EXTENSION CLOSURE OF ADJOINT COTORSIONFREE MODULES

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Abstract. Let R and S be rings and ${}_R\omega_S$ a semidualizing bimodule, and let $n \geqslant 1$. We characterize the extension closure of the category of adjoint k-cotorsionfree modules with respect to ω for any $1 \leqslant k \leqslant n$ in terms of the (strong) cograde conditions of certain modules.

1. Introduction

Throughout this paper, all rings are associative rings with units. For a ring R, we use $\operatorname{Mod} R$ to denote the category of left R-modules. Recall that a subcategory $\mathcal{X}$ of $\operatorname{Mod} R$ is called *extension closed* provided that for any exact sequence

$$0 \to A \to B \to C \to 0$$

in $\operatorname{Mod} R$, if A and C are in $\mathcal{X}$, then so is B. The extension closure of certain subcategories has been proved to be important in characterizing rings. We mention a well-known result about the extension closure of the category of n-syzygy modules over a Noetherian algebra R ([2, Theorem 4.7]), which shows that the category of finitely generated i-syzygy modules is extension closed for any $1 \leqslant i \leqslant n$ if and only if R is quasi n-Gorenstein in the sense of [9]. Applying this theorem, under Serre's condition, Goto and Takahashi characterized a commutative Noetherian local ring in height less than n to be Gorenstein in terms of the extension closure of the category of finitely generated n-syzygy modules ([6, Theorem B]). The extension closure of various subcategories has been studied extensively, see [3]–[4], [10]–[11], [15] and references therein.

In particular, Huang [10] initialed the study of extension closure of the category of n-torsionfree modules with respect to a semidualizing bimodule ${}_R\omega_S$ by using the properties of the (strong) grade of modules. In [16] and [18] we dualized the Auslander transpose and introduced the notions of n-ω-cotorsionfree modules and adjoint n-ω-cotorsionfree modules respectively. These two classes have many dual properties of relative n-torsionfree modules. It is thus natural to ask the following question:

Question 1.1. When are the category of n-ω-cotorsionfree modules and that of adjoint n-ω-cotorsionfree modules extension closed?

This question has been partially solved by Zhao and Zhang so far, and they proved that the category of i-ω-cotorsionfree modules is extension closed for any $1 \leqslant i \leqslant n$ if and only if the strong Tor-cograde of $\operatorname{Ext}_R^{i+1}(\omega, M)$ is at least i for any ω-i-syzygy module M and $1 \leqslant i \leqslant n$ ([22, Theorem 3.10]). The purpose of this

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paper is to proceed with the study of Question 1.1. Indeed, we will investigate the extension closure of the category of adjoint n-ω-cotorsionfree modules.

The organization of this paper is as follows. Section 2 contains some basic definitions and preliminary results. Let R, S be arbitrary rings and let ${}_R\omega_S$ be a semidualizing bimodule. In Section 3, we show that the categories of adjoint 1-cotorsionfree modules and adjoint 2-cotorsionfree modules are extension closed if and only if $\mathrm{Tor}^S_{k-1}(\omega, \mathrm{Ext}^k_R(\omega, M)) = 0$ for any left R-module M and $k = 1, 2$, and if and only if the Tor-cograde of $\mathrm{Ext}^k_R(\omega, M)$ with respect to ω is at least k for any left R-module M and $k = 1, 2$ (Theorem 3.10).

Let $\mathcal{A}_\omega(S)$ be the Auslander class with respect to ω. In Section 4, we show that the category of adjoint k-ω-cotorsionfree modules is extension closed for any $1 \leqslant k \leqslant n$, if and only if category of k-$\mathcal{A}_\omega(S)$-syzygy modules is extension closed for any $1 \leqslant k \leqslant n$, if and only if the strong Ext-cograde of $\mathrm{Tor}^S_{k+1}(\omega, N)$ with respect to ω is at least k for any left S-module N and $1 \leqslant k \leqslant n$, and if and only if the Tor-cograde of $\mathrm{Ext}^k_R(\omega, M)$ with respect to ω is at least k for any left module M and $1 \leqslant k \leqslant n$ (Theorem 4.6). As a consequence, we obtain some equivalent characterizations of right quasi n-Gorenstein rings (Corollary 4.8).

2. **Preliminaries**

This section is devoted to stating the definitions and basic properties of notions which are needed in the sequel.

Definition 2.1. ([1, 8]). Let R and S be rings. An (R, S)-bimodule ${}_R\omega_S$ is called *semidualizing* if the following conditions are satisfied.

(a1) ${}_R\omega$ admits a degreewise finite R-projective resolution.
(a2) ω_S admits a degreewise finite S^{op}-projective resolution.
(b1) The homothety map ${}_RR_R \stackrel{R\gamma}{\to} \mathrm{Hom}_{S^{op}}(\omega, \omega)$ is an isomorphism.
(b2) The homothety map ${}_SS_S \stackrel{\gamma_S}{\to} \mathrm{Hom}_R(\omega, \omega)$ is an isomorphism.
(c1) $\mathrm{Ext}^{\geqslant 1}_R(\omega, \omega) = 0$.
(c2) $\mathrm{Ext}^{\geqslant 1}_{S^{op}}(\omega, \omega) = 0$.

From now on, R and S are arbitrary rings and we fix a semidualizing bimodule ${}_R\omega_S$. For convenience, We write

$$(-)_* := \mathrm{Hom}(\omega, -),$$

$${}_R\omega^{\perp} := \{M \in \mathrm{Mod}\, R \mid \mathrm{Ext}^{\geqslant 1}_R(\omega, M) = 0\},$$

$${\omega_S}^{\top} := \{N \in \mathrm{Mod}\, S \mid \mathrm{Tor}^S_{\geqslant 1}(\omega, N) = 0\}.$$

Following [8], set

$$\mathcal{F}_\omega(R) := \{\omega \otimes_S F \mid F \text{ is flat in } \mathrm{Mod}\, S\},$$

$$\mathcal{P}_\omega(R) := \{\omega \otimes_S P \mid P \text{ is projective in } \mathrm{Mod}\, S\},$$

$$\mathcal{I}_\omega(S) := \{I_* \mid I \text{ is injective in } \mathrm{Mod}\, R\}.$$

The modules in $\mathcal{F}_\omega(R)$, $\mathcal{P}_\omega(R)$ and $\mathcal{I}_\omega(S)$ are called *ω-flat*, *ω-projective* and *ω-injective* respectively. For a subcategory $\mathcal{X}$ of $\mathrm{Mod}\, R$ (resp. $\mathrm{Mod}\, S$), we use $\mathrm{Add}\, \mathcal{X}$ (resp. $\mathrm{Prod}\, \mathcal{X}$) to denote the subcategory of $\mathrm{Mod}\, R$ (resp. $\mathrm{Mod}\, S$) consisting of modules isomorphic to direct summands of direct sums (resp. products) of modules in $\mathcal{X}$.

We write $(-)^+ := \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers. By [13, Proposition 2.4], we have

$$\mathcal{P}_\omega(R) = \mathrm{Add}_R\,\omega \ \text{ and } \ \mathcal{I}_\omega(S) = \mathrm{Prod}\,\omega^+.$$

Let $M \in \mathrm{Mod}\,R$ and $N \in \mathrm{Mod}\,S$. Then we have the following two canonical valuation homomorphisms

$$\theta_M : \omega \otimes_S M_* \to M$$

defined by $\theta_M(x \otimes f) = f(x)$ for any $x \in \omega$ and $f \in M_*$; and

$$\mu_N : N \to (\omega \otimes_S N)_*$$

defined by $\mu_N(y)(x) = x \otimes y$ for any $y \in N$ and $x \in \omega$. Recall that a module $M \in \mathrm{Mod}\,R$ is called *ω-cotorsionless* (resp. *ω-coreflexive*) if θ_M is an epimorphism (resp. an isomorphism) ([16]); and a module $N \in \mathrm{Mod}\,S$ is called *adjoint ω-cotorsionless* (resp. *adjoint ω-coreflexive*) if μ_N is a monomorphism (resp. an isomorphism) ([19]).

Definition 2.2. ([8]).

(1) The *Auslander class* $\mathcal{A}_\omega(S)$ with respect to ω consists of all left S-modules N satisfying the following conditions.
  (A1) $N \in {\omega_S}^\top$.
  (A2) $\omega \otimes_S N \in {}_R\omega^\perp$.
  (A3) N is adjoint ω-coreflexive.

(2) The *Bass class* $\mathcal{B}_\omega(R)$ with respect to ω consists of all left R-modules M satisfying the following conditions.
  (B1) $M \in {}_R\omega^\perp$.
  (B2) $M_* \in {\omega_S}^\top$.
  (B3) M is ω-coreflexive.

For a module $M \in \mathrm{Mod}\,R$, we use

$$0 \to M \to I^0(M) \xrightarrow{g^0} I^1(M) \tag{2.1}$$

to denote the minimal injective copresentation of M in $\mathrm{Mod}\,R$. For a module $N \in \mathrm{Mod}\,S$, we use

$$F_1(N) \xrightarrow{f_0} F_0(N) \to N \to 0 \tag{2.2}$$

to denote the minimal flat presentation of N in $\mathrm{Mod}\,S$.

Definition 2.3. ([16, 18]). Let $M \in \mathrm{Mod}\,R$ and $N \in \mathrm{Mod}\,S$, and let $n \geqslant 1$.

(1) $\mathrm{cTr}_\omega\, M := \mathrm{Coker}({g^0}_*)$ is called the *cotranspose* of M with respect to ω, where g^0 is as in (2.1).
(2) M is called *n-ω-cotorsionfree* if $\mathrm{Tor}^S_{1\leqslant i\leqslant n}(\omega, \mathrm{cTr}_\omega\, M) = 0$.
(3) $\mathrm{acTr}_\omega\, N := \mathrm{Ker}(1_\omega \otimes f_0)$ is called the *adjoint cotranspose* of N with respect to ω, where f_0 is as in (2.2).
(4) N is called *adjoint n-ω-cotorsionfree* if $\mathrm{Ext}_R^{1\leqslant i\leqslant n}(\omega, \mathrm{acTr}_\omega\, N) = 0$.

