Duality Pairs Induced by Auslander and Bass Classes[*] [†]

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Abstract

Let R and S be any rings and ${}_RC_S$ a semidualizing bimodule, and let $\mathcal{A}_C(R^{op})$ and $\mathcal{B}_C(R)$ be the Auslander and Bass classes respectively. Then both the pairs

$$(\mathcal{A}_C(R^{op}), \mathcal{B}_C(R)) \text{ and } (\mathcal{B}_C(R), \mathcal{A}_C(R^{op}))$$

are coproduct-closed and product-closed duality pairs and both $\mathcal{A}_C(R^{op})$ and $\mathcal{B}_C(R)$ are covering and preenveloping; in particular, the former duality pair is perfect. Moreover, if $\mathcal{B}_C(R)$ is enveloping in $\operatorname{Mod} R$, then $\mathcal{A}_C(S)$ is enveloping in $\operatorname{Mod} S$. Then some applications to the Auslander projective dimension of modules are given.

1 Introduction

In relative homological algebra, the theory of covers and envelopes is fundamental and important. Let R be a ring and $\operatorname{Mod} R$ the category of left R-modules. Given a subcategory of $\operatorname{Mod} R$, it is always worth studying whether or when it is (pre)covering or (pre)enveloping. This problem has been studied extensively, see [2]–[9] and references therein.

Let R be a commutative noetherian ring and C a semidualizing R-module, and let $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ be the Auslander and Bass classes respectively. By proving that both $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ are Kaplansky classes, Enochs and Holm got in [5, Theorems 3.11 and 3.12] that the pair $(\mathcal{A}_C(R), (\mathcal{A}_C(R))^{\perp})$ is a perfect cotorsion pair, $\mathcal{A}_C(R)$ is covering and preenveloping and $\mathcal{B}_C(R)$ is preenveloping. Holm and Jørgensen introduced the notion of duality pairs and proved the following remarkable result. Let R be an arbitrary ring, and let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of $\operatorname{Mod} R$ and $\operatorname{Mod} R^{op}$ respectively. When $(\mathscr{X}, \mathscr{Y})$ is a duality pair, the following assertions hold true: (1) If $\mathscr{X}$ is closed under coproducts, then $\mathscr{X}$ is covering; (2) if $\mathscr{X}$ is closed under products, then $\mathscr{X}$ is preenveloping; and (3) if ${}_RR \in \mathscr{X}$ and $\mathscr{X}$ is closed under coproducts and extensions, then $(\mathscr{X}, \mathscr{X}^{\perp})$ is a perfect cotorsion pair ([9, Theorem 3.1]). By using it, they generalized the above result of Enochs and Holm to the category of complexes, and Enochs and Iacob investigated in [6] the existence of Gorenstein injective envelopes over commutative noetherian rings.

Let R and S be arbitrary rings and ${}_RC_S$ a semidualizing bimodule, and let $\mathcal{A}_C(R^{op})$ be the Auslander class in $\operatorname{Mod} R^{op}$ and $\mathcal{B}_C(R)$ the Bass class in $\operatorname{Mod} R$. Our first main result is the following

Theorem 1.1. (Theorem 3.3)

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(1) Both the pairs

$$(\mathcal{A}_C(R^{op}), \mathcal{B}_C(R)) \text{ and } (\mathcal{B}_C(R), \mathcal{A}_C(R^{op}))$$

are coproduct-closed and product-closed duality pairs; and furthermore, the former one is perfect.

(2) $\mathcal{A}_C(R^{op})$ *is covering and preenveloping in* $\operatorname{Mod} R^{op}$ *and* $\mathcal{B}_C(R)$ *is covering and preenveloping in* $\operatorname{Mod} R$.

As a consequence of Theorem 1.1, we get that the pair

$$(\mathcal{A}_C(R^{op}), \mathcal{A}_C(R^{op})^{\perp})$$

is a hereditary perfect cotorsion pair and $\mathcal{A}_C(R^{op})$ is covering and preenveloping in $\operatorname{Mod} R^{op}$, where $\mathcal{A}_C(R^{op})^{\perp}$ is the right Ext-orthogonal class of $\mathcal{A}_C(R^{op})$ (Corollary 3.4). This result was proved in [5, Theorem 3.11] when R is a commutative noetherian ring and ${}_RC_S = {}_RC_R$.

By Theorem 1.1 and its symmetric result, we have that $\mathcal{B}_C(R)$ is preenveloping in $\operatorname{Mod} R$ and $\mathcal{A}_C(S)$ is preenveloping in $\operatorname{Mod} S$. Moreover, we prove the following

Theorem 1.2. (Theorem 3.7(2)) *If* $\mathcal{B}_C(R)$ *is enveloping in* $\operatorname{Mod} R$*, then* $\mathcal{A}_C(S)$ *is enveloping in* $\operatorname{Mod} S$.

Then we apply these results and their symmetric results to study the Auslander projective dimension of modules. We obtain some criteria for computing the Auslander projective dimension of modules in $\operatorname{Mod} S$ (Theorem 4.4). Furthermore, we get the following

Theorem 1.3. (Theorem 4.10) *If* ${}_RC$ *has an ultimately closed projective resolution, then*

$$\mathcal{A}_C(S) = {C_S}^{\top} = {}^{\perp}\mathcal{I}_C(S),$$

where ${C_S}^{\top}$ *is the* Tor*-orthogonal class of* C_S *and* ${}^{\perp}\mathcal{I}_C(S)$ *is the left* Ext*-orthogonal class of the subcategory* $\mathcal{I}_C(S)$ *of* $\operatorname{Mod} S$ *consisting of* C*-injective modules.*

As a consequence, we have that if ${}_RC$ has an ultimately closed projective resolution, then the projective dimension of C_S is at most n if and only if the Auslander projective dimension of any module in $\operatorname{Mod} S$ is at most n (Corollary 4.11).

2 Preliminaries

In this paper, all rings are associative with identities. Let R be a ring. We use $\operatorname{Mod} R$ to denote the category of left R-modules and all subcategories of $\operatorname{Mod} R$ are full and closed under isomorphisms. For a subcategory $\mathscr{X}$ of $\operatorname{Mod} R$, we write

$${}^{\perp}\mathscr{X} := \{A \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geq 1}(A, X) = 0 \text{ for any } X \in \mathscr{X}\},$$

$$\mathscr{X}^{\perp} := \{A \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geq 1}(X, A) = 0 \text{ for any } X \in \mathscr{X}\},$$

$${}^{\perp_1}\mathscr{X} := \{A \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{1}(A, X) = 0 \text{ for any } X \in \mathscr{X}\},$$

$$\mathscr{X}^{\perp_1} := \{A \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{1}(X, A) = 0 \text{ for any } X \in \mathscr{X}\}.$$

For subcategories $\mathscr{X}, \mathscr{Y}$ of $\operatorname{Mod} R$, we write $\mathscr{X} \perp \mathscr{Y}$ if $\operatorname{Ext}_R^{\geq 1}(X, Y) = 0$ for any $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.

Definition 2.1. ([4, 7]) Let $\mathscr{X} \subseteq \mathscr{Y}$ be subcategories of $\operatorname{Mod} R$. A homomorphism $f : X \to Y$ in $\operatorname{Mod} R$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$ is called an **$\mathscr{X}$-precover** of Y if $\operatorname{Hom}_R(X', f)$ is epic for any $X' \in \mathscr{X}$; and f is called **right minimal** if an endomorphism $h : X \to X$ is an automorphism whenever $f = fh$. An **$\mathscr{X}$-precover** $f : X \to Y$ is called an **$\mathscr{X}$-cover** of Y if it is right minimal. The subcategory $\mathscr{X}$ is called **(pre)covering** in $\mathscr{Y}$ if any object in $\mathscr{Y}$ admits an $\mathscr{X}$-(pre)cover. Dually, the notions of an **$\mathscr{X}$-(pre)envelope**, a **left minimal homomorphism** and a **(pre)enveloping subcategory** are defined.

Definition 2.2. ([7, 8]) Let $\mathscr{U}, \mathscr{V}$ be subcategories of $\operatorname{Mod} R$.

(1) The pair $(\mathscr{U}, \mathscr{V})$ is called a **cotorsion pair** in $\operatorname{Mod} R$ if $\mathscr{U} = {}^{\perp_1}\mathscr{V}$ and $\mathscr{V} = \mathscr{U}^{\perp_1}$.

(2) A cotorsion pair $(\mathscr{U}, \mathscr{V})$ is called **perfect** if $\mathscr{U}$ is covering and $\mathscr{V}$ is enveloping in $\operatorname{Mod} R$.

(3) A cotorsion pair $(\mathscr{U}, \mathscr{V})$ is called **hereditary** if one of the following equivalent conditions is satisfied.

(3.1) $\mathscr{U} \perp \mathscr{V}$.

(3.2) $\mathscr{U}$ is projectively resolving in the sense that $\mathscr{U}$ contains all projective modules in $\operatorname{Mod} R$, $\mathscr{U}$ is closed under extensions and kernels of epimorphisms.

(3.3) $\mathscr{V}$ is injectively coresolving in the sense that $\mathscr{V}$ contains all injective modules in $\operatorname{Mod} R$, $\mathscr{V}$ is closed under extensions and cokernels of monomorphisms.

Set $(-)^+ := \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers. The following is the definition of duality pairs (cf. [6, 9]).

