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ENDOMORPHISM ALGEBRAS AND IGUSA–TODOROV ALGEBRAS[*]

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Abstract. Let A be an Artin algebra. If $V \in \operatorname{mod} A$ such that the global dimension of $\operatorname{End}_A V$ is at most 3, then for any $M \in \operatorname{add}_A V$, both B and B^{op} are 2-Igusa–Todorov algebras, where $B = \operatorname{End}_A M$. Let $P \in \operatorname{mod} A$ be projective and $B = \operatorname{End}_A P$ such that the projective dimension of P as a right B-module is at most $n(< \infty)$. If A is an m-syzygy-finite algebra (resp. an m-Igusa–Todorov algebra), then B is an $(m+n)$-syzygy-finite algebra (resp. an $(m+n)$-Igusa–Todorov algebra); in particular, the finitistic dimension of B is finite in both cases. Some applications of these results are given.

1. Introduction

Let A be an Artin algebra. Recall that the (little) finitistic dimension of A, denoted by $\operatorname{fin.dim} A$, is defined to be the supremum of the projective dimensions of all finitely generated left A-modules with finite projective dimension. The famous (little) finitistic dimension conjecture states that $\operatorname{fin.dim} A < \infty$ for any Artin algebra A ([4,9]). This conjecture remains still open and is related to some other homological conjectures, such as the Gorenstein symmetry conjecture, the Wakamatsu tilting conjecture and the (generalized) Nakayama conjecture ([16]).

Many authors have studied the finitistic dimension conjecture, see [5–10, 15,17], and so on. In studying the finitistic dimension conjecture, Igusa and

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Todorov introduced in [10] a powerful function and proved that $\operatorname{fin.dim} A$ is finite provided that the representation dimension of A is at most 3. By using the Igusa–Todorov function, Xi developed in [15] some new ideas to prove the finiteness of finitistic dimension of some Artin algebras; Zhang and Zhang proved in [17] that the finitistic dimension conjecture holds for the endomorphism algebra of any projective module over an Artin algebra with representation dimension at most 3. In particular, motivated by the properties of the Igusa–Todorov function, Wei introduced in [13] the notion of n-Igusa–Todorov algebras, and showed that a 0-Igusa–Todorov algebra is left-right symmetric and is inherited by endomorphism algebras of projective modules. Furthermore, he proved in [14] that the finitistic dimension conjecture holds for all Igusa–Todorov algebras, and the class of 2-Igusa–Todorov algebras is closed under taking the endomorphism algebras of projective modules.

Based on the results mentioned above, in this paper we study further properties of Igusa–Todorov algebras. We first prove the following

THEOREM 1.1. *Let A be an Artin algebra and $V \in \operatorname{mod} A$ such that $\operatorname{gl.dim} \operatorname{End}_A V \leqq 3$. Then for any $M \in \operatorname{add}_A V$, both $\operatorname{End}_A M$ and $(\operatorname{End}_A M)^{op}$ are 2-Igusa–Todorov algebras; in particular, $\operatorname{fin.dim} \operatorname{End}_A M < \infty$ and $\operatorname{fin.dim} (\operatorname{End}_A M)^{op} < \infty$.*

Next we construct new Igusa–Todorov algebras from some given Igusa–Todorov algebras.

THEOREM 1.2. *Let A be an Artin algebra and $P \in \operatorname{mod} A$ projective, and let $B = \operatorname{End}_A P$ and $\operatorname{pd}_{B^{op}} P = n < \infty$. If A is an m-syzygy-finite algebra (resp. an m-Igusa–Todorov algebra), then B is an $(m+n)$-syzygy-finite algebra (resp. an $(m+n)$-Igusa–Todorov algebra); in particular, $\operatorname{fin.dim} B < \infty$.*

This paper is organized as follows. In Section 2, we give some notations and some known facts. In Section 3, we prove the main results and give some applications.

