Gorenstein Algebras and Recollements∗†‡§

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Abstract

Let $A$, $A'$ and $A''$ be artin algebras. We prove that if there is a recollement of the bounded Gorenstein derived category $D^b_{G(P(\text{Mod } A))}(\text{Mod } A)$ relative to the bounded Gorenstein derived categories $D^b_{G(P(\text{Mod } A'))}(\text{Mod } A')$ and $D^b_{G(P(\text{Mod } A''))}(\text{Mod } A'')$, then $A$ is Gorenstein if and only if so are $A'$ and $A''$. In addition, we prove that a virtually Gorenstein algebra $A$ is Gorenstein if and only if the bounded homotopy category of (finitely generated) projective left $A$-modules and that of (finitely generated) injective left $A$-modules coincide.

1 Introduction

Recollements play an important role in algebraic geometry and representation theory, see [1, 9, 25], which were introduced by Be˘ılinson, Bernstein and Deligne in [9]. It is known that there are many interesting homological properties or invariants in the framework of recollements. For instance, for a recollement of bounded derived categories over finite dimensional algebras, Wiedemann proved in [32] that the finiteness of the global dimension is invariant in the recollement, and Happel proved in [21] that the finiteness of the finitistic dimension is also invariant in the recollement, and so on.

For an artin algebra $A$, we use $\text{Mod } A$ and $\text{mod } A$ to denote the category of left $A$-modules and the category of finitely generated left $A$-modules respectively. Let $X = \text{Mod } A$ or $X = \text{mod } A$. We use $G(P(X))$ to denote the subcategory of $X$ consisting of Gorenstein projective modules, and use $D^b_{G(P(X))}(X)$ to denote the bounded Gorenstein derived categories of $X$ ([19]).

Recently, the Gorensteinness of algebras in recollements was studied by several authors. Let $A$, $A'$ and $A''$ be finite dimensional algebras and the bounded derived category $D^b(\text{mod } A)$ admit a recollement

$$D^b(\text{mod } A') \xleftarrow{i} \xrightarrow{j^*} D^b(\text{mod } A) \xleftarrow{i^*} \xrightarrow{j} D^b(\text{mod } A'').$$

Pan proved that if $A$ is Gorenstein, then $A'$ and $A''$ are also Gorenstein ([28, Theorem 3.1]). Later, Chen and König proved that $A$ is Gorenstein if and only if $A'$ and $A''$ are also Gorenstein plus an extra condition ([15, Proposition 3.4]). Qin and Han proved that $A$ is Gorenstein if and only if so are $A'$ and $A''$ provided the recollement is what they call a 4-recollement, where they used the unbounded derived category of $\text{Mod }$ ([29, Theorem III]). On the other hand, Asadollahi, Bahiraei, Hafezi and Vahed studied

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The Gorensteinness of algebras in recollements of relative derived categories and they showed that the Gorensteinness of certain artin algebras is an invariant of recollements; that is, for artin algebras $A$, $A'$ and $A''$ with $\mathcal{G}(\mathcal{P}^{\text{mod}}(A))$, $\mathcal{G}(\mathcal{P}^{\text{mod}}(A'))$ and $\mathcal{G}(\mathcal{P}^{\text{mod}}(A''))$ contravariantly finite in mod $A$, mod $A'$ and mod $A''$ respectively, if $D^b_{\mathcal{G}(\mathcal{P}^{\text{mod}}(A))}(\text{mod } A)$ admits a recollement

$$D^b_{\mathcal{G}(\mathcal{P}^{\text{mod}}(A'))}(\text{mod } A') \xrightarrow{\text{inj}} D^b_{\mathcal{G}(\mathcal{P}^{\text{mod}}(A))}(\text{mod } A) \xrightarrow{\text{proj}} D^b_{\mathcal{G}(\mathcal{P}^{\text{mod}}(A''))}(\text{mod } A''),$$

then $A$ is Gorenstein if and only if so are $A'$ and $A''$ ([3, Theorem 4.7]). Independently, using a different proof from that in [3], Gao showed in [18, Theorem 3.5] that the Gorensteinness of virtually Gorenstein artin algebras is an invariant of recollements; that is, for virtually Gorenstein artin algebras $A$, $A'$ and $A''$, if $D^b_{\mathcal{G}(\mathcal{P}^{\text{mod}}(A))}(\text{mod } A)$ admits a recollement as above, then $A$ is Gorenstein if and only if so are $A'$ and $A''$.

The following is one of the main results in this paper, which is a “big module” analogue of the Asadollahi, Bahiraei, Hafezi and Vahed’s result mentioned above as well as a Gorenstein analogue of a classical result of Wiedemann ([32, Lemma 2.1]).

**Theorem 1.1.** (Corollary 3.8) For artin algebras $A$, $A'$ and $A''$, if $D^b_{\mathcal{G}(\mathcal{P}^{\text{mod}}(A))}(\text{mod } A)$ admits a recollement

$$D^b_{\mathcal{G}(\mathcal{P}^{\text{mod}}(A'))}(\text{mod } A') \xrightarrow{\text{inj}} D^b_{\mathcal{G}(\mathcal{P}^{\text{mod}}(A))}(\text{mod } A) \xrightarrow{\text{proj}} D^b_{\mathcal{G}(\mathcal{P}^{\text{mod}}(A''))}(\text{mod } A''),$$

then $A$ is Gorenstein if and only if so are $A'$ and $A''$.

In fact, we prove this result in some more general setting (Theorem 3.7).