Definition 2.3. Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of $\operatorname{Mod} R$ and $\operatorname{Mod} R^{op}$ respectively.

(1) The pair $(\mathscr{X}, \mathscr{Y})$ is called a **duality pair** if the following conditions are satisfied.

(1.1) For a module $X \in \operatorname{Mod} R$, $X \in \mathscr{X}$ if and only if $X^+ \in \mathscr{Y}$.

(1.2) $\mathscr{Y}$ is closed under direct summands and finite direct sums.

(2) A duality pair $(\mathscr{X}, \mathscr{Y})$ is called **(co)product-closed** if $\mathscr{X}$ is closed under (co)products.

(3) A duality pair $(\mathscr{X}, \mathscr{Y})$ is called **perfect** if it is coproduct-closed, ${}_RR \in \mathscr{X}$ and $\mathscr{X}$ is closed under extensions.

We also recall the following remarkable result.

Lemma 2.4. ([6, p.7, Theorem] and [9, Theorem 3.1]) *Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of* $\operatorname{Mod} R$ *and* $\operatorname{Mod} R^{op}$ *respectively. If $(\mathscr{X}, \mathscr{Y})$ is a duality pair, then the following assertions hold true.*

(1) If $(\mathscr{X}, \mathscr{Y})$ is coproduct-closed, then $\mathscr{X}$ is covering.

(2) If $(\mathscr{X}, \mathscr{Y})$ is product-closed, then $\mathscr{X}$ is preenveloping.

(3) If $(\mathscr{X}, \mathscr{Y})$ is perfect, then $(\mathscr{X}, \mathscr{X}^{\perp})$ is a perfect cotorsion pair.

Definition 2.5. ([10]). Let R and S be rings. An (R, S)-bimodule ${}_RC_S$ is called **semidualizing** if the following conditions are satisfied.

(a1) ${}_RC$ admits a degreewise finite R-projective resolution.

(a2) C_S admits a degreewise finite S-projective resolution.

(b1) The homothety map ${}_RR_R \stackrel{R\gamma}{\to} \mathrm{Hom}_{S^{op}}(C,C)$ is an isomorphism.

(b2) The homothety map ${}_SS_S \stackrel{\gamma_S}{\to} \mathrm{Hom}_R(C,C)$ is an isomorphism.

(c1) $\mathrm{Ext}_R^{\geq 1}(C,C)=0$.

(c2) $\mathrm{Ext}_{S^{op}}^{\geq 1}(C,C)=0$.

Wakamatsu in [17] introduced and studied the so-called **generalized tilting modules**, which are usually called **Wakamatsu tilting modules**, see [2, 15]. Note that a bimodule ${}_RC_S$ is semidualizing if and only if it is Wakamatsu tilting ([19, Corollary 3.2]). Examples of semidualizing bimodules are referred to [10, 18].

3 Duality pairs

In this section, R and S are arbitrary rings and ${}_RC_S$ is a semidualizing bimodule. We write $(-)_* := \mathrm{Hom}(C,-)$ and

$${}_RC^{\perp} := \{M \in \mathrm{Mod}\, R \mid \mathrm{Ext}_R^{\geq 1}(C,M)=0\} \text{ and } {C_S}^{\perp} := \{B \in \mathrm{Mod}\, S^{op} \mid \mathrm{Ext}_{S^{op}}^{\geq 1}(C,B)=0\},$$

$${}^{\top}{}_RC := \{N \in \mathrm{Mod}\, R^{op} \mid \mathrm{Tor}^R_{\geq 1}(N,C)=0\} \text{ and } {C_S}^{\top} := \{A \in \mathrm{Mod}\, S \mid \mathrm{Tor}^S_{\geq 1}(C,A)=0\}.$$

Definition 3.1. ([10])

(1) The **Auslander class** $\mathcal{A}_C(R^{op})$ with respect to C consists of all modules N in $\mathrm{Mod}\, R^{op}$ satisfying the following conditions.

(a1) $N \in {}^{\top}{}_RC$.

(a2) $N \otimes_R C \in {C_S}^{\perp}$.

(a3) The canonical valuation homomorphism

$$\mu_N : N \to (N \otimes_R C)_*$$

defined by $\mu_N(x)(c) = x \otimes c$ for any $x \in N$ and $c \in C$ is an isomorphism in $\mathrm{Mod}\, R^{op}$.

(2) The **Bass class** $\mathcal{B}_C(R)$ with respect to C consists of all modules M in $\mathrm{Mod}\, R$ satisfying the following conditions.

(b1) $M \in {}_RC^{\perp}$.

(b2) $M_* \in {C_S}^{\top}$.

(b3) The canonical valuation homomorphism

$$\theta_M : C \otimes_S M_* \to M$$

defined by $\theta_M(c \otimes f) = f(c)$ for any $c \in C$ and $f \in M_*$ is an isomorphism in $\mathrm{Mod}\, R$.

(3) The **Auslander class** $\mathcal{A}_C(S)$ in $\operatorname{Mod} S$ and the **Bass class** $\mathcal{B}_C(S^{op})$ in $\operatorname{Mod} S^{op}$ are defined symmetrically.

The following result is crucial. From its proof, it is known that the conditions in the definitions of $\mathcal{A}_C(R^{op})$ and $\mathcal{B}_C(R)$ are dual item by item.

Proposition 3.2.

(1) For a module $N \in \operatorname{Mod} R^{op}$, $N \in \mathcal{A}_C(R^{op})$ if and only if $N^+ \in \mathcal{B}_C(R)$.

(2) For a module $M \in \operatorname{Mod} R$, $M \in \mathcal{B}_C(R)$ if and only if $M^+ \in \mathcal{A}_C(R^{op})$.

Proof. (1) Let $N \in \operatorname{Mod} R^{op}$. Then we have the following

(a)

$$\begin{aligned}
& N \in {}^{\top}{}_R C \\
\Leftrightarrow\ & \operatorname{Tor}^R_{\geq 1}(N, C) = 0 \\
\Leftrightarrow\ & [\operatorname{Tor}^R_{\geq 1}(N, C)]^+ = 0 \\
\Leftrightarrow\ & \operatorname{Ext}_R^{\geq 1}(C, N^+) = 0 \text{ (by [8, Lemma 2.16(b)])} \\
\Leftrightarrow\ & N^+ \in {}_R C^{\perp}.
\end{aligned}$$

(b)

$$\begin{aligned}
& N \otimes_R C \in {C_S}^{\perp} \\
\Leftrightarrow\ & \operatorname{Ext}^{\geq 1}_{S^{op}}(C, N \otimes_R C) = 0 \\
\Leftrightarrow\ & [\operatorname{Ext}^{\geq 1}_{S^{op}}(C, N \otimes_R C)]^+ = 0 \\
\Leftrightarrow\ & \operatorname{Tor}^S_{\geq 1}(C, (N \otimes_R C)^+) = 0 \text{ (by [8, Lemma 2.16(d)])} \\
\Leftrightarrow\ & \operatorname{Tor}^S_{\geq 1}(C, (N^+)_*) = 0 \text{ (by [8, Lemma 2.16(a)])} \\
\Leftrightarrow\ & (N^+)_* \in {C_S}^{\top}.
\end{aligned}$$

(c) By [8, Lemma 2.16(c)], the canonical valuation homomorphism

$$\alpha : C \otimes_S (N \otimes_R C)^+ \to [\operatorname{Hom}_{S^{op}}(C, N \otimes_R C)]^+$$

defined by $\alpha(c \otimes g)(f) = gf(c)$ for any $c \in C$, $g \in (N \otimes_R C)^+$ and $f \in \operatorname{Hom}_{S^{op}}(C, N \otimes_R C)$ is an isomorphism in $\operatorname{Mod} R$. By [8, Lemma 2.16(a)], the canonical valuation homomorphism

$$\beta : (N \otimes_R C)^+ \to \operatorname{Hom}_R(C, N^+)$$

defined by $\beta(g)(c)(x) = g(x \otimes c)$ for any $g \in (N \otimes_R C)^+$, $c \in C$ and $x \in N$ is an isomorphism in $\operatorname{Mod} S$. So

$$1_C \otimes \beta : C \otimes_S (N \otimes_R C)^+ \to C \otimes_S \operatorname{Hom}_R(C, N^+)$$

via $(1_C \otimes \beta)(c \otimes g) = c \otimes \beta(g)$ for any $c \in C$ and $g \in (N \otimes_R C)^+$ is an isomorphism in $\operatorname{Mod} R$.