2. Preliminaries

Throughout this paper, A is an Artin algebra. We denote by $\operatorname{mod} A$ the category of finitely generated left A-modules, and by $\operatorname{gl.dim} A$ the global dimension of A. We denote by $\mathbb{D} := \operatorname{Hom}_R\big(-, E\big(R/J(R)\big)\big)$ the standard duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{op}$, where R is the center of A, $J(R)$ is the radical of R and $E\big(R/J(R)\big)$ is the injective envelope of $R/J(R)$. For a module M in $\operatorname{mod} A$, we denote by $\operatorname{pd}_A M$ (resp. $\operatorname{id}_A M$) the projective (resp. injective) dimension of M, and use $\Omega_A^i(M)$ to denote the i-th syzygy of M, in particular $\Omega_A^0(M) = M$. For a full subcategory $\mathcal{C}$ of $\operatorname{mod} A$, we denote by $\operatorname{add}_A \mathcal{C}$ the full subcategory of $\operatorname{mod} A$ consisting of the modules isomorphic to the direct summands of finite direct sums of modules in $\mathcal{C}$, and if M is a module, we abbreviate $\operatorname{add}_A\{M\}$ as $\operatorname{add}_A M$.

Recall that A is said to be of *representation-finite type* if there exist only finitely many non-isomorphic indecomposable modules in $\operatorname{mod} A$. The *finitistic dimension* of A, denoted by $\operatorname{fin.dim} A$, is defined as $\operatorname{fin.dim} A = \sup\{\operatorname{pd}_A M \mid M \in \operatorname{mod} A \text{ and } \operatorname{pd}_A M < \infty\}$. The *finitistic dimension conjecture* states that the finitistic dimension of any Artin algebra is finite (see [9]). The *representation dimension of* A, denoted by $\operatorname{rep.dim} A$, is defined as $\operatorname{rep.dim} A = \inf\{\operatorname{gl.dim} \operatorname{End}_A M \mid M$ is a generator-cogenerator for $\operatorname{mod} A$, that is, $A \oplus \mathbb{D}A^{op} \in \operatorname{add}_A M\}$. Auslander showed in [2] that $\operatorname{rep.dim} A \leqq 2$ if and only if A is of representation-finite type. Igusa and Todorov introduced in [10] a function, which is called the *Igusa–Todorov function*. By using the properties of this function, they proved that the finitistic dimension conjecture holds for Artin algebras with representation dimension at most 3. However, though the representation dimension of an Artin algebra is always finite [11], there exists no upper bound for the representation dimension of an Artin algebra [12].

The Igusa–Todorov function is powerful in the study of the finitistic dimension conjecture (see [13–15,17]). The following are some elementary properties of this function.

PROPOSITION 2.1 [10]. *There exists a function Ψ which is defined on the objects of* $\operatorname{mod} A$ *and takes non-negative integers as values such that*

(1) $\Psi(M) = \operatorname{pd}_A M$ *if* $\operatorname{pd}_A M < \infty$.

(2) *For any* $X, Y \in \operatorname{mod} A$, $\Psi(X) \leqq \Psi(Y)$ *if* $\operatorname{add}_A X \subseteqq \operatorname{add}_A Y$. *The equality holds if* $\operatorname{add}_A X = \operatorname{add}_A Y$.

(3) *If* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\operatorname{mod} A$ *with* $\operatorname{pd}_A Z < \infty$, *then* $\operatorname{pd}_A Z \leqq \Psi(X \oplus Y) + 1$.

Motivated by the paper of Igusa and Todorov, Wei introduced in [13] the notion of n-Igusa–Todorov algebras as follows.

DEFINITION 2.2 [13]. For a non-negative integer n, A is called an *n-Igusa–Todorov algebra* if there exists a module $V \in \operatorname{mod} A$ such that for any $M \in \operatorname{mod} A$ there exists an exact sequence:

$$0 \to V_1 \to V_0 \to \Omega_A^n(M) \oplus P \to 0$$

in $\operatorname{mod} A$ with $V_0, V_1 \in \operatorname{add}_A V$ and P projective. When such a module V exists, it is called *an n-Igusa–Todorov module.*

Recall that a full subcategory $\mathcal{C}$ of $\operatorname{mod} A$ is called *add-finite* if $\mathcal{C} \subseteqq \operatorname{add}_A C$ for some $C \in \operatorname{mod} A$. For any $n \geqq 0$, $\mathcal{C}$ is called *n-syzygy-finite* if $\Omega_A^n(\mathcal{C})$ is add-finite; in particular, A is called an *n-syzygy-finite algebra* if $\operatorname{mod} A$ is n-syzygy-finite. An n-syzygy-finite algebra is called a *Ω^n-representation-finite algebra* in [13]. We call A a *syzygy-finite algebra* (resp.

an *Igusa–Todorov algebra*) if it is n-syzygy-finite (resp. n-Igusa–Todorov) for some n.