Let $A$ be an artin algebra and $X = \text{Mod } A$ or $X = \text{mod } A$. For a subcategory $\mathcal{C}$ of $X$, we use $K^b(\mathcal{C})$ to denote the bounded homotopy category of $\mathcal{C}$. Happel proved in [20, Lemma 1.5] that a finite-dimensional algebra $A$ is Gorenstein if and only if $K^b(\mathcal{I}(\text{mod } A)) = K^b(\mathcal{P}(\text{mod } A))$, where $\mathcal{I}(\text{mod } A)$ and $\mathcal{P}(\text{mod } A)$ are the subcategories of mod $A$ consisting of injective modules and projective modules respectively. On the other hand, as an important generalization of Gorenstein algebras, Beligiannis and Reiten introduced in [11] virtually Gorenstein algebras. Note that any Gorenstein algebra is virtually Gorenstein and the converse is not true in general ([11, 12]). We get some equivalent characterizations for virtually Gorenstein artin algebras being Gorenstein, which can be regarded as a Gorenstein analogue of the above result of Happel.

**Theorem 1.2.** (Theorem 4.6) For a virtually Gorenstein artin algebra $A$, the following statements are equivalent.

1. $A$ is Gorenstein.
2. $K^b(\mathcal{G}(\mathcal{P}(\text{mod } A))) = K^b(\mathcal{G}(\mathcal{I}(\text{mod } A)))$ in $D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A)$.
3. $K^b(\mathcal{G}(\mathcal{P}(\text{mod } A))) = K^b(\mathcal{G}(\mathcal{I}(\text{mod } A)))$ in $D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A)$.

## 2 Preliminaries

Throughout this paper, $\mathcal{A}$ is an abelian category and all subcategories of $\mathcal{A}$ are additive, full and closed under isomorphisms. We use $\mathcal{P}(\mathcal{A})$ and $\mathcal{I}(\mathcal{A})$ to denote the subcategories of $\mathcal{A}$ consisting of projective and injective objects, respectively.

We fix a subcategory $\mathcal{C}$ of $\mathcal{A}$. A complex $X$ in $\mathcal{A}$ is called $\mathcal{C}$-acyclic (resp. $\mathcal{C}$-coacyclic) if $H^i \text{Hom}_\mathcal{A}(C, X) = 0$ (resp. $H^i \text{Hom}_\mathcal{A}(X, C) = 0$) for any $C \in \mathcal{C}$ and $i \in \mathbb{Z}$ (the set of integers).
Definition 2.1. ([6]) Let $\mathcal{C} \subseteq \mathcal{D}$ be subcategories of $\mathcal{A}$. The morphism $f : C \to D$ in $\mathcal{A}$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$ is called a right $\mathcal{C}$-approximation of $D$ if the complex $C \overset{f}{\to} D \to 0$ is $\mathcal{C}$-acyclic. If every object in $\mathcal{D}$ admits a right $\mathcal{C}$-approximation, then $\mathcal{C}$ is called contravariantly finite in $\mathcal{D}$. Dually, the notions of left $\mathcal{C}$-approximations and covariantly finite subcategories are defined.

Let $K^*_\mathcal{C-acyc}(\mathcal{A})$ (resp. $K^*_\mathcal{C-coacyc}(\mathcal{A})$) denote the subcategory of the homotopy category $K^*(\mathcal{A})$ consisting of $\mathcal{C}$-acyclic (resp. $\mathcal{C}$-coacyclic) complexes, where $* \in \{\text{blank}, -, +, b\}$. A chain map $f : X \to Y$ is called a $\mathcal{C}$-quasi-isomorphism (resp. $\mathcal{C}$-coquasi-isomorphism) if $\text{Hom}_\mathcal{A}(C, f)$ (resp. $\text{Hom}_\mathcal{A}(f, C)$) is a quasi-isomorphism for any $C \in \mathcal{C}$. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then the relative derived category, denoted by $D^*_\mathcal{C}(\mathcal{A})$ (resp. $\text{co-}D^*_\mathcal{C}(\mathcal{A})$), is the Verdier quotient of the homotopy category $K^*(\mathcal{A})$ with respect to the thick subcategory $K^*_\mathcal{C-acyc}(\mathcal{A})$ (resp. $K^*_\mathcal{C-coacyc}(\mathcal{A})$) ([14]). Set

$$K^{-, \text{Cb}}(\mathcal{C}) := \{X \in K^{-}(\mathcal{C}) | \text{there exists } N \in \mathbb{Z} \text{ such that } H^i \text{Hom}_\mathcal{A}(C, X) = 0 \text{ for any } C \in \mathcal{C} \text{ and } i \leq N\}.$$

$$K^{+, \text{Cb}}(\mathcal{C}) := \{X \in K^{+}(\mathcal{C}) | \text{there exists } N \in \mathbb{Z} \text{ such that } H^i \text{Hom}_\mathcal{A}(X, C) = 0 \text{ for any } C \in \mathcal{C} \text{ and } i \geq N\}.$$

Lemma 2.2. ([19, Theorem 3.6] and [4, Theorem 3.3])

(1) If $\mathcal{C}$ is contravariantly finite in $\mathcal{A}$, then $D^*_\mathcal{C}(\mathcal{A}) \cong K^{-, \text{Cb}}(\mathcal{C})$.

(2) If $\mathcal{C}$ is covariantly finite in $\mathcal{A}$, then $\text{co-}D^*_\mathcal{C}(\mathcal{A}) \cong K^{+, \text{Cb}}(\mathcal{C})$.

Definition 2.3. ([31]) The Gorenstein subcategory $\mathcal{G}(\mathcal{C})$ of $\mathcal{A}$ is defined as $\mathcal{G}(\mathcal{C}) = \{M \in \mathcal{A} | \text{there exists an exact sequence:} \}

\[\cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots\]

in $\mathcal{A}$ with all $C_i, C^i$ in $\mathcal{C}$, which is both $\mathcal{C}$-acyclic and $\mathcal{C}$-coacyclic, such that $M \cong \text{Im}(C_0 \to C^0)$.

If $\mathcal{C} = \mathcal{P}(\mathcal{A})$ (resp. $\mathcal{I}(\mathcal{A})$), then $\mathcal{G}(\mathcal{P}(\mathcal{A}))$ (resp. $\mathcal{G}(\mathcal{I}(\mathcal{A}))$) is exactly the subcategory of $\mathcal{A}$ consisting of Gorenstein projective (resp. Gorenstein injective) objects ([16]).