We use $\mathrm{c}\mathcal{T}^n_\omega(R)$ (resp. $\mathrm{ac}\mathcal{T}^n_\omega(S)$) to denote the subcategory of $\mathrm{Mod}\,R$ (resp. $\mathrm{Mod}\,S$) consisting of n-ω-cotorsionfree (resp, adjoint n-ω-cotorsionfree) modules. By [16, Proposition 3.2], we have that a module in $\mathrm{Mod}\,R$ is ω-cotorsionless (resp. ω-coreflexive) if and only if it is in $\mathrm{c}\mathcal{T}^1_\omega(R)$ (resp. $\mathrm{c}\mathcal{T}^2_\omega(R)$). In particular, we have

$$\mathcal{F}_\omega(R) \subseteq \mathcal{B}_\omega(R) \subseteq \mathrm{c}\mathcal{T}^i_\omega(R)$$

for any $i \geqslant 1$ by [8, Corollary 6.1] and [16, Theorem 3.9]. On the other hand, by [18, Proposition 3.2], we have that a module in $\operatorname{Mod} S$ is adjoint ω-cotorsionless (resp. adjoint ω-coreflexive) if and only if it is in $\mathrm{ac}\mathcal{T}_{\omega}^{1}(S)$ (resp. $\mathrm{ac}\mathcal{T}_{\omega}^{2}(S)$). We have

$$\mathcal{I}_{\omega}(S) \subseteq \mathcal{A}_{\omega}(S) \subseteq \mathrm{ac}\mathcal{T}_{\omega}^{i}(S)$$

for any $i \geqslant 1$ by [8, Corollary 6.1] and [18, Proposition 3.4].

Definition 2.4. ([17])

(1) Let $M \in \operatorname{Mod} R$ and $n \geqslant 0$. The Ext-*cograde* of M with respect to ω is defined as $\text{E-cograde}_{\omega} M := \inf\{i \geqslant 0 \mid \operatorname{Ext}_{R}^{i}(\omega, M) \neq 0\}$; and *the strong* Ext-*cograde* of M with respect to ω, denoted by $\text{s.E-cograde}_{\omega} M$, is said to be at least n if $\text{E-cograde}_{\omega} X \geqslant n$ for any quotient module X of M.

(2) Let $N \in \operatorname{Mod} S$ and $n \geqslant 0$. The Tor-*cograde* of N with respect to ω is defined as $\text{T-cograde}_{\omega} N := \inf\{i \geqslant 0 \mid \operatorname{Tor}_{i}^{S}(\omega, N) \neq 0\}$; and *the strong* Tor-*cograde* of N with respect to ω, denoted by $\text{s.T-cograde}_{\omega} N$, is said to be at least n if $\text{T-cograde}_{\omega} Y \geqslant n$ for any submodule Y of N.

Definition 2.5. ([20]) Let $\mathcal{X}$ be a subcategory of an abelian category $\mathcal{E}$ and $n \geqslant 1$. If there exists an exact sequence

$$0 \to N \to X_0 \to \cdots \to X_{n-1} \to M \to 0$$

in $\mathcal{E}$ with all X_i in $\mathcal{X}$, then N is called an *n-$\mathcal{X}$-syzygy* of M and M is called an *n-$\mathcal{X}$-cosyzygy* of N.

For subcategories $\mathcal{X}, \mathcal{Y}$ of an abelian category $\mathcal{E}$ and $n \geqslant 1$, we write

$$\Omega_{\mathcal{X}}^{n}(\mathcal{Y}) := \{N \in \mathcal{E} \mid N \text{ is an } n\text{-}\mathcal{X}\text{-syzygy of some object in } \mathcal{Y}\},$$

$$\mathrm{co}\Omega_{\mathcal{X}}^{n}(\mathcal{Y}) := \{M \in \mathcal{E} \mid M \text{ is an } n\text{-}\mathcal{X}\text{-cosyzygy of some object in } \mathcal{Y}\}.$$

In particular, $\Omega_{\mathcal{X}}^{0}(\mathcal{Y}) = \mathcal{Y} = \mathrm{co}\Omega_{\mathcal{X}}^{0}(\mathcal{Y})$ and $\Omega_{\mathcal{X}}^{-1}(\mathcal{Y}) = 0 = \mathrm{co}\Omega_{\mathcal{X}}^{-1}(\mathcal{Y})$. For convenience, we write

$$\Omega_{\mathcal{A}}^{n}(S) := \Omega_{\mathcal{A}_{\omega}(S)}^{n}(\operatorname{Mod} S),\ \Omega_{\mathcal{I}_{\omega}}^{n}(S) := \Omega_{\mathcal{I}_{\omega}(S)}^{n}(\operatorname{Mod} S),\ \Omega_{\mathrm{ac}\mathcal{T}_{\omega}^{i}}^{n}(S) := \Omega_{\mathrm{ac}\mathcal{T}_{\omega}^{i}}^{n}(\operatorname{Mod} S),$$

$$\mathrm{co}\Omega_{\mathcal{B}}^{n}(R) := \mathrm{co}\Omega_{\mathcal{B}_{\omega}(R)}^{n}(\operatorname{Mod} R),\ \mathrm{co}\Omega_{\mathcal{F}_{\omega}}^{n}(R) := \mathrm{co}\Omega_{\mathcal{F}_{\omega}(R)}^{n}(\operatorname{Mod} R),$$

$$\mathrm{co}\Omega_{\mathcal{P}_{\omega}}^{n}(R) := \mathrm{co}\Omega_{\mathcal{P}_{\omega}(R)}^{n}(\operatorname{Mod} R),\ \mathrm{co}\Omega_{\mathrm{c}\mathcal{T}_{\omega}^{i}}^{n}(R) := \mathrm{co}\Omega_{\mathrm{c}\mathcal{T}_{\omega}^{i}}^{n}(\operatorname{Mod} R).$$

3. Tor-cograde and extension closure

Our aim in this section is to show how the extension closure of the subcategories $\mathrm{ac}\mathcal{T}_{\omega}^{1}(S)$ and $\mathrm{ac}\mathcal{T}_{\omega}^{2}(S)$ is connected with the Tor-cograde of $\operatorname{Ext}_{R}^{k}(\omega, M)$ for any $M \in \operatorname{Mod} R$ and $k = 1, 2$.

In what follows, for any $i \geqslant 1$, we use $\mathcal{C}_i$ (resp. $\mathcal{D}_i$) to denote a subcategory of $\operatorname{Mod} R$ (resp. $\operatorname{Mod} S$) satisfying

$$\mathcal{F}_{\omega}(R) \subseteq \mathcal{C}_i \subseteq \mathrm{c}\mathcal{T}_{\omega}^{i}(R) \text{ (resp. } \mathcal{I}_{\omega}(S) \subseteq \mathcal{D}_i \subseteq \mathrm{ac}\mathcal{T}_{\omega}^{i}(S)).$$

We begin by proving the following lemma.

Lemma 3.1. *For any $i \geqslant 1$, it holds that*

(1) $\Omega_{\mathcal{D}_i}^{1}(S) = \mathrm{ac}\mathcal{T}_{\omega}^{1}(S)$.

(2) $\mathrm{co}\Omega_{\mathcal{C}_i}^{1}(R) = \mathrm{c}\mathcal{T}_{\omega}^{1}(R)$.

Proof. (1) Since $\mathcal{I}_\omega(S) \subseteq \mathcal{D}_i$, we have $\mathrm{ac}\mathcal{T}^1_\omega(S) \subseteq \Omega^1_{\mathcal{D}_i}(S)$ by [18, Lemma 3.7(1)]. Now let $N \in \Omega^1_{\mathcal{D}_i}(S)$. We may assume that $f^0 : N \rightarrowtail H$ is a monomorphism in $\mathrm{Mod}\, S$ with $H \in \mathcal{D}_i$. As $\mathcal{D}_i \subseteq \mathrm{ac}\mathcal{T}^i_\omega(S) \subseteq \mathrm{ac}\mathcal{T}^1_\omega(S)$, we have that μ_H a monomorphism. Then from the following commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{f^0} & H \\ \downarrow{\scriptstyle \mu_N} & & \downarrow{\scriptstyle \mu_H} \\ (\omega \otimes_S N)_* & \xrightarrow{(1_\omega \otimes f^0)_*} & (\omega \otimes_S H)_*, \end{array}$$

we get that μ_N is a monomorphism. Thus $N \in \mathrm{ac}\mathcal{T}^1_\omega(S)$ and $\Omega^1_{\mathcal{D}_i}(S) \subseteq \mathrm{ac}\mathcal{T}^1_\omega(S)$.

(2) Since $\mathcal{P}_\omega(R) \subseteq \mathcal{C}_i$, we have $\mathrm{c}\mathcal{T}^1_\omega(R) \subseteq \mathrm{co}\Omega^1_{\mathcal{C}_i}(R)$ by [16, Lemma 3.6(1)]. Now let $M \in \mathrm{co}\Omega^1_{\mathcal{C}_i}(R)$. We may assume that $f_0 : L \twoheadrightarrow M$ is an epimorphism in $\mathrm{Mod}\, R$ with $L \in \mathcal{C}_i$. As $\mathcal{C}_i \subseteq \mathrm{c}\mathcal{T}^i_\omega(R) \subseteq \mathrm{c}\mathcal{T}^1_\omega(R)$, we have that θ_L is an epimorphism. Then from the following commutative diagram

$$\begin{array}{ccc} \omega \otimes_S L_* & \xrightarrow{1_\omega \otimes f_{0*}} & \omega \otimes_S M_* \\ \downarrow{\scriptstyle \theta_L} & & \downarrow{\scriptstyle \theta_M} \\ L & \xrightarrow{f_0} & M, \end{array}$$

we get that θ_M is an epimorphism. Thus $M \in \mathrm{c}\mathcal{T}^1_\omega(R)$ and $\mathrm{co}\Omega^1_{\mathcal{C}_i}(R) \subseteq \mathrm{c}\mathcal{T}^1_\omega(R)$. $\square$

Lemma 3.2. *The following statements are equivalent for any $i \geqslant 2$.*

(1) $M \in \mathrm{co}\Omega^2_{\mathcal{P}_\omega}(R)$.

(2) $M \in \mathrm{co}\Omega^2_{\mathcal{C}_i}(R)$.

