Coproper Coresolutions and Direct Limits

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Abstract

Let R be a ring and let M be a left R-module such that $M = \varinjlim_{i\in I} M_i$ with I a directed index set. For a class $\mathscr{X}$ of left R-modules, we construct certain coresolution of M from strong coproper $\mathscr{X}$-coresolutions of all M_i. As a consequence, we get that if $\mathscr{X}$ is coresolving and closed under direct limits, then the supremum of $\mathscr{X}$-injective dimensions of all left R-modules and that of all finitely presented left R-modules coincide. Some known results are obtained as corollaries. Moreover, we get some equivalent characterizations of weakly Gorenstein algebras.

1. **Introduction**

It is well known that the notion of direct limits is fundamental in homological theory, which plays a very important role in studying the structure and classification of modules and rings. For example, any module is a direct limit of its finitely presented submodules, and any flat module is a direct limit of finitely generated projective modules [28]. These results provide useful tools for investigating the transfer of certain homological properties between infinitely generated modules and finitely presented ones. In addition, a ring R is left Noetherian if and only if any direct limit of injective left R-modules is injective [5], and a ring R is left coherent if and only if any direct limit of FP-injective left R-modules is FP-injective [32]. These are partial classical results about direct limits. Recently, many authors studied certain properties of direct limits in (relative) homological theory, see [13, 15, 18, 20, 24, 30, 31, 34] and references therein. In particular, it was proved in [18] that if R is a left Noetherian ring and M is a left R-module such that $M = \varinjlim_{i\in I} M_i$ with I a directed index set, then a (minimal) injective coresolution of M can be constructed from those of all M_i. The aim of this paper is to generalize this result to a much more general setting and give some applications. This paper is organized as follows.

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In Section 2, some terminology and preliminary results are given. Let R be a ring and let M be a left R-module such that $M = \varinjlim_{i\in I} M_i$ with I a directed index set. In Section 3, we prove that certain coresolution of M can be constructed from injective coresolutions of all M_i; and if $\mathscr{X}$ is a class of left R-modules closed under direct limits, then a strong coproper $\mathscr{X}$-coresolution of M can be constructed from those of all M_i (Theorem 3.5). Let $\mathscr{X}'$ be the class of finitely presented submodules of modules in $\mathscr{X}$, and let $M = \varinjlim_{i\in I} M_i$ with all M_i finitely presented. As a consequence of the above result, we get that if both $\mathscr{X}$ and the left 1-orthogonal class of $\mathscr{X}$ are closed under direct limits and each M_i admits a strong coproper n-$\mathscr{X}'$-coresolution, where $\mathscr{X}'$ is the class of finitely presented submodules of modules in $\mathscr{X}$, then M admits a strong coproper n-$\mathscr{X}$-coresolution (Proposition 3.6).

In Section 4, we give some applications of the results obtained in Section 3. We prove that if $\mathscr{X}$ is a class of left R-modules which is coresolving and closed under direct limits, then the supremum of $\mathscr{X}$-injective dimensions of all left R-modules and that of all finitely presented left R-modules coincide (Theorem 4.1). Some known results are obtained as corollaries. Finally, we obtain some equivalent characterizations of weakly Gorenstein algebras (Theorem 4.8).

2. **Preliminaries**

In this paper, R is an arbitrary associative ring with unit. We use $\mathrm{Mod}R$ to denote the class of left R-modules, and use $\mathrm{mod}R$ to denote the class of finitely presented left R-modules. Let $\mathscr{C}$ be a subclass of $\mathrm{Mod}R$. We write

$$^{\perp_1}\mathscr{C} := \{M \in \mathrm{Mod}R \mid \mathrm{Ext}^1_R(M,C) = 0 \text{ for any } C \in \mathscr{C}\},$$

$$^{\perp}\mathscr{C} := \{M \in \mathrm{Mod}R \mid \mathrm{Ext}^{\geq 1}_R(M,C) = 0 \text{ for any } C \in \mathscr{C}\},$$

$$\mathscr{C}^{\perp_1} := \{M \in \mathrm{Mod}R \mid \mathrm{Ext}^1_R(C,M) = 0 \text{ for any } C \in \mathscr{C}\},$$

$$\mathscr{C}^{\perp} := \{M \in \mathrm{Mod}R \mid \mathrm{Ext}^{\geq 1}_R(C,M) = 0 \text{ for any } C \in \mathscr{C}\}.$$

Definition 2.1. ([10, 11]) Let $\mathscr{C} \subseteq \mathscr{D}$ be two subclasses of $\mathrm{Mod}R$. A homomorphism $f : D \to C$ in $\mathrm{Mod}R$ with $D \in \mathscr{C}$ and $C \in \mathscr{C}$ is called a *$\mathscr{C}$-preenvelope* of D if $\mathrm{Hom}_R(f, C')$ is epic for any $C' \in \mathscr{C}$. A homomorphism $f : D \to C$ in $\mathrm{Mod}R$ is called *left minimal* if any endomorphism $h : C \to C$ is an automorphism whenever $f = hf$. A $\mathscr{C}$-preenvelope $f : D \to C$ of D is called a *$\mathscr{C}$-envelope* of D if f is left minimal. A $\mathscr{C}$-preenvelope $f : D \to C$ of D is called *special* if f is monic and $\mathrm{Coker} f \in {}^{\perp_1}\mathscr{C}$. The subclass $\mathscr{C}$ is said to be *(pre)enveloping* in $\mathscr{D}$ if any module in $\mathscr{D}$ admits a $\mathscr{C}$-(pre)envelope, and it is said to be *special preenveloping* in $\mathscr{D}$ if any module in $\mathscr{D}$ admits a special $\mathscr{C}$-preenvelope. Dually, the notions of a *(special) $\mathscr{C}$-precover* of D and a *special precovering subclass* are defined.

By the Wakamatsu lemma (cf. [11, Proposition 7.2.4]), if $\mathscr{C}$ is closed under extensions, then any monic $\mathscr{C}$-envelope of a left R-module M is a special $\mathscr{C}$-preenvelope of M, and any epic $\mathscr{C}$-cover of a left R-module M is a special $\mathscr{C}$-precover of M.

Let $\mathscr{C}$ be a subclass of Mod$R$. The *$\mathscr{C}$-injective dimension* $\mathscr{C}$-id$M$ of M is defined as $\inf\{n \mid$ there exists an exact sequence

$$0 \to M \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

in Mod$R$ with all $C^i \in \mathscr{C}\}$, and set $\mathscr{C}$-id$M = \infty$ if no such integer exists. Recall that $\mathscr{C}$ is called *coresolving* if $\mathscr{C}$ contains all injective left R-modules, and $\mathscr{C}$ is closed under extensions and cokernels of monomorphisms. A sequence

$$\mathbb{S} : \cdots \to S_1 \to S_2 \to S_3 \to \cdots$$

in Mod$R$ is called $\mathrm{Hom}_R(-, \mathscr{C})$*-exact* if $\mathrm{Hom}_R(\mathbb{S}, C)$ is exact for any $C \in \mathscr{C}$; dually, the notion of $\mathrm{Hom}_R(\mathscr{C}, -)$*-exact sequences* is defined [11].

Definition 2.2. Let $\mathscr{C}$ be a subclass of Mod$R$ and $n \geq 0$. A module $M \in$ Mod$R$ is said to *admit a coproper n-$\mathscr{C}$-coresolution* if there exists a $\mathrm{Hom}_R(-, \mathscr{C})$-exact exact sequence

$$0 \to M \xrightarrow{f^0} C^0 \xrightarrow{f^1} C^1 \xrightarrow{f^2} \cdots \xrightarrow{f^n} C^n \tag{2.1}$$

in Mod$R$ with all C^i in $\mathscr{C}\}$, and M is said to *admit a coproper ∞-$\mathscr{C}$-coresolution* if M admits a coproper n-$\mathscr{C}$-coresolution for all $n \geq 0$.

A coproper n-$\mathscr{C}$-coresolution of M as in (2.1) is called *minimal* if all $\mathrm{Im} f^i \rightarrowtail C^i$ are left minimal. A coproper n-$\mathscr{C}$-coresolution of M as in (2.1) is called *strong* if all $\mathrm{Coker} f^i$ are in ${}^{\perp_1}\mathscr{C}$. If (2.1) is a strong coproper n-$\mathscr{C}$-coresolution of M, then

$$0 \to M \xrightarrow{f^0} C^0 \xrightarrow{f^1} C^1 \xrightarrow{f^2} \cdots \xrightarrow{f^n} C^n \to \mathrm{Coker} f^n \to 0$$

is called a *partial strong coproper n-$\mathscr{C}$-coresolution* of M.