The following result establishes the relation between the Igusa–Todorov algebras and the finitistic dimension conjecture as well as the syzygy-finite algebras.

LEMMA 2.3. *For any* $n \geqq 0$, *we have*

(1) $\operatorname{fin.dim} A < \infty$ *for any* n*-Igusa–Todorov algebra* A ([13, Theorem 1.1]).

(2) *If* A *is* $(n+1)$*-syzygy-finite, then* A *is* n*-Igusa–Todorov* ([13, Proposition 2.5]).

Given an additive category $\mathcal{A}$, recall that a functor $F : \mathcal{A}^{op} \to \mathrm{Ab}$ (the category of all abelian groups) is called *coherent* if there exists an exact sequence $\operatorname{Hom}_{\mathcal{A}}(-, A_1) \to \operatorname{Hom}_{\mathcal{A}}(-, A_0) \to F \to 0$ with $A_i \in \mathcal{A}$ for $i = 0, 1$. The full subcategory of the functor category $(\mathcal{A}^{op}, \mathrm{Ab})$ consisting of all coherent functors is denoted by $\widehat{\mathcal{A}}$. By the Yoneda lemma, the projective objects in $\widehat{\mathcal{A}}$ are of the form $\operatorname{Hom}_{\mathcal{A}}(-, X)$ with X an object in $\mathcal{A}$, and each coherent functor F can be determined by a morphism $f : A_1 \to A_0$, that is, there exists an exact sequence $\operatorname{Hom}_{\mathcal{A}}(-, A_1) \xrightarrow{(-,f)} \operatorname{Hom}_{\mathcal{A}}(-, A_0) \to F \to 0$ in $\widehat{\mathcal{A}}$. As in the case of a module category, the *global dimension* of the category $\widehat{\mathcal{A}}$, denoted by $\operatorname{gl.dim} \widehat{\mathcal{A}}$, is defined as the supremum of the projective dimensions of all functors in $\widehat{\mathcal{A}}$. For more information on coherent functors we refer to [1,2]. The precise connection between the global dimension of an Artin algebra and that of the coherent functor category is recorded in the following lemma due to Auslander.

LEMMA 2.4 [1,2]. *Let* $M \in \operatorname{mod} A$. *Then* $\widehat{\operatorname{add}_A M}$ *and* $\operatorname{mod} \operatorname{End}_A M$ *are equivalent*; *in particular*, $\operatorname{gl.dim} \operatorname{End}_A M = \operatorname{gl.dim} \widehat{\operatorname{add}_A M}$.

3. Main results

Let $M \in \operatorname{mod} A$ with $B = \operatorname{End}_A M$, and let $F = M \otimes_B - : \operatorname{mod} B \to \operatorname{mod} A$ and $G = \operatorname{Hom}_A(M, -) : \operatorname{mod} A \to \operatorname{mod} B$. Then (F, G) is an adjoint pair. Let $\sigma : 1_B \to GF$ be the counit of (F, G). It is easy to see that σ_P is an isomorphism for any projective module $P \in \operatorname{mod} B$. For a module X, we use

$$\cdots \to P_i(X) \to \cdots \to P_1(X) \to P_0(X) \to X \to 0$$

to denote the minimal projective resolution of X.

LEMMA 3.1. *Let* $M \in \operatorname{mod} A$ *and* $B = \operatorname{End}_A M$, *and let* $X \in \operatorname{mod} B$ *and*

$$P_1(X) \xrightarrow{f} P_0(X) \to X \to 0$$

be a minimal projective presentation of X *in* $\operatorname{mod} B$. *Then we have*

(1) $\Omega_B^2(X) \cong \operatorname{Hom}_A(M, Y)$, *where* $Y = \operatorname{Ker}(1_M \otimes f)$.