(3) *There is a module $N \in \mathrm{Mod}\, S$ such that $M \cong \omega \otimes_S N$.*

Proof. (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) Let $M \in \mathrm{co}\Omega^2_{\mathcal{C}_i}(R)$ and let

$$L^0 \xrightarrow{f} L^1 \to M \to 0$$

be an exact sequence in $\mathrm{Mod}\, R$ with $L^0, L^1 \in \mathcal{C}_i \subseteq \mathrm{c}\mathcal{T}^i_\omega(R)$. As $\mathrm{c}\mathcal{T}^i_\omega(R) \subseteq \mathrm{c}\mathcal{T}^2_\omega(R)$, we have that θ_{L^0} and θ_{L^1} are isomorphisms. Then from the following commutative diagram with exact rows

$$\begin{array}{ccccccc} \omega \otimes_S {L^0}_* & \xrightarrow{1_\omega \otimes f_*} & \omega \otimes_S L^1_* & \longrightarrow & \omega \otimes_S \mathrm{Coker}\, f_* & \longrightarrow & 0 \\ \downarrow{\scriptstyle \theta_{L^0}} & & \downarrow{\scriptstyle \theta_{L^1}} & & \downarrow{\scriptstyle h} & & \\ L^0 & \xrightarrow{f} & L^1 & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

we get that the induced homomorphism h is an isomorphism, and thus $M \cong \omega \otimes_S \mathrm{Coker}\, f_*$.

(3) $\Rightarrow$ (1) Suppose $M \cong \omega \otimes_S N$ for some $N \in \mathrm{Mod}\, S$, and let

$$Q_1 \to Q_0 \to N \to 0$$

be a projective presentation of N. Applying the functor $\omega \otimes_S -$ to it yields an exact sequence

$$\omega \otimes_S Q_1 \to \omega \otimes_S Q_0 \to \omega \otimes_S N \to 0.$$

Since $\omega\otimes_S Q_1, \omega\otimes_S Q_0 \in \mathcal{P}_\omega(R)$, we have $M \in \mathrm{co}\Omega^2_{\mathcal{P}_\omega(R)}$. □

We give an analogue of Lemma 3.2.

Lemma 3.3. *The following statements are equivalent for any* $i \geqslant 2$.

(1) $N \in \Omega^2_{\mathcal{I}_\omega}(S)$.

(2) $N \in \Omega^2_{\mathcal{D}_i}(S)$.

(3) *There is a module* $M \in \mathrm{Mod}\, R$ *such that* $N \cong M_*$.

Proof. (1) ⇒ (2) It is obvious.

(2) ⇒ (3) Let $N \in \Omega^2_{\mathcal{D}_i}(R)$ and let

$$0 \to N \to H^0 \xrightarrow{g} H^1$$

be an exact sequence in $\mathrm{Mod}\, S$ with $H^0, H^1 \in \mathcal{D}_i \subseteq \mathrm{ac}\mathcal{T}^i_\omega(S)$. As $\mathrm{ac}\mathcal{T}^i_\omega(S) \subseteq \mathrm{ac}\mathcal{T}^2_\omega(S)$, we have that μ_{H^0} and μ_{H^1} are isomorphisms. Then from the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & \longrightarrow & H^0 & \xrightarrow{g} & H^1 \\
 & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle \mu_{H^0}} & & \Big\downarrow{\scriptstyle \mu_{H^1}} \\
0 & \longrightarrow & (\mathrm{Ker}(1_\omega\otimes g))_* & \longrightarrow & (\omega\otimes_S H^0)_* & \xrightarrow{(1_\omega\otimes g)_*} & (\omega\otimes_S H^1)_*,
\end{array}$$

we get that the induced homomorphism h is an isomorphism, and thus $N \cong (\mathrm{Ker}(1_\omega\otimes g))_*$.

(3) ⇒ (1) Suppose $N \cong M_*$ for some $M \in \mathrm{Mod}\, R$, and let

$$0 \to M \to I^0 \to I^1$$

be an injective copresentation of M. Applying the functor $(-)_*$ to it yields an exact sequence

$$0 \to M_* \to I^0_* \to I^1_*.$$

Since $I^0_*, I^1_* \in \mathcal{I}_\omega(S)$, we have $N \in \Omega^2_{\mathcal{I}_\omega(S)}$. □

Proposition 3.4. *The following statements are equivalent for any* $i \geqslant 2$.

(1) $M_* \in \mathrm{ac}\mathcal{T}^2_\omega(S)$ *for any* $M \in \mathrm{Mod}\, R$.

(2) $\omega\otimes_S N \in \mathrm{c}\mathcal{T}^2_\omega(R)$ *for any* $N \in \mathrm{Mod}\, S$.

(3) $\mathrm{c}\mathcal{T}^2_\omega(R) = \mathrm{co}\Omega^2_{\mathcal{C}_i}(R)$.

(4) $\mathrm{ac}\mathcal{T}^2_\omega(S) = \Omega^2_{\mathcal{D}_i}(S)$.

Proof. (1) ⇒ (4) Let $N \in \Omega^2_{\mathcal{D}_i}(S)$. Then by Lemma 3.3 and (1) there is a module $M \in \mathrm{Mod}\, R$ such that $N \cong M_* \in \mathrm{ac}\mathcal{T}^2_\omega(S)$, and so $\Omega^2_{\mathcal{D}_i}(S) \subseteq \mathrm{ac}\mathcal{T}^2_\omega(S)$. The inclusion $\mathrm{ac}\mathcal{T}^2_\omega(S) \subseteq \Omega^2_{\mathcal{D}_i}(S)$ follows from [18, Lemma 3.7(2)].

(4) ⇒ (1) Let $M \in \mathrm{Mod}\, R$. Then by Lemma 3.3 and (4), we have $M_* \in \Omega^2_{\mathcal{D}_i}(S) = \mathrm{ac}\mathcal{T}^2_\omega(S)$.

Similarly, we get (2) ⇔ (3) by Lemma 3.2 and [16, Lemma 3.6(2)].

(1) ⇔ (2) It follows from [20, Lemma 4.18]. □

Proposition 3.5. *For any* $n \geqslant 1$, *the following statements are equivalent.*

(1) $\text{T-cograde}_\omega\, \mathrm{Ext}^k_R(\omega, M) \geqslant k-1$ *for any* $M \in \mathrm{Mod}\, R$ *and* $1 \leqslant k \leqslant n$.

(2) $\text{T-cograde}_\omega\, \mathrm{Ext}^k_R(\omega, M) \geqslant k-1$ *for any* $M \in \Omega^k_{\mathcal{C}_n}(R)$ *and* $1 \leqslant k \leqslant n$.

(3) $\text{E-cograde}_\omega\, \mathrm{Tor}^S_k(\omega, N) \geqslant k-1$ *for any* $N \in \mathrm{Mod}\, S$ *and* $1 \leqslant k \leqslant n$.

(4) $\text{E-cograde}_\omega \operatorname{Tor}_k^S(\omega, N) \geqslant k-1$ *for any* $N \in \mathrm{co}\Omega^k_{\mathcal{D}_n}(S)$ *and* $1 \leqslant k \leqslant n$.
(5) $\mathrm{c}\mathcal{T}^k_\omega(R) = \mathrm{co}\Omega^k_{\mathcal{C}_n}(R)$ *for any* $1 \leqslant k \leqslant n$.
(6) $\mathrm{ac}\mathcal{T}^k_\omega(S) = \Omega^k_{\mathcal{D}_n}(S)$ *for any* $1 \leqslant k \leqslant n$.

Proof. (2) $\Rightarrow$ (5) By [16, Proposition 3.7], it suffices to prove $\mathrm{co}\Omega^k_{\mathcal{C}_n}(R) \subseteq \mathrm{c}\mathcal{T}^k_\omega(R)$ for any $1 \leqslant k \leqslant n$. We proceed by induction on n. The cases for $n = 1$ follows from Lemma 3.1.

Now let $M \in \mathrm{co}\Omega^n_{\mathcal{C}_n}(R)$ with $n \geqslant 2$ and let

$$W_{n-1} \xrightarrow{f_{n-1}} \cdots \to W_1 \xrightarrow{f_1} W_0 \xrightarrow{f_0} M \to 0 \tag{3.1}$$

be an exact sequence in $\operatorname{Mod} R$ with all W_i in $\mathcal{C}_n$. By the induction hypothesis, we have $\operatorname{Im} f_1 \in \mathrm{c}\mathcal{T}^{n-1}_\omega(R)$ and there is an exact sequence

$$V_{n-1} \xrightarrow{g_{n-1}} \cdots \to V_1 \xrightarrow{g_1} W_0 \xrightarrow{f_0} M \to 0 \tag{3.2}$$

in $\operatorname{Mod} R$ with all V_i in $\mathcal{P}_\omega(R)$ by [16, Proposition 3.7]. Applying the functor $(-)_*$ to (3.2) gives an exact sequence

$$0 \to (\operatorname{Im} g_1)_* \to W_{0*} \xrightarrow{f_{0*}} M_* \to \operatorname{Ext}^n_R(\omega, \operatorname{Ker} g_{n-1}) \to 0. \tag{3.3}$$

Set $N := \operatorname{Im}(f_{0*})$ and let $f_{0*} := \alpha\pi$ (where $\pi : W_{0*} \twoheadrightarrow N$ and $\alpha : N \hookrightarrow M_*$) be the natural epic-monic decompositions of f_{0*}. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\operatorname{Tor}_1^S(\omega, N) & \xrightarrow{h} & \omega \otimes_S (\operatorname{Im} g_1)_* & \longrightarrow & \omega \otimes_S W_{0*} & \xrightarrow{1_\omega \otimes \pi} & \omega \otimes_S N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_{\operatorname{Im} g_1}} & & \downarrow{\scriptstyle \theta_{W_0}} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & \operatorname{Im} g_1 & \longrightarrow & W_0 & \xrightarrow{f_0} & M & \longrightarrow & 0.
\end{array}$$

Diagram (3.1)

So we have

$$\theta_M(1_\omega \otimes \alpha)(1_\omega \otimes \pi) = \theta_M(1_\omega \otimes f_{0*}) = f_0\theta_{W_0} = g(1_\omega \otimes \pi).$$

Because $1_\omega \otimes \pi$ is epic, we have $\theta_M \cdot (1_\omega \otimes \alpha) = g$ and the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\omega \otimes_S N & \xrightarrow{1_\omega \otimes \alpha} & \omega \otimes_S M_* & \longrightarrow & \omega \otimes_S \operatorname{Ext}^n_R(\omega, \operatorname{Ker} g_{n-1}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \theta_M} & & & & \\
M & = & M. & & & &
\end{array}$$

Diagram (3.2)

Since $\operatorname{Im} g_1 = \operatorname{Im} f_1 \in \mathrm{c}\mathcal{T}^{n-1}_\omega(R)$, we have that $\theta_{\operatorname{Im} g_1}$ is an epimorphism. So g is an isomorphism by the snake lemma, and hence $1_\omega \otimes \alpha$ is a monomorphism. Since $\omega \otimes_S \operatorname{Ext}^n_R(\omega, \operatorname{Ker} g_{n-1}) = 0$ by assumption, we see that θ_M is an isomorphism and $M \in \mathrm{c}\mathcal{T}^2_\omega(R)$ by the diagram (3.2). This shows that the assertion holds true for $n = 2$.