Dually, the notions of *proper ∞-$\mathscr{C}$-resolutions* and (*partial*) *strong proper n-$\mathscr{C}$-resolutions* are defined.

It is easy to see that (2.1) is a coproper n-$\mathscr{C}$-coresolution of M if and only if each $\mathrm{Im} f^i \rightarrowtail C^i$ is a monic $\mathscr{C}$-preenvelope of $\mathrm{Im} f^i$, and that (2.1) is a strong coproper n-$\mathscr{C}$-coresolution of M if and only if each $\mathrm{Im} f^i \rightarrowtail C^i$ is a special $\mathscr{C}$-preenvelope of $\mathrm{Im} f^i$.

The following observation might be known.

Lemma 2.3. *Let $\mathscr{C}$ be an enveloping class of left R-modules and $n \geq 0$. If M admits a coproper n-$\mathscr{C}$-coresolution*

$$0 \to M \xrightarrow{f^0} C^0 \to C^1 \to \cdots \to C^n, \tag{2.2}$$

then M admits a minimal coproper n-$\mathscr{C}$-coresolution in the following form:

$$0 \to M \xrightarrow{f'^0} C'^0 \xrightarrow{f'^1} C'^1 \xrightarrow{f'^2} \cdots \xrightarrow{f'^n} C'^n, \tag{2.3}$$

where C'^i is a direct summand of C^i for any $0 \leq i \leq n$.

Proof. Since $\mathscr{C}$ is an enveloping class, we have that M admits a monic $\mathscr{C}$-envelope $f'^0 : M \to C'^0$ by (2.2). Set $M^1 := \mathrm{Coker} f^0$ and $M'^1 := \mathrm{Coker} f'^0$. Then we get an exact commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{f'^0} & C'^0 & \longrightarrow & M'^1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow {\scriptstyle g^1} & & \\ 0 & \longrightarrow & M & \xrightarrow{f^0} & C^0 & \longrightarrow & M^1 & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & C''^0 & = = = & C''^0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array}$$

By [11, Proposition 6.1.2], the middle column splits (and hence C'^0 is a direct summand of C^0). It follows that the rightmost column splits. Thus we get a $\mathscr{C}$-preenvelope $M'^1 \rightarrowtail C^1$, which is the composition $M'^1 \overset{g^1}{\rightarrowtail} M^1 \hookrightarrow C^1$. Similar to above, we get a monic $\mathscr{C}$-envelope $f'^1 : M'^1 \to C'^1$ of M'^1 such that C'^1 is a direct summand of C^1. Continuing this procedure, the assertion follows. □

We write

$$\mathcal{P}(\mathrm{Mod}R) := \text{the class of projective left } R\text{-modules},$$

$$\mathcal{P}(\mathrm{mod}R) := \text{the class of finitely generated projective left } R\text{-modules},$$

$$\mathcal{I}(\mathrm{Mod}R) := \text{the class of injective left } R\text{-modules}.$$

Definition 2.4. ([11])

(1) A module $M \in \mathrm{Mod}R$ is called *Gorenstein projective* if $M \in {}^{\perp}\mathcal{P}(\mathrm{Mod}R)$ and M admits a coproper ∞-$\mathcal{P}(\mathrm{Mod}R)$-coresolution.

(2) A module $N \in \mathrm{Mod}R^{op}$ is called *Gorenstein injective* if $N \in \mathcal{I}(\mathrm{Mod}R^{op})^{\perp}$ and N admits a proper ∞-$\mathcal{I}(\mathrm{Mod}R^{op})$-resolution.

We write

$$\mathcal{GP}(\mathrm{Mod}R) := \text{the class of Gorenstein projective left } R\text{-modules},$$

$$\mathcal{GP}(\mathrm{mod}R) := \text{the class of finitely generated Gorenstein projective left } R\text{-modules},$$

$$\mathcal{GI}(\mathrm{Mod}R^{op}) := \text{the class of Gorenstein injective right } R\text{-modules},$$

$$\mathcal{GI}(\mathrm{mod}R^{op}) := \text{the class of finitely generated Gorenstein injective right } R\text{-modules}.$$

3. Constructions of coproper coresolutions

We begin with the following lemma.

Lemma 3.1. *Let κ be a limit ordinal number. Suppose that*

$$\mathbb{S}_\alpha : = 0 \to M_\alpha \to X_\alpha^0 \to X_\alpha^1 \to \cdots \xrightarrow{f_\alpha^n} X_\alpha^n$$

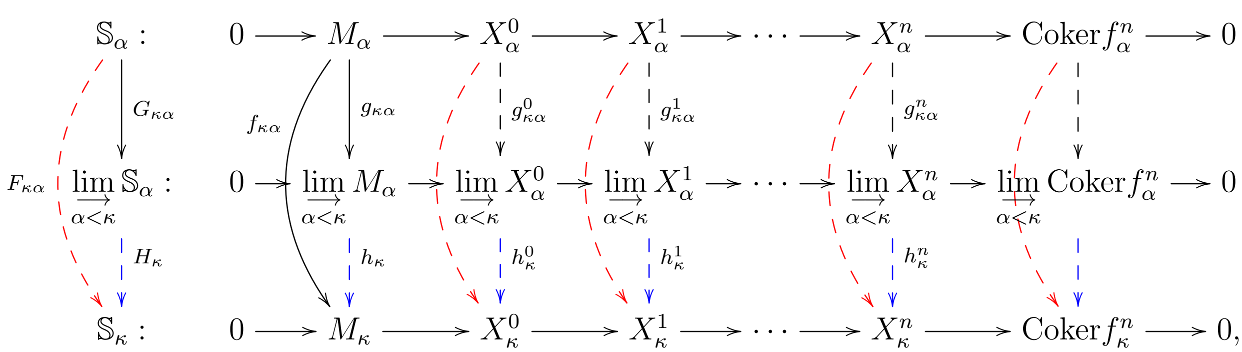
is an exact sequence in Mod$R$ *for any $\alpha \leq \kappa$ and $\{\mathbb{S}_\alpha, F_{\beta\alpha} : \mathbb{S}_\alpha \to \mathbb{S}_\beta \mid \alpha \leq \beta < \kappa\}$ is a direct system of exact sequences. If there exists a chain map from $\varinjlim_{\alpha<\kappa} \mathbb{S}_\alpha$ to $\mathbb{S}_\kappa$, that is,*

$$\begin{array}{cccccccccccccccc}
\varinjlim_{\alpha<\kappa}\mathbb{S}_\alpha : & 0 & \to & \varinjlim_{\alpha<\kappa} M_\alpha & \to & \varinjlim_{\alpha<\kappa} X_\alpha^0 & \to & \varinjlim_{\alpha<\kappa} X_\alpha^1 & \to & \cdots & \to & \varinjlim_{\alpha<\kappa} X_\alpha^n & \to & \varinjlim_{\alpha<\kappa} \mathrm{Coker} f_\alpha^n & \to & 0 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\mathbb{S}_\kappa : & 0 & \to & M_\kappa & \to & X_\kappa^0 & \to & X_\kappa^1 & \to & \cdots & \to & X_\kappa^n & \to & \mathrm{Coker} f_\alpha^n & \to & 0,
\end{array}$$

Diagram (3.1)

then $\{\mathbb{S}_\alpha, F_{\beta\alpha} : \mathbb{S}_\alpha \to \mathbb{S}_\beta \mid \alpha \leq \beta \leq \kappa\}$ is also a direct system of exact sequences.

Proof. Consider the following commutative diagram:



where $G_{\kappa\alpha}$ is the colimit map and H_κ is obtained by assumption. For each $\alpha < \kappa$, set $F_{\kappa\alpha} := H_\kappa G_{\kappa\alpha}$. It follows that $F_{\kappa\alpha} = F_{\kappa\beta}F_{\beta\alpha}$ for any $\alpha \leq \beta < \kappa$. As a consequence, $\{\mathbb{S}_\alpha, F_{\beta\alpha} : \mathbb{S}_\alpha \to \mathbb{S}_\beta \,|\, \alpha \leq \beta \leq \kappa\}$ is a direct system. □

As a consequence, we obtain the following result, which plays a crucial role in proving the main result.