Consider the following diagram

$$\begin{array}{ccc} C\otimes_S (N\otimes_R C)^+ & \xrightarrow{\alpha} & [\mathrm{Hom}_{S^{op}}(C, N\otimes_R C)]^+ \\ \downarrow{\scriptstyle 1_C\otimes\beta} & & \downarrow{\scriptstyle (\mu_N)^+} \\ C\otimes_S \mathrm{Hom}_R(C, N^+) & \xrightarrow{\theta_{N^+}} & N^+, \end{array}$$

where

$$(\mu_N)^+ : [\mathrm{Hom}_{S^{op}}(C, N\otimes_R C)]^+ \to N^+$$

via $(\mu_N)^+(f^{'}) = f^{'}\mu_N$ for any $f^{'} \in [\mathrm{Hom}_{S^{op}}(C, N\otimes_R C)]^+$ is a natural homomorphism in $\mathrm{Mod}\, R$, and

$$\theta_{N^+} : C\otimes_S \mathrm{Hom}_R(C, N^+) \to N^+$$

defined by $\theta_{N^+}(c\otimes f^{''}) = f^{''}(c)$ for any $c\in C$ and $f^{''} \in \mathrm{Hom}_R(C, N^+)$ is a canonical valuation homomorphism in $\mathrm{Mod}\, R$. Then for any $c\in C$, $g\in (N\otimes_R C)^+$ and $x\in N$, we have

$$(\mu_N)^+\alpha(c\otimes g)(x) = \alpha(c\otimes g)\mu_N(x) = g\mu_N(x)(c) = g(x\otimes c)$$

$$\theta_{N^+}(1_C\otimes\beta)(c\otimes g)(x) = \theta_{N^+}(c\otimes\beta(g))(x) = \beta(g)(c)(x) = g(x\otimes c),$$

Thus

$$(\mu_N)^+\alpha = \theta_{N^+}(1_C\otimes\beta),$$

and therefore μ_N is an isomorphism $\Leftrightarrow$ $(\mu_N)^+$ is an isomorphism $\Leftrightarrow$ θ_{N^+} is an isomorphism.

We conclude that $N\in\mathcal{A}_C(R^{op}) \Leftrightarrow N^+\in\mathcal{B}_C(R)$.

(2) Let $M\in \mathrm{Mod}\, R$. Then we have the following

(a)

$$\begin{aligned} & M\in {}_RC^{\perp} \\ \Leftrightarrow\ & \mathrm{Ext}_R^{\geq 1}(C, M) = 0 \\ \Leftrightarrow\ & [\mathrm{Ext}_R^{\geq 1}(C, M)]^+ = 0 \\ \Leftrightarrow\ & \mathrm{Tor}_{\geq 1}^R(M^+, C) = 0 \text{ (by [8, Lemma 2.16(d)])} \\ \Leftrightarrow\ & M^+\in {}^{\top}{}_RC. \end{aligned}$$

(b)

$$\begin{aligned} & M_*\in {C_S}^{\top} \\ \Leftrightarrow\ & \mathrm{Tor}_{\geq 1}^S(C, M_*) = 0 \\ \Leftrightarrow\ & [\mathrm{Tor}_{\geq 1}^S(C, M_*)]^+ = 0 \\ \Leftrightarrow\ & \mathrm{Ext}_{S^{op}}^{\geq 1}(C, (M_*)^+) = 0 \text{ (by [8, Lemma 2.16(b)])} \\ \Leftrightarrow\ & \mathrm{Ext}_{S^{op}}^{\geq 1}(C, M^+\otimes_R C) = 0 \text{ (by [8, Lemma 2.16(c)])} \\ \Leftrightarrow\ & M^+\otimes_R C\in {C_S}^{\perp}. \end{aligned}$$

(c) By [8, Lemma 2.16(a)], the canonical valuation homomorphism

$$\tau : [C\otimes_S \mathrm{Hom}_R(C, M)]^+ \to \mathrm{Hom}_{S^{op}}(C, [\mathrm{Hom}_R(C, M)]^+)$$

defined by $\tau(g^{'})(c)(f) = g^{'}(c \otimes f)$ for any $g^{'} \in [C \otimes_S \mathrm{Hom}_R(C, M)]^+$, $c \in C$ and $f \in \mathrm{Hom}_R(C, M)$ is an isomorphism in $\mathrm{Mod}\, R^{op}$. By [8, Lemma 2.16(c)], the canonical valuation homomorphism

$$\sigma : M^+ \otimes_R C \to [\mathrm{Hom}_R(C, M)]^+$$

defined by $\sigma(g \otimes c)(f) = gf(c)$ for any $g \in M^+$, $c \in C$ and $f \in \mathrm{Hom}_R(C, M)$ is an isomorphism in $\mathrm{Mod}\, S^{op}$. So

$$\mathrm{Hom}_{S^{op}}(C, \sigma) : \mathrm{Hom}_{S^{op}}(C, M^+ \otimes_R C) \to \mathrm{Hom}_{S^{op}}(C, [\mathrm{Hom}_R(C, M)]^+)$$

via $\mathrm{Hom}_{S^{op}}(C, \sigma)(g^{''}) = \sigma g^{''}$ for any $g^{''} \in \mathrm{Hom}_{S^{op}}(C, M^+ \otimes_R C)$ is an isomorphism in $\mathrm{Mod}\, R^{op}$.

Consider the following diagram

$$\begin{array}{ccc} M^+ & \xrightarrow{(\theta_M)^+} & [C \otimes_S \mathrm{Hom}_R(C, M)]^+ \\ \downarrow{\scriptstyle \mu_{M^+}} & & \downarrow{\scriptstyle \tau} \\ \mathrm{Hom}_{S^{op}}(C, M^+ \otimes_R C) & \xrightarrow{\mathrm{Hom}_{S^{op}}(C,\sigma)} & \mathrm{Hom}_{S^{op}}(C, [\mathrm{Hom}_R(C, M)]^+), \end{array}$$

where

$$(\theta_M)^+ : M^+ \to [C \otimes_S \mathrm{Hom}_R(C, M)]^+$$

via $(\theta_M)^+(g) = g\theta_M$ for any $g \in M^+$ is a natural homomorphism in $\mathrm{Mod}\, R^{op}$, and

$$\mu_{M^+} : M^+ \to \mathrm{Hom}_{S^{op}}(C, M^+ \otimes_R C)$$

defined by $\mu_{M^+}(g)(c) = g \otimes c$ for any $g \in M^+$ and $c \in C$ is a canonical valuation homomorphism in $\mathrm{Mod}\, R^{op}$. Then for any $g \in M^+$, $c \in C$ and $f \in \mathrm{Hom}_R(C, M)$, we have

$$\tau(\theta_M)^+(g)(c)(f) = (\theta_M)^+(g)(c \otimes f) = g\theta_M(c \otimes f) = gf(c),$$

$$\mathrm{Hom}_{S^{op}}(C, \sigma)\mu_{M^+}(g)(c)(f) = \sigma\mu_{M^+}(g)(c)(f) = \sigma(g \otimes c)(f) = gf(c),$$

Thus

$$\tau(\theta_M)^+ = \mathrm{Hom}_{S^{op}}(C, \sigma)\mu_{M^+},$$

and therefore θ_M is an isomorphism $\Leftrightarrow$ $(\theta_M)^+$ is an isomorphism $\Leftrightarrow$ μ_{M^+} is an isomorphism.

We conclude that $M \in \mathcal{B}_C(R) \Leftrightarrow M^+ \in \mathcal{A}_C(R^{op})$. □

As a consequence, we get the following

Theorem 3.3.

(1) The pair

$$(\mathcal{A}_C(R^{op}), \mathcal{B}_C(R))$$

is a perfect coproduct-closed and product-closed duality pair and $\mathcal{A}_C(R^{op})$ is covering and preenveloping in $\mathrm{Mod}\, R^{op}$.

(2) The pair

$$(\mathcal{B}_C(R), \mathcal{A}_C(R^{op}))$$

is a coproduct-closed and product-closed duality pair and $\mathcal{B}_C(R)$ is covering and preenveloping in $\mathrm{Mod}\, R$.

Proof. It follows from [10, Proposition 4.2(a)] that both $\mathcal{A}_C(R^{op})$ and $\mathcal{B}_C(R)$ are closed under direct summands, coproducts and products. So by Lemma 2.4(1)(2) and Proposition 3.2, we have that both the pairs

$$(\mathcal{A}_C(R^{op}), \mathcal{B}_C(R)) \text{ and } (\mathcal{B}_C(R), \mathcal{A}_C(R^{op}))$$

are coproduct-closed and product-closed duality pairs, $\mathcal{A}_C(R^{op})$ is covering and preenveloping in $\operatorname{Mod} R^{op}$ and $\mathcal{B}_C(R)$ is covering and preenveloping in $\operatorname{Mod} R$. Moreover, $\mathcal{A}_C(R^{op})$ is projectively resolving by [10, Theorem 6.2], so the duality pair $(\mathcal{A}_C(R^{op}), \mathcal{B}_C(R))$ is perfect. □

We write

$$\mathcal{A}_C(R^{op})^{\perp} := \{Y \in \operatorname{Mod} R^{op} \mid \operatorname{Ext}^{\geq 1}_{R^{op}}(N, Y) = 0 \text{ for any } N \in \mathcal{A}_C(R^{op})\}.$$

The following corollary was proved in [5, Theorem 3.11] when R is a commutative noetherian ring and ${}_RC_S = {}_RC_R$.

Corollary 3.4. *The pair*

$$(\mathcal{A}_C(R^{op}), \mathcal{A}_C(R^{op})^{\perp})$$

is a hereditary perfect cotorsion pair and $\mathcal{A}_C(R^{op})$ *is covering and preenveloping in* $\operatorname{Mod} R^{op}$.

Proof. It follows from Theorem 3.3(1) and Lemma 2.4(3). □

The following two results are the symmetric versions of Theorem 3.3 and Corollary 3.4 respectively.

Theorem 3.5.

(1) The pair

$$(\mathcal{A}_C(S), \mathcal{B}_C(S^{op}))$$

is a perfect coproduct-closed and product-closed duality pair and $\mathcal{A}_C(S)$ *is covering and preenveloping in* $\operatorname{Mod} S$.