If $n > 2$, then $\theta_{\operatorname{Im} g_1}$ is an isomorphism as $\operatorname{Im} g_1 \in \mathrm{c}\mathcal{T}^{n-1}_\omega(R)$, we also have $\operatorname{Tor}_1^S(\omega, W_{0*}) = 0$ by [16, Corollary 3.4(3)]. So h is monic and $\operatorname{Tor}_1^S(\omega, N) = 0$ by the diagram (3.1). Moreover, it is clear that $\operatorname{Tor}^S_{1\leqslant k\leqslant n-3}(\omega, (\operatorname{Im} g_1)_*) = 0$ by [16, Corollary 3.4(3)]. Because $\text{T-cograde}_\omega \operatorname{Ext}^n_R(\omega, \operatorname{Ker} f_{n-1}) \geqslant n-1$ by assumption, applying the dimension shifting to (3.3) yields $\mathrm{Tor}^S_{1\leqslant k\leqslant n-2}(\omega, M_*)=0$. Therefore $M\in \mathrm{c}\mathcal{T}^n_\omega(R)$ by [16, Corollary 3.4(3)] again.

(5) $\Rightarrow$ (1) For any $1\leqslant k\leqslant n$, since $\mathrm{c}\mathcal{T}^k_\omega(R)\subseteq \mathrm{co}\Omega^k_{\mathcal{P}_\omega}(R)\subseteq \mathrm{co}\Omega^k_{\mathcal{C}_n}(R)$, we have $\mathrm{c}\mathcal{T}^k_\omega(R)=\mathrm{co}\Omega^k_{\mathcal{P}_\omega}(R)$ by (5), and hence $\mathrm{c}\mathcal{T}^k_\omega(R)=\mathrm{co}\Omega^k_{\mathcal{B}}(R)$ by [20, Proposition 4.17]. Now (1) follows from [20, Theorem 4.19].

The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are obvious.

(1) $\Leftrightarrow$ (3) It follows from [20, Theorem 4.19].

The proofs of (4) $\Rightarrow$ (6) and (6) $\Rightarrow$ (2) are similar to that of (2) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1) respectively. □

Lemma 3.6. *For any $M\in \mathrm{Mod}\,R$, there are two exact sequences*

$$0\to \mathrm{Ext}^1_R(\omega,M)\to \mathrm{cTr}_\omega M\xrightarrow{\pi} H\to 0,$$

$$0\to H\xrightarrow{\lambda}(\omega\otimes_S \mathrm{cTr}_\omega M)_*\to \mathrm{Ext}^2_R(\omega,M)\to 0$$

in $\mathrm{Mod}\,S$ *such that* $\mathrm{Hom}_S(\pi,\omega^+)$ *is an isomorphism.*

Proof. By [17, Corollary 6.8], there is an exact sequence

$$0\to \mathrm{Ext}^1_R(\omega,M)\to \mathrm{cTr}_\omega M\xrightarrow{\mu_{\mathrm{cTr}_\omega M}}(\omega\otimes_S \mathrm{cTr}_\omega M)_*\to \mathrm{Ext}^2_R(\omega,M)\to 0$$

in $\mathrm{Mod}\,S$. Put $H:=\mathrm{Im}\,\mu_{\mathrm{cTr}_\omega M}$ and assume that $\mu_{\mathrm{cTr}_\omega M}=\lambda\pi$, where $\pi:\mathrm{cTr}_\omega M\to H$ is an epimorphism and $\lambda: H\to (\omega\otimes_S \mathrm{cTr}_\omega M)_*$ is a monomorphism. Then we have the following exact sequences

$$0\to \mathrm{Ext}^1_R(\omega,M)\to \mathrm{cTr}_\omega M\xrightarrow{\pi} H\to 0,$$

$$0\to H\xrightarrow{\lambda}(\omega\otimes_S \mathrm{cTr}_\omega M)_*\to \mathrm{Ext}^2_R(\omega,M)\to 0.$$

In view of [17, Lemma 6.1(2)], $1_\omega\otimes\mu_{\mathrm{cTr}_\omega M}$ is a monomorphism, and so $1_\omega\otimes\pi$ is an isomorphism. It follows from the adjoint isomorphism theorem that $\mathrm{Hom}_S(\pi,\omega^+)\cong(1_\omega\otimes\pi)^+$ is also an isomorphism. □

Lemma 3.7. *The following statements are equivalent for any $i\geqslant 1$.*

(1) $\mathrm{ac}\mathcal{T}^1_\omega(S)$ *is extension closed.*
(2) $\mathrm{T\text{-}cograde}_\omega\,\mathrm{Ext}^1_R(\omega,M)\geqslant 1$ *for any* $M\in\Omega^2_{\mathcal{C}_i}(R)$.
(3) $\mathrm{T\text{-}cograde}_\omega\,\mathrm{Ext}^1_R(\omega,M)\geqslant 1$ *for any* $M\in \mathrm{Mod}\,R$.

Proof. (2) $\Rightarrow$ (1) Let

$$0\to A\xrightarrow{f} B\xrightarrow{g} C\to 0$$

be an exact sequence in $\mathrm{Mod}\,S$ with $A,C\in \mathrm{ac}\mathcal{T}^1_\omega(S)$. By [18, Proposition 3.2], $\mathrm{Ker}\,\mu_B\cong \mathrm{Ext}^1_R(\omega,\mathrm{acTr}_\omega B)$. Notice that $\mathrm{acTr}_\omega B\in\Omega^2_{\mathcal{C}_i}(R)$, so

$$\omega\otimes_S \mathrm{Ext}^1_R(\omega,\mathrm{acTr}_\omega B)=0$$

by (2), and hence $\omega\otimes_S\mathrm{Ker}\,\mu_B=0$. Moreover, since $\mu_C g=(1_\omega\otimes g)_*\mu_B$ and μ_C is a monomorphism, we get $\mathrm{Ker}\,\mu_B\subseteq \mathrm{Ker}\,g\cong A$. Note that $A\in \mathrm{ac}\mathcal{T}^1_\omega(S)$ and

$$\mathrm{Hom}_S(\mathrm{Ker}\,\mu_B,\omega^+)\cong(\omega\otimes_S\mathrm{Ker}\,\mu_B)^+=0.$$

It follows from [18, Lemma 3.7] and [13, Proposition 3.7] that $\mathrm{Hom}_S(\mathrm{Ker}\,\mu_B,A)=0$, which implies $\mathrm{Ker}\,\mu_B=0$, and thus $B\in\mathrm{ac}\mathcal{T}^1_\omega(S)$.

(1) $\Rightarrow$ (3) By Lemma 3.6, there is an exact sequence

$$0\to \mathrm{Ext}^1_R(\omega,M)\to \mathrm{cTr}_\omega M\xrightarrow{\pi} H\to 0$$

in $\operatorname{Mod} S$ such that $\operatorname{Hom}_S(\pi, \omega^+)$ is an isomorphism. Then

$$\operatorname{Ker}\operatorname{Ext}^1_S(\pi, \omega^+) \cong \operatorname{Hom}_S(\operatorname{Ext}^1_R(\omega, M), \omega^+) \cong (\omega \otimes_S \operatorname{Ext}^1_R(\omega, M))^+.$$

Suppose

$$\alpha : 0 \to \omega^+ \to X \xrightarrow{f} H \to 0$$

is an element in $\operatorname{Ker}\operatorname{Ext}^1_S(\pi, \omega^+)$, that is, $\operatorname{Ext}^1_S(\pi, \omega^+)(\alpha) = 0$. Then we have the following pull-back diagram with the first row splitting:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega^+ & \longrightarrow & Y & \xrightarrow{u} & \operatorname{cTr}_\omega M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & \omega^+ & \longrightarrow & X & \xrightarrow{f} & H & \longrightarrow & 0.
\end{array}$$

Diagram (3.3)

So there is a homomorphism $u' : \operatorname{cTr}_\omega M \to Y$ such that $uu' = 1_{\operatorname{cTr}_\omega M}$. Since $\pi u = ft$, we have $\pi = ftu'$. Note that $(\omega \otimes_S \operatorname{cTr}_\omega M)_* \in \operatorname{ac}\mathcal{T}^1_\omega(S)$ by [17, Lemma 6.1(1)]. Thus $H \in \operatorname{ac}\mathcal{T}^1_\omega(S)$ since H is a submodule of $(\omega \otimes_S \operatorname{cTr}_\omega M)_*$ by Lemma 3.6. So $X \in \operatorname{ac}\mathcal{T}^1_\omega(S)$ by (1), and hence there is a monomorphism $0 \to X \to U^0$ in $\operatorname{Mod} S$ with $U^0 \in \mathcal{I}_\omega(S) = \operatorname{Prod}\omega^+$. As $\operatorname{Hom}_S(\pi, \omega^+)$ is an isomorphism, we have that $\operatorname{Hom}_S(\pi, U^0)$, and hence $\operatorname{Hom}_S(\pi, X)$, is an isomorphism by [11, Lemma 2.1]. Then there is a homomorphism $f' : H \to X$ such $f'\pi = tu'$, and so $\pi = ff'\pi$. But π is an epimorphism, thus $ff' = 1_H$, which implies that α splits, and thus $\omega \otimes_S \operatorname{Ext}^1_R(\omega, M) = 0$.

(3) $\Rightarrow$ (2) It is trivial. □

Lemma 3.8. *For any $N \in \operatorname{Mod} S$, the following statements are equivalent.*

(1) $\omega \otimes_S N \in \operatorname{c}\mathcal{T}^2_\omega(R)$.

(2) $\omega \otimes_S \operatorname{Coker}\mu_N = 0$.

