). Then there exists an exact sequence
\[ 0 \to \Ext^1_R(\omega, \Coker \omega, N) \to N \stackrel{\mu_N}{\longrightarrow} (\omega \otimes S N)_{\ast} \to \Ext^2_R(\omega, \Coker \omega, N) \to 0 \]
in \text{Mod } S.

**Proof.** The assertion (1) follows from Lemma 2.5 and Proposition 5.1, and the assertion (2) follows from Lemma 2.5 and [TH2, Proposition 6.7].

**Corollary 5.3.**

(1) Let \( N \in \text{Mod } S \). Then there exists an exact sequence
\[ 0 \to \Tor^S_2(\omega, N) \to \omega \otimes S (\acTr \omega, N)_{\ast} \stackrel{\theta_{\omega \otimes S}}{\longrightarrow} \acTr \omega, N \to \Tor^S_1(\omega, N) \to 0 \]
in \text{Mod } R.

(2) Let \( M \in \text{Mod } R \). Then there exists an exact sequence
\[ 0 \to \Ext^1_R(\omega, M) \to \acTr \omega, M \stackrel{\mu_{\omega \otimes S}}{\longrightarrow} (\omega \otimes S \acTr \omega, M)_{\ast} \to \Ext^2_R(\omega, \acTr \omega, M) \to 0 \]
in \text{Mod } S.

**Proof.** (1) Let \( N \in \text{Mod } S \). Then we have the following exact sequence
\[ 0 \to \acTr \omega, N \to \omega \otimes S F_1(N) \stackrel{1 \otimes f_0(N)}{\longrightarrow} \omega \otimes S F_0(N) \to \omega \otimes S N \to 0 \]
in \text{Mod } R with both \( \omega \otimes S F_1(N) \) and \( \omega \otimes S F_0(N) \) in \( F_\omega(R) \). By Lemma 2.5(1), we have that both \( \omega \otimes S F_1(N) \) and \( \omega \otimes S F_0(N) \) are in \( \text{Cor}_\omega(R) \). On the other hand, by Lemma 2.5(2), we have that \((\omega \otimes S F)_\ast \cong F \) for any flat module \( F \) in \( \text{Mod } S \). So we have
\[ \Tor^S_{\geq 1}(\omega, (\omega \otimes S F_0(N))_{\ast}) = 0 = \Tor^S_{\geq 1}(\omega, (\omega \otimes S F_1(N))_{\ast}). \]
Now the assertion follows from Proposition 5.1.

(2) See [TH2, Corollary 6.8].

For the case \( n = 0 \), the first assertion in the following result is exactly Corollary 5.2.

**Proposition 5.4.** Let \( M \in \text{Mod } R \) be \( n \)-\( \omega \)-cospherical with \( n \geq 0 \). Then we have

(1) There exists an exact sequence
\[ 0 \to \Tor^S_{n+2}(\omega, \Coker f^n(M))_{\ast} \to \omega \otimes S M_{\ast} \stackrel{\theta_{\omega \otimes S}}{\longrightarrow} M \to \Tor^S_{n+1}(\omega, \Coker f^n(M))_{\ast} \to 0 \]
in \text{Mod } R.

(2) \( \Coker f^n(M)_{\ast} \) is adjoint \( n \)-\( \omega \)-cospherical.

**Proof.** Let \( M \in \text{Mod } R \) be \( n \)-\( \omega \)-cospherical. Then \( \Ext^1_R(\omega, M, M) = 0 \) and we get the following exact sequence
\[ 0 \to M_{\ast} \to I^0(M)_{\ast} \stackrel{f^0(M)}{\longrightarrow} I^1(M)_{\ast} \stackrel{f^1(M)}{\longrightarrow} \cdots \]
\[ \stackrel{f^{n-1}(M)}{\longrightarrow} I^n(M)_{\ast} \stackrel{f^n(M)}{\longrightarrow} I^{n+1}(M)_{\ast} \to \Coker f^n(M)_{\ast} \to 0 \]
in \text{Mod } S with \( \acTr \omega, M = \Coker f^0(M)_{\ast} \).
(1) Because $\text{Tor}^S_{i+1}(\omega, f^n(M)_+) = 0$ for any injective module in $\text{Mod} R$ by Lemma 2.5(2), we have $\text{Tor}^S_i(\omega, c\text{Tr}_\omega M) \cong \text{Tor}^S_{n+1}(\omega, \text{Coker } f^n(M)_+)$ for any $i \geq 1$. Now the assertion follows from Corollary 5.2.

(2) Applying the functor $\omega \otimes_S -$ to (5.2) we get the following commutative diagram

\[
\begin{array}{ccccccc}
\omega \otimes_S f^n(M) & \otimes \omega & f^n(M) & \ldots & \otimes \omega & f^n(M) & \omega \otimes_S \text{Coker } f^n(M)_+ & 0 \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\
\omega \otimes_S f^n(M) & \otimes \omega & f^n(M) & \ldots & \otimes \omega & f^n(M) & \omega \otimes_S \text{Coker } f^n(M)_+ & 0 \\
\end{array}
\]

All columns in this diagram are isomorphisms by Lemma 2.5(1). So the upper row is exact, which implies $\text{Tor}^S_{1\leq i \leq n}(\omega, \text{Coker } f^n(M)_+) = 0$ and $\text{Coker } f^n(M)_+$ is adjoin $n$-$\omega$-cospherical.

Let $N \in \text{Mod } S^{op}$ and let

\[
\ldots \rightarrow g_{n+1} P_n \rightarrow g_n \rightarrow \ldots \rightarrow g_2 P_1 \rightarrow g_1 P_0 \rightarrow g_0 N \rightarrow 0
\]

be a projective resolution of $N$ in $\text{Mod } S^{op}$. If there exists $n \geq 1$ such that $\text{Im } g_n \cong \oplus_{j=1}^m U_j$ with each $U_j$ isomorphic to a direct summand of some $\text{Im } g_i$ with $i_j < n$, then we say $N$ has a projective resolution ultimately closed at $n$ (see [J2]).

We now are in a position to prove the following

**Theorem 5.5.** Let $n \geq 1$. Then any $n$-$\omega$-cospherical module in $\text{Mod } R$ is $\omega$-coreflexive provided that one of the following conditions is satisfied.

1. $\text{pd}_{S^{op}} \omega \leq n$.
2. $\omega_S$ admits a projective resolution ultimately closed at $n$.

**Proof.** (1) It follows directly from Proposition 5.4(1).

(2) Let

\[
\ldots \rightarrow g_{n+1} P_n \rightarrow g_n \rightarrow \ldots \rightarrow g_2 P_1 \rightarrow g_1 P_0 \rightarrow g_0 \omega \rightarrow 0
\]

be a projective resolution of $\omega$ in $\text{Mod } S^{op}$ ultimately closed at $n$. Then $\text{Im } g_n \cong \oplus_{i=1}^m U_j$ with each $U_j$ isomorphic to a direct summand of some $\text{Im } g_i$ with $i_j < n$.