Lemma 3.2. *Let $\mathscr{X}$ be a subclass of* Mod$R$*, and let κ be an ordinal number. Suppose that $\{M_\alpha, f_{\beta\alpha} : M_\alpha \to M_\beta \mid \alpha \leq \beta < \kappa\}$ is a direct system in* Mod$R$ *and*

$$\mathbb{S}_\alpha : = 0 \to M_\alpha \xrightarrow{\varphi_\alpha^0} X_\alpha^0 \to X_\alpha^1 \to \cdots \xrightarrow{\varphi_\alpha^n} X_\alpha^n$$

is an exact sequence in Mod$R$ *with all X_α^i in $\mathscr{X}$. If one of the conditions is satisfied:*

(1) $\mathbb{S}_\alpha$ *is an injective coresolution of* M_α,

(2) $\mathbb{S}_\alpha$ *is a strong coproper* n-$\mathscr{X}$*-coresolution of* M_α *and both* $\mathscr{X}$ *and* ${}^{\perp_1}\mathscr{X}$ *are closed under direct limits,*

then these exact sequences $\mathbb{S}_\alpha$ *are the members of a direct system indexed by* $\alpha < \kappa$ *in such a way that if* $\alpha \leq \beta < \kappa$*, the map from the sequence indexed by* α *into that indexed by* β *with the origin map* $f_{\beta\alpha} : M_\alpha \to M_\beta$*. In particular, we obtain an exact sequence*

$$\varinjlim_{\alpha<\kappa} \mathbb{S}_\alpha : \ 0 \to \varinjlim_{\alpha<\kappa} M_\alpha \to \varinjlim_{\alpha<\kappa} X^0_\alpha \to \varinjlim_{\alpha<\kappa} X^1_\alpha \to \cdots \to \varinjlim_{\alpha<\kappa} X^n_\alpha \to \varinjlim_{\alpha<\kappa} \mathrm{Coker}\varphi^n_\alpha \to 0.$$

In Case (2)*, the sequence* $\varinjlim_{\alpha<\kappa} \mathbb{S}_\alpha$ *is a partial strong coproper* n-$\mathscr{X}$*-coresolution of* $\varinjlim_{\alpha<\kappa} M_\alpha$*.*

Proof. We need construct a direct system $\mathbb{S} = \{\mathbb{S}_\alpha, F_{\beta\alpha} : \mathbb{S}_\alpha \to \mathbb{S}_\beta \,|\, \alpha \leq \beta < \kappa\}$ indexed by κ, where each $\mathbb{S}_\alpha$ is a coproper n-$\mathscr{X}$-coresolution of M_α and $F_{\beta\alpha}$ is a sequence of maps $(f_{\beta\alpha}, f^0_{\beta\alpha}, \cdots, f^n_{\beta\alpha})$ such that the following diagram

$$\begin{array}{ccccccccccc}
\mathbb{S}_\alpha : & & 0 & \longrightarrow & M_\alpha & \longrightarrow & X^0_\alpha & \longrightarrow & X^1_\alpha & \longrightarrow \cdots \longrightarrow & X^n_\alpha \\
\downarrow{\scriptstyle F_{\beta\alpha}} & & & & \downarrow{\scriptstyle f_{\beta\alpha}} & & \downarrow{\scriptstyle f^0_{\beta\alpha}} & & \downarrow{\scriptstyle f^1_{\beta\alpha}} & & \downarrow{\scriptstyle f^n_{\beta\alpha}} \\
\mathbb{S}_\beta : & & 0 & \longrightarrow & M_\beta & \longrightarrow & X^0_\beta & \longrightarrow & X^1_\beta & \longrightarrow \cdots \longrightarrow & X^n_\beta
\end{array}$$

commutes. In the following, we use transfinite induction on $\beta < \kappa$ to construct $F_{\beta\alpha} : \mathbb{S}_\alpha \to \mathbb{S}_\beta$ with $\alpha \leq \beta < \kappa$.

(i) For the successor case, assume that we have constructed $F_{\gamma\alpha}$ for any $\alpha \leq \gamma \leq \beta$. Since $\mathbb{S}_\beta$ is a strong coproper $\mathscr{X}$-n-coresolution of M_β, there exists $f^i_{\beta+1,\beta} : X^i_\beta \to X^i_{\beta+1}$ for any $0 \leq i \leq n$, such that the following diagram

$$\begin{array}{ccccccccccc}
\mathbb{S}_\beta : & & 0 & \longrightarrow & M_\beta & \longrightarrow & X^0_\beta & \longrightarrow & X^1_\beta & \longrightarrow \cdots \longrightarrow & X^n_\beta \\
\downarrow{\scriptstyle F_{\beta+1,\beta}} & & & & \downarrow{\scriptstyle f_{\beta+1,\beta}} & & \downarrow{\scriptstyle f^0_{\beta+1,\beta}} & & \downarrow{\scriptstyle f^1_{\beta+1,\beta}} & & \downarrow{\scriptstyle f^n_{\beta+1,\beta}} \\
\mathbb{S}_{\beta+1} : & & 0 & \longrightarrow & M_{\beta+1} & \longrightarrow & X^0_{\beta+1} & \longrightarrow & X^1_{\beta+1} & \longrightarrow \cdots \longrightarrow & X^n_{\beta+1}.
\end{array}$$

commutes. Let $F_{\beta+1,\beta} := (f_{\beta+1,\beta}, f^0_{\beta+1,\beta}, f^1_{\beta+1,\beta}, \cdots, f^n_{\beta+1,\beta})$ and $F_{\beta+1,\alpha} := F_{\beta+1,\beta}F_{\beta\alpha}$ for any ordinal $\alpha < \beta$. Then we complete the proof for the successor case.

(ii) For the limit case, let $\beta < \kappa$ be a limit ordinal. Assume that we have constructed $F_{\gamma\alpha}$ for any $\alpha \leq \gamma < \beta$. Now we need construct $F_{\beta\alpha}$ for any $\alpha < \beta$. Note that $\{\mathbb{S}_\alpha, F_{\gamma\alpha} : \mathbb{S}_\alpha \to \mathbb{S}_\gamma \,|\, \alpha \leq \gamma < \beta\}$ is a direct subsystem of $\mathbb{S}$. We need to find the chain map in Diagram (3.1).

For Case (1), since X^i_β is injective, it is clear.

For Case (2), we get an exact sequence $\varinjlim_{\alpha<\beta} \mathbb{S}_\alpha$ and a colimt map $G_{\beta\alpha} = (g_{\beta\alpha}, g^0_{\beta\alpha}, g^1_{\beta\alpha}, \cdots, g^n_{\beta\alpha})$. Set $K^1_\alpha := \mathrm{Coker}\varphi^0_\alpha$ for any $\alpha < \beta$. Then $\{K^1_\alpha\}_{\alpha<\beta}$ is also a direct system. Since $K_\alpha \in {}^{\perp_1}\mathscr{X}$ and ${}^{\perp_1}\mathscr{X}$ is closed under direct limits, we have $\varinjlim_{\alpha<\beta} K_\alpha \in {}^{\perp_1}\mathscr{X}$ and $\varinjlim_{\alpha<\beta} M_\alpha \rightarrowtail \varinjlim_{\alpha<\beta} X^0_\alpha$

is a special $\mathscr{X}$-preenvelope of $\varinjlim_{\alpha<\beta} M_\alpha$. Since $K_\alpha^1 \rightarrowtail X_\alpha^1$ is a special $\mathscr{X}$-preenvelope of K_α^1, we get that $\varinjlim_{\alpha<\beta} K_\alpha^1 \rightarrowtail \varinjlim_{\alpha<\beta} X_\alpha^1$ is a special $\mathscr{X}$-preenvelope of $\varinjlim_{\alpha<\beta} K_\alpha^1$ by using an argument similar to that as above. Continuing this procedure, we have that $\varinjlim_{\alpha<\beta} \mathbb{S}_\alpha$ is a strong coproper n- $\mathscr{X}$-coresolution. By using the universal property of $\varinjlim_{\alpha<\beta} M_\alpha$, there exists a unique h_β such that $h_\beta g_{\beta\alpha} = f_{\beta\alpha}$, and we get $H_\beta = (h_\beta, h_\beta^0, \cdots, h_\beta^n)$ induced by the origin map h_β. Then by Lemma 3.1 and transfinite induction, we get the desired system $\mathbb{S}$. $\square$

Definition 3.3. ([21]) Let β be an ordinal number. A set S is called a *continuous union* of a family of subsets indexed by ordinals α with $\alpha < \beta$ if for each such α we have a subset $S_\alpha \subset S$ such that if $\alpha \leq \alpha'$ then $S_\alpha \subset S_{\alpha'}$, and such that if $\gamma < \beta$ is a limit ordinal then $S_\gamma = \cup_{\alpha<\gamma} S_\alpha$.

The following lemma is [14, Lemma 2.14].