(2) *If* $M \in \operatorname{mod} A$ *is projective, then there exists a projective module* $Q \in \operatorname{mod} A$ *such that* $\Omega_B^2(X) \cong \operatorname{Hom}_A\left(M, \Omega_A^2(M \otimes_B X) \oplus Q\right)$.

PROOF. (1) Let $X \in \operatorname{mod} B$. Then from the exact sequence $0 \to \Omega_B^2(X) \to P_1(X) \xrightarrow{f} P_0(X) \to X \to 0$ in $\operatorname{mod} B$, we get an exact sequence

$$0 \to Y \to M \otimes_B P_1(X) \xrightarrow{1_M \otimes f} M \otimes_B P_0(X) \to M \otimes_B X \to 0 \tag{3.1}$$

in $\operatorname{mod} A$, where $Y = \operatorname{Ker}(1_M \otimes f)$. Applying the functor $\operatorname{Hom}_A(M, -)$ to (3.1), we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Omega_B^2(X) & \longrightarrow & P_1(X) & \xrightarrow{f} & P_0(X) \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \sigma_{P_1(X)}} & & \downarrow{\scriptstyle \sigma_{P_0(X)}} \\
0 & \longrightarrow & \operatorname{Hom}_A(M, Y) & \longrightarrow & \operatorname{Hom}_A(M, M \otimes_B P_1(X)) & \xrightarrow{(M, 1_M \otimes f)} & \operatorname{Hom}_A(M, M \otimes_B P_0(X))
\end{array}$$

Since $\sigma_{P_0(X)}, \sigma_{P_1(X)}$ are isomorphisms, ϕ is an isomorphism and $\Omega_B^2(X) \cong \operatorname{Hom}_A(M, Y)$.

(2) If $M \in \operatorname{mod} A$ is projective, then in (3.1) both $M \otimes_B P_1(X)$ and $M \otimes_B P_0(X)$ are projective in $\operatorname{mod} A$. So there exists a projective module $Q \in \operatorname{mod} A$ such that $Y \cong \Omega_A^2(M \otimes_B X) \oplus Q$, and the assertion follows from (1). □

As a consequence of Lemma 3.1(1), we get the following

THEOREM 3.2. *Let* $V \in \operatorname{mod} A$ *such that* $\operatorname{gl.dim} \operatorname{End}_A V \leqq 3$. *Then for any* $M \in \operatorname{add}_A V$, *both* $\operatorname{End}_A M$ *and* $(\operatorname{End}_A M)^{op}$ *are* 2*-Igusa–Todorov algebras*; *in particular*, $\operatorname{fin.dim} \operatorname{End}_A M < \infty$ *and* $\operatorname{fin.dim} (\operatorname{End}_A M)^{op} < \infty$.

PROOF. Let $V \in \operatorname{mod} A$ and $M \in \operatorname{add}_A V$, and let $B = \operatorname{End}_A M$ and $X \in \operatorname{mod} B$. From the exact sequence $0 \to \Omega_B^2(X) \to P_1(X) \xrightarrow{f} P_0(X) \to X \to 0$, we get an exact sequence

$$0 \to Y \to M \otimes_B P_1(X) \xrightarrow{1_M \otimes f} M \otimes_B P_0(X) \to M \otimes_B X \to 0$$

where $Y = \operatorname{Ker}(1_M \otimes f)$ and $M \otimes_B P_i(X) \in \operatorname{add}_A M \subseteqq \operatorname{add}_A V$ for $i = 0, 1$. Then we have an exact sequence

$$0 \to \operatorname{Hom}_A(-, Y) \to \operatorname{Hom}_A\left(-, M \otimes_B P_1(X)\right)$$

$$\xrightarrow{(-, 1_M \otimes f)} \operatorname{Hom}_A\left(-, M \otimes_B P_0(X)\right) \to H \to 0$$