Now let $M \in \text{Mod } R$ be $n$-$\omega$-cospherical. Then $\text{Ext}^{1\leq i \leq n}_R(\omega, M) = 0$ and we have

\[
\text{Tor}^S_{n+1}(\omega, \text{Coker } f^n(M)_+)
\cong \text{Tor}^S_i(\text{Im } g_n, \text{Coker } f^n(M)_+)
\cong \text{Tor}^S_i(\oplus_{j=1}^m U_j, \text{Coker } f^n(M)_+)
\cong \oplus_{j=1}^m \text{Tor}^S_i(U_j, \text{Coker } f^n(M)_+).
\]

By Proposition 5.4(2), we have

\[
\text{Tor}^S_i(\text{Im } g_i, \text{Coker } f^n(M)_+) \cong \text{Tor}^S_{i+1}(\omega, \text{Coker } f^n(M)_+) = 0.
\]

Note that $U_j$ isomorphic to a direct summand of some $\text{Im } g_i$. Then we have $\text{Tor}^S_i(U_j, \text{Coker } f^n(M)_+) = 0$ for any $1 \leq j \leq m$, and so $\text{Tor}^S_{n+1}(\omega, \text{Coker } f^n(M)_+) = 0$. By Proposition 5.4(2), we conclude that $\text{Tor}^S_{1\leq i \leq n+1}(\omega, \text{Coker } f^n(M)_+) = 0$. Similar to the above argument we get $\text{Tor}^S_{n+2}(\omega, \text{Coker } f^n(M)_+) = 0$. Consequently, by Proposition 5.4(1), we have that $\theta_M$ is an isomorphism and $M$ is $\omega$-coreflexive. \qed
Corollary 5.6. For any $n \geq 1$, a module $M \in \text{Mod } R$ satisfying $\text{Ext}_{R}^{0 \leq i \leq n}(\omega, M) = 0$ implies $M = 0$ provided that one of the following conditions is satisfied.

1. $\text{pd}_{S}\omega \leq n$.
2. $\omega$ satisfies a projective resolution ultimately closed at $n$.

Proof. If $M \in \text{Mod } R$ satisfies $\text{Ext}_{R}^{0 \leq i \leq n}(\omega, M) = 0$, then $M \in \text{Cor}_{\omega}(R)$ by Theorem 5.5. So $M \cong \omega \otimes_{S} M_{*} = 0$.

Example 5.7. Let $R$ be a finite-dimensional algebra over an algebraically closed field given by the quiver:

![Quiver Diagram]

modulo the ideal generated by $\{\alpha_{i+1} \alpha_{i}, \alpha_{1} \alpha_{n} \mid 1 \leq i \leq n-1\}$. For any $1 \leq i \leq n$, we use $S(i)$ and $P(i)$ to denote the simple $R$-module and the indecomposable projective $R$-module corresponding to the vertex $i$, respectively. Then $R$ is a self-injective algebra with infinite global dimension. For any $1 \leq i \leq n$, the following exact sequence

$$\cdots \to P(i) \to P(i-1) \to \cdots \to P(1) \to P(n) \to P(n-1) \to \cdots \to P(i) \to S(i) \to 0 \quad (5.3)$$

is a minimal projective resolution of $S(i)$ with $\text{Im}(P(i) \to P(i-1)) \cong S(i)$. So $\text{pd}_{R} S(i) = \infty$ and (5.3) is ultimately closed at $m$ for any $m \geq n$.

From (5.3), we know that

$$\cdots \to \oplus_{i=1}^{n} P(i) \to \cdots \to \oplus_{i=1}^{n} P(i) \to \oplus_{i=1}^{n} P(i) \to \oplus_{i=1}^{n} S(i) \to 0 \quad (5.4)$$

is a minimal projective resolution of $\oplus_{i=1}^{n} S(i)$ with $\text{Im}(\oplus_{i=1}^{n} P(i) \to \oplus_{i=1}^{n} P(i)) \cong \oplus_{i=1}^{n} S(i)$. So $\text{pd}_{R} \oplus_{i=1}^{n} S(i) = \infty$ and (5.4) is ultimately closed at $m$ for any $m \geq 1$.

6. $\omega$-coreflexive modules and small projective dimension

In this section, by investigating the relationship between $\omega$-coreflexive modules and adjoint $\omega$-coreflexive modules, we give some equivalent characterizations for $\omega$ having projective dimension at most two. We begin with the following

Proposition 6.1. The following statements are equivalent.

1. Any $2$-$\omega$-cospherical module in $\text{Mod } R$ is $\omega$-coreflexive.
2. Any adjoint $\omega$-coreflexive module in $\text{Mod } S$ is adjoint $2$-$\omega$-cospherical.
Proof. (1) ⇒ (2) Let \( N \in Acor_{\omega}(S) \). Then \( \text{acTr}_\omega N \in \text{Mod} R \) is \( 2\)-\( \omega \)-cospherical. So by (1), we have that \( \text{acTr}_\omega N \in \text{Cor}_\omega(R) \). By Corollary 5.3, there exists an exact sequence

\[
0 \to \text{Tor}_2^S(\omega, N) \to \omega \otimes_S \text{acTr}_\omega N \xrightarrow{\theta_{\text{acTr}}^\omega} \text{acTr}_\omega N \to \text{Tor}_1^S(\omega, N) \to 0.
\]

It induces that

\[
\text{Tor}_1^S(\omega, N) = 0 = \text{Tor}_2^S(\omega, N)
\]

and \( N \) is adjoint \( 2\)-\( \omega \)-cospherical.

(2) ⇒ (1) Let \( M \in \text{Mod} R \) be \( 2\)-\( \omega \)-cospherical. Then

\[
\text{Ext}_1^R(\omega, M) = 0 = \text{Ext}_2^R(\omega, M).
\]

By Corollary 5.3(2), there exists an exact sequence

\[
0 \to \text{Ext}_1^R(\omega, M) \to \text{cTr}_\omega M \xrightarrow{\mu_{\text{cTr}}} \omega \otimes_S \text{cTr}_\omega M \to \text{Ext}_2^R(\omega, M) \to 0.
\]

So \( \mu_{\text{cTr}} \) is an isomorphism and \( \text{cTr}_\omega M \in Acor_\omega(S) \). Hence by (2), we have

\[
\text{Tor}_1^S(C, \text{cTr}_\omega M) = 0 = \text{Tor}_2^S(\omega, \text{cTr}_\omega M).
\]

It follows from Corollary 5.2 that \( \theta_M \) is an isomorphism and \( M \in \text{Cor}_\omega(R) \).

Dually, we have the following

**Proposition 6.2.** The following statements are equivalent.

1. Any adjoint \( 2\)-\( \omega \)-cospherical module in \( \text{Mod} S \) is adjoint \( \omega \)-coreflexive.
2. Any \( \omega \)-coreflexive module in \( \text{Mod} R \) is \( 2\)-\( \omega \)-cospherical.