Lemma 3.4. *If $\mathscr{X}$ is a class of left R-modules closed under direct limits of well-ordered chains, then $\mathscr{X}$ is closed under direct limits.*

In Lemma 3.4, if $M = \varinjlim_{i\in I} M_i$ with I infinite, then I can be written as a continuous union $I = \cup_{\alpha<\beta} I_\alpha$ for some ordinal β, where each I_α is a directed index set with the order induced by that of I and where $|I_\alpha| < |I|$ for each $\alpha < \beta$. Set $N_\alpha := \varinjlim_{i\in I_\alpha} M_i$. Then $\varinjlim_{\alpha<\beta} N_\alpha = \varinjlim_{i\in I} M_i$, see [14, Lemma 2.14] and its proof.

Our main result is the following theorem.

Theorem 3.5. *Let $M \in \mathrm{Mod} R$ such that $M = \varinjlim_{i\in I} M_i$ with I a directed index set. Keep the notations as above.*

(1) *If*

$$\mathbb{S}_i : = 0 \to M_i \to E_i^0 \to E_i^1 \to \cdots \xrightarrow{f_i^n} E_i^n$$

is an injective coresolution of M_i for any $i \in I$, then we have an exact sequence

$$\varinjlim_{i\in I_\alpha} \mathbb{S}_i : = 0 \to N_\alpha \to \varinjlim_{i\in I_\alpha} E_i^0 \to \varinjlim_{i\in I_\alpha} E_i^1 \to \cdots \to \varinjlim_{i\in I_\alpha} E_i^n \to \varinjlim_{i\in I_\alpha} \mathrm{Coker} f_i^n \to 0. \tag{3.1}$$

Furthermore, if

$$\mathbb{S}'_\alpha : = 0 \to N_\alpha \to E_\alpha^0 \to E_\alpha^1 \to \cdots \xrightarrow{f_\alpha^n} E_\alpha^n$$

is an injective coresolution of N_α, then we have the following exact sequence

$$\mathbb{S} : = 0 \to M \to \varinjlim_{\alpha<\beta} E_\alpha^0 \to \varinjlim_{\alpha<\beta} E_\alpha^1 \to \cdots \to \varinjlim_{\alpha<\beta} E_\alpha^n \to \varinjlim_{\alpha<\beta} \mathrm{Coker} f_\alpha^n \to 0. \tag{3.2}$$

(2) *Let* $\mathscr{X}$ *be a subclass of* $\mathrm{Mod}R$ *such that both* $\mathscr{X}$ *and* ${}^{\perp_1}\mathscr{X}$ *are closed under direct limits. If*

$$\mathbb{S}_i := 0 \to M_i \to X_i^0 \to X_i^1 \to \cdots \xrightarrow{g_i^n} X_i^n$$

is a strong coproper n-$\mathscr{X}$*-coresolution of* M_i *for any* $i \in I$*, then* N_α *admits a partial strong coproper* n-$\mathscr{X}$*-coresolution*

$$\varinjlim_{i\in I_\alpha}\mathbb{S}_i := 0 \to N_\alpha \to \varinjlim_{i\in I_\alpha}X_i^0 \to \varinjlim_{i\in I_\alpha}X_i^1 \to \cdots \to \varinjlim_{i\in I_\alpha}X_i^n \to \varinjlim_{i\in I_\alpha}\mathrm{Coker}g_i^n \to 0; \qquad (3.3)$$

furthermore, M *admits a partial strong coproper* n-$\mathscr{X}$*-coresolution*

$$\mathbb{S} := 0 \to M \to \varinjlim_{\alpha<\beta}Y_\alpha^0 \to \varinjlim_{\alpha<\beta}Y_\alpha^1 \to \cdots \to \varinjlim_{\alpha<\beta}Y_\alpha^n \to \varinjlim_{\alpha<\beta}C_\alpha^n \to 0, \qquad (3.4)$$

where $Y_\alpha^j = \varinjlim_{i\in I_\alpha}X_i^j$ *and* $C_\alpha^n = \varinjlim_{i\in I_\alpha}\mathrm{Coker}g_i^n$ *for any* $0 \le j \le n$*.*

Proof. We prove it by transfinite induction on $|I|$. The case for $|I| < \infty$ is clear.

Suppose that $|I| = \aleph_0$ and $I = \{i_n \mid n \in \mathbb{N}\}$ with $\mathbb{N}$ the set of non-negative integers. We construct a sequence $j_0, j_1, \cdots$ of elements in I by letting $j_0 = i_0$, then we choose j_1 such that $j_1 \ge j_0, i_1$ by the upper directed set I. By induction, we choose $j_n \ge j_{n-1}, i_n$. Let $J = \{j_n \mid n \in \mathbb{N}\}$. Then J is cofinal well-ordered subset of I and

$$M = \varinjlim_{i\in I} M_i = \varinjlim_{j\in J} M_j.$$

The assertions follow from Lemma 3.2.

When $|I| > \aleph_0$, using Lemma 3.4, we may write $I = \cup_{\alpha<\beta}I_\alpha$ for some ordinal β and we have

$$M = \varinjlim_{i\in I} M_i = \varinjlim_{\alpha<\beta} N_\alpha,$$

where $N_\alpha = \varinjlim_{i\in I_\alpha} M_i$. Since $|I_\alpha| < |I|$ for each α, we get (3.1) and (3.3) by induction hypothesis.

For (1), there exists a chain map from $\varinjlim_{i\in I_\alpha} \mathbb{S}_i$ to $\mathbb{S}'_\alpha$ as follows:

$$\begin{array}{ccccccccccccc}
\mathbb{S}_i: & 0 \to & M_i & \to & E_i^0 & \to & E_i^1 & \to \cdots \to & E_i^n & \to & \mathrm{Coker}f_i^n & \to & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\varinjlim_{i\in I_\alpha}\mathbb{S}_i: & 0 \to & N_\alpha & \to & \varinjlim_{i\in I_\alpha}E_i^0 & \to & \varinjlim_{i\in I_\alpha}E_i^1 & \to \cdots \to & \varinjlim_{i\in I_\alpha}E_i^n & \to & \varinjlim_{i\in I_\alpha}\mathrm{Coker}f_i^n & \to & 0\\
\downarrow & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{S}'_\alpha: & 0 \to & N_\alpha & \to & E_\alpha^0 & \to & E_\alpha^1 & \to \cdots \to & E_\alpha^n & \to & \mathrm{Coker}f_\alpha^n & \to & 0.
\end{array}$$

For (2), note that each N_α admits a strong coproper n-$\mathscr{X}$-coresolution. Thus we get (3.2) and (3.4) from Lemma 3.2. □

Let R be a left Noetherian but not left Artinian ring with global dimension at least two (for example, the polynomial ring in n indeterminates over the ring of integers with $n \geq 1$). Then there exists a flat left R-module M which is not torsionless by [8, Theorem 4.1]. Note that $M = \varinjlim_{i \in I} M_i$ with all M_i finitely generated projective left R-modules by [28, Thmorem 5.40]. Set $\mathscr{X} := \mathcal{P}(\mathrm{Mod}R)$. It is easy to see that each M_i admits a strong coproper 0-$\mathscr{X}$-coresolution. But M is not torsionless, so M does not admit a (strong) coproper 0-$\mathscr{X}$-coresolution. On the other hand, we have that R is not left perfect by [1, Corollary 15.23 and Theorem 28.4], and hence $\mathscr{X}$ is not closed under direct limits. This means that the condition that the class $\mathscr{X}$ is closed under direct limits in Theorem 3.5(2) is necessary.

Proposition 3.6. *Let $\mathscr{X}$ be a subclass of* Mod$R$*, and let $M \in \mathrm{Mod}R$ such that $M = \varinjlim_{i \in I} M_i$ with all M_i in* mod$R$*. Set $\mathscr{X}' := \{$all finitely presented submodules of modules in $\mathscr{X}\}$. If both $\mathscr{X}$ and ${}^{\perp_1}\mathscr{X}$ are closed under direct limits and each M_i admits a strong coproper n-$\mathscr{X}'$-coresolution, then M admits a strong coproper n-$\mathscr{X}$-coresolution.*

Proof. Let

$$0 \to M_i \xrightarrow{\varphi_i^0} X_i^0 \xrightarrow{\varphi_i^1} X_i^1 \longrightarrow \cdots \xrightarrow{\varphi_i^n} X_i^n \tag{3.5}$$

be a strong coproper n-$\mathscr{X}'$-coresolution of M_i for any $i \in I$. Set $K_i^j := \mathrm{Coker}\varphi_i^j$ for any $0 \leq j \leq n$. Then all K_i^j are in ${}^{\perp_1}\mathscr{X}'$. Notice that all K_i^j are in mod$R$, they are in ${}^{\perp_1}\mathscr{X}$ by [14, Lemma 6.6]. Thus (3.5) is a strong coproper n-$\mathscr{X}$-coresolution of M_i for any $i \in I$. Now the assertion follows from Theorem 3.5(2). □

By [14, Lemma 2.5], we have that any module is a direct limit of its finitely presented submodules. Thus by Theorem 3.5(2) and Proposition 3.6, we obtain the following result.