Proof. By [17, Lemma 6.1(2)], we have $\theta_{\omega\otimes_S N}(1_\omega \otimes \mu_N) = 1_{\omega\otimes_S N}$. It follows that $\theta_{\omega\otimes_S N}$ is a split epimorphism and

$$\operatorname{Ker}\theta_{\omega\otimes_S N} \cong \operatorname{Coker}(1_\omega \otimes \mu_N) \cong \omega \otimes_S \operatorname{Coker}\mu_N.$$

Now the assertion follows easily. □

Lemma 3.9. *If $\operatorname{ac}\mathcal{T}^1_\omega(S)$ is extension closed, then $\omega \otimes_S N \in \operatorname{c}\mathcal{T}^2_\omega(R)$ for any $N \in \operatorname{Mod} S$.*

Proof. By the definition of adjoint cotranspose, there is an exact sequence

$$0 \to \operatorname{acTr}_\omega N \to \omega \otimes_S F_1 \to \omega \otimes_S F_0 \to \omega \otimes_S N \to 0$$

in $\operatorname{Mod} R$ with F_0, F_1 flat. Let $K := \operatorname{Im}(\omega \otimes_S F_1 \to \omega \otimes_S F_0)$. We have

$$\operatorname{Coker}\mu_N \cong \operatorname{Ext}^2_R(\omega, \operatorname{acTr}_\omega N) \cong \operatorname{Ext}^1_R(\omega, K)$$

by [18, Proposition 3.2]. By assumption and Lemma 3.7, we have $\omega \otimes_S \operatorname{Coker}\mu_N = 0$. Thus $\omega \otimes_S N \in \operatorname{c}\mathcal{T}^2_\omega(R)$ by Lemma 3.8. □

We are now in a position to prove the main result of this section.

Theorem 3.10. *The following statements are equivalent for any $i \geqslant 1$.*

(1) $\operatorname{ac}\mathcal{T}^k_\omega(S)$ *is extension closed for $k = 1, 2$.*

(2) $\operatorname{Tor}^S_{k-1}(\omega, \operatorname{Ext}^k_R(\omega, M)) = 0$ *for any $M \in \Omega^2_{\mathcal{C}_i}(R)$ and $k = 1, 2$.*

(3) T-cograde$_\omega$ $\mathrm{Ext}^k_R(\omega, M) \geqslant k$ *for any* $M \in \Omega^2_{\mathcal{C}_i}(R)$ *and* $k = 1, 2$.
(4) $\mathrm{Tor}^S_{k-1}(\omega, \mathrm{Ext}^k_R(\omega, M)) = 0$ *for any* $M \in \mathrm{Mod}\, R$ *and* $k = 1, 2$.
(5) T-cograde$_\omega$ $\mathrm{Ext}^k_R(\omega, M) \geqslant k$ *for any* $M \in \mathrm{Mod}\, R$ *and* $k = 1, 2$.

Proof. (1) $\Rightarrow$ (5) By Lemma 3.6, there are two exact sequences

$$0 \to \mathrm{Ext}^1_R(\omega, M) \to \mathrm{cTr}_\omega M \xrightarrow{\pi} H \to 0, \tag{3.4}$$

$$0 \to H \xrightarrow{\lambda} (\omega \otimes_S \mathrm{cTr}_\omega M)_* \xrightarrow{\beta} \mathrm{Ext}^2_R(\omega, M) \to 0 \tag{3.5}$$

in $\mathrm{Mod}\, S$ such that $\mathrm{Hom}_S(\pi, \omega^+)$ is an isomorphism and $\lambda\pi = \mu_{\mathrm{cTr}_\omega M}$. Since $\mathrm{Hom}_S(\pi, \omega^+) \cong (\omega \otimes \pi)^+$ by the adjoint isomorphism theorem, it follows that $(\omega \otimes \pi)^+$ and $\omega \otimes \pi$ are isomorphisms.

By Lemma 3.7, it is easy to see that T-cograde$_\omega$ $\mathrm{Ext}^1_R(\omega, M) \geqslant 1$ and $\omega \otimes_S \mathrm{Ext}^1_R(\omega, M) = 0$. Then by the adjoint isomorphism theorem, we have that

$$\mathrm{Hom}_S(\mathrm{Ext}^1_R(\omega, M), \omega^+) \cong (\omega \otimes_S \mathrm{Ext}^1_R(\omega, M))^+ = 0$$

and $\mathrm{Ext}^1_S(\pi, \omega^+)$ is a monomorphism. We know from [17, Lemma 6.1(2)] that $\mathrm{Hom}_S(\mu_{\mathrm{cTr}_\omega M}, \omega^+)$ is an epimorphism. Then the fact that

$$\mathrm{Hom}_S(\mu_{\mathrm{cTr}_\omega M}, \omega^+) = \mathrm{Hom}_S(\pi, \omega^+)\,\mathrm{Hom}_S(\lambda, \omega^+)$$

in which $\mathrm{Hom}_S(\pi, \omega^+)$ is an isomorphism (by Lemma 3.6) implies that $\mathrm{Hom}_S(\lambda, \omega^+)$ is also an epimorphism. On the other hand, note that

$$\mathrm{Ext}^1_S(\mu_{\mathrm{cTr}_\omega M}, \omega^+) = \mathrm{Ext}^1_S(\pi, \omega^+)\,\mathrm{Ext}^1_S(\lambda, \omega^+)$$

and $\mathrm{Ext}^1_S(\pi, \omega^+)$ is a monomorphism by the above argument. Applying the functor $\mathrm{Hom}_S(-, \omega^+)$ to (3.5) gives

$$\mathrm{Ker}\,\mathrm{Ext}^1_S(\mu_{\mathrm{cTr}_\omega M}, \omega^+) \cong \mathrm{Ker}\,\mathrm{Ext}^1_S(\lambda, \omega^+) \cong \mathrm{Ext}^1_S(\mathrm{Ext}^2_R(\omega, M), \omega^+).$$

Let

$$\alpha : 0 \to \omega^+ \to X \xrightarrow{f} (\omega \otimes_S \mathrm{cTr}_\omega M)_* \to 0$$

be an element in $\mathrm{Ker}\,\mathrm{Ext}^1_S(\mu_{\mathrm{cTr}_\omega M}, \omega^+)$, that is, $\mathrm{Ext}^1_S(\mu_{\mathrm{cTr}_\omega M}, \omega^+)(\alpha) = 0$. Then we have the following pull-back diagram with the first row splitting:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega^+ & \longrightarrow & Y & \xrightarrow{u} & \mathrm{cTr}_\omega M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \mu_{\mathrm{cTr}_\omega M}} & & \\
0 & \longrightarrow & \omega^+ & \longrightarrow & X & \longrightarrow & (\omega \otimes_S \mathrm{cTr}_\omega M)_* & \longrightarrow & 0.
\end{array}$$

So there is a homomorphism $u' : \mathrm{cTr}_\omega M \to Y$ such that $uu' = 1_{\mathrm{cTr}_\omega M}$. Since $\mu_{\mathrm{cTr}_\omega M} u = ft$, we have

$$\mu_{\mathrm{cTr}_\omega M} = ftu'.$$

By Lemma 3.9, we have $\omega \otimes_S \mathrm{cTr}_\omega M \in \mathrm{c}\mathcal{T}^2_\omega(R)$. It follows from [19, Proposition 6.4] that $(\omega \otimes_S \mathrm{cTr}_\omega M)_* \in \mathrm{ac}\mathcal{T}^2_\omega(S)$. Since $\mathrm{ac}\mathcal{T}^2_\omega(S)$ is extension closed by (1), we have $X \in \mathrm{ac}\mathcal{T}^2_\omega(S)$. As $\mu_X tu' = (1_\omega \otimes tu')_* \mu_{\mathrm{cTr}_\omega M}$, we have

$$\mu_{\mathrm{cTr}_\omega M} = ftu' = f\mu_X^{-1}(1_\omega \otimes tu')_* \mu_{\mathrm{cTr}_\omega M} \text{ and}$$

$$(1_{(\omega \otimes_S \mathrm{cTr}_\omega M)_*} - f\mu_X^{-1}(1_\omega \otimes tu')_*)\mu_{\mathrm{cTr}_\omega M} = 0,$$

and hence

$$(1_{(\omega \otimes_S \mathrm{cTr}_\omega M)_*} - f\mu_X^{-1}(1_\omega \otimes tu')_*)\lambda = 0.$$

By the universal property of cokernels, there is a homomorphism $g : \operatorname{Ext}^2_R(\omega, M) \to (\omega \otimes_S \operatorname{cTr}_\omega M)_*$ such that

$$1_{(\omega\otimes_S \operatorname{cTr}_\omega M)_*} - f\mu_X^{-1}(1_\omega \otimes tu')_* = g\beta.$$

In addition, Since $\operatorname{Ext}^2_R(\omega, M) \cong \operatorname{Coker} \mu_{\operatorname{cTr}_\omega M}$, we have

$$\omega \otimes_S \operatorname{Ext}^2_R(\omega, M) = 0$$

by Lemma 3.8. It follows from the adjoint isomorphism theorem that

$$\operatorname{Hom}_S(\operatorname{Ext}^2_R(\omega, M), \omega^+) \cong (\omega \otimes_S \operatorname{Ext}^2_R(\omega, M))^+ = 0.$$

Moreover, since $(\omega \otimes_S \operatorname{cTr}_\omega M)_* \in \operatorname{ac}\mathcal{T}^2_\omega(S)$, we have that $(\omega \otimes_S \operatorname{cTr}_\omega M)_*$ is isomorphic to a submodule of some module in $\operatorname{Prod}\omega^+$ by [18, Lemma 3.7]. Then

$$\operatorname{Hom}_S(\operatorname{Ext}^2_R(\omega, M), (\omega \otimes_S \operatorname{cTr}_\omega M)_*) = 0.$$

So $g = 0$ and

$$1_{(\omega\otimes_S \operatorname{cTr}_\omega M)_*} = f\mu_X^{-1}(1_\omega \otimes tu')_*,$$

which means $\alpha = 0$. It follows from [5, Theorem 3.2.1] that $\operatorname{Tor}^S_1(\omega, \operatorname{Ext}^2_R(\omega, M)) = 0$, and thus $\text{T-cograde}_\omega \operatorname{Ext}^2_R(\omega, M) \geqslant 2$.

The implications (3) $\Rightarrow$ (2), (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3) + (4) are trivial.