By Propositions 6.1 and 6.2, we have the following

**Corollary 6.3.** The following statements are equivalent.

1. A module in \( \text{Mod} R \) is \( 2\)-\( \omega \)-cospherical if and only if it is \( \omega \)-coreflexive.
2. A module in \( \text{Mod} S \) is adjoint \( \omega \)-coreflexive if and only if it is adjoint \( 2\)-\( \omega \)-cospherical.

In the following, we establish a direct connection between \( \omega \)-coreflexive modules and adjoint \( \omega \)-coreflexive modules.

**Proposition 6.4.** For any \( N \in \text{Mod} S \), the following statements are equivalent.

1. \( \omega \otimes_S N \in \text{Cor}_\omega(R) \).
2. \( (\omega \otimes_S N)_* \in Acor_\omega(S) \).

Proof. (1) ⇒ (2) By Lemma 2.4(3).

(2) ⇒ (1) By Lemma 2.4(2), we have

\[
\theta_{\omega \otimes_S N} : (1_\omega \otimes \mu_N) \to 1_{\omega \otimes_S N}.
\]

So \( \theta_{\omega \otimes_S N} \) is an epimorphism and

\[
\text{Ker} \theta_{\omega \otimes_S N} \cong \text{Coker}(1_\omega \otimes \mu_N) \cong \omega \otimes_S \text{Coker} \mu_N.
\]

On the other hand, since \( (\theta_{\omega \otimes_S N})_* : \mu(\omega \otimes_S N)_* = 1_{(\omega \otimes_S N)_*} \), by Lemma 2.4(1), we have

\[
(\text{Ker} \theta_{\omega \otimes_S N})_* \cong \text{Ker} \theta_{(\omega \otimes_S N)_*} \cong \text{Ker} \mu(\omega \otimes_S N)_*.
\]

So \( (\omega \otimes_S \text{Coker} \mu_N)_* \cong \text{Coker} \mu(\omega \otimes_S N)_* = 0 \) by (2). Thus \( \omega \otimes_S \text{Coker} \mu_N = 0 \) by [TH2, Corollary 6.6(2)], and therefore \( \theta_{\omega \otimes_S N} \) is a monomorphism. Consequently, we conclude that \( \theta_{\omega \otimes_S N} \) is an isomorphism and \( \omega \otimes_S N \in \text{Cor}_\omega(R) \).
For any $A$ module in $\omega$ Tor $R$, the following statements are equivalent.

1. $M_\ast \in \text{Acot}_\omega(S)$.
2. $\omega \otimes S M_\ast \in \text{Cor}_\omega(R)$.

As a consequence of Propositions 6.4 and 6.5, we have the following.

**Corollary 6.6.** The following statements are equivalent.

1. $\omega \otimes S N \in \text{Cor}_\omega(R)$ for any $N \in \text{Mod} S$.
2. $M_\ast \in \text{Acot}_\omega(S)$ for any $M \in \text{Mod} R$.

**Proof.** (1) $\Rightarrow$ (2) Let $M \in \text{Mod} R$. Then $\omega \otimes S M_\ast \in \text{Cor}_\omega(R)$ by (1). Thus $M_\ast \in \text{Acot}_\omega(S)$ by Proposition 6.5.

(2) $\Rightarrow$ (1) Let $N \in \text{Mod} S$. Then $(\omega \otimes S N)_\ast \in \text{Acot}_\omega(S)$ by (2). Thus $\omega \otimes S N \in \text{Cor}_\omega(R)$ by Proposition 6.4.

**Lemma 6.7.** If $\text{pd}_R \omega \leq 2$, then $\text{Ext}^{2\,1}_R(\omega, \omega \otimes S N) = 0$ for any $N \in \text{Mod} S$.

**Proof.** Let $N \in \text{Mod} S$. Then we have the following exact sequence

$$0 \rightarrow \text{acTr}_\omega N \rightarrow \omega \otimes S F_1(N) \xrightarrow{1 - \otimes f_0(N)} \omega \otimes S F_0(N) \rightarrow \omega \otimes S N \rightarrow 0$$

in $\text{Mod} R$. By Lemma 2.5(2), we have

$$\text{Ext}^{2\,1}_R(\omega, \omega \otimes S F_0(N)) = 0 = \text{Ext}^{2\,1}_R(\omega, \omega \otimes S F_1(N)).$$

Because $\text{pd}_R C \leq 2$ by assumption, we have

$$\text{Ext}^i_R(\omega, \omega \otimes S N) \cong \text{Ext}^{i+2}_R(\omega, \text{acTr}_\omega N) = 0$$

for any $i \geq 1$.

The following is the main result in this section.

**Theorem 6.8.** If $\text{pd}_R \omega \leq 2$, then the following statements are equivalent.

1. $\text{pd}_{S\omega} \omega \leq 2$.
2. Any $2$-$\omega$-cospherical module in $\text{Mod} R$ is $\omega$-coreflexive.
3. A module in $\text{Mod} R$ is $2$-$\omega$-cospherical if and only if it is $\omega$-coreflexive.
4. Any adjoint $\omega$-coreflexive module in $\text{Mod} S$ is adjoint $2$-$\omega$-cospherical.
5. A module in $\text{Mod} S$ is adjoint $\omega$-coreflexive if and only if it is adjoint $2$-$\omega$-cospherical.
6. Any module of $\omega$-$\text{P(S)}$-class $2$ in $\text{Mod} R$ is $\omega$-coreflexive.
7. Any module of $\omega$-$\text{T(S)}$-class $2$ in $\text{Mod} R$ is $\omega$-coreflexive.
8. $\text{Tor}_S^2(\omega, V) = 0$ for any $V \in \text{Acot}_\omega(S)$.
9. $\text{Tor}_S^1(\omega, N) = 0$ for any $N \in \text{Mod} S$.
10. $\text{Tor}_S^1(\omega, U_\ast) = 0$ or any $U \in \text{Cot}_\omega(R)$.

**Proof.** By Theorems 4.5 and 4.7, we have (1) $\iff$ (6) $\iff$ (7) $\iff$ (8) $\iff$ (9) $\iff$ (10). The assertions (1) $\Rightarrow$ (2) $\iff$ (4) and (3) $\iff$ (5) follow from Theorem 5.5, Proposition 6.1 and Corollary 6.3, respectively. The implications (3) $\Rightarrow$ (2) and (5) $\Rightarrow$ (4) are trivial.

(2) $\Rightarrow$ (1) Let $N \in \text{Mod} S$. Then $\text{Ext}^{2\,1}_R(\omega, \omega \otimes S N) = 0$ by Lemma 6.7. So $\omega \otimes S N \in \text{Cor}_\omega(R)$ by (2). Then it follows from Corollary 6.6 that $(\text{acTr}_\omega N)_\ast \in \text{Acot}_\omega(S)$. So $\text{Tor}_S^1(\omega, (\text{acTr}_\omega N)_\ast) = 0$ by (4). Since $(\omega \otimes S F_1(N))_\ast \cong F_1(N)$ and
$(\omega \otimes_S F_0(N))_* \cong F_0(N)$ by Lemma 2.5(2), it induces that Ker $f_0(N) \cong (acTr_\omega N)_*$. So we have that $Tor^S_3(\omega, N) \cong Tor^1_1(\omega, (acTr_\omega N)_*) = 0$ and $pd_{S^{op}} \omega \leq 2$.