Corollary 3.7. *Let $\mathscr{X}$ be a subclass of* Mod$R$ *such that both $\mathscr{X}$ and ${}^{\perp_1}\mathscr{X}$ are closed under direct limits. If one of the following conditions is satisfied, then $\mathscr{X}$ is special preenveloping in* Mod$R$*.*

(1) *Any module in* mod$R$ *admits a special $\mathscr{X}$-preenvelope.*
(2) *$\mathscr{X}'$ is special preenveloping in* mod$R$*, where $\mathscr{X}' = \{$all finitely presented submodules of modules in $\mathscr{X}\}$.*

In the following, we list the dual counterparts of Lemma 3.2, Theorem 3.5(2), Proposition 3.6 and Corollary 3.7. Since their proofs are completely dual to those of the previous corresponding results, we omit them.

Lemma 3.8. (Dual to Lemma 3.2) *Let $\mathcal{X}$ be a subclass of* Mod$R$*, and let κ be an ordinal number. Suppose that $\{M_\alpha, f_{\beta\alpha} : M_\alpha \to M_\beta \mid \alpha \leq \beta < \kappa\}$ is a direct system in* Mod$R$ *and*

$$\mathbb{S}_\alpha := X_\alpha^n \xrightarrow{\varphi_\alpha^n} \cdots \to X_\alpha^1 \to X_\alpha^0 \xrightarrow{\varphi_\alpha^0} M_\alpha \to 0$$

is an exact sequence in Mod$R$ *with all* X^i_α *in* $\mathcal{X}$. *If* $\mathbb{S}_\alpha$ *is a strong proper* n-$\mathcal{X}$*-resolution of* M_α *and both* $\mathcal{X}$ *and* $\mathcal{X}^{\perp_1}$ *are closed under direct limits, then these exact sequences* $\mathbb{S}_\alpha$ *are the members of a direct system indexed by* $\alpha < \kappa$ *in such a way that if* $\alpha \leq \beta < \kappa$, *the map from the sequence indexed by* α *into that indexed by* β *with the origin map* $f_{\beta\alpha} : M_\alpha \to M_\beta$. *In particular, we obtain a partial strong proper* n-$\mathcal{X}$*-resolution of* $\varinjlim_{\alpha<\kappa} M_\alpha$*:*

$$\varinjlim_{\alpha<\kappa} \mathbb{S}_\alpha :\ 0 \to \varinjlim_{\alpha<\kappa} \mathrm{Ker}\varphi^n_\alpha \to \varinjlim_{\alpha<\kappa} X^n_\alpha \to \cdots \to \varinjlim_{\alpha<\kappa} X^1_\alpha \to \varinjlim_{\alpha<\kappa} X^0_\alpha \to \varinjlim_{\alpha<\kappa} M_\alpha \to 0.$$

Theorem 3.9. (Dual to Theorem 3.5(2)) *Let* $M \in$ Mod$R$ *such that* $M = \varinjlim_{i\in I} M_i$ *with* I *a directed index set. Keep the notations as above. Let* $\mathcal{X}$ *be a subclass of* Mod$R$ *such that both* $\mathcal{X}$ *and* $\mathcal{X}^{\perp_1}$ *are closed under direct limits. If*

$$\mathbb{S}_i := X^n_i \xrightarrow{g^n_i} \cdots \to X^1_i \to X^0_i \to M_i \to 0$$

is a strong proper n-$\mathcal{X}$*-resolution of* M_i *for any* $i \in I$, *then* N_α *admits a partial strong proper* n-$\mathcal{X}$*-resolution*

$$\varinjlim_{i\in I_\alpha} \mathbb{S}_i := 0 \to \varinjlim_{i\in I_\alpha} \mathrm{Ker} g^n_i \to \varinjlim_{i\in I_\alpha} X^0_i \to \cdots \to \varinjlim_{i\in I_\alpha} X^1_i \to \varinjlim_{i\in I_\alpha} X^0_i \to N_\alpha \to 0; \qquad (5.1)$$

furthermore, M *admits a partial strong proper* n-$\mathcal{X}$*-resolution*

$$\mathbb{S} := 0 \to \varinjlim_{\alpha<\beta} C^n_\alpha \to \varinjlim_{\alpha<\beta} Y^0_\alpha \to \cdots \to \varinjlim_{\alpha<\beta} Y^1_\alpha \to \varinjlim_{\alpha<\beta} Y^n_\alpha \to M \to 0, \qquad (5.2)$$

where $Y^j_\alpha = \varinjlim_{i\in I_\alpha} X^j_i$ *and* $C^n_\alpha = \varinjlim_{i\in I_\alpha} \mathrm{Ker} g^n_i$ *for any* $0 \leq j \leq n$.

Note that all finitely presented modules are pure injective over Artin algebras.

Proposition 3.10. (Dual to Proposition 3.6) *Let* R *be an Artin algebra and* $\mathcal{X}$ *be a subclass of* Mod$R$ *which is closed under direct limits, and let* $M \in$ Mod$R$ *such that* $M = \varinjlim_{i\in I} M_i$ *with all* M_i *in* mod$R$. *Set* $\mathcal{X}' := \{$*all finitely presented submodules of modules in* $\mathcal{X}\}$. *If* $\mathcal{X}^{\perp_1}$ *is closed under direct limits and each* M_i *admits a strong proper* n-$\mathcal{X}'$*-resolution, then* M *admits a strong proper* n-$\mathcal{X}$*-resolution.*

Corollary 3.11. (Dual to Corollary 3.7) *Let* R *be an Artin algebra,* $\mathcal{X}$ *be a subclass of* Mod$R$ *such that both* $\mathcal{X}$ *and* $\mathcal{X}^{\perp_1}$ *are closed under direct limits. If one of the following conditions is satisfied, then* $\mathcal{X}$ *is special precovering in* Mod$R$.

(1) *Any module in* mod$R$ *admits a special* $\mathcal{X}$*-precover.*
(2) $\mathcal{X}'$ *is special precovering in* mod$R$, *where* $\mathcal{X}' = \{$*all finitely presented submodules of modules in* $\mathcal{X}\}$.

In the final of this section, we raise the following question.

Question 3.12. Is there a dual counterpart of Theorem 3.5(1)?

4. **Applications**

In this section, we give some applications of the results obtained in Section 3.

4.1. **Relative injective dimension.** As an application of Theorem 3.5(1), we get the following result.

Theorem 4.1. *Let* $\mathscr{X}$ *be a subclass of* Mod$R$ *which is coresolving and closed under direct limits. Then*

$$\sup\{\mathscr{X}\text{-id}M \mid M \in \mathrm{Mod}R\} = \sup\{\mathscr{X}\text{-id}M \mid M \in \mathrm{mod}R\}.$$

Proof. It is trivial that $\sup\{\mathscr{X}\text{-id}M \mid M \in \mathrm{Mod}R\} \geq \sup\{\mathscr{X}\text{-id}M \mid M \in \mathrm{mod}R\}$.

Now suppose $\sup\{\mathscr{X}\text{-id}M \mid M \in \mathrm{mod}R\} = n < \infty$. Let $M \in \mathrm{Mod}R$. Then $M = \varinjlim_{i\in I} M_i$ with all M_i finitely presented submodules of M by [14, Lemma 2.5]. For any $i \in I$, we have $\mathscr{X}\text{-id}M_i \leq n$. Since $\mathscr{X}$ is coresolving, there exists an exact sequence

$$0 \to M_i \to E_i^0 \to E_i^1 \to \cdots \to E_i^n \to X_i^{n+1} \to 0$$

in Mod$R$ with all E_i^j injective and X_i^{n+1} in $\mathscr{X}$ by the dual version of [35, Lemma 2.1] (cf. the dual version of [4, Lemma 3.12]). Keep the notations N_α and I_α as in Theorem 3.5(1). Then we get an exact sequence

$$0 \to N_\alpha \to \varinjlim_{i\in I_\alpha} E_i^0 \to \varinjlim_{i\in I_\alpha} E_i^1 \to \cdots \to \varinjlim_{i\in I_\alpha} E_i^n \to \varinjlim_{i\in I_\alpha} X_i^{n+1} \to 0.$$