(2) The pair

$$(\mathcal{B}_C(S^{op}), \mathcal{A}_C(S))$$

is a coproduct-closed and product-closed duality pair and $\mathcal{B}_C(S^{op})$ *is covering and preenveloping in* $\operatorname{Mod} S^{op}$.

We write

$$\mathcal{A}_C(S)^{\perp} := \{X \in \operatorname{Mod} S \mid \operatorname{Ext}^{\geq 1}_{S}(N', X) = 0 \text{ for any } N' \in \mathcal{A}_C(S)\}.$$

Corollary 3.6. *The pair*

$$(\mathcal{A}_C(S), \mathcal{A}_C(S)^{\perp})$$

is a hereditary perfect cotorsion pair and $\mathcal{A}_C(S)$ *is covering and preenveloping in* $\operatorname{Mod} S$.

Holm and White proved in [10, Proposition 4.1] that there exist the following (Foxby) equivalences of categories

$$\mathcal{A}_C(S) \underset{\operatorname{Hom}_R(C,-)}{\overset{C\otimes_S -}{\rightleftarrows}} \mathcal{B}_C(R), \quad (\sim)$$

$$\mathcal{A}_C(R^{op}) \underset{\operatorname{Hom}_{S^{op}}(C,-)}{\overset{-\otimes_R C}{\rightleftarrows}} \mathcal{B}_C(S^{op}). \quad (\sim)$$

Compare this result with Theorems 3.3 and 3.5.

By Theorems 3.3(2) and 3.5(1), $\mathcal{B}_C(R)$ is preenveloping in $\operatorname{Mod} R$ and $\mathcal{A}_C(S)$ is preenveloping in $\operatorname{Mod} S$. In the following result, we construct an $\mathcal{A}_C(S)$-preenvelope of a given module in $\operatorname{Mod} S$ from a $\mathcal{B}_C(R)$-preenvelope of some module in $\operatorname{Mod} R$.

Theorem 3.7.

(1) Let $N \in \operatorname{Mod} S$ *and*

$$f : C \otimes_S N \to B$$

be a $\mathcal{B}_C(R)$*-preenvelope of* $C \otimes_S N$ *in* $\operatorname{Mod} R$*. Then we have*

(1.1)

$$f_*\mu_N : N \to B_*$$

is an $\mathcal{A}_C(S)$*-preenvelope of* N *in* $\operatorname{Mod} S$*.*

(1.2) If f *is a* $\mathcal{B}_C(R)$*-envelope of* $C \otimes_S N$*, then* $f_*\mu_N$ *is an* $\mathcal{A}_C(S)$*-envelope of* N*.*

(2) If $\mathcal{B}_C(R)$ *is enveloping in* $\operatorname{Mod} R$*, then* $\mathcal{A}_C(S)$ *is enveloping in* $\operatorname{Mod} S$*.*

Proof. (1.1) Let $N \in \operatorname{Mod} S$ and

$$f : C \otimes_S N \to B$$

be a $\mathcal{B}_C(R)$-preenvelope in $\operatorname{Mod} R$. By [10, Proposition 4.1], we have $B_* \in \mathcal{A}_C(S)$. Let $g \in \operatorname{Hom}_S(N, A)$ with $A \in \mathcal{A}_C(S)$. By [10, Proposition 4.1] again, we have $C \otimes_S A \in \mathcal{B}_C(R)$. So there exists $h \in \operatorname{Hom}_R(B, C \otimes_S A)$ such that $1_C \otimes g = hf$, that is, the following diagram

$$\begin{array}{ccc} C \otimes_S N & \xrightarrow{f} & B \\ {\scriptstyle 1_C\otimes g}\downarrow & \swarrow {\scriptstyle h} & \\ C \otimes_S A & & \end{array}$$

commutes. From the following commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{g} & A \\ {\scriptstyle \mu_N}\downarrow & & \downarrow{\scriptstyle \mu_A} \\ (C \otimes_S N)_* & \xrightarrow{(1_C\otimes g)_*} & (C \otimes_S A)_*, \end{array}$$

we get $\mu_A g = (1_C \otimes g)_*\mu_N$. Because μ_A is an isomorphism, we have

$$g = {\mu_A}^{-1}(1_C \otimes g)_*\mu_N = ({\mu_A}^{-1}h_*)(f_*\mu_N),$$

that is, the following diagram

$$\begin{array}{ccc} N & \xrightarrow{f_*\mu_N} & B \\ {\scriptstyle g}\downarrow & \swarrow {\scriptstyle {\mu_A}^{-1}h_*} & \\ A & & \end{array}$$

commutes. Thus $f_*\mu_N : N \to B_*$ is an $\mathcal{A}_C(S)$-preenvelope of N.

(1.2) By (1.1), it suffices to prove that if f is left minimal, then so is $f_*\mu_N$.

Let f be left minimal and $h \in \mathrm{Hom}_S(B_*, B_*)$ such that $f_*\mu_N = h(f_*\mu_N)$. Then we have

$$(1_C \otimes f_*)(1_C \otimes \mu_N) = 1_C \otimes (f_*\mu_N) = 1_C \otimes (h(f_*\mu_N)) = (1_C \otimes h)(1_C \otimes f_*)(1_C \otimes \mu_N). \tag{3.1}$$

From the following commutative diagram

$$\begin{array}{ccc} C \otimes_S (C \otimes_S N)_* & \xrightarrow{1_C \otimes f_*} & C \otimes_S B_* \\ {\scriptstyle \theta_{C\otimes_S N}}\downarrow & & \downarrow{\scriptstyle \theta_B} \\ C \otimes_S N & \xrightarrow{f} & B, \end{array}$$

we get

$$f\theta_{C\otimes_S N} = \theta_B(1_C \otimes f_*). \tag{3.2}$$

So we have

$$\begin{aligned} f &= f 1_{C\otimes_S N} \\ &= f(\theta_{C\otimes_S N}(1_C \otimes \mu_N)) \text{ (by [20, Proposition 2.2(1)])} \\ &= \theta_B(1_C \otimes f_*)(1_C \otimes \mu_N) \text{ (by (3.2))} \\ &= \theta_B(1_C \otimes h)(1_C \otimes f_*)(1_C \otimes \mu_N) \text{ (by (3.1))} \\ &= \theta_B(1_C \otimes h)({\theta_B}^{-1}\theta_B)(1_C \otimes f_*)(1_C \otimes \mu_N) \text{ (because } \theta_B \text{ is an isomorphism)} \\ &= \theta_B(1_C \otimes h){\theta_B}^{-1} f\theta_{C\otimes_S N}(1_C \otimes \mu_N) \text{ (by (3.2))} \\ &= \theta_B(1_C \otimes h){\theta_B}^{-1} f 1_{C\otimes_S N} \text{ (by [20, Proposition 2.2(1)])} \\ &= \theta_B(1_C \otimes h){\theta_B}^{-1} f. \end{aligned}$$

Because f is left minimal, $\theta_B(1_C \otimes h){\theta_B}^{-1}$ is an isomorphism, which implies that $1_C \otimes h$ and $(1_C \otimes h)_*$ are also isomorphisms. From the following commutative diagram

$$\begin{array}{ccc} B_* & \xrightarrow{h} & B_* \\ {\scriptstyle \mu_{B_*}}\downarrow & & \downarrow{\scriptstyle \mu_{B_*}} \\ (C \otimes_S B_*)_* & \xrightarrow{(1_C \otimes h)_*} & (C \otimes_S B_*)_*, \end{array}$$

we get

$$(1_C \otimes h)_*\mu_{B_*} = \mu_{B_*}h.$$

Because $B_* \in \mathcal{A}_C(S)$ by [10, Proposition 4.1], μ_{B_*} is an isomorphism. It follows that h is also an isomorphism and $f_*\mu_N$ is left minimal.

(2) It follows from the assertion (1.2) immediately. □

We do not know whether a $\mathcal{B}_C(R)$-preenvelope of given module in $\operatorname{Mod} R$ can be constructed from an $\mathcal{A}_C(S)$-preenvelope of some module in $\operatorname{Mod} S$, and do not know whether the converse of Theorem 3.7(2) holds true.

By Theorems 3.3(2) and 3.5(1), $\mathcal{B}_C(R)$ is covering in $\operatorname{Mod} R$ and $\mathcal{A}_C(S)$ is covering in $\operatorname{Mod} S$. In the following result, we construct a $\mathcal{B}_C(R)$-cover of a given module in $\operatorname{Mod} R$ from an $\mathcal{A}_C(S)$-cover of some module in $\operatorname{Mod} S$.