(2) $\Rightarrow$ (3) Let $M \in Cor_\omega(R)$. Then $M \cong \omega \otimes_S M_0$. By Lemma 6.7, we have $\text{Ext}^1_R(\omega, M) \cong \text{Ext}^1_R(\omega, \omega \otimes_S M_0) = 0$ for any $i \geq 1$.

As a consequence of Theorem 6.8, we have the following

**Corollary 6.9.** pd$_R \omega = pd_{S^{op}} \omega \leq 2$ if and only if for $M \in \text{Mod } R$, the following statements are equivalent.

1. $M \in \text{Cor}_\omega(R)$.
2. There exists an exact sequence

$$U_1 \to U_0 \to M \to 0$$

in $\text{Mod } R$ with all $U_i \in \text{Add}_R \omega \cup \text{Inj } R$.
3. $M$ is 2-$\omega$-cospherical.

**Proof.** Let pd$_R \omega = pd_{S^{op}} \omega \leq 2$. Then (1) $\Leftrightarrow$ (3) by Theorem 6.8, and (1) $\Rightarrow$ (2) by [TH1, Lemma 3.6]. Now let

$$U_1 \to U_0 \to M \to 0$$

be an exact sequence in $\text{Mod } R$ with all $U_i \in \text{Add}_R \omega \cup \text{Inj } R$, and let $K = \text{Ker}(U_1 \to U_0)$. Then by Lemma 2.5(1), we have $\text{Ext}^3_R(\omega, M) \cong \text{Ext}^2_R(\omega, K) = 0$ for any $i \geq 1$. So we have (2) $\Rightarrow$ (3).

Conversely, for any $K \in \text{Mod } R$, consider the following exact sequence

$$0 \to K \to I^0(K) \to I^0(K) \to M \to 0,$$

where $M = \text{Coker } f_0^0$. Then by the equivalence between (2) and (3), we have $	ext{Ext}^3_R(\omega, K) \cong \text{Ext}^4_R(\omega, M) = 0$. It implies pd$_R \omega \leq 2$. So by Theorem 6.8 and assumption, we have pd$_{S^{op}} \omega \leq 2$. It follows from [TH3, Proposition 5.1] that pd$_R \omega = pd_{S^{op}} \omega$. \hfill \Box

In the following result, we give some equivalent characterizations for $\omega_S$ or $R^\omega$ being projective.

**Proposition 6.10.**

(1) The following statements are equivalent.

1a) $\omega_S$ is projective.
1b) Any module in $\text{Mod } R$ is $\omega$-coreflexive.
1c) Any module in $\text{Mod } R$ is $\omega$-cotorsionless.

(2) The following statements are equivalent.

2a) $R^\omega$ is projective.
2b) Any module in $\text{Mod } S$ is adjoint $\omega$-coreflexive.
2c) Any module in $\text{Mod } S$ is adjoint $\omega$-cotorsionless.

**Proof.** (1) The implication (1a)$\Rightarrow$(1b) follows from Corollary 5.2(1), and the implication (1b)$\Rightarrow$(1c) is trivial.

(1c)$\Rightarrow$(1a) Let $N \in \text{Mod } S$. By (1c), acTr$_\omega N \in \text{Cot}_\omega(R)$ and $\theta_{acTr_\omega} N$ is an epimorphism. So by Corollary 5.3(1), we have that $\text{Tor}^2_R(\omega, N) = 0$ and $\omega_S$ is flat, and hence projective.
(2) The implication (2a)⇒(2b) follows from Corollary 5.2(2), and the implication (2b)⇒(2c) is trivial.

(2c)⇒(2a) Let \( M \in \text{Mod} \). By (2c), \( cTr_\omega M \in \text{Acot}_\omega(S) \) and \( \mu_{cTr_\omega} \) is a monomorphism. So by Corollary 5.3(2), we have that \( \text{Ext}^1_R(\omega, M) = 0 \) and \( R\omega \) is projective.

Let \( R \) be an artin algebra and \( \mathbb{D} \) its ordinary duality. Then we have the following facts: (1) \( R\mathbb{D}(R)R \) is a semidualizing bimodule; and (2) \( R \) is selfinjective if and only if \( \mathbb{D}(R) \) is projective as a left (or right) \( R \)-module. The following result is an immediate consequence of Proposition 6.10. Compare it with [J1, Corollary 1.2], which states that a left and right noetherian ring \( R \) is self-injective if and only if any finitely generated left (or right) \( R \)-module \( A \) is reflexive, that is, \( \text{Hom}_R(\text{Hom}_R(A, R), R) \cong A \).

**Corollary 6.11.** For an artin algebra \( R \), the following statements are equivalent.

1. \( R \) is selfinjective.
2. Any module in \( \text{Mod} \) is \( D(R) \)-coreflexive.
3. Any module in \( \text{Mod} \) is \( D(R) \)-cotorsionless.
4. Any module in \( \text{Mod} \) is adjoint \( D(R) \)-coreflexive.
5. Any module in \( \text{Mod} \) is adjoint \( D(R) \)-cotorsionless.

In the following result, we give some equivalent characterizations for \( \omega_S \) having projective dimension at most one.

**Theorem 6.12.** The following statements are equivalent.

1. \( \text{pd}_{\omega_S} \omega \leq 1 \).
2. Any 1-\( \omega \)-cospherical module in \( \text{Mod} \) is \( \omega \)-cotorsionless.
3. Any 1-\( \omega \)-cospherical module in \( \text{Mod} \) is \( \omega \)-coreflexive.
4. Any \( \omega \)-cotorsionless module in \( \text{Mod} \) is \( \omega \)-coreflexive.
5. \( \text{Tor}^1_S(\omega, V) = 0 \) for any \( V \in \text{Acot}_\omega(S) \).
6. \( \text{Tor}^2_S(\omega, N) = 0 \) for any \( N \in \text{Mod} S \).

**Proof.** By Theorem 4.5 and Lemma 4.3, we have (1) ⇔ (4) ⇔ (5) ⇔ (6). The implication (3) ⇒ (2) is trivial.

(2) ⇒ (4) Let \( M \in \text{Cot}_{\omega}(R) \). Then \( \theta_M \) is an epimorphism. By [TH1, Corollary 3.8] and Lemma 2.5(1), there exists an exact sequence

\[
0 \to N \to W \to M \to 0
\]

in \( \text{Mod} R \) with \( W \in \mathcal{P}_{\omega}(R) \) and \( N \) 1-\( \omega \)-cospherical. Then we get the following commutative diagram with exact rows

\[
\begin{array}{cccccccc}
\omega \otimes_S N & \xrightarrow{\theta_N} & \omega \otimes_S W & \xrightarrow{\theta_W} & \omega \otimes_S M & \rightarrow 0 \\
0 & \xrightarrow{\theta_M} & N & \xrightarrow{\theta_W} & W & \rightarrow 0,
\end{array}
\]

where \( \theta_W \) is an isomorphism by Lemma 2.5(1). Because \( N \in \text{Cot}_{\omega}(R) \) and \( \theta_N \) is an epimorphism by (2), we have that \( \theta_M \) is a monomorphism, and hence an isomorphism. Thus \( M \in \text{Cor}_{\omega}(R) \).