(2) $\Rightarrow$ (1) Let

$$0 \to A \to B \xrightarrow{f} C \to 0$$

be an exact sequence in $\operatorname{Mod} S$ with $A, C \in \operatorname{ac}\mathcal{T}^2_\omega(S)$. Then $B \in \operatorname{ac}\mathcal{T}^1_\omega(S)$ by (2) and Lemma 3.7. It follows from [18, Proposition 3.2] that

$$\operatorname{Coker} \mu_B \cong \operatorname{Ext}^2_R(\omega, \operatorname{acTr}_\omega B).$$

Consider the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{f} & C & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle \mu_B} & & \Big\downarrow{\scriptstyle \mu_C} & & \\
0 & \longrightarrow & A' & \longrightarrow & (\omega \otimes_S B)_* & \xrightarrow{(1_\omega \otimes f)_*} & (\omega \otimes_S C)_*, & &
\end{array}$$

where $A' \cong \operatorname{Ker}(1_\omega \otimes f)_*$ and h is an induced homomorphism. By the snake lemma, we get an exact sequence

$$0 \to A \xrightarrow{h} A' \to \operatorname{Ext}^2_R(\omega, \operatorname{acTr}_\omega B) \to 0. \tag{3.6}$$

Since $\operatorname{acTr}_\omega B \in \Omega^2_{\mathcal{C}_i}(R)$, we have

$$\omega \otimes_S \operatorname{Coker} \mu_B \cong \omega \otimes_S \operatorname{Ext}^2_R(\omega, \operatorname{acTr}_\omega B) = 0$$

by Lemmas 3.8 and 3.9. Also we have $\operatorname{Tor}^S_1(\omega, \operatorname{Ext}^2_R(\omega, \operatorname{acTr}_\omega B)) = 0$ by (2). Applying the functor $\omega \otimes -$ to (3.6) yields that $1_\omega \otimes h$ is an isomorphism. Since A' is a submodule of $(\omega \otimes_S B)_*$, we have $A' \in \operatorname{ac}\mathcal{T}^1_\omega(S)$, and thus $\mu_{A'}$ is a monomorphism. On the other hand, since μ_A is an isomorphism and

$$(1_\omega \otimes h)_* \mu_A = \mu_{A'} h,$$

we get that $\mu_{A'}$ is an epimorphism, and hence an isomorphism, which implies that h is also an isomorphism. Thus μ_B is an isomorphism and $B \in \operatorname{ac}\mathcal{T}^2_\omega(S)$. $\square$

4. **Strong Ext-cograde and extension closure**

In this section, we characterize the extension closure of the class $\mathrm{ac}\mathcal{T}_\omega^k(S)$ for any $1 \leqslant k \leqslant n$ in terms of the strong cograde of $\mathrm{Tor}_{k+1}^S(\omega, C)$ for any $C \in \mathrm{co}\Omega_{\mathcal{D}_i}^k(S)$ and $1 \leqslant k \leqslant n$.

Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be an exact sequence in $\mathrm{Mod}\, R$. Then one gets two exact sequences

$$0 \to \mathrm{Im}(1_\omega \otimes f) \to \omega \otimes_S B \xrightarrow{1_\omega \otimes g} \omega \otimes_S C \to 0, \tag{4.1}$$

$$0 \to \mathrm{Ker}(1_\omega \otimes f) \to \omega \otimes_S A \to \mathrm{Im}(1_\omega \otimes f) \to 0. \tag{4.2}$$

Applying the functor $(-)_*$ to (4.1) yields the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mu_B} & & \downarrow{\scriptstyle \mu_C} & & \\ 0 & \longrightarrow & (\mathrm{Im}(1_\omega \otimes f))_* & \longrightarrow & (\omega \otimes_S B)_* & \longrightarrow & (\omega \otimes_S C)_*, & & \end{array}$$

Diagram (4.1)

where α is an induced homomorphism. It is easy to check that the following diagram

$$\begin{array}{ccccccccc} & & & & A & = & A & & \\ & & & & \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & (\mathrm{Ker}(1_\omega \otimes f))_* & \longrightarrow & (\omega \otimes_S A)_* & \longrightarrow & (\mathrm{Im}(1_\omega \otimes f))_* & \longrightarrow & \mathrm{Ext}_R^1(\omega, \mathrm{Ker}(1_\omega \otimes f)) \end{array}$$

Diagram (4.2)

is commutative with the bottom row exact.

The following two lemmas are useful in this section.

Lemma 4.1. *The following statements are equivalent for any* $C \in \mathrm{ac}\mathcal{T}_\omega^1(S)$.

(1) $\text{s.E-cograde}_\omega\, \mathrm{Tor}_1^S(\omega, C) \geqslant 1$.

(2) *If*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an exact sequence in $\mathrm{Mod}\, R$ *with* $A \in \mathrm{ac}\mathcal{T}_\omega^1(S)$, *then* $B \in \mathrm{ac}\mathcal{T}_\omega^1(S)$.

Proof. (1) $\Rightarrow$ (2) Since $\mathrm{Ker}(1_\omega \otimes f)$ is a quotient module of $\mathrm{Tor}_1^S(\omega, C)$, we have $(\mathrm{Ker}(1_\omega \otimes f))_* = 0$ by (1). Then it follows from the diagram (4.2) that α is a monomorphism. Applying the snake lemma to diagram (4.1) gives that μ_B is also a monomorphism, and so $B \in \mathrm{ac}\mathcal{T}_\omega^1(S)$.

(2) $\Rightarrow$ (1) Let L be a quotient module of $\mathrm{Tor}_1^S(\omega, C)$. Then L^+ is a submodule of $\mathrm{Ext}_S^1(C, \omega^+)(\cong [\mathrm{Tor}_1^S(\omega, C)]^+)$. Since $\omega \in \mathcal{B}_\omega(S^{op})$, for any cardinal ζ, we have $\omega^{+^\zeta} \in \mathcal{A}_\omega(S)$ by [12, Theorem 3.3]. Thus

$$\mathrm{Ext}_S^1(\omega^{+^\zeta}, \omega^+) \cong [\mathrm{Tor}_S^1(\omega, \omega^{+^\zeta})]^+ = 0.$$

It follows from the proof of [21, Lemma 6.9] or the dual result of [22, Lemma 3.4] that there is a cardinal λ such that there is an exact sequence

$$0 \to \omega^{+^\lambda} \xrightarrow{f} D \to C \to 0 \tag{4.3}$$

in $\operatorname{Mod} R$ with $\operatorname{Coker} \operatorname{Hom}_S(f, \omega^+) \cong L^+$. Then we get the following commutative diagram with the top and bottom rows exact:

$$\begin{array}{ccccccc}
\operatorname{Hom}_S(D, \omega^+) \otimes_R \omega & \xrightarrow{g} & \operatorname{Hom}_S(\omega^{+^\lambda}, \omega^+) \otimes_R \omega & \longrightarrow & L^+ \otimes_R \omega & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & & & \\
[(\omega \otimes_S D)_*]^+ & \longrightarrow & [(\omega \otimes_S \omega^{+^\lambda})_*]^+ & & & & \\
\downarrow (\mu_D)^+ & & \downarrow (\mu_{\omega^{+^\lambda}})^+ & & & & \\
D^+ & \longrightarrow & (\omega^{+^\lambda})^+ & \longrightarrow & 0. & &
\end{array}$$

Diagram (4.3)

Since $\omega^{+^\lambda} \in \mathrm{ac}\mathcal{T}_\omega^2(S)$, we have $D \in \mathrm{ac}\mathcal{T}_\omega^1(S)$ by (2). It follows that g is an epimorphism and $L^+ \otimes_R \omega = 0$. Since ω admits a degreewise finite projective resolution, we have

$$L^+ \otimes_R \omega \cong \operatorname{Hom}_R(\omega, L)^+ = 0 \tag{4.4}$$

by [5, Theorem 3.2.11]. Thus $\operatorname{Hom}_R(\omega, L) = 0$ and $\text{s.E-cograde}_\omega \operatorname{Tor}_1^S(\omega, C) \geqslant 1$. □

Lemma 4.2. *The following statements are equivalent for any $C \in \mathrm{ac}\mathcal{T}_\omega^2(S)$.*

(1) $\text{s.E-cograde}_\omega \operatorname{Tor}_1^S(\omega, C) \geqslant 2$.
(2) *If*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an exact sequence in $\operatorname{Mod} R$ with $A \in \mathrm{ac}\mathcal{T}_\omega^2(S)$, then $B \in \mathrm{ac}\mathcal{T}_\omega^2(S)$.

Proof. (1) ⇒ (2) Since $\operatorname{Ker}(1_\omega \otimes f)$ is a quotient module of $\operatorname{Tor}_1^S(\omega, C)$, we have

$$[\operatorname{Ker}(1_\omega \otimes f)]_* = 0 = \operatorname{Ext}_R^1(\omega, \operatorname{Ker}(1_\omega \otimes f))$$

by (1). Since $A \in \mathrm{ac}\mathcal{T}_\omega^2(S)$ by assumption, from the diagram (4.2) we know that α is an isomorphism. Notice that $C \in \mathrm{ac}\mathcal{T}_\omega^2(S)$, so $B \in \mathrm{ac}\mathcal{T}_\omega^2(S)$ by the diagram (4.1).

(2) ⇒ (1) By [12, Theorem 3.3], we have that $\omega^{+^\lambda} \in \mathcal{A}_\omega(S)$ with λ same as that in the proof of (2) ⇒ (1) in Lemma 4.1. Then the middle term D in the exact sequence (4.3) is in $\mathrm{ac}\mathcal{T}_\omega^2(S)$ by (2). It follows from the diagram (4.3) that $L^+ \otimes_R \omega = 0$. Thus

$$\operatorname{Hom}_R(\omega, L) = 0$$

by the exact sequence (4.4).