Since $\mathscr{X}$ is closed under direct limits, we have that all $\varinjlim_{i\in I_\alpha} E_i^j$ and $\varinjlim_{i\in I_\alpha} X_i^{n+1}$ are in $\mathscr{X}$, and thus $\mathscr{X}\text{-id}N_\alpha \leq n$. As above, there exists an exact sequence

$$0 \to N_\alpha \to E_\alpha^0 \to E_\alpha^1 \to \cdots \to E_\alpha^n \to X_\alpha^{n+1} \to 0$$

in Mod$R$ with all E_α^j injective and X_α^{n+1} in $\mathscr{X}$, which induces an exact sequence

$$0 \to M \to \varinjlim_{\alpha<\beta} E_\alpha^0 \to \varinjlim_{\alpha<\beta} E_\alpha^1 \to \cdots \to \varinjlim_{\alpha<\beta} E_\alpha^n \to \varinjlim_{\alpha<\beta} X_\alpha^{n+1} \to 0$$

with all $\varinjlim_{\alpha<\beta} E_\alpha^j$ and $\varinjlim_{i\in I_\alpha} X_\alpha^{n+1}$ are in $\mathscr{X}$. Thus $\mathscr{X}\text{-id}M \leq n$, and the assertion follows. □

Recall that a module $M \in \mathrm{Mod}R$ is called *weak injective* [12], or *absolutely clean* [6], if $\mathrm{Ext}_R^1(A, M) = 0$ for any left R-module A admitting a degreewise finite R-projective resolution. We use $\mathcal{WI}(\mathrm{Mod}R)$ to denote the class of weak injective left R-modules. Recall from [32] that a module $M \in \mathrm{Mod}R$ is called *FP-injective* (or *absolutely pure*) if $\mathrm{Ext}_R^1(A, M) = 0$ for any finitely presented left R-module A. If R is a left Noetherian ring, then the class $\mathcal{I}(\mathrm{Mod}R)$ of injective left R-modules coincides with $\mathcal{WI}(\mathrm{Mod}R)$, and if R is a left coherent ring, then the class $\mathcal{FI}(\mathrm{Mod}R)$ of FP-injective left R-modules coincides with $\mathcal{WI}(\mathrm{Mod}R)$.

Recall that a ring R is called *left* Π*-coherent* if any finitely generated torsionless left R-module is finitely presented, and a module $M \in \text{Mod}R$ is called *FGT-injective* if $\text{Ext}_R^1(A, M) = 0$ for any finitely generated torsionless left R-module A [7, 9]. We use $\mathcal{FTI}(\text{Mod}R)$ to denote the class of FGT-injective left R-modules.

Let R and S be rings. An (R, S)-bimodule ${}_RC_S$ is called *semidualizing* if the following conditions are satisfied: (1) ${}_RC$ admits a degreewise finite R-projective resolution and C_S admits a degreewise finite S^{op}-projective resolution, (2) the homothety maps ${}_RR_R \overset{R\gamma}{\to} \text{Hom}_{S^{op}}(C, C)$ and ${}_SS_S \overset{\gamma_S}{\to} \text{Hom}_R(C, C)$ are isomorphisms, and (3) $\text{Ext}_R^{\geq 1}(C, C) = 0 = \text{Ext}_{S^{op}}^{\geq 1}(C, C)$. The *Bass class* $\mathcal{B}_C(R)$ with respect to C consists of all left R-modules M satisfying the following conditions: (1) $\text{Ext}_R^{\geq 1}(C, M) = 0$, (2) $\text{Tor}_{\geq 1}^S(C, \text{Hom}_R(C, M)) = 0$, and (3) the canonical evaluation homomorphism $\theta_M : C \otimes_S \text{Hom}_R(C, M) \to M$ defined by $\theta_M(x \otimes f) = f(x)$ for any $x \in C$ and $f \in \text{Hom}_R(C, M)$ is an isomorphism of left R-modules [16].

We collect some known facts that we need to use.

Fact 4.2. It holds that

(1) By [6, Lemma 2.7(3)(4)], the class $\mathcal{WI}(\text{Mod}R)$ is coresolving and closed under direct limits.
(2) If R is a left Π-coherent ring, then the class $\mathcal{FTI}(\text{Mod}R)$ is coresolving and closed under direct limits by [9, Propositions 1.4 and 2.2].
(3) The class $\mathcal{GI}(\text{Mod}R)$ of Gorenstein injective left R-modules is coresolving by [15, Theorem 2.6]. If R is a left Artinian ring such that the injective envelope of every simple left R-module is finitely generated (in particular, if R is an Artin algebra), then $\mathcal{GI}(\text{Mod}R)$ is closed under direct limits by [20, Theorem 2] and [24, Theorem 2.3].
(4) Recall that a module $T \in \text{Mod}R$ is called *tilting* if the following conditions are satisfied: (i) the projective dimension of T is finite; (ii) $\text{Ext}_R^{\geq 1}(T, T^{(I)}) = 0$ for any set I; and (iii) there exists an exact sequence

$$0 \to R \to T^0 \to T^1 \to \cdots \to T^n \to 0$$

in $\text{Mod}R$ with all T^i direct summands of direct sums of copies of T. Let $T \in \text{Mod}R$ be tilting. Then $T^{\perp}$ is clearly coresolving, and it is closed under direct limits by [14, Corollary 13.42].
(5) Let $\mathcal{PP}(R)$ be the class of pure projective left R-modules, then $\mathcal{PP}(R)^{\perp}$ is coresolving by [33, Proposition 39]. If R is left coherent, then $\mathcal{PP}(R)^{\perp}$ is closed under direct limits [33, Theorem 47].
(6) The Bass class $\mathcal{B}_C(R)$ with respect to a semidualizing bimodule ${}_RC_S$ is coresolving and closed under direct limits [16, Theorem 6.2 and Proposition 4.2(a)].

Following the usual customary notation, we write

$$\text{w-id}_R M := \mathcal{WI}(\text{Mod}R)\text{-id}M, \quad \text{id}_R M := \mathcal{I}(\text{Mod}R)\text{-id}M,$$

$$\text{FP-id}_R M := \mathcal{FI}(\text{Mod}R)\text{-id}M, \quad \text{FGT-id}_R M := \mathcal{FTI}(\text{Mod}R)\text{-id}M,$$

$$\text{G-id}_R M := \mathcal{GI}(\text{Mod}R)\text{-id}M.$$

Let $M \in \text{Mod}R$. If R is a left Noetherian ring, then $\text{w-id}_R M = \text{id}_R M$. If R is a left coherent ring, then $\text{w-id}_R M = \text{FP-id}_R M$.

By Theorem 4.1 and Fact 4.2, we obtain the following result.

Corollary 4.3. *It holds that*

(1) $\sup\{\text{w-id}_R M \mid M \in \text{Mod}R\} = \sup\{\text{w-id}_R M \mid M \in \text{mod}R\}$. *In particular, we have*

  (a) ([26, Theorem C]) *If R is a left Noetherian ring, then*

$$\sup\{\text{id}_R M \mid M \in \text{Mod}R\} = \sup\{\text{id}_R M \mid M \in \text{mod}R\}.$$

  (b) ([32, Theorem 3.3]) *If R is a left coherent ring, then*

$$\sup\{\text{FP-id}_R M \mid M \in \text{Mod}R\} = \sup\{\text{FP-id}_R M \mid M \in \text{mod}R\}.$$

(2) *If R is a left Π-coherent ring, then*

$$\sup\{\text{FGT-id}_R M \mid M \in \text{Mod}R\} = \sup\{\text{FGT-id}_R M \mid M \in \text{mod}R\}.$$

(3) *If R is a left Artinian ring such that the injective envelope of every simple left R-module is finitely generated* (*in particular, if R is an Artin algebra*), *then*

$$\sup\{\text{G-id}_R M \mid M \in \text{Mod}R\} = \sup\{\text{G-id}_R M \mid M \in \text{mod}R\}.$$

(4) *If $T \in \text{Mod}R$ is a tilting module, then*

$$\sup\{T^{\perp}\text{-id}M \mid M \in \text{Mod}R\} = \sup\{T^{\perp}\text{-id}M \mid M \in \text{mod}R\}.$$

(5) *If R is a left coherent ring, then*

$$\sup\{\mathcal{PP}(R)^{\perp}\text{-id}M \mid M \in \text{Mod}R\} = \sup\{\mathcal{PP}(R)^{\perp}\text{-id}M \mid M \in \text{mod}R\}.$$

(6) *We have*

$$\sup\{\mathcal{B}_C(R)\text{-id}M \mid M \in \text{Mod}R\} = \sup\{\mathcal{B}_C(R)\text{-id}M \mid M \in \text{mod}R\}.$$

Recall from [25] that a module $M \in \text{Mod}R$ is called *strong Gorenstein injective*, which is usually called *Ding injective* [13, 20], if $M \in \mathcal{FI}(\text{Mod}R)^{\perp}$ and there exists a $\text{Hom}_R(\mathcal{FI}(\text{Mod}R), -)$-exact exact sequence

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^i \to \cdots$$

in $\text{Mod}R$ with all E^i in $\mathcal{I}(\text{Mod}R)$. Recall from [11] that a module $N \in \text{Mod}R^{op}$ is called *Gorenstein flat* if there exists an exact sequence

$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots \to F^i \to \cdots$$

in $\mathrm{Mod}R^{op}$ with all F^i flat, such that it remains exact after applying the functor $-\otimes_R E$ for any $E \in \mathcal{I}(\mathrm{Mod}R)$, and $N \cong \mathrm{Im}(F_0 \to F^0)$. For a module $M \in \mathrm{Mod}R$, we call $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ its *character module*, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers.