(4) ⇒ (3) Let \( M \in \text{Mod} R \) be 1-\( \omega \)-cospherical. Then the following exact sequence

\[
0 \to M \to I^0(M) \to M_1 \to 0
\]

is an exact sequence. The proof is similar to the previous cases.
Let $M_1 \in \text{Cot}_\omega(R)$. By (4), we have that $M_1 \in \text{Cor}_\omega(R)$ and $\theta_{M_1}$ is an isomorphism. Thus $\theta_M$ is an epimorphism and $M \in \text{Cot}_\omega(R)$. □

7. Wakamatsu tilting conjecture over artinian rings

In this section, we aim at studying the Wakamatsu tilting conjecture in some special cases.

Let $N \in \text{Mod}_S$. In the minimal flat resolution (2.1) of $N$ in $\text{Mod}_S$, for any $i \geq 1$, put $\text{Im} f_i(N) = N_i$, and let $f_i(N) = \alpha_i \cdot \pi_i$ be the natural epic-monic decomposition of $f_i(N)$ with $\pi_i : F_{i+1}(N) \twoheadrightarrow N_i$ and $\alpha_i : N_i \hookrightarrow F_i(N)$.

**Lemma 7.1.** Let $N \in \text{Mod}_S$. Then for any $i \geq 1$, we have

$$\text{Ext}^i_R(\omega, \text{acTr}\omega N_i) = 0.$$ 

**Proof.** For any $i \geq 1$, we have the following two exact sequences

$$0 \to N_{i+1} \xrightarrow{\alpha_{i+1}} F_{i+1}(N) \xrightarrow{\pi_i} N_i \to 0,$$

$$0 \to \text{acTr}_\omega N_{i-1} \xrightarrow{\beta_{i-1}} \omega \otimes_S F_{i+1}(N) \xrightarrow{L \otimes f_i(N)} \omega \otimes_S F_i(N) \xrightarrow{L \otimes \pi_i} \omega \otimes_S N_{i-1} \to 0.$$ 

Then we get the following commutative diagram with exact rows

Diag (7.1)

where $h$ is an induced homomorphism. Note that $\mu_{F_{i+1}(N)}$ and $\mu_{F_i(N)}$ are isomorphisms by Lemma 2.5(2). So $h$ is an isomorphism and $(\text{acTr}_\omega N_{i-1})_\ast \cong N_{i-1}$. Because $N_i$ is isomorphic to a submodule of the adjoint $\omega$-coreflexive module $F_i(N)$, $N_i$ is adjoint $\omega$-cotorsionless. It follows from Corollary 5.2(2) that $\text{Ext}^i_R(\omega, \text{acTr}_\omega N_i) = 0$. □

**Lemma 7.2.** Let $N \in \text{Mod}_S$. Then for any $i \geq 1$, there exists an exact sequence

$$\eta_i : 0 \to \text{acTr}_\omega N_i \to \omega \otimes_S F_{i+2}(N) \xrightarrow{\varphi_i} \text{acTr}_\omega N_{i-1} \to \text{Tor}^S_{i+1}(\omega, N) \to 0.$$ (7.1)

**Proof.** Let $g_i$ be the composition

$$\omega \otimes_S F_{i+2}(N) \xrightarrow{L \otimes \pi_{i+1}} \omega \otimes_S N_{i+1} \xrightarrow{L \otimes h} \omega \otimes_S (\text{acTr}_\omega N_{i-1})_\ast \xrightarrow{\theta_{\text{acTr}_\omega N_{i-1}}} \text{acTr}_\omega N_{i-1},$$

where $h$ is as in Diagram (7.1). Since $L \otimes \pi_{i+1}$ is an epimorphism and $L \otimes h$ is an isomorphism, we have

$$\text{Im} g_i = \text{Im}(\theta_{\text{acTr}_\omega N_{i-1}} : (1 \omega \otimes h) : (1 \omega \otimes \pi_{i+1})) = \text{Im} \theta_{\text{acTr}_\omega N_{i-1}}.$$
Let \( \omega \) be an exact sequence and \( \omega \) be an assumption, it is immediate that \( \text{E-cograde}_\omega S \) is left coherent and \( \text{E-cograde}_\omega S \). By Lemma 2.4(2), we have
\[
\beta_i + 1 \cdot h = \mu F_{i+1}(N) \cdot \alpha_i + 1,
\]
so we have
\[
(1_\omega \otimes \beta_{i+1}) \cdot (1_\omega \otimes h) = (1_\omega \otimes \mu F_{i+1}(N)) \cdot (1_\omega \otimes \alpha_{i+1}).
\]
Note that
\[
f_{i+1}(N) = \alpha_{i+1} \cdot \pi_{i+1} + 1 \text{ and } \beta_{i+1} \cdot \theta_{\text{acTr}_\omega N_{i-1}} = \theta_{\omega \otimes \theta F_{i+1}(N)} \cdot (1_\omega \otimes \beta_{i+1}).
\]
So by Lemma 2.4(2), we have
\[
1_\omega \otimes f_{i+1}(N)
= \theta_{\omega \otimes \theta F_{i+1}(N)} \cdot (1_\omega \otimes \mu F_{i+1}(N)) \cdot (1_\omega \otimes f_{i+1}(N))
= \theta_{\omega \otimes \theta F_{i+1}(N)} \cdot (1_\omega \otimes \mu F_{i+1}(N)) \cdot (1_\omega \otimes \alpha_{i+1}) \cdot (1_\omega \otimes \pi_{i+1})
= \beta_{i+1} \cdot \theta_{\text{acTr}_\omega N_{i-1}} \cdot (1_\omega \otimes h) \cdot (1_\omega \otimes \pi_{i+1})
= \beta_{i+1} \cdot g.
\]
Since \( \beta_{i+1} \) is a monomorphism, we have
\[
\text{Ker} \, g_i \cong \text{Ker}(1_\omega \otimes f_{i+1}(N)) = \text{acTr}_\omega N_i.
\]
The proof is finished. \( \square \)

Following [TH2, Definition 6.2], the Ext-cograde of a module \( M \) in \( \text{Mod}_R \) with respect to \( \omega \) is defined as \( \text{E-cograde}_\omega M := \inf \{ i \geq 0 \mid \text{Ext}_{R}^{i}(\omega, M) \neq 0 \} \). If \( \text{Ext}_{R}^{\geq 0}(\omega, M) = 0 \), then set \( \text{E-cograde}_\omega M = \infty \).

In the following, \( m \) and \( n \) are positive integers. We use \( \text{mod} \, S \) to denote the class of finitely presented left \( S \)-modules.