in $(\operatorname{add}_A V, \mathrm{Ab})$, where $H = \operatorname{Coker} \operatorname{Hom}_A(-, 1_M \otimes f)(\in \widehat{\operatorname{add}_A V})$. Because $\operatorname{gl.dim} \widehat{\operatorname{add}_A V} = \operatorname{gl.dim} \operatorname{End}_A V \leqq 3$ by assumption and Lemma 2.4, $\operatorname{pd}_{(\operatorname{add}_A V, \mathrm{Ab})} \operatorname{Hom}_A(-, Y) \leqq 1$ and there exists an exact sequence

$$0 \to \operatorname{Hom}_A(M, V_1) \to \operatorname{Hom}_A(M, V_0) \to \operatorname{Hom}_A(M, Y) \to 0$$

in $\operatorname{mod} B$ with $V_0, V_1 \in \operatorname{add}_A V$. On the other hand, $\Omega_B^2(X) \cong \operatorname{Hom}_A(M, Y)$ by Lemma 3.1(1). Thus B is a 2-Igusa–Todorov algebra with $\operatorname{Hom}_A(M, V)$ a 2-Igusa–Todorov module.

Note that $\mathbb{D}V \in \operatorname{mod} A^{op}$ and $\operatorname{End}_{A^{op}} \mathbb{D}V \cong (\operatorname{End}_A V)^{op}$. So

$$\operatorname{gl.dim} \operatorname{End}_{A^{op}} \mathbb{D}V = \operatorname{gl.dim} (\operatorname{End}_A V)^{op} = \operatorname{gl.dim} \operatorname{End}_A V \leqq 3.$$

Hence, repeating the above process, we have that $B^{op} = \operatorname{End}_{A^{op}} \mathbb{D}M$ is a 2-Igusa–Todorov algebra with $\operatorname{Hom}_{A^{op}}(\mathbb{D}M, \mathbb{D}V) \cong \operatorname{Hom}_A(V, M)$ a 2-Igusa–Todorov module. □

Putting ${}_AV = {}_AA$ in Theorem 3.2, we get the following

Corollary 3.3. *Let* $\operatorname{gl.dim} A \leqq 3$ *and* $P \in \operatorname{mod} A$ *be projective. Then both* $\operatorname{End}_A P$ *and* $(\operatorname{End}_A P)^{op}$ *are* 2*-Igusa–Todorov algebras.*

If $\operatorname{rep.dim} A \leqq 3$, then there exists a generator-cogenerator V for $\operatorname{mod} A$ such that $\operatorname{gl.dim} \operatorname{End}_A V \leqq 3$ [2]. Such a module V is called an *Auslander generator* of A [13]. As another application of Theorem 3.2, we have the following

Corollary 3.4 [13, Proposition 3.7]. *Let* $\operatorname{rep.dim} A \leqq 3$ *and* $V \in \operatorname{mod} A$ *be an Auslander generator. Then for any* $M \in \operatorname{add}_A V$, *both* $\operatorname{End}_A M$ *and* $(\operatorname{End}_A M)^{op}$ *are* 2*-Igusa–Todorov algebras.*

In the following, we investigate the relation between the syzygy-finite algebras (resp. the Igusa–Todorov algebras) and the endomorphism algebras of projective modules. Firstly, we prove the following lemma.

Lemma 3.5. *Let* $P \in \operatorname{mod} A$ *be projective and* $B = \operatorname{End}_A P$ *such that* $\operatorname{pd}_{B^{op}} P = n < \infty$. *Then for any* $X \in \operatorname{mod} B$ *and* $m \geqq 0$, *there exist* $Y, Q \in \operatorname{mod} A$ *with* Q *projective such that*

$$P \otimes_B \Omega_B^{n+m}(X) \cong \Omega_A^m(Y) \oplus Q.$$

Proof. If $\operatorname{pd}_B X \leqq n + m$, then $\Omega_B^{n+m}(X) \in \operatorname{mod} B$ is projective, and so $P \otimes_B \Omega_B^{n+m}(X) \in \operatorname{mod} A$ is projective.