Let $\mathcal{A}$ be an Artin algebra. Recall from [10, 11] that $\mathcal{A}$ is called virtually Gorenstein if $\mathcal{G}(\mathcal{P}(\text{Mod } \mathcal{A}))^\perp = \perp \mathcal{G}(\mathcal{I}(\text{Mod } \mathcal{A}))$,

where $\mathcal{G}(\mathcal{P}(\text{Mod } \mathcal{A}))^\perp = \{M \in \text{Mod } \mathcal{A} | \text{Ext}_\mathcal{A}^{\geq 1}(G, M) = 0 \text{ for any } G \in \mathcal{G}(\mathcal{P}(\text{Mod } \mathcal{A}))\}$

and $\perp \mathcal{G}(\mathcal{I}(\text{Mod } \mathcal{A})) = \{M \in \text{Mod } \mathcal{A} | \text{Ext}_\mathcal{A}^{\geq 1}(M, H) = 0 \text{ for any } H \in \mathcal{G}(\mathcal{I}(\text{Mod } \mathcal{A}))\}$.

Definition 2.4. ([5, 7]) Let $M \in \mathcal{A}$ and $n \geq 0$. The $\mathcal{C}$-dimension $\mathcal{C}$-dim $M$ of $M$ is said to be at most $n$ if there exists an exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to M \to 0 \quad (2.1)$$

in $\mathcal{A}$ with all $C_i$ objects in $\mathcal{C}$. Moreover, the sequence (2.1) is called a proper $\mathcal{C}$-resolution of $M$ if it is $\mathcal{C}$-acyclic. Dually, The $\mathcal{C}$-codimension $\mathcal{C}$-codim $M$ of $M$ is said to be at most $n$ if there exists an exact sequence

$$0 \to M \to C^0 \to \cdots \to C^{n-1} \to C^n \to 0 \quad (2.2)$$

in $\mathcal{A}$ with all $C^i$ objects in $\mathcal{C}$. Moreover, the sequence (2.2) is called a coproper $\mathcal{C}$-coresolution of $M$ if it is $\mathcal{C}$-coacyclic.
**Definition 2.5.** ([9]) Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be triangulated categories. A **recollement** of $\mathcal{T}$ relative to $\mathcal{T}'$ and $\mathcal{T}''$ is a diagram

\[
\begin{array}{c}
\mathcal{T}' \xrightarrow{i^*} \mathcal{T} \xrightarrow{j_*} \mathcal{T}''
\end{array}
\]

of triangle functors such that

1. $(i^*, i_*)$, $(i^*, i^!)$, $(j_!, j^*)$ and $(j^*, j_*)$ are adjoint pairs.
2. $j^*i_* = 0$.
3. $i_*$, $j_!$ and $j_*$ are fully faithful.
4. For any object $X$ in $\mathcal{T}$, there exist triangles

\[
\begin{array}{c}
i_*i^!X \longrightarrow X \longrightarrow j_*j^*X \longrightarrow (i_*i^!X)[1]
\end{array}
\]

and

\[
\begin{array}{c}
j_*j^*X \longrightarrow X \longrightarrow i_*i^*X \longrightarrow (j_*j^*X)[1],
\end{array}
\]

where the maps are given by adjunctions.

3 Relative Singularity Categories and Recollements

**Definition 3.1.** ([27]) Let $\mathcal{D}$ be a triangulated category. An object $P \in \mathcal{D}$ is called **perfect** if for any object $Y \in \mathcal{D}$, $\text{Hom}_{\mathcal{D}}(P, Y[i]) = 0$ except for finitely many $i \in \mathbb{Z}$. Dually, the notion of **coperfect objects** is defined.

We use $\mathcal{D}_{\text{perf}}$ (resp. $\mathcal{D}_{\text{coperf}}$) to denote the triangulated subcategory consisting of perfect (resp. coperfect) objects, which is called the **perfect (resp. coperfect) subcategory** of $\mathcal{D}$ ([27]). Obviously, $\mathcal{D}_{\text{perf}}$ and $\mathcal{D}_{\text{coperf}}$ are thick subcategories of $\mathcal{D}$, and both of them are invariants of triangle equivalence.

Let $\mathcal{C}$ be a contravariantly finite subcategory of $\mathcal{A}$. Li and Huang introduced in [26] the **relative singularity category** by the following Verdier quotient

\[
\mathcal{D}^b_{\mathcal{C}}(\mathcal{A}) = \mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{C}).
\]

On the other hand, Orlov [27] defined it in a different way.

**Definition 3.2.** ([27, Definition 1.7]) Let $\mathcal{D}$ be a triangulated category, the **singularity category** of $\mathcal{D}$ is defined by the Verdier quotient $\mathcal{D}/\mathcal{D}_{\text{perf}}$. In particular, let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a subcategory of $\mathcal{A}$. The **relative singularity category** is defined by the following Verdier quotient

\[
\mathcal{D}^b_{\mathcal{C}}(\mathcal{A}) = \mathcal{D}^b(\mathcal{C})/\mathcal{D}^b_{\text{perf}}(\mathcal{A}).
\]

In general, we have $K^b(\mathcal{C}) \subseteq \mathcal{D}^b(\mathcal{A})_{\text{perf}}$. In the following, we will give some sufficient condition such that they are identical. In this case, the two definitions of the relative singularity categories mentioned above coincide.

Rickard proved in [30, Proposition 6.2] that $\mathcal{D}^b(\text{Mod } A)_{\text{perf}} = K^b(\mathcal{P}(\text{Mod } A))$ for any ring $A$. We generalize this result and [13, Lemma 1.2.1] as follows.

**Proposition 3.3.** Assume that

(a) $\mathcal{A}$ has enough projective objects; and

(b) $\mathcal{C}$ is a contravariantly finite subcategory of $\mathcal{A}$ closed under direct summands, such that $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}$.