Proposition 3.8. *Let* $M \in \operatorname{Mod} R$ *and*

$$g : A \to M_*$$

be an $\mathcal{A}_C(S)$*-cover of* M_* *in* $\operatorname{Mod} S$*. Then*

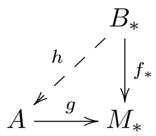
$$\theta_M(1_C \otimes g) : C \otimes_S A \to M$$

is a $\mathcal{B}_C(R)$*-cover of* M *in* $\operatorname{Mod} R$*.*

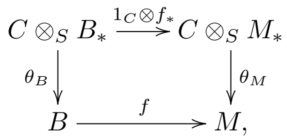
Proof. Let $M \in \operatorname{Mod} R$ and

$$g : A \to M_*$$

be an $\mathcal{A}_C(S)$-cover of M_* in $\operatorname{Mod} S$. By [10, Proposition 4.1], we have $C \otimes_S A \in \mathcal{B}_C(R)$. Let $f \in \operatorname{Hom}_R(B, M)$ with $B \in \mathcal{B}_C(R)$. By [10, Proposition 4.1] again, we have $B_* \in \mathcal{A}_C(S)$. So there exists $h \in \operatorname{Hom}_S(B_*, A)$ such that $f_* = gh$, that is, the following diagram



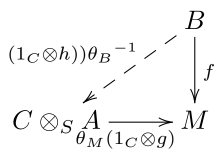
commutes. From the following commutative diagram



we get $f\theta_B = \theta_M(1_C \otimes f_*)$. Because θ_B is an isomorphism, we have

$$f = \theta_M(1_C \otimes f_*){\theta_B}^{-1} = \theta_M(1_C \otimes (gh)){\theta_B}^{-1} = (\theta_M(1_C \otimes g))((1_C \otimes h)){\theta_B}^{-1}),$$

that is, the following diagram



commutes. Thus $\theta_M(1_C \otimes g) : C \otimes_S A \to M$ is a $\mathcal{B}_C(R)$-precover of M.

In the following, it suffices to prove that $\theta_M(1_C \otimes g)$ is right minimal.

Let $h \in \operatorname{Hom}_R(C \otimes_S A, C \otimes_S A)$ such that $\theta_M(1_C \otimes g) = (\theta_M(1_C \otimes g))h$. Then we have

$$(\theta_M)_*(1_C \otimes g)_* = (\theta_M(1_C \otimes g))_* = ((\theta_M(1_C \otimes g))h)_* = (\theta_M)_*(1_C \otimes g)_* h_*. \tag{3.3}$$

From the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & M_* \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_{M_*}} \\ (C\otimes_S A)_* & \xrightarrow{(1_C\otimes g)_*} & (C\otimes_S M_*)_*, \end{array}$$

we get

$$\mu_{M_*}g = (1_C \otimes g)_*\mu_A. \tag{3.4}$$

So we have

$$\begin{aligned} g &= 1_{M_*}g \\ &= (\theta_M)_*\mu_{M_*}g \text{ (by [20, Proposition 2.2(1)])} \\ &= (\theta_M)_*(1_C\otimes g)_*\mu_A \text{ (by (3.4))} \\ &= (\theta_M)_*(1_C\otimes g)_*h_*\mu_A \text{ (by (3.3))} \\ &= (\theta_M)_*(1_C\otimes g)_*\mu_A{\mu_A}^{-1}h_*\mu_A \text{ (because } \mu_A \text{ is an isomorphism)} \\ &= (\theta_M)_*\mu_{M_*}g{\mu_A}^{-1}h_*\mu_A \text{ (by (3.4))} \\ &= 1_{M_*}g{\mu_A}^{-1}h_*\mu_A \text{ (by [20, Proposition 2.2(1)])} \\ &= g{\mu_A}^{-1}h_*\mu_A. \end{aligned}$$

Because g is right minimal, ${\mu_A}^{-1}h_*\mu_A$ is an isomorphism, which implies that h_* and $1_C\otimes h_*$ are also isomorphisms. From the following commutative diagram

$$\begin{array}{ccc} C\otimes_S(C\otimes_S A)_* & \xrightarrow{1_C\otimes h_*} & C\otimes_S(C\otimes_S A)_* \\ {\scriptstyle \theta_{C\otimes_S A}}\downarrow & & \downarrow{\scriptstyle \theta_{C\otimes_S A}} \\ C\otimes_S A & \xrightarrow{h} & C\otimes_S A, \end{array}$$

we get

$$h\theta_{C\otimes_S A} = \theta_{C\otimes_S A}(1_C\otimes h_*).$$

Because $C\otimes_S A\in\mathcal{B}_C(R)$ by [10, Proposition 4.1], $\theta_{C\otimes_S A}$ is an isomorphism. It follows that h is also an isomorphism and $\theta_M(1_C\otimes g)$ is right minimal. □

We do not know whether an $\mathcal{A}_C(S)$-cover of a given module in $\operatorname{Mod} S$ can be constructed from a $\mathcal{B}_C(R)$-cover of some module in $\operatorname{Mod} R$.

4 The Auslander projective dimension of modules

For a subcategory $\mathscr{X}$ of $\operatorname{Mod} S$ and $N\in\operatorname{Mod} S$, the **$\mathscr{X}$-projective dimension** $\mathscr{X}\text{-pd}_S N$ of N is defined as $\inf\{n \mid$ there exists an exact sequence

$$0\to X_n\to\cdots\to X_1\to X_0\to N\to 0$$

in $\operatorname{Mod} S$ with all $X_i\in\mathscr{X}\}$, and we set $\mathscr{X}\text{-pd}_S N$ infinite if no such integer exists. We call $\mathcal{A}_C(S)\text{-pd}_S N$ the **Auslander projective dimension** of N. For any $n\geq 0$, we use $\Omega^n(N)$ to denote the n-th syzygy of N (note: $\Omega^0(N)=N$).

Lemma 4.1. *Let $N \in \operatorname{Mod} S$ and $n \geq 0$. If $\mathcal{A}_C(S)\text{-pd}_S N \leq n$ and*

$$0 \to K_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to N \to 0$$

be an exact sequence in $\operatorname{Mod} S$ with all A_i in $\mathcal{A}_C(S)$, then $K_n \in \mathcal{A}_C(S)$; in particular, $\Omega^n(N) \in \mathcal{A}_C(S)$.

Proof. Because $\mathcal{A}_C(S)$ is projectively resolving and is closed under direct summands and coproducts by [10, Theorem 6.2 and Proposition 4.2(a)], the assertion follows from [1, Lemma 3.12]. □

We use $\mathcal{A}_C(S)\text{-pd}^{<\infty}$ to denote the subcategory of $\operatorname{Mod} S$ consisting of modules with finite Auslander projective dimension.

Proposition 4.2. *$\mathcal{A}_C(S)\text{-pd}^{<\infty}$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms.*

Proof. Let

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

be an exact sequence in $\operatorname{Mod} S$ and $n \geq 0$. If $\max\{\mathcal{A}_C(S)\text{-pd}_S N_1, \mathcal{A}_C(S)\text{-pd}_S N_3\} \leq n$, then by Lemma 4.1, there exist exact sequences

$$0 \to \Omega^n(N_1) \to P_1^{n-1} \to \cdots \to P_1^1 \to P_1^0 \to N_1 \to 0,$$

$$0 \to \Omega^n(N_3) \to P_3^{n-1} \to \cdots \to P_3^1 \to P_3^0 \to N_3 \to 0$$

in $\operatorname{Mod} S$ with all P_i^j projective and $\Omega^n(N_1), \Omega^n(N_3) \in \mathcal{A}_C(S)$. Then we get exact sequences

$$0 \to K_n \to P_1^{n-1} \oplus P_3^{n-1} \to \cdots \to P_1^1 \oplus P_3^1 \to P_1^0 \oplus P_3^0 \to N_2 \to 0,$$

$$0 \to \Omega^n(N_1) \to K_n \to \Omega^n(N_3) \to 0$$

in $\operatorname{Mod} S$. By [10, Theorem 6.2], we have $K_n \in \mathcal{A}_C(S)$ and $\mathcal{A}_C(S)\text{-pd}_S N_2 \leq n$.

If $\max\{\mathcal{A}_C(S)\text{-pd}_S N_1, \mathcal{A}_C(S)\text{-pd}_S N_2\} \leq n$, then by Corollary 3.6 and Lemma 4.1, there exist $\operatorname{Hom}_S(\mathcal{A}_C(S), -)$-exact exact sequences

$$0 \to A_1^n \to A_1^{n-1} \to \cdots \to A_1^1 \to A_1^0 \to N_1 \to 0,$$

$$0 \to A_2^n \to A_2^{n-1} \to \cdots \to A_2^1 \to A_2^0 \to N_2 \to 0$$

in $\operatorname{Mod} S$ with all A_i^j in $\mathcal{A}_C(S)$. By [11, Theorem 3.6], we get an exact sequence

$$0 \to A_1^n \to A_1^{n-1} \oplus A_2^n \to \cdots \to A_1^0 \oplus A_2^1 \to A_2^0 \to N_3 \to 0$$

in $\operatorname{Mod} S$, and so $\mathcal{A}_C(S)\text{-pd}_S N_3 \leq n+1$.

If $\max\{\mathcal{A}_C(S)\text{-pd}_S N_2, \mathcal{A}_C(S)\text{-pd}_S N_3\} \leq n$, then by Corollary 3.6 and Lemma 4.1, there exist $\operatorname{Hom}_S(\mathcal{A}_C(S), -)$-exact exact sequences

$$0 \to A_2^n \to A_2^{n-1} \to \cdots \to A_2^1 \to A_2^0 \to N_2 \to 0,$$

$$0 \to A_3^n \to A_3^{n-1} \to \cdots \to A_3^1 \to A_3^0 \to N_3 \to 0$$

in $\operatorname{Mod} S$ with all A_i^j in $\mathcal{A}_C(S)$. By [11, Theorem 3.2], we get exact sequences

$$0 \to A_2^n \to A_2^{n-1} \oplus A_3^n \to \cdots \to A_2^1 \oplus A_3^2 \to A \to N_1 \to 0,$$

$$0 \to A \to A_2^0 \oplus A_3^1 \to A_3^0 \to 0$$

in $\operatorname{Mod} S$. By [10, Theorem 6.2], we have $A \in \mathcal{A}_C(S)$, and so $\mathcal{A}_C(S)\text{-pd}_S N_1 \leq n$. □

We write

$$\mathcal{I}_C(S) := \{I_* \mid I \text{ is injective in } \operatorname{Mod} R\}.$$

The modules in $\mathcal{I}_C(S)$ is called C**-injective** ([10]). Let Q be an injective cogenerator for $\operatorname{Mod} R$. Then

$$\mathcal{I}_C(S) = \operatorname{Prod}_S Q_*$$

by [14, Proposition 2.4(2)], where $\operatorname{Prod}_S Q_*$ is the subcategory of $\operatorname{Mod} S$ consisting of direct summands of products of copies of Q_*. By [8, Lemma 2.16(b)], we have the following isomorphism of functors

$$\operatorname{Hom}_R(\operatorname{Tor}_i^S(C, -), Q) \cong \operatorname{Ext}_S^i(-, Q_*)$$

for any $i \geq 1$. This gives the following

Lemma 4.3. ${C_S}^{\top} = {}^{\perp}\mathcal{I}_C(S)$.