Applying the functor $\operatorname{Hom}_S(-, \omega^+)$ to the exact sequence (4.3) gives an exact sequence

$$0 \to \operatorname{Hom}_S(C, \omega^+) \xrightarrow{\delta} \operatorname{Hom}_S(D, \omega^+) \xrightarrow{\theta} \operatorname{Hom}_S(\omega^{+^\lambda}, \omega^+) \to L^+ \to 0 \tag{4.5}$$

in $\mathrm{Mod}\, R^{op}$. Then we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \mathrm{Hom}_S(C,\omega^+)\otimes_R\omega & \xrightarrow{\delta\otimes 1_\omega} & \mathrm{Hom}_S(D,\omega^+)\otimes_R\omega & \longrightarrow & \mathrm{Im}\,\theta\otimes_R\omega & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \| & & \\
 & & ((\omega\otimes_S C)_*)^+ & \longrightarrow & ((\omega\otimes_S D)_*)^+ & \longrightarrow & \mathrm{Im}\,\theta\otimes_R\omega & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle(\mu_C)^+} & & \downarrow{\scriptstyle(\mu_D)^+} & & \vdots{\scriptstyle h} & & \\
0 & \longrightarrow & C^+ & \longrightarrow & D^+ & \longrightarrow & \omega^{+^{\lambda}+} & \longrightarrow & 0,
\end{array}$$

Diagram (4.4)

where h is an induced homomorphism. As $C, D \in \mathrm{ac}\mathcal{T}^2_\omega(S)$, we have that both $(\mu_C)^+$ and $(\mu_D)^+$ are isomorphisms, and so h is also an isomorphism. On the other hand, from the exact sequence (4.5) we get the following commutative diagram with the top row exact:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Tor}_1^S(L^+,\omega) & \longrightarrow & \mathrm{Im}\,\theta\otimes_R\omega & \longrightarrow & ((\omega\otimes_S\omega^{+^{\lambda}})_*)^+(\cong \mathrm{Hom}_S(\omega^{+^{\lambda}},\omega^+)\otimes_R\omega) \\
 & & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle(\mu_{\omega^{+^{\lambda}}})^+} \\
 & & & (\omega^{+^{\lambda}})^+ & = & (\omega^{+^{\lambda}})^+.
\end{array}$$

Diagram (4.5)

Because both h and $(\mu_{\omega^{+^\lambda}})^+$ are isomorphisms, we have $\mathrm{Tor}_1^S(L^+,\omega) = 0$, and thus $\mathrm{Ext}^1_R(\omega, L) = 0$. The proof is finished. □

Theorem 4.3. *The following statements are equivalent for any* $C \in \mathrm{ac}\mathcal{T}^n_\omega(S)$ *and* $n \geqslant 1$.

(1) s.E-cograde$_\omega$ $\mathrm{Tor}_1^S(\omega, C) \geqslant n$.

(2) *If*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an exact sequence in $\mathrm{Mod}\, S$ *with* $A \in \mathrm{ac}\mathcal{T}^n_\omega(S)$, *then* $B \in \mathrm{ac}\mathcal{T}^n_\omega(S)$.

Proof. The cases for $n = 1$ and $n = 2$ follow from Lemmas 4.1 and 4.2 respectively. Now suppose $n \geqslant 3$,

(1) ⇒ (2) By Lemma 4.2, we have $B \in \mathrm{ac}\mathcal{T}^2_\omega(S)$. Since $C \in \mathrm{ac}\mathcal{T}^n_\omega(S)$, we have $\mathrm{Ext}^i_R(\omega, \omega\otimes_S C) = 0$ for any $1 \leqslant i \leqslant n-2$ by [18, Corollary 3.3(3)]. On the other hand, since $\mathrm{Ker}(1_\omega \otimes f)$ is a quotient module of $\mathrm{Tor}_1^S(\omega, C)$, we have $\mathrm{Ext}^i_R(\omega, \mathrm{Ker}(1_\omega\otimes f)) = 0$ for any $0 \leqslant i \leqslant n-1$ by (1). Then it is induced from the exact sequence (4.2) that $\mathrm{Ext}^i_R(\omega, \mathrm{Im}(1_\omega \otimes f)) = 0$ for any $1 \leqslant i \leqslant n-2$. Thus it follows from the exact sequence (4.1) that $\mathrm{Ext}^i_R(\omega, \omega\otimes_S B) = 0$ for any $1 \leqslant i \leqslant n-2$, and so $B \in \mathrm{ac}\mathcal{T}^n_\omega(S)$.

(2) ⇒ (1) It follows from the exact sequence (4.4) that there are two exact sequences

$$0 \to \mathrm{Hom}_S(C,\omega^+) \xrightarrow{\delta} \mathrm{Hom}_S(D,\omega^+) \to \mathrm{Im}\,\theta \to 0,$$

$$0 \to \mathrm{Im}\,\theta \to \mathrm{Hom}_S(\omega^{+^{\lambda}},\omega^+) \to L^+ \to 0.$$

By (2), we have $D \in \mathrm{ac}\mathcal{T}^n_\omega(S)$. From the proof of Lemma 4.2, we know that $\delta \otimes_R 1_\omega$ is a monomorphism and

$$\mathrm{Tor}_1^S(L^+,\omega) = 0 = L^+ \otimes_R \omega.$$

Moreover, it is easy to verify that $\mathrm{Tor}^S_{2\leqslant i\leqslant n-1}(L^+,\omega)=0$. Since

$$[\mathrm{Ext}^i_R(\omega,L)]^+\cong \mathrm{Tor}^S_i(L^+,\omega)$$

for any $i\geqslant 0$ by [5, Theorem 3.2.1], it follows that $[\mathrm{Ext}^{0\leqslant i\leqslant n-1}_R(\omega,L)]^+=0$, and hence $\mathrm{Ext}^{0\leqslant i\leqslant n-1}_R(\omega,L)=0$. The proof is finished. □

Proposition 4.4. *Suppose $n\geqslant 1$ and $\mathcal{D}_n\subseteq {\omega_S}^{\top}$. Then the following assertions hold.*

(1) *If $\Omega^k_{\mathcal{D}_n}(S)$ is extension closed for any $1\leqslant k<n$, then $\mathrm{ac}\mathcal{T}^k_\omega(S)=\Omega^k_{\mathcal{D}_n}(S)$ for any $1\leqslant k\leqslant n$, and hence $\mathrm{Add}\,\Omega^k_{\mathcal{D}_n}(S)=\Omega^k_{\mathcal{D}_n}(S)$ for any $1\leqslant k\leqslant n$.*

(2) *If $\mathrm{Add}\,\Omega^k_{\mathcal{D}_n}(S)$ is extension closed for any $1\leqslant k\leqslant n$, then $\mathrm{Add}\,\Omega^k_{\mathcal{D}_n}(S)=\Omega^k_{\mathcal{D}_n}(S)$ for any $1\leqslant k\leqslant n$.*

Proof. (1) We proceed by induction on n. When $n=1$, the assertion follows from Lemma 3.1.

Now suppose $n\geqslant 2$ and $\mathrm{ac}\mathcal{T}^k_\omega(S)=\Omega^k_{\mathcal{D}_n}(S)$ for any $1\leqslant k\leqslant n-1$. Let $N\in \mathrm{co}\Omega^n_{\mathcal{D}_n}(S)$ and let

$$A_{n-1}\to\cdots\to A_0\to N\to 0$$

be an exact sequence in $\mathrm{Mod}\,S$ with all A_j in $\mathcal{D}_n$. Set $L:=\mathrm{Im}(A_{n-1}\to A_{n-2})$. Then $L\in\Omega^{n-1}_{\mathcal{D}_n}(S)=\mathrm{ac}\mathcal{T}^{n-1}_\omega(S)$. By Theorem 4.3, we have

$$\text{s.E-cograde}_\omega\,\mathrm{Tor}^S_n(\omega,N)=\text{s.E-cograde}_\omega\,\mathrm{Tor}^S_1(\omega,L)\geqslant n-1.$$

Then $\mathrm{ac}\mathcal{T}^k_\omega(S)=\Omega^k_{\mathcal{D}_n}(S)$ for any $1\leqslant k\leqslant n$ by Proposition 3.5. Because $\mathrm{ac}\mathcal{T}^k_\omega(S)$ is closed under direct sums and direct summands for any $k\geqslant 1$, we have $\mathrm{Add}\,\Omega^k_{\mathcal{D}_n}(S)=\Omega^k_{\mathcal{D}_n}(S)$ for any $1\leqslant k\leqslant n$.

(2) We proceed by induction on n. By Lemma 3.1, we have $\Omega^1_{\mathcal{D}_n}(S)=\mathrm{ac}\mathcal{T}^1_\omega(S)$. Since $\mathrm{Add}\,\mathrm{ac}\mathcal{T}^1_\omega(S)=\mathrm{ac}\mathcal{T}^1_\omega(S)$, the case for $n=1$ follows. Let $n\geqslant 2$. By the induction hypothesis, we have that $\mathrm{Add}\,\Omega^k_{\mathcal{D}_n}(S)=\Omega^k_{\mathcal{D}_n}(S)$ is extension closed for any $1\leqslant k<n$, it follows from (1) that $\mathrm{ac}\mathcal{T}^n_\omega(S)=\Omega^n_{\mathcal{D}_n}(S)$. Thus $\Omega^n_{\mathcal{D}_n}(S)$ is closed under direct sums and direct summands and $\mathrm{Add}\,\Omega^n_{\mathcal{D}_n}(S)=\Omega^n_{\mathcal{D}_n}(S)$. □

Theorem 4.5. *Suppose $n\geqslant 1$ and $\mathcal{D}_n\subseteq {\omega_S}^{\top}$. Then the following statements are equivalent.*

(1) $\text{s.E-cograde}_\omega\,\mathrm{Tor}^S_{k+1}(\omega,N)\geqslant k$ *for any $N\in\mathrm{co}\Omega^k_{\mathcal{D}_n}(S)$ and $1\leqslant k\leqslant n$.*

(2) $\Omega^k_{\mathcal{D}_n}(S)$ *is extension closed for any $1\leqslant k\leqslant n$.*

(3) $\Omega^k_{\mathcal{D}_n}(S)$ *is extension closed and $\mathrm{ac}\mathcal{T}^k_\omega(S)=\Omega^k_{\mathcal{D}_n}(S)$ for any $1\leqslant k\leqslant n$.*

(4) $\mathrm{ac}\mathcal{T}^k_\omega(S)$ *is extension closed for any $1\leqslant k\leqslant n$.*

(5) $\mathrm{Add}\,\Omega^k_{\mathcal{D}_n}(S)$ *is extension closed for any $1\leqslant k\leqslant n$.*

Proof. (1) ⇒ (2) By Proposition 3.5, we have $\mathrm{ac}\mathcal{T}^k_\omega(S)=\Omega^k_{\mathcal{D}_n}(S)$ for any $1\leqslant k\leqslant n$. Let $N\in\mathrm{ac}\mathcal{T}^k_\omega(S)$. Then there is an exact sequence

$$0\to N\to U_0\to U_1\to\cdots\to U_{k-1}\to L\to 0$$

in $\mathrm{Mod}\,S$ with all U_i in $\mathcal{D}_n$ by [18, Proposition 3.8]. Notice that $L\in\mathrm{co}\Omega^k_{\mathcal{D}_n}(S)$, so

$$\text{s.E-cograde}_\omega\,\mathrm{Tor}^S_1(\omega,N)=\text{s.E-cograde}_\omega\,\mathrm{Tor}^S_{k+1}(\omega,L)\geqslant k.$$

It follows from Theorem 4.3 that $\Omega^k_{\mathcal{D}_n}(S)$ is extension closed for any $1\leqslant k\leqslant n$.