Then for any $G \in \mathcal{D}^b_{\text{perf}}(\mathcal{A})$, the following statements are equivalent.

1. $G \in K^b(\mathcal{C})$. 

(2) There exists $i(G) \in \mathbb{Z}$, such that $\text{Hom}_{D^b_c(A)}(G, M[j]) = 0$ for any $M \in \mathcal{A}$ and $j \geq i(G)$.

(3) There exists a finite set $I(G) \subseteq \mathbb{Z}$, such that $\text{Hom}_{D^b_c(A)}(G, M[j]) = 0$ for any $M \in \mathcal{A}$ and $j \notin I(G)$.

Furthermore, if $\mathcal{A}$ is closed under direct sums, then $D^b_c(A)_{\text{perf}} = K^b_c(\mathcal{C})$.

Proof. The implications $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$ are obvious.

$(2) \Rightarrow (1)$ Let $G \in D^b_c(A)$. Then there exists a $C$-quasi-isomorphism $Q \to G$ with $Q \in K^{-b}(\mathcal{C})$ by Lemma 2.2, and hence there exists $N \in \mathbb{Z}$ such that $H^i \text{Hom}_A(C, Q) = 0$ for any $C \in \mathcal{C}$ and $i \leq N$. It follows that $H^iQ = 0$ for any $i \leq N$ since $\mathcal{P}(A) \subseteq \mathcal{C}$.

We claim that there exists $n \leq N$ such that $\text{Im} d^b_Q \notin \mathcal{C}$. Otherwise, there exists $-n \geq i(G)$ such that $\text{Im} d^b_Q \notin \mathcal{C}$. Put $M := \text{Im} d^b_Q$. Then there exists a non-zero epimorphism $\overline{d}^Q : Q^n \to M$ in $\mathcal{A}$, such that $\overline{d}^Q$ is identified with the composition of $Q^n \xrightarrow{d^b_Q} M \xrightarrow{\lambda} Q^{n+1}$. It induces the following chain map

\[
\begin{array}{cccccccc}
Q & & & & \cdots & & & & \text{Im} \overline{d}^Q \\
\downarrow f & & & & \downarrow & & & & \downarrow \\
M[-n] & & & & \cdots & & & & 0 \\
\end{array}
\]

Notice that $\text{Im} d^b_Q \notin \mathcal{C}$, so $f$ is not null homotopic. Otherwise, there exists a morphism $h : Q^{n+1} \to M$ such that $\overline{d}^Q = hd^b_Q = h\lambda d^b_Q$. Since $d^b_Q$ is epic, we have that $1_M = h\lambda$ and $M$ is a isomorphic to a direct summand of $Q^{n+1}$ in $\mathcal{C}$. Since $\mathcal{C}$ is closed under direct summands by assumption, it follows that $M \in \mathcal{C}$, a contradiction. So $\text{Hom}_{K^{-b}(A)}(Q, M[-n]) \neq 0$ and we have

\[
\text{Hom}_{D^b_c(A)}(G, M[-n]) \cong \text{Hom}_{D^b_c(A)}(Q, M[-n]) \cong \text{Hom}_{K^{-b}(A)}(Q, M[-n]) \neq 0,
\]

which contradicts the assumption. The claim is proved.

Now we have a $C$-quasi-isomorphism

\[
\begin{array}{cccccccc}
Q & & & & \cdots & & & & \text{Im} d^b_Q \\
\downarrow \tau_{\geq n+1}Q & & & & \downarrow & & & & \downarrow \\
\cdots & & & & 0 & & & & \cdots \\
\end{array}
\]

with $\tau_{\geq n+1}Q \in K^b_c(\mathcal{C})$. Then $G \cong Q \cong \tau_{\geq n+1}Q$ in $D^b_c(A)$, and it follows that $G \cong \tau_{\geq n+1}Q \in K^b_c(\mathcal{C})$ in $D^b_c(A)$.

Similarly, we get $(3) \Rightarrow (1)$.

Furthermore, the inclusion $K^b_c(\mathcal{C}) \subseteq D^b_c(A)_{\text{perf}}$ is obvious. Conversely, let $P \in D^b_c(A)_{\text{perf}}$. Then there exists a $C$-quasi-isomorphism $Q \to P$ with $Q \in K^{-b}(\mathcal{C})$ by Lemma 2.2, and hence there exists $N \in \mathbb{Z}$ such that $H^i \text{Hom}_A(C, Q) = 0$ for any $C \in \mathcal{C}$ and $i \leq N$. Since $\mathcal{P}(A) \subseteq \mathcal{C}$, we have $H^iQ = 0$ for any $i \leq N$.

We claim that there exists $n \leq N$ such that $\text{Im} d^b_Q \notin \mathcal{C}$. Otherwise, suppose that $\text{Im} d^b_Q \notin \mathcal{C}$ for any $n \leq N$. Take $M := \bigoplus_{i \leq N} \text{Im} d^b_Q \in \mathcal{C}$ and the non-zero morphism

\[
\overline{d}^Q : Q^n \longrightarrow M = \text{Im} d^b_Q \oplus (\bigoplus_{i \leq N, i \neq n} \text{Im} d^b_Q),
\]
such that $d^n_Q$ is identified with the composition of $Q^n \xrightarrow{d^n_Q} M \xrightarrow{\lambda} Q^{n+1}$. It induces the following chain map

$$
\begin{array}{cccccccc}
Q & \rightarrow & Q^{n-1} & \xrightarrow{d^{n-1}_Q} & Q^n & \xrightarrow{d^n_Q} & Q^{n+1} & \rightarrow & \cdots \\
M[-n] & \rightarrow & 0 & \rightarrow & M & \rightarrow & 0 & \rightarrow & \cdots
\end{array}
$$

Note that $\text{Im} d^n_Q \not\subset C$, so $f$ is not null homotopic. Otherwise, by an argument similar to the above, we have $\text{Im} d^n_Q \subset C$, a contradiction. So $\text{Hom}_{K^-(\mathcal{A})}(Q, M[-n]) \neq 0$ and we have

$$
\text{Hom}_{D^\mathcal{C}(\mathcal{A})}(P, M[-n]) \cong \text{Hom}_{D^\mathcal{C}(\mathcal{A})}(Q, M[-n]) \\
\cong \text{Hom}_{K^-(\mathcal{A})}(Q, M[-n]) \neq 0.
$$

So there are infinitely many $n$ such that $\text{Hom}_{D^\mathcal{C}(\mathcal{A})}(P, M[-n]) \neq 0$, which contradicts the assumption that $P \in D^b_{\mathcal{C}(\mathcal{A})}\text{perf}$. The claim is proved.