For a subcategory $\mathscr{X}$ of $\operatorname{Mod} S$, a sequence in $\operatorname{Mod} S$ is called $\operatorname{Hom}_S(-, \mathscr{X})$**-exact** if it is exact after applying the functor $\operatorname{Hom}_S(-, X)$ for any $X \in \mathscr{X}$. Now we give some criteria for computing the Auslander projective dimension of modules.

Theorem 4.4. *Let $N \in \operatorname{Mod} S$ with $\mathcal{A}_C(S)\text{-pd}_S N < \infty$ and $n \geq 0$. Then the following statements are equivalent.*

(1) $\mathcal{A}_C(S)\text{-pd}_S N \leq n$.

(2) $\Omega^n(N) \in \mathcal{A}_C(S)$.

(3) $\operatorname{Tor}_{\geq n+1}^S(C, N) = 0$.

(4) There exists an exact sequence

$$0 \to H \to A \to N \to 0$$

in $\operatorname{Mod} S$ *with* $A \in \mathcal{A}_C(S)$ *and* $\mathcal{I}_C(S)\text{-pd}_S H \leq n-1$.

(5) There exists a ($\operatorname{Hom}_S(-, \mathcal{I}_C(S))$*-exact) exact sequence*

$$0 \to N \to H' \to A' \to 0$$

in $\operatorname{Mod} S$ *with* $A' \in \mathcal{A}_C(S)$ *and* $\mathcal{I}_C(S)\text{-pd}_S H' \leq n$.

Proof. By Lemma 4.1 and the dimension shifting, we have (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2) Because $\mathrm{Tor}^S_{\geq n+1}(C,N)=0$ by (3), we have $\Omega^n(N)\in {C_S}^{\top}$, and so $\Omega^n(N)\in {}^{\perp}\mathcal{I}_C(S)$ by Lemma 4.3. Note that all projective modules in $\mathrm{Mod}\, S$ are in $\mathcal{A}_C(S)$ by [10, Theorem 6.2]. Because $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, N<\infty$ by assumption, we have $\mathcal{A}_C(S)$-$\mathrm{pd}_S\,\Omega^n(N)<\infty$ by Proposition 4.2.

Assume that $\mathcal{A}_C(S)$-$\mathrm{pd}_S\,\Omega^n(N)=m(<\infty)$ and

$$0\to A_m\to\cdots\to A_1\to A_0\to\Omega^n(N)\to 0 \tag{4.1}$$

is an exact sequence in $\mathrm{Mod}\, S$ with all A_j in $\mathcal{A}_C(S)$. Because $\mathcal{A}_C(S)\subseteq {C_S}^{\top}={}^{\perp}\mathcal{I}_C(S)$ by Lemma 4.3, the exact sequence (4.1) is $\mathrm{Hom}_S(-,\mathcal{I}_C(S))$-exact. By [16, Theorem 3.11(1)], we have the following $\mathrm{Hom}_S(-,\mathcal{I}_C(S))$-exact exact sequence

$$0\to A_j\to U_j^0\to U_j^1\to\cdots\to U_j^i\to\cdots$$

in $\mathrm{Mod}\, S$ with all U_j^i in $\mathcal{I}_C(S)$ for any $0\leq j\leq m$ and $i\geq 0$. It follows from [11, Corollary 3.5] that there exist the following two exact sequences

$$0\to\Omega^n(N)\to U\to\oplus_{i=0}^m U_i^{i+1}\to\oplus_{i=0}^m U_i^{i+2}\to\oplus_{i=0}^m U_i^{i+3}\to\cdots,$$

$$0\to U_m^0\to U_m^1\oplus U_{m-1}^0\to\cdots\to\oplus_{i=2}^m U_i^{i-2}\to\oplus_{i=1}^m U_i^{i-1}\to\oplus_{i=0}^m U_i^i\to U\to 0,$$

and the former one is $\mathrm{Hom}_S(-,\mathcal{I}_C(S))$-exact. Because $\mathcal{I}_C(S)$ is closed under finite direct sums and cokernels of monomorphisms by [10, Proposition 5.1(c) and Corollary 6.4], we have $U\in\mathcal{I}_C(S)$. By [16, Theorem 3.11(1)] again, we have $\Omega^n(N)\in\mathcal{A}_C(S)$.

(1) $\Rightarrow$ (4) By [10, Theorem 6.2], $\mathcal{A}_C(S)$ is closed under extensions. By [16, Theorem 3.11(1)], we have that $\mathcal{I}_C(S)$ is an $\mathcal{I}_C(S)$-coproper cogenerator for $\mathcal{A}_C(S)$ in the sense of [12]. Then the assertion follows from [12, Theorem 4.7].

(4) $\Rightarrow$ (5) Let

$$0\to H\to A\to N\to 0$$

be an exact sequence in $\mathrm{Mod}\, S$ with $A\in\mathcal{A}_C(S)$ and $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H\leq n-1$. By [16, Theorem 3.11(1)], there exists a $\mathrm{Hom}_S(-,\mathcal{I}_C(S))$-exact exact sequence

$$0\to A\to U\to A'\to 0$$

in $\mathrm{Mod}\, S$ with $U\in\mathcal{I}_C(S)$ and $A'\in\mathcal{A}_C(S)$. Consider the following push-out diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & H & \to & A & \to & N & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & H & \to & U & \to & H' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & A' & = & A' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

By the middle row in this diagram, we have $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H^{'} \leq n$. Because the middle column in the above diagram is $\mathrm{Hom}_S(-, \mathcal{I}_C(S))$-exact, the rightmost column is also $\mathrm{Hom}_S(-, \mathcal{I}_C(S))$-exact by [11, Lemma 2.4(2)] and it is the desired exact sequence.

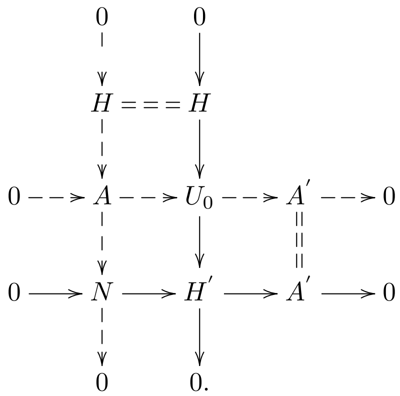
(5) $\Rightarrow$ (1) Let

$$0 \to N \to H^{'} \to A^{'} \to 0$$

be an exact sequence in $\mathrm{Mod}\, S$ with $A^{'} \in \mathcal{A}_C(S)$ and $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H^{'} \leq n$. Then there exists an exact sequence

$$0 \to U_n \to \cdots \to U_1 \to U_0 \to H^{'} \to 0$$

in $\mathrm{Mod}\, S$ with all U_i in $\mathcal{I}_C(S)$. Set $H := \mathrm{Ker}(U_0 \to H^{'})$. Then $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H \leq n-1$. Consider the following pull-back diagram



Applying [10, Theorem 6.2] to the middle row in this diagram yields $A \in \mathcal{A}_C(S)$. Thus $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, N \leq n$ by the leftmost column in the above diagram. $\square$

The only place where the assumption $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, N < \infty$ in Theorem 4.4 is used is in showing (3) $\Rightarrow$ (2). By Theorem 4.4, it is easy to get the following standard observation.

Corollary 4.5. *Let*

$$0 \to L \to M \to K \to 0$$

be an exact sequence in $\mathrm{Mod}\, S$. *Then we have*

(1) $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, K \leq \max\{\mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, M, \mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, L + 1\}$, *and the equality holds true if* $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, M \neq \mathcal{A}_C(S)$-$\mathrm{pd}_S\, L$.

(2) $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, L \leq \max\{\mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, M, \mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, K - 1\}$, *and the equality holds true if* $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, M \neq \mathcal{A}_C(S)$-$\mathrm{pd}_S\, K$.

(3) $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, M \leq \max\{\mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, L, \mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, K\}$, *and the equality holds true if* $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, K \neq \mathcal{A}_C(S)$-$\mathrm{pd}_S\, L + 1$.

The following corollary is an addendum to the implications (1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) in Theorem 4.4.

Corollary 4.6. *Let* $N \in \mathrm{Mod}\, S$ *with* $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, N = n (< \infty)$. *Then there exist exact sequences*

$$0 \to H \to A \to N \to 0,$$

$$0 \to N \to H' \to A' \to 0$$

in $\operatorname{Mod} S$ *with* $A, A' \in \mathcal{A}_C(S)$ *and* $\mathcal{I}_C(S)\text{-pd}_S H = \mathcal{I}_C(S)\text{-pd}_S H' = n$.