Note that the class $\mathcal{SGI}(\mathrm{Mod}R)$ of strong Gorenstein injective left R-modules is coresolving by [17, Remark 4.4(4)(b)]. About its direct limit closure, we have the following result, which extends [20, Theorem 2].

Proposition 4.4. *The following statements are equivalent.*

(1) $\mathcal{SGI}(\mathrm{Mod}R)$ *is closed under direct limits.*
(2) $\mathcal{GI}(\mathrm{Mod}R)$ *is closed under direct limits.*
(3) R *is a left Noetherian ring and the character module of any Gorenstein injective left R-module is Gorenstein flat.*

Proof. The equivalence (2) $\Longleftrightarrow$ (3) has been proved in [20, Theorem 2]. It is well known that $\mathcal{FI}(\mathrm{Mod}R) = \mathcal{I}(\mathrm{Mod}R)$ if R is a left Noetherian ring, thus we have (2)+(3) $\Longrightarrow$ (1).

Now suppose that the assertion (1) holds true, to prove that (2) also holds true, it suffices to prove that R is a left Noetherian ring by the above argument. Since $\mathcal{SGI}(\mathrm{Mod}R)$ is closed under direct limits by (1), we have that $\mathcal{SGI}(\mathrm{Mod}R)$ is closed under direct products and pure submodules by [13, Theorem 44] and [30, Theorem 3.5]. Notice that the direct sum of modules is a pure submodule of the direct product of the modules, so $\mathcal{SGI}(\mathrm{Mod}R)$ is closed under direct sums.

Let $\{E_i \mid i \in I\}$ be a family of injective left R-modules. Then $\oplus_{i\in I}E_i \in \mathcal{SGI}(\mathrm{Mod}R)$, and thus there exists a $\mathrm{Hom}_R(\mathcal{FI}(\mathrm{Mod}R), -)$-exact exact sequence

$$E \xrightarrow{\varphi} \oplus_{i\in I}E_i \to 0$$

in $\mathrm{Mod}R$ with $E \in \mathcal{I}(\mathrm{Mod}R)$. For each standard embedding $\lambda_i : E_i \rightarrowtail \bigoplus_{i\in I} E_i$, there exists $f_i \in \mathrm{Hom}_R(E_i, E)$ such that $\varphi f_i = \lambda_i$. By the universal property of direct sums, there exists $\varphi' \in \mathrm{Hom}_R(\oplus_{i\in I}E_i, E)$ such that $\varphi'\lambda_i = f_i$, and thus

$$(\varphi\varphi')\lambda_i = \varphi f_i = \lambda_i.$$

It yields that $\varphi\varphi'$ is the identity homomorphism of $\oplus_{i\in I}E_i$ and φ is a split epimorphism. So $\oplus_{i\in I}E_i$ is a direct summand of E, and hence it is injective. It follows from [5, Theorem 1.1] that R is a left Noetherian ring. $\square$

For a module $M \in \mathrm{Mod}R$, we use $\mathrm{fd}_R M$ to denote the flat dimension of M. The assertion (1) in the following result generalizes [18, Theorem 3.1].

Proposition 4.5. *Let $M \in \mathrm{Mod}R$ such that $M = \varinjlim_{i\in I} M_i$ with I a directed index set, and let $\mathscr{X}$ be a subclass of* $\mathrm{Mod}R$ *such that both $\mathscr{X}$ and ${}^{\perp_1}\mathscr{X}$ are closed under direct limits. Assume that*

$$0 \to M_i \to X_i^0 \to X_i^1 \to \cdots \to X_i^n$$

is a strong coproper n-$\mathscr{X}$-coresolution of M_i for any $i \in I$. Then M admits a strong coproper n-$\mathscr{X}$-coresolution

$$0 \to M \to X^0 \to X^1 \to \cdots \to X^n$$

such that for any $j \geq 0$, it holds that

(1) $\mathrm{fd}_R X^j = \sup\{\mathrm{fd}_R X_i^j \mid i \in I\}$.

(2) $\text{w-id}_R X^j = \sup\{\text{w-id}_R X_i^j \mid i \in I\}$.

In particular, if $\mathscr{X}$ is enveloping, then a minimal coproper n-$\mathscr{X}$-coresolution of M as (3.3) *also satisfies* (1) *and* (2).

Proof. (1) Since the functor Tor commutes with direct limits, the assertion follows from Theorem 3.5.

(2) For any $X \in \mathrm{Mod}R$ and $n \geq 0$, it is easy to see that $\text{w-id}_R X = n$ if and only if $n = \inf\{i \mid \mathrm{Ext}_R^{\geq i+1}(A, X) = 0$ for any left R-module A admitting a degreewise finite R-projective resolution$\}$. Now the assertion follows from Theorem 3.5(2) and [14, Lemma 6.6].

According to (1) and (2), the last assertion follows from Lemma 2.3. □

4.2. **Weakly Gorenstein algebras.** As an application of Proposition 3.6, we have the following result.

Proposition 4.6. *If R is a right coherent and left perfect ring, then the following statements are equivalent.*

(1) $\mathcal{GP}(\mathrm{mod}R) = {}^{\perp}{}_R R \cap \mathrm{mod}R$.

(2) $\mathcal{GP}(\mathrm{Mod}R) = {}^{\perp}{}_R R$.

Proof. The implication (2) $\Longrightarrow$ (1) is clear.

(1) $\Longrightarrow$ (2) Let $M \in {}^{\perp}{}_R R$. Then $M = \varinjlim_{i \in I} M_i$ with all M_i finitely presented submodules of M by [14, Lemma 2.5]. Since R is a right coherent and left perfect ring, then any projective left R-module has a decomposition as a direct sum of indecomposable projective submodules by [1, Theorem 27.11]. It follows from [29, Theorem 5] and [22, Corollary 2.7] that any projective left R-module is pure injective. Then ${}^{\perp_1}\mathcal{P}(\mathrm{Mod}R)$ is closed under direct limits and

$$\varprojlim_{i \in I} \mathrm{Ext}_R^j(M_i, R) \cong \mathrm{Ext}_R^j(\varinjlim_{i \in I} M_i, R) = \mathrm{Ext}_R^j(M, R) = 0$$

for any $j \geq 1$ by [3, Proposition I.10.1]. Then for any $i \in I$ and $j \geq 1$, we have $\mathrm{Ext}_R^j(M_i, R) = 0$, that is, $M_i \in {}^{\perp}{}_R R \cap \mathrm{mod}R$, and hence $M_i \in \mathcal{GP}(\mathrm{mod}R)$ by (1). Thus each M_i admits a strong coproper ∞-$\mathcal{P}(\mathrm{mod}R)$-coresolution. Since R is left perfect, a left R-module is flat if and only if it is projective by [1, Theorem 28.4], and so $\mathcal{P}(\mathrm{Mod}R)$ is closed under direct limits by [23, Proposition 4.4]. It follows from Proposition 3.6 that M admits a strong coproper ∞-$\mathcal{P}(\mathrm{Mod}R)$-coresolution and $M \in \mathcal{GP}(\mathrm{Mod}R)$. □

For an Artin algebra R, we use $\mathbb{D}$ to denote the usual duality between mod$R$ and modR^{op}. We need the following easy observation.