Consider the following $\mathcal{C}$-quasi-isomorphism

$$
\begin{array}{cccccccc}
Q & \rightarrow & Q^{n-1} & \xrightarrow{d^{n-1}_Q} & Q^n & \xrightarrow{d^n_Q} & Q^{n+1} & \rightarrow & \cdots \\
\tau_{\geq n+1}Q & \rightarrow & 0 & \rightarrow & \text{Im} d^n_Q & \rightarrow & Q^{n+1} & \rightarrow & \cdots
\end{array}
$$

with $\tau_{\geq n+1}Q \in K^b(\mathcal{C})$. Then $P \cong Q \cong \tau_{\geq n+1}Q$ in $D^\mathcal{C}(\mathcal{A})$, and so $P \cong \tau_{\geq n+1}Q \in K^b(\mathcal{C})$ in $D^b_{\mathcal{C}(\mathcal{A})}$. It follows that $P \in K^b(\mathcal{C})$. The proof is finished.

Because $\mathcal{G}(\mathcal{C})$ is closed under direct summands by [23, Theorem 4.6(2)], the following is an immediate consequence of Proposition 3.3.

**Corollary 3.4.** Assume that

(a) $\mathcal{A}$ has enough projective objects; and

(b) $\mathcal{G}(\mathcal{C})$ is a contravariantly finite subcategory of $\mathcal{A}$ and $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}$.

Then for any $G \in D^b_{\mathcal{G}(\mathcal{C})}(\mathcal{A})$, the following statements are equivalent.

1. $G \in K^b(\mathcal{G}(\mathcal{C}))$.

2. There exists $i(G) \in \mathbb{Z}$, such that $\text{Hom}_{D^b_{\mathcal{G}(\mathcal{C})}(\mathcal{A})}(G, M[j]) = 0$ for any $M \in \mathcal{A}$ and $j \geq i(G)$.

3. There exists a finite set $I(G) \subseteq \mathbb{Z}$, such that $\text{Hom}_{D^b_{\mathcal{G}(\mathcal{C})}(\mathcal{A})}(G, M[j]) = 0$ for any $M \in \mathcal{A}$ and $j \not\in I(G)$.

Furthermore, if $\mathcal{A}$ is closed under direct sums, then $D^b_{\mathcal{G}(\mathcal{C})}(\mathcal{A})\text{perf} = K^b(\mathcal{G}(\mathcal{C}))$.

Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated subcategories of triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ respectively. Recall from [2] that a triangle functor $F : \mathcal{D} \rightarrow \mathcal{D}'$ is said to restrict to $\mathcal{D}$ if $F$ restricts to a triangle functor $\mathcal{D} \rightarrow \mathcal{D}'$.

**Lemma 3.5.** Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be a triangle functor of triangulated categories. If $F$ admits a right adjoint $G$, then $F$ restricts to $\mathcal{D}_\text{perf}$.

**Proof.** Let $X \in \mathcal{D}_\text{perf}$. For any $Y \in \mathcal{D}'$, we have

$$
\text{Hom}_{\mathcal{D}'}(FX, Y[i]) \cong \text{Hom}_{\mathcal{D}}(X, GY[i]) \cong \text{Hom}_{\mathcal{D}}(X, (GY)[i]).
$$

By the definition of perfect objects, we have $FX \in \mathcal{D'}_{\text{perf}}$, and hence $F$ restricts to $\mathcal{D}_{\text{perf}}$. \qed
Proposition 3.6. Assume that
(1) both $\mathcal{A}$ and $\mathcal{A}'$ are abelian categories having enough projective objects and closed under direct sums.
(2) $\mathcal{C}$ and $\mathcal{C}'$ are contravariantly finite subcategories of $\mathcal{A}$ and $\mathcal{A}'$ closed under direct summands respectively, such that $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}$ and $\mathcal{P}(\mathcal{A}') \subseteq \mathcal{C}'$.

If $F : D^b_{\mathcal{C}}(\mathcal{A}) \to D^b_{\mathcal{C}'}(\mathcal{A}')$ is a triangle functor admitting a right adjoint $G$, then $F$ restricts to $K^b(\mathcal{C})$.

Proof. It follows from Proposition 3.3 and Lemma 3.5. □

Our main result is the following

Theorem 3.7. Assume that
(1) $\mathcal{A}, \mathcal{A}'$ and $\mathcal{A}''$ are abelian categories having enough projective objects and closed under direct sums;
(2) $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$ are contravariantly finite subcategories of $\mathcal{A}, \mathcal{A}'$ and $\mathcal{A}''$ closed under direct summands respectively, such that $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}$, $\mathcal{P}(\mathcal{A}') \subseteq \mathcal{C}'$ and $\mathcal{P}(\mathcal{A}'') \subseteq \mathcal{C}''$.