Proof. Let $N \in \operatorname{Mod} S$ with $\mathcal{A}_C(S)\text{-pd}_S N = n(< \infty)$. By Theorem 4.4, there exists an exact sequence

$$0 \to H \to A \to N \to 0$$

in $\operatorname{Mod} S$ with $A \in \mathcal{A}_C(S)$ and $(\mathcal{A}_C(S)\text{-pd}_S H \leq)\mathcal{I}_C(S)\text{-pd}_S H \leq n-1$. By Theorem 4.4 again, we have $\sup\{i \geq 0 \mid \operatorname{Tor}_i^S(C,N) \neq 0\} = n$. So $\sup\{i \geq 0 \mid \operatorname{Tor}_i^S(C,H) \neq 0\} = n-1$, and hence $\mathcal{A}_C(S)\text{-pd}_S H = n-1$ by Theorem 4.4. It follows that $\mathcal{I}_C(S)\text{-pd}_S H = n-1$.

By Theorem 4.4, there exists an exact sequence

$$0 \to N \to H' \to A' \to 0$$

in $\operatorname{Mod} S$ with $A' \in \mathcal{A}_C(S)$ and $(\mathcal{A}_C(S)\text{-pd}_S H \leq)\mathcal{I}_C(S)\text{-pd}_S H' \leq n$. By Corollary 4.5(3), we have $\mathcal{A}_C(S)\text{-pd}_S H = \mathcal{A}_C(S)\text{-pd}_S N = n$, and so $\mathcal{I}_C(S)\text{-pd}_S H' = n$. □

Let $N \in \operatorname{Mod} S$. Bican, El Bashir and Enochs proved in [3] that N has a flat cover. We use

$$\cdots \xrightarrow{f_{n+1}} F_n(N) \xrightarrow{f_n} \cdots \xrightarrow{f_2} F_1(N) \xrightarrow{f_1} F_0(N) \xrightarrow{f_0} N \to 0 \tag{4.2}$$

to denote a minimal flat resolution of N in $\operatorname{Mod} S$, where each $F_i(N) \to \operatorname{Im} f_i$ is a flat cover of $\operatorname{Im} f_i$.

Lemma 4.7. *Let* $N \in \operatorname{Mod} S$ *and* $n \geq 0$. *If* $\operatorname{Tor}^S_{1\leq i\leq n}(C,N) = 0$, *then we have*

(1) There exists an exact sequence

$$0 \to \operatorname{Ext}_R^{n+1}(C, \operatorname{Ker}(1_C \otimes f_{n+1})) \to N \xrightarrow{\mu_N} (C \otimes_S N)_* \to \operatorname{Ext}_R^{n+2}(C, \operatorname{Ker}(1_C \otimes f_{n+1})) \to 0$$

in $\operatorname{Mod} S$.

(2) $\operatorname{Ext}_R^{1\leq i\leq n}(C, \operatorname{Ker}(1_C \otimes f_{n+1})) = 0$.

Proof. (1) The case for $n = 0$ follows from [16, Proposition 3.2]. Now suppose $n \geq 1$. If $\operatorname{Tor}^S_{1\leq i\leq n}(C,N) = 0$, then the exact sequence (4.2) yields the following exact sequence

$$\begin{aligned} 0 \to \operatorname{Ker}(1_C \otimes f_{n+1}) \to C \otimes_S F_{n+1}(N) &\xrightarrow{1_C\otimes f_{n+1}} C \otimes_S F_n(N) \xrightarrow{1_C\otimes f_n} \cdots \\ &\xrightarrow{1_C\otimes f_2} C \otimes_S F_1(N) \xrightarrow{1_C\otimes f_1} C \otimes_S F_0(N) \xrightarrow{1_C\otimes f_0} C \otimes_S N \to 0 \end{aligned} \tag{4.3}$$

in $\operatorname{Mod} R$. Because all $C \otimes_S F_i(N)$ are in ${}_RC^{\perp}$ by [16, Lemma 2.3(1)], we have

$$\operatorname{Ext}_R^1(C, \operatorname{Ker}(1_C \otimes f_1)) \cong \operatorname{Ext}_R^{n+1}(C, \operatorname{Ker}(1_C \otimes f_n)),$$

$$\operatorname{Ext}_R^2(C, \operatorname{Ker}(1_C \otimes f_1)) \cong \operatorname{Ext}_R^{n+2}(C, \operatorname{Ker}(1_C \otimes f_n)).$$

Now the assertion follows from [16, Proposition 3.2].

(2) Applying the functor $(-)_*$ to the exact sequence (4.3) we get the following commutative diagram

$$\begin{array}{ccccccccc} & & F_{n+1}(N) & \xrightarrow{f_{n+1}} & F_n(N) & \xrightarrow{f_n} \cdots \xrightarrow{f_1} & F_0(N) \\ & & \downarrow{\scriptstyle \mu_{F_{n+1}(N)}} & & \downarrow{\scriptstyle \mu_{F_n(N)}} & & \downarrow{\scriptstyle \mu_{F_0(N)}} \\ 0 \longrightarrow (\operatorname{Ker}(1_C \otimes f_{n+1}))_* & \longrightarrow & (C \otimes_S F_{n+1}(N))_* & \xrightarrow{(1_C\otimes f_{n+1})_*} & (C \otimes_S F_n(N))_* & \xrightarrow{(1_C\otimes f_n)_*} \cdots \xrightarrow{(1_C\otimes f_1)_*} & (C \otimes_S F_0(N))_*. \end{array}$$

All columns are isomorphisms by [10, Lemma 4.1]. So the bottom row in this diagram is exact. Because all $C \otimes_S F_i(N)$ are in ${}_RC^{\perp}$, we have $\operatorname{Ext}_R^{1\leq i\leq n}(C, \operatorname{Ker}(1_C \otimes f_{n+1})) = 0$. □

Let $X \in \operatorname{Mod} R$ and let

$$\cdots \xrightarrow{g_{n+1}} P_n \xrightarrow{g_n} \cdots \xrightarrow{g_2} P_1 \xrightarrow{g_1} P_0 \xrightarrow{g_0} X \to 0$$

be a projective resolution of X in $\operatorname{Mod} R$. If there exists $n \geq 1$ such that $\operatorname{Im} g_n \cong \oplus_j W_j$, where each W_j is isomorphic to a direct summand of some $\operatorname{Im} g_{i_j}$ with $i_j < n$, then we say that X **has an ultimately closed projective resolution at** n; and we say that X **has an ultimately closed projective resolution** if it has an ultimately closed projective resolution at some n ([13]). It is trivial that if $\operatorname{pd}_R X$ (the projective dimension of X) $\leq n$, then X has an ultimately closed projective resolution at $n+1$. Let R be an artin algebra. If either R is of finite representation type or the square of the radical of R is zero, then any finitely generated left R-module has an ultimately closed projective resolution ([13, p.341]). Following [20], a module $N \in \operatorname{Mod} S$ is called C**-adstatic** if μ_N is an isomorphism.

Proposition 4.8. *Let $N \in \operatorname{Mod} S$ and $n \geq 1$. If $\operatorname{Tor}^S_{1\leq i\leq n}(C, N) = 0$, then N is C-adstatic provided that one of the following conditions is satisfied.*

(1) $\operatorname{pd}_R C \leq n$.

(2) ${}_R C$ has an ultimately closed projective resolution at n.

Proof. (1) It follows directly from Lemma 4.7(1).

(2) Let

$$\cdots \xrightarrow{g_{n+1}} P_n \xrightarrow{g_n} \cdots \xrightarrow{g_2} P_1 \xrightarrow{g_1} P_0 \xrightarrow{g_0} C \to 0$$

be a projective resolution of C in $\operatorname{Mod} R$ ultimately closed at n. Then $\operatorname{Im} g_n \cong \oplus_j W_j$ such that each W_j is isomorphic to a direct summand of some $\operatorname{Im} g_{i_j}$ with $i_j < n$. Let $N \in \operatorname{Mod} S$ with $\operatorname{Tor}^S_{1\leq i\leq n}(C, N) = 0$. By Lemma 4.7(2), we have

$$\operatorname{Ext}^1_R(\operatorname{Im} g_{i_j}, \operatorname{Ker}(1_C \otimes f_{n+1})) \cong \operatorname{Ext}^{i_j+1}_R(C, \operatorname{Ker}(1_C \otimes f_{n+1})) = 0.$$

Because W_j is isomorphic to a direct summand of some $\operatorname{Im} g_{i_j}$, we have $\operatorname{Ext}^1_R(W_j, \operatorname{Ker}(1_C \otimes f_{n+1})) = 0$ for any j, which implies

$$\begin{aligned} & \operatorname{Ext}^{n+1}_R(C, \operatorname{Ker}(1_C \otimes f_{n+1})) \\ \cong\ & \operatorname{Ext}^1_R(\operatorname{Im} g_n, \operatorname{Ker}(1_C \otimes f_{n+1})) \\ \cong\ & \operatorname{Ext}^1_R(\oplus_j W_j, \operatorname{Ker}(1_C \otimes f_{n+1})) \\ \cong\ & \Pi_j \operatorname{Ext}^1_R(W_j, \operatorname{Ker}(1_C \otimes f_{n+1})) \\ =\ & 0. \end{aligned}$$

Then by Lemma 4.7(2), we conclude that $\operatorname{Ext}^{1\leq i\leq n+1}_R(C, \operatorname{Ker}(1_C \otimes f_{n+1})) = 0$. Similar to the above argument we get $\operatorname{Ext}^{n+2}_R(C, \operatorname{Ker}(1_C \otimes f_{n+1})) = 0$. It follows from Lemma 4.7(1) that μ_N is an isomorphism and N is C-adstatic. □

Corollary 4.9. *For any $n \geq 1$, a module $N \in \operatorname{Mod} S$ satisfying $\operatorname{Tor}^S_{0\leq i\leq n}(C, N) = 0$ implies $N = 0$ provided that one of the following conditions is satisfied.*

(1) $\operatorname{pd}_R C \leq n$.