(5) ⇔ (2) ⇒ (3) It follows from Proposition 4.4.

(3) ⇒ (4) It is obvious.

(4) ⇒ (1) We proceed by induction on n. Let $N \in \mathrm{co}\Omega^1_{\mathcal{D}_n}(S)$ and let

$$0 \to L \to D \to N \to 0$$

be an exact sequence in $\mathrm{Mod}\, S$ with $D \in \mathcal{D}_n$. By Lemma 3.1, we have $L \in \mathrm{ac}\mathcal{T}^1_\omega(S)$. Thus

$$\text{s.E-cograde}_\omega \operatorname{Tor}^S_2(\omega, N) = \text{s.E-cograde}_\omega \operatorname{Tor}^S_1(\omega, L) \geqslant 1$$

by Theorem 4.3. The case for $n = 1$ follows.

Suppose $n \geqslant 2$. By the induction hypothesis, we have s.E-cograde$_\omega \operatorname{Tor}^S_{k+1}(\omega, N) \geqslant k$ for any $N \in \mathrm{co}\Omega^k_{\mathcal{D}_n}(S)$ and $1 \leqslant k \leqslant n-1$. From Proposition 3.5, we know that $\mathrm{ac}\mathcal{T}^k_\omega(S) = \Omega^k_{\mathcal{D}_n}(S)$ for any $1 \leqslant k \leqslant n$. Now let $N \in \mathrm{co}\Omega^n_{\mathcal{D}_n}(S)$ and let

$$0 \to L \to D_0 \to D_1 \to \cdots \to D_{n-1} \to N \to 0$$

be an exact sequence in $\mathrm{Mod}\, S$ with all D_i in $\mathcal{D}_n$. Then $L \in \mathrm{ac}\mathcal{T}^n_\omega(S)$, and hence

$$\text{s.E-cograde}_\omega \operatorname{Tor}^S_{n+1}(\omega, N) = \text{s.E-cograde}_\omega \operatorname{Tor}^S_1(\omega, L) \geqslant n$$

by Theorem 4.3. □

Since $\mathcal{A}_\omega(S)$ contains all projective left S-modules by [8, Lemma 4.1], any left S-module is in $\mathrm{co}\Omega^k_{\mathcal{A}}(S)$ for any $k \geqslant 1$. Moreover, we have $\mathcal{A}_\omega(S) = \mathrm{ac}\mathcal{T}(S) \cap {\omega_S}^\top$ by [18, Theorem 3.11].

Theorem 4.6. *The following statements are equivalent.*

1. (1) s.E-cograde$_\omega \operatorname{Tor}^S_{k+1}(\omega, N) \geqslant k$ *for any* $N \in \mathrm{Mod}\, S$ *and* $1 \leqslant k \leqslant n$.
2. (2) T-cograde$_\omega \operatorname{Ext}^k_R(\omega, M) \geqslant k$ *for any* $M \in \mathrm{Mod}\, R$ *for any* $1 \leqslant k \leqslant n$.
3. (3) $\Omega^k_{\mathcal{A}}(S)$ *is extension closed for any* $1 \leqslant k \leqslant n$.
4. (4) $\Omega^k_{\mathcal{A}}(S)$ *is extension closed and* $\mathrm{ac}\mathcal{T}^k_\omega(S) = \Omega^k_{\mathcal{A}}(S)$ *for any* $1 \leqslant k \leqslant n$.
5. (5) $\mathrm{ac}\mathcal{T}^k_\omega(S)$ *is extension closed for any* $1 \leqslant k \leqslant n$.
6. (6) $\mathrm{Add}\,\Omega^k_{\mathcal{A}}(S)$ *is extension closed for any* $1 \leqslant k \leqslant n$.

Proof. By [20, Proposition 4.12], we have (1) ⇔ (2). The other implications follow from Theorem 4.5 by replacing $\mathcal{D}_n$ with $\mathcal{A}_\omega(S)$. □

Recall from [14] that a ring R is called *semiregular* if $R/J(R)$ is von Neumann regular and idempotents can be lifted modulo $J(R)$, where $J(R)$ is the Jacobson radical of R. The class of semiregular rings includes: (i) von Neumann regular rings; (ii) semiperfect rings; (iii) left cotorsion rings; and (iv) right cotorsion rings. See [7] for the definitions of left cotorsion rings and right cotorsion rings.

Corollary 4.7. *Let* R *be semiregular and* $n \geqslant 1$. *Then the following statements are equivalent.*

1. (1) s.E-cograde$_\omega \operatorname{Tor}^S_{k+1}(\omega, N) \geqslant k$ *for any* $N \in \mathrm{Mod}\, S$ *and* $1 \leqslant k \leqslant n$.
2. (2) s.T-cograde$_\omega \operatorname{Ext}^{k+1}_{S^{op}}(\omega, N') \geqslant k$ *for any* $N' \in \mathrm{Mod}\, S^{op}$ *and* $1 \leqslant k \leqslant n$.
3. (3) $\mathrm{c}\mathcal{T}^k_\omega(S^{op})$ *is extension closed for any* $1 \leqslant k \leqslant n$.
4. (4) $\mathrm{ac}\mathcal{T}^k_\omega(S)$ *is extension closed for any* $1 \leqslant k \leqslant n$.

Proof. By the dual proof of Theorem 4.6, we get (2) ⇔ (3). The assertions (1) ⇔ (2) and (1) ⇔ (4) follow from from [20, Theorem 4.14] and Theorem 4.6 respectively. □

Recall that an artin algebra R is called *right quasi n-Gorenstein* if the projective dimension of the i-term in a minimal injective resolution of R_R is at most i for any $1 \leqslant i \leqslant n$ ([9]). Let D be the ordinary duality between $\operatorname{mod} R$ and $\operatorname{mod} R^{op}$. Then $D(R)$ is a semidualizing (R, R)-bimodule. It is induced from [20, Example 4.20] that R is right quasi n-Gorenstein if and only if $\text{s.E-cograde}_{\omega} \operatorname{Tor}^R_{i+1}(D(R), N) \geqslant i$ for any $N \in \operatorname{Mod} R$ and $1 \leqslant i \leqslant n$.

Corollary 4.8. *Let R be an artin algebra and $n \geqslant 1$. Then the following statements are equivalent.*

(1) *R is right quasi n-Gorenstein.*

(2) $\mathrm{c}\mathcal{T}^k_{D(R)}(R^{op})$ *is extension closed for any* $1 \leqslant k \leqslant n$.

(3) $\mathrm{ac}\mathcal{T}^k_{D(R)}(R)$ *is extension closed for any* $1 \leqslant k \leqslant n$.

Proof. It is a consequence of Corollary 4.7. □

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References

[1] T. Araya, R. Takahashi and Y. Yoshino, *Homological invariants associated to semi-dualizing bimodules*, J. Math. Kyoto Univ. **45** (2005), 287–306.

[2] M. Auslander and I. Reiten, *Syzygy modules for noetherian rings*, J. Algebra **183** (1996), 167–185.

[3] B. Bhmler and R. Marczinzik, *On the Extension-Closed Property for the Subcategory* $\operatorname{Tr}(\Omega^2(\operatorname{mod} - A))$, Algebras Represent. Theory (2022), https://doi.org/10.1007/s10468-022-10140-7.

[4] S. Dey and R. Takahashi, *On the subcategories of n-torsionfree modules and related modules*, Collect. Math. (2021), https://doi.org/10.1007/s13348-021-00338-1.

[5] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra. de Gruyter Exp. in Math. **30**, Walter de Gruyter, Berlin, New York, 2000.

[6] S. Goto and R. Takahashi, *Extension closedness of syzygies and local Gorensteinness of commutative rings*, Algebras Represent. Theory **19** (2016), 511–521.

[7] P. A. Guil Asensio and I. Herzog, *Left cotorsion rings*, Bull. London Math. Soc. **36** (2004), 303–309.

[8] H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), 781–808.

[9] Z. Y. Huang, *Syzygy modules for quasi k-Gorenstein rings*, J. Algebra **299** (2006), 21–32.

[10] Z. Y. Huang, *Extension closure of k-torsionfree modules*, Comm. Algebra **27** (1999), 1457–1464.

[11] Z. Y. Huang, *Extension closure of relative syzygy modules*, Sci China Ser A: Math **46** (2003), 611–620.

[12] Z. Y. Huang, *Duality pairs induced by Auslander and Bass classes*, Georgian Math. J. **28** (2021), 867–882.

[13] Z. F. Liu, Z. Y. Huang and A. M. Xu, *Gorenstein projective dimension relative to a semidualizing bimodule*, Comm. Algebra **41** (2013), 1–18.

[14] W. K. Nicholson, *Semiregular modules and rings*, Canad. J. Math. **28** (1976), 1105–1120.

[15] R. Takahashi, *When is there a nontrivial extension-closed subcategory?*, J. Algebra **331** (2011), 388–399.

[16] X. Tang and Z. Y. Huang, *Homological aspects of the dual Auslander transpose*, Forum Math. **27** (2015), 3717–3743.

[17] X. Tang and Z. Y. Huang, *Homological aspects of the dual Auslander transpose II*, Kyoto J. Math. **57** (2017), 17–53.

[18] X. Tang and Z. Y. Huang, *Homological aspects of the adjoint cotranspose*, Colloq. Math. **150** (2017), 293–311.
[19] X. Tang and Z. Y. Huang, *Coreflexive modules and semidualizing modules with finite projective dimension*, Taiwanese J. Math. **21** (2017), 1283–1324.
[20] X. Tang and Z. Y. Huang, *Cograde conditions and cotorsion pairs*, Publ. Res. Inst. Math. Sci. **56** (2020), 445–502.
[21] J. Trlifaj, *Whitehead test modules*, Trans. Amer. Math. Soc. **348** (1996), 1521–1554.
[22] G. Q. Zhao and B. Zhang, *Homological behavior of relative cotorsionfree and cosyzygy modules*, Algebra Colloq. **26** (2019), 467–478.

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