If $D^b_{\mathcal{C}}(\mathcal{A})$ admits a recollement

$$
\begin{array}{c}
D^b_{\mathcal{C}'}(\mathcal{A}') \\
\downarrow \quad j^* \quad \downarrow \quad i^* \quad \downarrow \quad j^* \quad \downarrow \quad i^* \\
D^b_{\mathcal{C}}(\mathcal{A}) \\
\downarrow \quad j^* \quad \downarrow \quad i^* \quad \downarrow \quad j^* \\
D^b_{\mathcal{C}''}(\mathcal{A}'')
\end{array}
$$

then $D^b_{\mathcal{C}-sg}(\mathcal{A}) = 0$ if and only if $D^b_{\mathcal{C}-sg}(\mathcal{A}') = 0 = D^b_{\mathcal{C}''-sg}(\mathcal{A}'')$.

Proof. By assumption, we have $D^b_{\mathcal{C}}(\mathcal{A}) \cong K^b(\mathcal{C})$. Let $X', Y \in D^b_{\mathcal{C}'}(\mathcal{A}')$. Since $i_*$ is fully faithful and $i_* X' \in D^b_{\mathcal{C}'}(\mathcal{A}')(\cong K^b(\mathcal{C}'))$, it follows from Proposition 3.3 that

$$
\text{Hom}_{D^b_{\mathcal{C}}}(X', Y[i]) \cong \text{Hom}_{D^b_{\mathcal{C}}}(i_* X', i_* Y[i]) \cong \text{Hom}_{D^b_{\mathcal{C}}}(i_* X', (i_* Y)[i]) = 0
$$

except for finitely many $i \in \mathbb{Z}$ and so $X' \in K^b(\mathcal{C}')$. Thus $K^b(\mathcal{C}') \cong D^b_{\mathcal{C}'}(\mathcal{A}')$ and $D^b_{\mathcal{C}-sg}(\mathcal{A}') = 0$. Similarly, we have $D^b_{\mathcal{C}''-sg}(\mathcal{A}'') = 0$.

Conversely, let $X \in D^b_{\mathcal{C}}(\mathcal{A})$. Then there exists a triangle

$$
\begin{array}{c}
j_* j^* X \\
\quad \longrightarrow \\
i_* i^* X \quad \longrightarrow \\
\quad \longrightarrow \\
X \quad \longrightarrow \\
j_* j^* X + [1]
\end{array}
$$

By assumption, $j^* X \in K^b(\mathcal{C}'')$ and $i^* X \in K^b(\mathcal{C}')$. Since $i_*$ restricts to $K^b(\mathcal{C}')$ and $j_*$ restricts to $K^b(\mathcal{C}'')$ by Proposition 3.6, we have $j_* j^* X \in K^b(\mathcal{C})$ and $i_* i^* X \in K^b(\mathcal{C})$, and hence $X \in K^b(\mathcal{C})$, which implies that $K^b(\mathcal{C}) \cong D^b_{\mathcal{C}}(\mathcal{A})$ and $D^b_{\mathcal{C}-sg}(\mathcal{A}) = 0$. □

Recall that an artin algebra $A$ is called Gorenstein if its left self-injective dimension id$_A A$ and right self-injective dimension id$_{A^e} A$ are finite. We have the following facts: (1) An artin algebra $A$ is Gorenstein if and only if $D^b_G(\mathcal{P}(\text{Mod } A))_{-sg}(\text{Mod } A) = 0$ ([8]); (2) $G(\mathcal{P}(\text{Mod } A))$ is contravariantly finite in $\text{Mod } A$ ([10, Theorem 3.5]). So, by taking $A = \text{Mod } A$ and $\mathcal{C} = G(\mathcal{P}(\text{Mod } A))$ in Theorem 3.7, we have the following result, which is a Gorenstein analogue of [32, Lemma 2.1] as well as a “big module” analogue of [3, Theorem 4.7] and [18, Theorem 3.5].

Corollary 3.8. Let $A, A'$ and $A''$ be artin algebras. If $D^b_G(\mathcal{P}(\text{Mod } A'))(\text{Mod } A)$ admits a recollement

$$
\begin{array}{c}
D^b_G(\mathcal{P}(\text{Mod } A'))(\text{Mod } A') \\
\downarrow \quad i^* \quad \downarrow \quad j^* \\
D^b_G(\mathcal{P}(\text{Mod } A))(\text{Mod } A) \\
\downarrow \quad i^* \quad \downarrow \quad j^* \\
D^b_G(\mathcal{P}(\text{Mod } A''))(\text{Mod } A''),
\end{array}
$$

then $A$ is Gorenstein if and only if so are $A'$ and $A''$.
It is well known that a ring $A$ has finite left global dimension if and only if $\text{D}^b(\text{Mod} A) = K^b(\text{P}(\text{Mod} A))$ (that is, $D^b_{\text{P}(\text{Mod} A)-\text{sg}}(\text{Mod} A) = 0$). So, by taking $A = \text{Mod} A$ and $C = \text{P}(\text{Mod} A)$ in Theorem 3.7, we have the following “big module” analogue of [32, Lemma 2.1].

Corollary 3.9. (cf. [25, Corollaries 5 and 6]) Let $A$, $A'$ and $A''$ be rings. If $\text{D}^b(\text{Mod} A)$ admits a recollement

$$
\begin{array}{c}
\text{D}^b(\text{Mod} A') & \xrightarrow{\cdot} & \text{D}^b(\text{Mod} A) & \xleftarrow{\cdot} & \text{D}^b(\text{Mod} A''),
\end{array}
$$

then $A$ has finite left global dimension if and only if so do $A'$ and $A''$.

We remark that the dual counterparts of all results in this section also hold true.

4 Characterizing Gorenstein Algebras

In this section, $A$ is an artin algebra.

Lemma 4.1. We have

(1) For any $X \in K^-(\text{G}(\text{P}(\text{mod} A)))$, the following statements are equivalent.

(1.1) $X \in K^b(\text{G}(\text{P}(\text{mod} A)))$.

(1.2) For any $Y \in K^+(\text{G}(\text{I}(\text{mod} A)))$,

$$
\text{Hom}_{K^+(\text{G}(\text{I}(\text{mod} A)))}^i(X, Y[i]) = 0
$$

except for finitely many $i \in \mathbb{Z}$.