(2) $_RC$ *has an ultimately closed projective resolution at* n*.*

Proof. Let $N \in \operatorname{Mod} S$ with $\operatorname{Tor}^S_{0\leq i\leq n}(C, N) = 0$. By Proposition 4.8, we have that N is C-adstatic and $N \cong (C \otimes_S N)_* = 0$. $\square$

We now are in a position to give the following

Theorem 4.10. *If* $_RC$ *has an ultimately closed projective resolution, then*

$$\mathcal{A}_C(S) = {C_S}^{\top} = {}^{\perp}\mathcal{I}_C(S).$$

Proof. By the definition of $\mathcal{A}_C(S)$ and Lemma 4.3, we have $\mathcal{A}_C(S) \subseteq {C_S}^{\top} = {}^{\perp}\mathcal{I}_C(S)$.

Now let $N \in {}^{\perp}\mathcal{I}_C(S)$ and let $f : C \otimes_S N \to B$ be a $\mathcal{B}_C(R)$-preenvelope of $C \otimes_S N$ in $\operatorname{Mod} R$ as in Theorem 3.7. Because $\mathcal{B}_C(R)$ is injectively coresolving in $\operatorname{Mod} R$ by [10, Theorem 6.2], f is monic. By Proposition 4.8, μ_N is an isomorphism. Then by Theorem 3.7(1), we have a monic $\mathcal{A}_C(S)$-preenvelope

$$f^0 : N \rightarrowtail A^0$$

of N, where $f^0 = f_*\mu_N$ and $A^0 = B_*$. So we have a $\operatorname{Hom}_S(-, \mathcal{A}_C(S))$-exact exact sequence

$$0 \to N \xrightarrow{f^0} A^0 \to N^1 \to 0$$

in $\operatorname{Mod} S$, where $N^1 = \operatorname{Coker} f^0$. Because $A^0 \in {}^{\perp}\mathcal{I}_C(S)$, we have $N^1 \in {}^{\perp}\mathcal{I}_C(S)$. Similar to the above argument, we get a $\operatorname{Hom}_S(-, \mathcal{A}_C(S))$-exact exact sequence

$$0 \to N^1 \xrightarrow{f^1} A^1 \to N^2 \to 0$$

in $\operatorname{Mod} S$ with $A^1 \in \mathcal{A}_C(S)$ and $N^2 \in {}^{\perp}\mathcal{I}_C(S)$. Repeating this procedure, we get a $\operatorname{Hom}_S(-, \mathcal{A}_C(S))$-exact exact sequence

$$0 \to N \xrightarrow{f^0} A^0 \xrightarrow{f^1} A^1 \xrightarrow{f^2} \cdots \xrightarrow{f^i} A^i \xrightarrow{f^{i+1}} \cdots$$

in $\operatorname{Mod} S$ with all A^i in $\mathcal{A}_C(S)$. Because $\mathcal{I}_C(S) \subseteq \mathcal{A}_C(S)$ by [10, Corollary 6.1], this exact sequence is $\operatorname{Hom}_S(-, \mathcal{I}_C(S))$-exact. By [16, Theorem 3.11(1)], there exists a $\operatorname{Hom}_S(-, \mathcal{A}_C(S))$-exact exact sequence

$$0 \to A^i \to U^i_0 \to U^i_1 \to \cdots \to U^i_j \to \cdots$$

in $\operatorname{Mod} S$ with all U^i_j in $\mathcal{I}_C(S)$ for any $i, j \geq 0$. Then by [11, Corollary 3.9], we get the following $\operatorname{Hom}_S(-, \mathcal{A}_C(S))$-exact exact sequence

$$0 \to N \to U^0_0 \to U^0_1 \oplus U^1_0 \to \cdots \to \oplus^n_{i=0} U^i_{n-i} \to \cdots$$

in $\operatorname{Mod} S$ with all terms in $\mathcal{I}_C(S)$. It follows from [16, Theorem 3.11(1)] that $N \in \mathcal{A}_C(S)$. The proof is finished. $\square$

We use $\operatorname{pd}_{S^{op}} C$ and $\operatorname{fd}_{S^{op}} C$ to denote the projective and flat dimensions of C_S respectively.

Corollary 4.11. *If* $_RC$ *has an ultimately closed projective resolution, then the following statements are equivalent for any* $n \geq 0$*.*

(1) $\operatorname{pd}_{S^{op}} C \leq n$*.*

(2) $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, N \leq n$ *for any* $N \in \mathrm{Mod}\, S$.

Proof. Assume that ${}_RC$ has an ultimately closed projective resolution. By Theorem 4.10, we have $\mathcal{A}_C(S) = {C_S}^{\top}$. Then it is easy to see that C_S is flat (equivalently, projective) if and only if $\mathcal{A}_C(S) = \mathrm{Mod}\, S$, so the assertion for the case $n = 0$ follows. Now let $N \in \mathrm{Mod}\, S$ and $n \geq 1$.

(2) $\Rightarrow$ (1) By (2) and Theorem 4.4, we have $\Omega^n(N) \in \mathcal{A}_C(S)(\subseteq {C_S}^{\top})$. Then by the dimension shifting, we have $\mathrm{Tor}^S_{\geq n+1}(C, N) = 0$, and so $\mathrm{pd}_{S^{op}}\, C = \mathrm{fd}_{S^{op}}\, C \leq n$.

(1) $\Rightarrow$ (2) If $\mathrm{pd}_{S^{op}}\, C \leq n$, then $\Omega^n(N) \in {C_S}^{\top}$ by the dimension shifting. By Theorem 4.10, we have $\Omega^n(N) \in \mathcal{A}_C(S)$ and $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, N \leq n$. □

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References

[1] M. Auslander and M. Bridger, Stable Module Theory, Memoirs Amer. Math. Soc. **94**, Amer. Math. Soc., Providence, RI, 1969.

[2] A. Beligiannis and I. Reiten, Homological and Homotopical Aspects of Torsion Theories, Mem. Amer. Math. Soc. **188** (883), Amer. Math. Soc., Providence, RI, 2007.

[3] L. Bican, R. El Bashir and E. E. Enochs, *All modules have flat covers*, Bull. London Math. Soc. **33** (2001), 385–390.

[4] E. E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (1981), 189–209.

[5] E. E. Enochs and H. Holm, *Cotorsion pairs associated with Auslander categories*, Israel J. Math. **174** (2009), 253–268.

[6] E. E. Enochs and A. Iacob, *Gorenstein injective covers and envelopes over noetherian rings*, Proc. Amer. Math. Soc. **143** (2015), 5–12.

[7] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, de Gruyter Expositions in Math. **30**, Walter de Gruyter GmbH & Co. KG, Berlin, 2000.

[8] R. Göbel and J. Trlifaj, Approximations and Endomorphism Algebras of Modules, de Gruyter Expositions in Math. **41**, 2nd revised and extended edition, Walter de Gruyter GmbH & Co. KG, Berlin-Boston, 2012.

[9] H. Holm and P. Jørgensen, *Cotorsion pairs induced by duality pairs*, J. Commut. Algebra **1** (2009), 621–633.

[10] H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), 781–808.

[11] Z. Y. Huang, *Proper resolutions and Gorenstein categories*, J. Algebra **393** (2013), 142–169.

[12] Z.Y. Huang, *Homological dimensions relative to preresolving subcategories*, Kyoto J. Math. **54** (2014), 727–757.

[13] J. P. Jans, *Some generalizations of finite projective dimension*, Illinois J. Math. **5** (1961), 334–344.

[14] Z. F. Liu, Z. Y. Huang and A. M. Xu, *Gorenstein projective dimension relative to a semidualizing bimodule*, Comm. Algebra **41** (2013), 1–18.

[15] F. Mantese and I. Reiten, *Wakamatsu tilting modules*, J. Algebra **278** (2004), 532–552.

[16] X. Tang and Z. Y. Huang, *Homological aspects of the adjoint cotranspose*, Colloq. Math. **150** (2017), 293–311.

[17] T. Wakamatsu, *On modules with trivial self-extensions*, J. Algebra **114** (1988), 106–114.

[18] T. Wakamatsu, *Stable equivalence for self-injective algebras and a generalization of tilting modules*, J. Algebra **134** (1990), 298–325.

[19] T. Wakamatsu, *Tilting modules and Auslander's Gorenstein property*, J. Algebra **275** (2004), 3–39.

[20] R. Wisbauer, *Static modules and equivalences*, Interactions Between Ring Theory and Representations of Algebras (Murcia), Lecture Notes in Pure and Appl. Math. **210**, Dekker, New York, 2000, pp.423–449.