**Lemma 7.3.** Let \( S \) be a left coherent ring. If \( \text{E-cograde}_\omega \text{Tor}^S_m(\omega, N) \geq n - 1 \) for any \( N \in \text{mod} \, S \), then \( \text{Ext}_{R}^{i}(\omega, \text{acTr}_\omega N_{i+j-2}) = 0 \) for any \( i \geq m \) and \( 1 \leq j \leq n \).

**Proof.** (1) The case for \( n = 1 \) follows from Lemma 7.1. Now suppose \( n \geq 2 \). Since \( S \) is left coherent and \( \text{E-cograde}_\omega \text{Tor}^S_m(\omega, N) \geq n - 1 \) for any \( N \in \text{mod} \, S \) by assumption, it is immediate that \( \text{E-cograde}_\omega \text{Tor}^S_1(\omega, N) \geq n - 1 \) for any \( N \in \text{mod} \, S \) and \( i \geq m \). We divide the exact sequence (7.1) in Lemma 7.2 into the following two exact sequences

\[
0 \to \text{acTr}_\omega N_i \to \omega \otimes S F_{i+2}(N) \xrightarrow{\nu_i} K_i \to 0,
\]
\[
0 \to K_i \xrightarrow{\lambda_i} \text{acTr}_\omega N_{i-1} \to \text{Tor}^S_{i+1}(\omega, N) \to 0,
\]
where \( K_i = \text{Im} \, g_i \) and \( g_i = \lambda_i \cdot \nu_i \) is the natural epic-monic decomposition of \( g_i \). For \( i \geq m \), applying the functor \((-)_* \) to (7.2) yields
\[
\text{Ext}_{R}^{j}(\omega, K_i) \cong \text{Ext}_{R}^{j+1}(\omega, \text{acTr}_\omega N_i)
\]
for any \( j \geq 1 \) by Lemma 2.5(1); and then applying the functor \((-)_* \) to (7.3) gives a monomorphism
\[
\text{Ext}_{R}^{j}(\omega, \text{acTr}_\omega N_i)(\cong \text{Ext}_{R}^{1}(\omega, K_i)) \to \text{Ext}_{R}^{j}(\omega, \text{acTr}_\omega N_{i-1}).
\]
Let $N$ denote any $\omega$-coherent ring. If $\text{pd}_R \omega \leq n$ and $E$-cograde $\omega, \text{Tor}_n^S(\omega, N) \geq n - 1$ for any $N \in \text{mod } S$, then we have

(1) $\text{Ext}_i^R(\omega, \text{acTr}_\omega N_i) = 0$ for any $i \geq m + n - 2$.
(2) $N_i$ is adjoint $\omega$-coreflexive for any $i \geq m + n - 2$.
(3) $\text{Ext}_i^R(\omega, \omega \otimes_S N_i) = 0$ for any $i \geq m + n - 2$.
(4) $E$-cograde $\omega, \text{Tor}_{i+1}^R(\omega, N) = \infty$ for any $i \geq m + n - 1$.

Proof. (1) Let $i \geq m + n - 2$. It follows from Lemma 7.3 that $\text{Ext}_i^{1 \leq j \leq n}(\omega, \text{acTr}_\omega N_i) = 0$. Since $\text{pd}_R \omega \leq n$, we have $\text{Ext}_i^{\geq n+1}(\omega, \text{acTr}_\omega N_i) = 0$.
(2) It follows from (1) and Corollary 5.2(2).
(3) Since there exists an exact sequence

$0 \rightarrow \text{acTr}_\omega N_i \rightarrow \omega \otimes_S F_{i+2}(N) \rightarrow \omega \otimes_S F_{i+1}(N) \rightarrow \omega \otimes_S N_i \rightarrow 0$,

the assertion follows from (1) and Lemma 2.5(1).
(4) Let $g_i$ be as in the proof of Lemma 7.2 with $i \geq m + n - 1$, that is,

$g_i = \theta_{\text{acTr}_\omega N_{i-1}} \cdot (1_\omega \otimes h) \cdot (1_\omega \otimes \pi_{i+1})$

Then we have

$g_\ast = (\theta_{\text{acTr}_\omega N_{i-1}})_\ast \cdot (1_\omega \otimes h)_\ast \cdot (1_\omega \otimes \pi_{i+1})_\ast$.

Because both $\mu_{N_{i+1}}$ and $\mu_{F_{i+2}(N)}$ are isomorphisms by (2) and Lemma 2.5(2), the equality

$(1_\omega \otimes \pi_{i+1})_\ast \cdot \mu_{F_{i+2}(N)} = \mu_{N_{i+1}} \cdot \pi_{i+1}$

implies that $(1_\omega \otimes \pi_{i+1})_\ast$ is an epimorphism. Because $(\theta_{\text{acTr}_\omega N_{i-1}})_\ast$ is an epimorphism by Lemma 2.4(1), we have that $g_\ast$ is also an epimorphism.

Consider the exact sequences (7.1)–(7.3) in Lemmas 7.2 and 7.3. Because $g_\ast = \lambda_\ast \cdot \nu_\ast$, we have that $\lambda_\ast$ is an epimorphism, and hence an isomorphism. Applying the functor $(-)_\ast$ to the exact sequence (7.2) we have

$\text{Ext}_R^j(\omega, K_i) \cong \text{Ext}_R^{j+1}(\omega, \text{acTr}_\omega N_i) = 0$

for any $j \geq 1$ by (1) and Lemma 2.5(1). Moreover, applying the functor $(-)_\ast$ to the exact sequence (7.3) we get a long exact sequence

$0 \rightarrow K_i \xrightarrow{\lambda_i} \text{acTr}_\omega N_{i-1} \rightarrow (\text{Tor}_{i+1}^S(\omega, N))_\ast \rightarrow \cdots$

$\cdots \rightarrow \text{Ext}_R^j(\omega, K_i) \rightarrow \text{Ext}_R^j(\omega, \text{acTr}_\omega N_{i-1}) \rightarrow \text{Ext}_R^j(\omega, \text{Tor}_{i+1}^S(\omega, N)) \rightarrow \cdots$. \hfill (7.4)

Notice that $i \geq m + n - 1$, so also by (1) we have $\text{Ext}_R^{\geq 1}(\omega, \text{acTr}_\omega N_{i-1}) = 0$. Then from the exact sequence (7.4) we get $\text{Ext}_R^1(\omega, \text{Tor}_{i+1}^S(\omega, N)) = 0$. Because $\lambda_\ast$ is an isomorphism, we have that $(\text{Tor}_{i+1}^S(\omega, N))_\ast = 0$ and $E$-cograde $\omega, \text{Tor}_{i+1}^S(\omega, N) = \infty$.

The main result in this section is the following

**Theorem 7.5.** Let $S$ be a left artinian ring and $\omega_S = S \omega_S$. If $\text{pd}_S \omega \leq n$ and $E$-cograde $\omega, \text{Tor}_n^S(\omega, N) \geq n - 1$ for any $N \in \text{mod } S$, then $\text{pd}_S \omega = \text{pd}_{S^{op}} \omega \leq n$. 