(2) For any $G \in K^+(\text{G}(\text{I}(\text{mod} A)))$, the following statements are equivalent.

(2.1) $G \in K^b(\text{G}(\text{I}(\text{mod} A)))$.

(2.2) for any $Y \in K^+(\text{G}(\text{I}(\text{mod} A)))$,

$$
\text{Hom}_{K^+(\text{G}(\text{I}(\text{mod} A)))}^i(Y[i], G) = 0
$$

except for finitely many $i \in \mathbb{Z}$.

Proof. The first assertion is [3, Lemma 4.5], and the second one is its dual.

Compare the following result with the last assertion in Corollary 3.4 and its dual counterpart.

Proposition 4.2. Let $A$ be virtually Gorenstein. Then we have

(1) $D^b_{\text{G}(\text{P}(\text{mod} A))}(\text{mod} A)_{\text{perf}} = K^b(\text{G}(\text{P}(\text{mod} A)))$.

(2) $\text{co-D}^b_{\text{G}(\text{I}(\text{mod} A))}(\text{mod} A)_{\text{copert}} = K^b(\text{G}(\text{I}(\text{mod} A)))$.

Proof. (1) Let $A$ be virtually Gorenstein. It follows from [12, Theorem 5] that $\text{G}(\text{P}(\text{mod} A))$ is contravariantly finite in $\text{mod} A$. Then by Lemma 2.2, we have

$$
D^b_{\text{G}(\text{P}(\text{mod} A))}(\text{mod} A)_{\text{perf}} = K^-(\text{G}(\text{P}(\text{mod} A)))b(\text{G}(\text{P}(\text{mod} A))).
$$

Now the assertion follows from Lemma 4.1(1).

(2) It is dual to (1).

We have the following easy observation.

Lemma 4.3. The following statements are equivalent.

(1) $A$ is Gorenstein.
(2) Any object in $G(\mathcal{P}(\text{Mod} A))$ has finite $G(\mathcal{I}(\text{Mod} A))$-codimension and any object in $G(\mathcal{I}(\text{Mod} A))$ has finite $G(\mathcal{P}(\text{Mod} A))$-dimension.

(3) Any object in $G(\mathcal{P}(\text{mod} A))$ has finite $G(\mathcal{I}(\text{mod} A))$-codimension and any object in $G(\mathcal{I}(\text{mod} A))$ has finite $G(\mathcal{P}(\text{mod} A))$-dimension.

Proof. (1) $\Rightarrow$ (2) + (3) Let $A$ be Gorenstein. Then any object in $\text{Mod} A$ (resp. $\text{mod} A$) has finite $G(\mathcal{P}(\text{Mod} A))$-dimension (resp. $G(\mathcal{P}(\text{mod} A))$-dimension) by [17, Theorem 11.5.1] (resp. [22, Theorem]). Dually, any object in $\text{Mod} A$ (resp. $\text{mod} A$) has finite $G(\mathcal{I}(\text{Mod} A))$-dimension (resp. $G(\mathcal{I}(\text{mod} A))$-dimension).

(2) $\Rightarrow$ (1) (resp. (3) $\Rightarrow$ (1)) By assumption, we have that $D(A_A)$ has finite $G(\mathcal{P}(\text{Mod} A))$-dimension (resp. $G(\mathcal{P}(\text{mod} A))$-dimension). So $D(A_A)$ has finite projective dimension by [24, Corollary 3.12], and hence $\text{id}_{A^*} A < \infty$. Dually, because $A_A$ has finite $G(\mathcal{I}(\text{Mod} A))$-dimension (resp. $G(\mathcal{I}(\text{mod} A))$-dimension) by assumption, we have $\text{id}_{A^*} A < \infty$ by [24, Corollary 4.12]. Thus $A$ is Gorenstein.

Lemma 4.4. Let $A$ be virtually Gorenstein. Then there exist triangle equivalences

$$D^b_G(\mathcal{P}(\text{mod} A))(\text{mod} A) \cong \text{co-}D^b_G(\mathcal{I}(\text{mod} A))(\text{mod} A),$$

$$D^b_G(\mathcal{P}(\text{mod} A))(\text{Mod} A) \cong \text{co-}D^b_G(\mathcal{I}(\text{mod} A))(\text{Mod} A).$$

Proof. The first equivalence follows from [3, Theorem 6.3].

By [33, Proposition 3.2], we have that $(G(\mathcal{P}(\text{Mod} A)), G(\mathcal{I}(\text{Mod} A)))$ is a balanced pair. Then by [14, Proposition 2.2], we have that an exact sequence is $G(\mathcal{P}(\text{Mod} A))$-acyclic if and only if it is $G(\mathcal{I}(\text{Mod} A))$-coacyclic, which yields the second equivalence.

As a consequence, we have the following

Proposition 4.5. Let $A$ be virtually Gorenstein and either $X = \text{Mod} A$ or $X = \text{mod} A$. Then we have

1. The following statements are equivalent.
   1.1 Any object in $G(\mathcal{P}(X))$ has finite $G(\mathcal{I}(X))$-codimension.
   1.2 $K^h(G(\mathcal{P}(X))) \subseteq K^h(G(\mathcal{I}(X)))$ in $D^b_G(\mathcal{P}(X))(X)$.

2. The following statements are equivalent.
   2.1 Any object in $G(\mathcal{I}(X))$ has finite $G(\mathcal{P}(X))$-dimension.
   2.2 $K^h(G(\mathcal{I}(X))) \subseteq K^h(G(\mathcal{P}(X)))$ in $D^b_G(\mathcal{P}(X))(X)$.