Lemma 4.7. *Let R be an Artin algebra, and let $M \in \text{mod}R$ and $M' \in \text{Mod}R$. Then for any $i \geq 1$, it holds that*

$$\text{Ext}^i_{R^{op}}(\mathbb{D}(M'), \mathbb{D}(M)) \cong \mathbb{D}^2\text{Ext}^i_R(M, M'),$$

in particular, if $M' \in \text{mod}R$, then

$$\text{Ext}^i_{R^{op}}(\mathbb{D}(M'), \mathbb{D}(M)) \cong \text{Ext}^i_R(M, M').$$

Proof. For any $M \in \text{mod}R$, $M' \in \text{Mod}R$ and $i \geq 1$, we have

$$\begin{aligned} & \text{Ext}^i_{R^{op}}(\mathbb{D}(M'), \mathbb{D}(M)) \\ \cong & \ \mathbb{D}\text{Tor}^R_i(\mathbb{D}(M'), M) \ \ \text{(by [14, Lemma 2.16(b)])} \\ \cong & \ \mathbb{D}^2\text{Ext}^i_R(M, M'). \ \ \text{(by [14, Lemma 2.16(d)])} \end{aligned}$$

If $M, M' \in \text{mod}R$, then $\mathbb{D}^2\text{Ext}^i_R(M, M') \cong \text{Ext}^i_R(M, M')$, and thus the latter assertion follows. □

Recall from [27] that an Artin algebra R is called *left weakly Gorenstein* if $\mathcal{GP}(\text{mod}R) = {}^{\perp}{}_RR \cap \text{mod}R$. A Gorenstein algebra R (that is, $\text{id}_RR = \text{id}_{R^{op}}R < \infty$) is left weakly Gorenstein, but the converse does not holds true in general [19, 27]. In the following result, we give some equivalent characterizations of weakly Gorenstein algebras, which generalizes part of [19, Theorem 4.9] (that is, the equivalence (4) $\Longleftrightarrow$ (5) there).

Theorem 4.8. *For an Artin algebra R, the following statements are equivalent.*

(1) *R is left weakly Gorenstein, that is, $\mathcal{GP}(\text{mod}R) = {}^{\perp}{}_RR \cap \text{mod}R$.*
(2) *$\mathcal{GP}(\text{Mod}R) = {}^{\perp}{}_RR$.*
(3) *$\mathcal{GP}(\text{Mod}R) = {}^{\perp}\mathcal{P}(\text{Mod}R)$.*
(4) *$\mathcal{GI}(\text{mod}R^{op}) = \mathbb{D}({}_RR)^{\perp} \cap \text{mod}R^{op}$.*
(5) *$\mathcal{GI}(\text{Mod}R^{op}) = \mathbb{D}({}_RR)^{\perp}$.*
(6) *$\mathcal{GI}(\text{Mod}R^{op}) = \mathcal{I}(\text{Mod}R^{op})^{\perp}$.*

Proof. The equivalence (1) $\Longleftrightarrow$ (2) follows from Proposition 4.6, and the implication (5) $\Longrightarrow$ (4) is clear. Since

$$\mathcal{GP}(\text{Mod}R) \subseteq {}^{\perp}\mathcal{P}(\text{Mod}R) \subseteq {}^{\perp}{}_RR \ \text{ and } \ \mathcal{GI}(\text{Mod}R^{op}) \subseteq \mathcal{I}(\text{Mod}R^{op})^{\perp} \subseteq \mathbb{D}({}_RR)^{\perp},$$

we have (2) $\Longrightarrow$ (3) and (5) $\Longrightarrow$ (6).

(3) $\Longrightarrow$ (1) Let $M \in {}^{\perp}{}_RR \cap \text{mod}R$. Then $M \in {}^{\perp}\mathcal{P}(\text{Mod}R)$ by [32, Theorem 3.2], and hence $M \in \mathcal{GP}(\text{Mod}R) \cap \text{mod}R = \mathcal{GP}(\text{mod}R)$ by (3).

(4) $\Longrightarrow$ (1) Let $M \in {}^{\perp}{}_RR \cap \text{mod}R$. Then we have

$$\text{Ext}^i_{R^{op}}(\mathbb{D}({}_RR), \mathbb{D}(M)) \cong \text{Ext}^i_R(M, {}_RR) = 0$$

for any $i \geq 1$ by Lemma 4.7, so $\mathbb{D}(M) \in \mathbb{D}({}_RR)^{\perp} \cap \operatorname{mod} R^{op} = \mathcal{GI}(\operatorname{mod} R^{op})$ by (4). Thus $M \in \mathcal{GP}(\operatorname{mod} R)$ by [15, Theorem 3.6] and [34, Corollary 3.7].

(1) $\Longrightarrow$ (4) Let $N \in \mathbb{D}({}_RR)^{\perp} \cap \operatorname{mod} R^{op}$. Then we have

$$\operatorname{Ext}^i_R(\mathbb{D}(N), {}_RR) \cong \operatorname{Ext}^i_R(\mathbb{D}(N), \mathbb{D}^2({}_RR)) \cong \operatorname{Ext}^i_{R^{op}}(\mathbb{D}({}_RR), N) = 0$$

for any $i \geq 1$ by Lemma 4.7, so $\mathbb{D}(N) \in {}^{\perp}{}_RR \cap \operatorname{mod} R = \mathcal{GP}(\operatorname{mod} R)$ by (1). It follows from [15, Theorem 3.6] and [34, Corollary 3.7] that $N \cong \mathbb{D}^2(N) \in \mathcal{GI}(\operatorname{mod} R^{op})$.

(4) $\Longrightarrow$ (5) Let $N \in \mathbb{D}({}_RR)^{\perp}$. Then $N = \varinjlim_{i \in I} N_i$ with all N_i finitely presented submodules of N by [14, Lemma 2.5]. Since

$$\varinjlim_{i \in I} \operatorname{Ext}^j_{R^{op}}(\mathbb{D}({}_RR), N_i) \cong \operatorname{Ext}^j_{R^{op}}(\mathbb{D}({}_RR), \varinjlim_{i \in I} N_i) = \operatorname{Ext}^j_{R^{op}}(\mathbb{D}({}_RR), N) = 0$$

for any $j \geq 1$ by [14, Lemma 6.6], we have $\operatorname{Ext}^{\geq 1}_{R^{op}}(\mathbb{D}({}_RR), N_i) = 0$, and hence $N_i \in \mathbb{D}({}_RR)^{\perp} \cap \operatorname{mod} R^{op} = \mathcal{GI}(\operatorname{mod} R^{op})$ for any $i \in I$ by (4). It follows from Fact 4.2(3) that $N \in \mathcal{GI}(\operatorname{Mod} R^{op})$.

(6) $\Longrightarrow$ (4) Let $N \in \mathbb{D}({}_RR)^{\perp} \cap \operatorname{mod} R^{op}$. Then for any $i \geq 1$, we have

$$\operatorname{Ext}^i_R(\mathbb{D}(N), {}_RR) \cong \operatorname{Ext}^i_{R^{op}}(\mathbb{D}({}_RR), \mathbb{D}^2(N)) \cong \operatorname{Ext}^i_{R^{op}}(\mathbb{D}({}_RR), N) = 0$$

by Lemma 4.7. It follows from [32, Theorem 3.2] that $\operatorname{Ext}^i_R(\mathbb{D}(N), \mathcal{P}(\operatorname{Mod} R)) = 0$. Then for any set J, we have

$$\begin{aligned}
&\operatorname{Ext}^i_{R^{op}}(\mathbb{D}({}_RR)^J, N) \cong \operatorname{Ext}^i_{R^{op}}(\mathbb{D}({}_RR^{(J)}), N) \cong \operatorname{Ext}^i_{R^{op}}(\mathbb{D}({}_RR^{(J)}), \mathbb{D}^2(N)) \\
&\cong \mathbb{D}^2\operatorname{Ext}^i_R(\mathbb{D}(N), {}_RR^{(J)}) \ \ \text{(by Lemma 4.7)} \\
&= 0.
\end{aligned}$$

Since any modules in $\mathcal{I}(\operatorname{Mod} R^{op})$ is a direct summand of $\mathbb{D}({}_RR)^J$ for some set J, we have that $N \in \mathcal{I}(\operatorname{Mod} R^{op})^{\perp}$, and hence $N \in \mathcal{GI}(\operatorname{Mod} R^{op}) \cap \operatorname{mod} R = \mathcal{GI}(\operatorname{mod} R^{op})$ by (6). $\square$

As a consequence, we obtain the following result.

Corollary 4.9. *If R is an Artin algebra with $\operatorname{id}_R R < \infty$, then the following statements are equivalent.*

(1) *R is Gorenstein.*
(2) *R is left weakly Gorenstein.*
(3) *$\mathcal{GP}(\operatorname{Mod} R) = {}^{\perp}{}_RR$.*
(4) *$\mathcal{GI}(\operatorname{Mod} R^{op}) = \mathbb{D}({}_RR)^{\perp}$.*

Proof. The assertion (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) follows from Theorem 4.8, and the assertion (1) $\Longleftrightarrow$ (3) follows from [2, Proposition 3.10]. $\square$

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