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Define a linear map
\[ \gamma : K_0(\text{mod } S) \to K_0(\text{mod } S) \] via \[ \gamma([M]) = \Sigma_{i \geq 0} (-1)^i [\text{Ext}_S^i(\omega, M)]. \]
Since pd$_S \omega \leq n$, this map is well defined. By Lemmas 2.5 and 7.4(2), for any $N \in \text{mod } S$ and $i \geq m + n - 1$ we have
\[ [N] = \sum_{j=0}^{i-1} (-1)^j [F_j(N)] + (-1)^i [N_{i-1}] \]
\[ = \sum_{j=0}^{i-1} (-1)^j ([\omega \otimes_S F_j(N)]_*) + (-1)^i ([\omega \otimes_S N_{i-1}]_*) \]
\[ = \sum_{j=0}^{i-1} (-1)^j \gamma([\omega \otimes_S F_j(N)]) + (-1)^i \gamma([\omega \otimes_S N_{i-1}]) \]
\[ = \gamma(\sum_{j=0}^{i-1} (-1)^j [\omega \otimes_S F_j(N)] + (-1)^i [\omega \otimes_S N_{i-1}]), \]
which implies that $\gamma$ is surjective. Because $S$ is left artinian by assumption, it follows from [ARS, p.5, Theorem 1.7] that $K_0(\text{mod } S)$ is a finitely generated free abelian group and $\gamma$ is bijective. On the other hand, for any $Y \in \text{mod } S$, we have that $[Y] = 0$ if and only if $Y = 0$. Since Ext$_S^0(\omega, \text{Tor}_S^m(\omega, N)) = 0$ by Lemma 7.4(4), we have $\gamma([\text{Tor}_S^m(\omega, N)]) = 0$ and $[\text{Tor}_S^m(\omega, N)] = 0$. So Tor$_S^m(\omega, N) = 0$ and pd$_S \omega \leq m + n - 1$. Now it follows from [TH3, Proposition 5.1] that pd$_S \omega = \text{pd}_S \omega \leq n$. □

In the following, we study when the Ext-cograde condition in Theorem 7.5 is satisfied. We need the following

**Lemma 7.6.** Let $Q \in \text{Mod } R$ be finitely generated projective and $t \geq 0$. Then fd$_S$ Hom$_R(Q, \omega) \leq t$ if and only if Hom$_R(Q, \text{Tor}_S^t(\omega, N)) = 0$ for any $N \in \text{mod } S$.

**Proof.** Let $N \in \text{mod } S$ and
\[ P =: \cdots \to P_t \to \cdots \to P_1 \to P_0 \to N \to 0 \]
be a projective resolution of $N$ in $\text{mod } S$. Because $Q \in \text{Mod } R$ is finitely generated projective by assumption, the functor Hom$_R(Q, -)$ is exact. Then we have
\[ \text{Tor}_S^t(\text{Hom}_R(Q, \omega), N) \]
\[ \cong H_{t+1}(\text{Hom}_R(Q, \omega) \otimes_S P)) \]
\[ \cong H_{t+1}(\text{Hom}_R(Q, \omega) \otimes_S P)) \]
\[ \cong \text{Hom}_R(Q, H_{t+1}(\omega \otimes_S P)) \] (by [EJ, p.33, Exercise 3])
\[ \cong \text{Hom}_R(Q, \text{Tor}_S^t(\omega, N)). \]
Now the assertion follows easily. □

Let $R$ be a semiperfect ring. Then any finitely generated left or right $R$-module has a projective cover. In this case, since $R\omega$ admits a degreewise finite $R$-projective resolution by Definition 2.1, we may assume that
\[ \cdots \xrightarrow{g_{t+1}(\omega)} P_t(\omega) \xrightarrow{g_1(\omega)} \cdots \xrightarrow{g_1(\omega)} P_1(\omega) \xrightarrow{g_0(\omega)} P_0(\omega) \xrightarrow{g_{t+1}(\omega)} R\omega \to 0 \]
is a minimal projective resolution of $R\omega$ in $\text{Mod } R$ with all $P_i(\omega)$ finitely generated. Put $\omega_i := \text{Im } g_i(\omega)$ for any $i \geq 1$ (in particular, $\omega_{-1} = \omega$). Let $n \geq 0$. Recall from [TH2, Definition 6.2] that the strong Ext-cograde of a module $M \in \text{Mod } R$ with respect to $\omega$, denoted by $\text{s.E-cograde}_\omega M$, is said to be at least $n$ if E-cograde $X \geq n$ for any quotient module $X$ of $M$.

**Proposition 7.7.** Let $R$ be a semiperfect ring. Then the following statements are equivalent.

1. $\text{s.E-cograde}_\omega \text{Tor}_m^S(\omega, N) \geq n - 1$ for any $N \in \text{Mod } S$.
2. $\text{fd}_{S^{op}} \text{Hom}_R(P_i(\omega), \omega) \leq m - 1$ for any $0 \leq i \leq n - 2$.

**Proof.** The case for $n = 1$ is trivial. Now suppose $n \geq 2$.

(1) $\Rightarrow$ (2) We proceed by using induction on $i$.

When $i = 0$, we will prove $\text{fd}_{S^{op}} \text{Hom}_R(P_0(\omega), \omega) \leq m - 1$. Let $N \in \text{Mod } S$. Because $S \text{E-cograde}_\omega \text{Tor}_m^S(\omega, N) \geq n - 1$ by (1), we have $\text{Hom}_R(\omega, \text{Tor}_m^S(\omega, N)) = 0$. Let $f \in \text{Hom}_R(P_0(\omega), \text{Tor}_m^S(\omega, N))$. Then $f$ induces a homomorphism

$$f : \omega \otimes P_0(\omega) \to \text{Tor}_m^S(\omega, N) / f(\omega_0)$$

in $\text{Mod } R$. Since $\text{s.E-cograde}_\omega \text{Tor}_m^S(\omega, N) \geq n - 1$ by (1), we have $\bar{f} = 0$. So $P_0(\omega) = \text{Ker } f + \omega_0$. Notice that $P_0(\omega)$ is the projective cover of $\omega$, so $\omega_0$ is superfluous in $P_0(\omega)$. It induces that $\text{Ker } f = P_0(\omega)$ and $f = 0$. Thus we have $\text{Hom}_R(P_0(\omega), \text{Tor}_m^S(\omega, N)) = 0$, and therefore $\text{fd}_{S^{op}} \text{Hom}_R(P_0(\omega), \omega) \leq m - 1$ by Lemma 7.6.