Proof. (1.1) $\Rightarrow$ (1.2) Let $X \in K^h(G(\mathcal{P}(X)))(\subseteq D^b_G(\mathcal{P}(X))(X))$. Then $X \in \text{co-}D^b_G(\mathcal{I}(X))(X)$ by Lemma 4.4. We will prove $X \in K^h(G(\mathcal{I}(X)))$ by induction on the width $w(X)$ of $X$. Let $w(X) = 1$. By the dual version of [23, Corollary 5.12], there exists a finite coproper $G(\mathcal{I}(X))$-coresolution

$$0 \longrightarrow X \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots \longrightarrow G^{n-1} \longrightarrow G^n \longrightarrow 0$$

of $X$, which can be viewed as a $G(\mathcal{I}(X))$-coquasi-isomorphism

$$0 \longrightarrow X \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots \longrightarrow G^{n-1} \longrightarrow G^n \longrightarrow 0.$$
It follows that $X \in K^b(\mathcal{G}(\mathcal{I}(\mathcal{X})))$. Now suppose $w(X) = n \geq 2$. From the following diagram

\[
\begin{array}{c}
X_1 \rightarrow X^0 \rightarrow X^1 \rightarrow \cdots \rightarrow X^{n-2} \rightarrow 0 \rightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
X \rightarrow X^0 \rightarrow X^1 \rightarrow \cdots \rightarrow X^{n-2} \rightarrow X^{n-1} \rightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
X_2 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow X^{n-1} \rightarrow 0,
\end{array}
\]

we get a triangle

\[
X_1 \rightarrow X \rightarrow X_2 \rightarrow X_1[1]
\]

with $w(X_1) = n - 1$ and $w(X_2) = 1$. By the induction hypothesis, we have $X_1 \in K^b(\mathcal{G}(\mathcal{I}(\mathcal{X})))$ and $X_2 \in K^b(\mathcal{G}(\mathcal{I}(\mathcal{X})))$, and hence $X \in K^b(\mathcal{G}(\mathcal{I}(\mathcal{X})))$.

(1.2) $\implies$ (1.1) Let $G \in \mathcal{G}(\mathcal{P}(\mathcal{X}))$, as a stalk complex, be an object in $K^b(\mathcal{G}(\mathcal{P}(\mathcal{X})))$. Then $G \in K^b(\mathcal{G}(\mathcal{I}(\mathcal{X}))) \subseteq \text{co-}D^b_{\mathcal{G}(\mathcal{I}(\mathcal{X}))}(\mathcal{X})$ by (1.2). Then there exists a $\mathcal{G}(\mathcal{I}(\mathcal{X}))$-coquasi-isomorphism

\[
\begin{array}{c}
0 \rightarrow G \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots \rightarrow G^{n-1} \rightarrow G^n \rightarrow 0;
\end{array}
\]

in particular, $0 \rightarrow G \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots \rightarrow G^n \rightarrow 0$ is exact, and hence $G$ has finite $\mathcal{G}(\mathcal{I}(\mathcal{X}))$-codimension.

Dually, we get (2.1) $\iff$ (2.2).

Now we are in a position to prove the following

**Theorem 4.6.** For a virtually Gorenstein algebra $A$, the following statements are equivalent.

1. $A$ is Gorenstein.
2. $K^b(\mathcal{G}(\mathcal{P}(\text{Mod } A))) = K^b(\mathcal{G}(\mathcal{I}(\text{Mod } A)))$ in $D^b_{\mathcal{G}(\mathcal{P}(\text{Mod } A))}(\text{Mod } A)$.
3. $D^b_{\mathcal{G}(\mathcal{P}(\text{Mod } A))}(\text{Mod } A)_{\text{perf}} = \text{co-}D^b_{\mathcal{G}(\mathcal{I}(\text{Mod } A))}(\text{Mod } A)_{\text{coperf}}$ in $D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{Mod } A)$.
4. $K^b(\mathcal{G}(\mathcal{P}(\text{mod } A))) = K^b(\mathcal{G}(\mathcal{I}(\text{mod } A)))$ in $D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A)$.
5. $D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A)_{\text{perf}} = \text{co-}D^b_{\mathcal{G}(\mathcal{I}(\text{mod } A))}(\text{mod } A)_{\text{coperf}}$ in $D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A)$.

*Proof.* By Lemma 4.3 and Proposition 4.5, we have (2) $\iff$ (1) $\iff$ (4).

By Corollary 3.4 and its dual counterpart, we have $D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A)_{\text{perf}} = K^b(\mathcal{G}(\mathcal{P}(\text{mod } A)))$ and $\text{co-}D^b_{\mathcal{G}(\mathcal{I}(\text{mod } A))}(\text{mod } A)_{\text{coperf}} = K^b(\mathcal{G}(\mathcal{I}(\text{mod } A)))$. So the assertion (2) $\iff$ (3) follows. The assertion (4) $\iff$ (5) follows from Proposition 4.2. \(\square\)
In summary, if $A$ is Gorenstein, then we have the following diagram of identifications

$$
\begin{align*}
D^b_{\mathcal{G}(\mathcal{P}(\text{mod}A))}(\text{mod}A)_{\text{perf}} & \quad \xrightarrow{\text{Theorem 4.6(3)}} \quad D^b_{\mathcal{G}(\mathcal{I}(\text{mod}A))}(\text{mod}A)_{\text{copperf}} \\
K^b(\mathcal{G}(\mathcal{P}(\text{mod}A))) & \quad \xrightarrow{\text{Theorem 4.6(2)}} \quad K^b(\mathcal{G}(\mathcal{I}(\text{mod}A))) \\
K^b(\mathcal{G}(\mathcal{P}(\text{mod}A))) & \quad \xrightarrow{\text{Theorem 4.6(4)}} \quad K^b(\mathcal{G}(\mathcal{I}(\text{mod}A))) \\
D^b_{\mathcal{G}(\mathcal{P}(\text{mod}A))}(\text{mod}A)_{\text{perf}} & \quad \xrightarrow{\text{Proposition 4.2(1)}} \quad D^b_{\mathcal{G}(\mathcal{I}(\text{mod}A))}(\text{mod}A)_{\text{copperf}}.
\end{align*}
$$

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**References**