Now suppose that $i \geq 1$ and $N \in \text{Mod } S$. Let $X$ be a quotient module of $\text{Tor}_m^S(\omega, N)$. By (1), we have $\text{Ext}_{R_{S^{op}}}^{pd+1}(\omega, X) = 0$. Then

$$\text{Ext}_{R}^i (\omega_{i-2}, X) \cong \text{Ext}_{R}^i (\omega, X) = 0$$

for any $1 \leq i \leq n - 2$. From the exact sequence

$$0 \to \omega_{i-1} \to P_{i-1}(\omega) \to \omega_{i-2} \to 0,$$

we get the following exact sequence

$$\text{Hom}_R(P_{i-1}(\omega), X) \to \text{Hom}_R(\omega_{i-1}, X) \to \text{Ext}_{R}^i (\omega_{i-2}, X) \to 0.$$  \hspace{1cm} (7.5)

By the induction hypothesis, we have $\text{fd}_{S^{op}} \text{Hom}_R(P_{i-1}(\omega), \omega) \leq m - 1$. Then it follows from Lemma 7.6 that $\text{Hom}_R(P_{i-1}(\omega), \text{Tor}_m^S(\omega, N)) = 0$ and $\text{Hom}_R(P_{i-1}(\omega), X) = 0$. So it is derived from (7.5) that $\text{Hom}_R(\omega_{i-1}, X) = 0$. Note that $P_i(\omega)$ is the projective cover of $\omega_{i-1}$. Then by using an argument similar to that in the proof of the case for $i = 0$, we get $\text{Hom}_R(P_i(\omega), \text{Tor}_m^S(\omega, N)) = 0$. Thus $\text{fd}_{S^{op}} \text{Hom}_R(P_i(\omega), \omega) \leq m - 1$ by Lemma 7.6.

(2) $\Rightarrow$ (1) Let $X$ be a quotient module of $\text{Tor}_m^S(\omega, N)$. Then by (2) and Lemma 7.6, we have $\text{Hom}_R(\oplus_{i=0}^{n-2} P_i(\omega), \text{Tor}_m^S(\omega, N)) = 0$ and $\text{Hom}_R(\oplus_{i=0}^{n-2} P_i(\omega), X) = 0$. Since $\omega_{i-1}$ is a quotient module of $P_i(\omega)$ for any $i \geq 0$, we then have $\text{Hom}_R(\oplus_{i=0}^{n-2} \omega_{i-1}, X) = 0$. So from (7.5) we get $\text{Ext}_{R}^i (\oplus_{i=0}^{n-2} \omega_{i-1}, X) = 0$. Since $\text{Ext}_{R}^i (\omega, X) \cong \text{Ext}_{R}^i (\omega_{i-1}, X)$ for any $i \geq 0$, we have that $\text{Ext}_{R}^i (\omega, X) = 0$ and $\text{s.E-cograde}_\omega \text{Tor}_m^S(\omega, N) \geq n - 1$.

By applying Theorem 7.5 and Proposition 7.7, we get the following

**Theorem 7.8.** Let $S$ be a left artinian ring and $R\omega_S = S\omega_S$. If $\text{pd}_{S^{op}} \text{Hom}_S(P_i(\omega), \omega) < \infty$ for any $0 \leq i \leq n - 2$, then $\text{pd}_{S^{op}} \omega = \text{pd}_{S} \omega \leq n$. 

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Proof. Without loss of generality, assume \( \text{pd}_{\text{opp}} \text{Hom}_S(P_i(\omega), \omega) \leq m(< \infty) \) for any \( 0 \leq i \leq n - 2 \). By Proposition 7.7, \( S \)-E-cograde, \( \text{Tor}_{m+1}^S(\omega, N) \geq n - 1 \) for any \( N \in \text{Mod} S \). Then it follows from Theorem 7.5 that \( \text{pd}_{\text{opp}} \omega = \text{pd}_S \omega \leq n \). \( \square \)

Note that in the case for \( n = 1 \), the condition “\( \text{pd}_{\text{opp}} \text{Hom}_S(P_i(\omega), \omega) < \infty \) for any \( 0 \leq i \leq n - 2 \)” in Theorem 7.8 is automatically satisfied. So we immediately have the following

**Corollary 7.9.** Let \( S \) be a left artinian ring and \( R\omega_S = s\omega_S \). If \( \text{pd}_S \omega \leq 1 \), then \( \text{pd}_{\text{opp}} \omega = \text{pd}_S \omega \leq 1 \).

We do not know whether the statements (1a) and (2a) in Proposition 6.10 are equivalent in general. However, by Corollary 7.9, we have the following

**Corollary 7.10.** Let \( S \) be a left artinian ring and \( R\omega_S = s\omega_S \). If \( s \omega \) is projective, then \( \omega_S \) is projective.

Let \( S \) an artin algebra over a commutative artinian ring and \( \mathbb{D} \) the usual Matlis duality between \( \text{mod} S \) and \( \text{mod} S^{\text{opp}} \). Then \( s\mathbb{D}(S)_S \) is a semidualizing bimodule and \( \text{Hom}(\cdot, \mathbb{D}(S)) \) maps minimal injective (resp. projective) resolutions of modules in \( \text{mod} S \) to minimal projective (resp. injective) resolutions of modules in \( \text{mod} S^{\text{opp}} \). Let

\[
0 \to S_S \to I^0(S_S) \to I^1(S_S) \to \cdots \to I^i(S_S) \to \cdots
\]

be a minimal injective resolution of \( S_S \) in \( \text{mod} S^{\text{opp}} \). Note that \( s\mathbb{D}(S) \) and \( \mathbb{D}(S)_S \) are injective cogenerators for \( \text{Mod} S \) and \( \text{Mod} S^{\text{opp}} \), respectively. So \( \text{pd}_S \mathbb{D}(S) = \text{id}_{S^{\text{opp}}} S \) and \( \text{pd}_{\text{opp}} \mathbb{D}(S) = \text{id}_S S \) by [EJ, Theorem 3.2.19]. Now, by putting \( s\omega_S = s\mathbb{D}(S)_S \) in Theorem 7.8, we get the following

**Corollary 7.11.** Let \( S \) be an artin algebra and \( \text{id}_{S^{\text{opp}}} S \leq n \). If \( \text{pd}_{S^{\text{opp}}} I^i(S_S) < \infty \) for any \( 0 \leq i \leq n - 2 \), then \( \text{id}_S S = \text{id}_{S^{\text{opp}}} S \leq n \).

The following corollary is well known, which is a dual version of Corollary 7.9.

**Corollary 7.12.** (cf. [BrB, Theorem 1]) Let \( S \) be an artin algebra. If \( \text{id}_{S^{\text{opp}}} S \leq 1 \), then \( \text{id}_S S = \text{id}_{S^{\text{opp}}} S \leq 1 \).

Putting \( n = 2 \) in Corollary 7.11, we have the following

**Corollary 7.13.** Let \( S \) be an artin algebra and \( \text{id}_{S^{\text{opp}}} S \leq 2 \). If \( \text{pd}_{S^{\text{opp}}} I^0(S_S) < \infty \), then \( \text{id}_S S = \text{id}_{S^{\text{opp}}} S \leq 2 \).

**Acknowledgements.** This research was partially supported by NSFC (Grant Nos. 11571164, 11501144), a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions and NSF of Guangxi Province of China (Grant No. 2013GXNSFBA019005).

**References**