Now let $\operatorname{pd}_B X > n + m$. Since $\operatorname{pd}_{B^{op}} P = n$, $\operatorname{Tor}_i^B(P, \Omega_B^{n+m}(X)) \cong \operatorname{Tor}_{n+m+i}^B(P, X) = 0$ for any $i \geqq 1$. From the exact sequence

$$0 \to P_n(P_B) \to \cdots \to P_1(P_B) \to P_0(P_B) \to P_B \to 0$$

in $\bmod B^{op}$, we get an exact sequence

$$0 \to P_n(P_B) \otimes_B \Omega_B^{n+m}(X) \to \cdots \to P_1(P_B) \otimes_B \Omega_B^{n+m}(X) \tag{3.2}$$

$$\to P_0(P_B) \otimes_B \Omega_B^{n+m}(X) \to P \otimes_B \Omega_B^{n+m}(X) \to 0.$$

On the other hand, we have the following complex:

$$\mathbf{X}_j : 0 \to P_j(P_B) \otimes_B \Omega_B^{n+m}(X) \to P_j(P_B) \otimes_B P_{n+m-1}({}_BX) \to \cdots$$

$$\to P_j(P_B) \otimes_B P_1({}_BX) \to P_j(P_B) \otimes_B P_0({}_BX) \to 0$$

in $\bmod A$ for $0 \leqq j \leqq n$. Then we get an exact sequence of chain complexes

$$0 \to \mathbf{X}_n \to \mathbf{X}_{n-1} \to \cdots \to \mathbf{X}_1 \to \mathbf{Y} \to 0,$$

where $\mathbf{Y}$ denotes the complex

$$0 \to P \otimes_B \Omega_B^{n+m}(X) \to P \otimes_B P_{n+m-1}(X) \to \cdots \to P \otimes_B P_0(X) \to 0$$

in $\bmod A$. Since $H_j(\mathbf{X}_i) = 0$ for any $i, j \geqq 1$, $H_j(\mathbf{Y}) = 0$ for any $j \geqq n+1$ by [15, Lemma 3.12]. Thus we have the following exact sequence:

$$0 \to P \otimes_B \Omega_B^{n+m}(X) \to P \otimes_B P_{n+m-1}(X) \to \cdots$$

$$\to P \otimes_B P_{n+1}(X) \to P \otimes_B P_n(X) \to Y \to 0$$

in $\bmod A$, where $Y = \operatorname{Coker}\big(P \otimes_B P_{n+1}(X) \to P \otimes_B P_n(X)\big)$. Therefore there exists a projective module $Q \in \bmod A$ such that $P \otimes_B \Omega_B^{n+m}(X) \cong \Omega_A^m(Y) \oplus Q$. □

We also need the following observation.

Lemma 3.6. *For a non-negative integer n, A is an n-Igusa–Todorov algebra if and only if there exists a module $V \in \bmod A$ such that for any $M \in \bmod A$ there exists an exact sequence*:

$$0 \to V_1 \to V_0 \to \Omega_A^n(M) \to 0$$

in $\bmod A$ *with* $V_0, V_1 \in \operatorname{add}_A V$.

Proof. The sufficiency is trivial. We next prove the necessity.

Let A be an n-Igusa–Todorov algebra and $M \in \bmod A$. Then there exists a module $V \in \bmod A$ and an exact sequence

$$0 \to V_1 \to V_0 \to \Omega_A^n(M) \oplus P \to 0$$

in $\operatorname{mod} A$ with $V_0, V_1 \in \operatorname{add}_A V$ and P projective. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_1 & \longrightarrow & V_0' & \longrightarrow & \Omega_A^n(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_1 & \longrightarrow & V_0 & \longrightarrow & \Omega_A^n(M) \oplus P & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & P & = & P & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Because the middle column in the above diagram is split, the first row is the desired exact sequence. □

We are now in a position to prove the following

THEOREM 3.7. *Let $P \in \operatorname{mod} A$ be projective and $B = \operatorname{End}_A P$ such that $\operatorname{pd}_{B^{op}} P = n < \infty$. Then for any $m \geqq 0$, we have*

(1) *If A is an m-syzygy-finite algebra, then B is an $(m+n)$-syzygy-finite algebra.*

(2) *If A is an m-Igusa–Todorov algebra, then B is an $(m+n)$-Igusa–Todorov algebra.*

In particular, $\operatorname{fin.dim} B < \infty$ *in both cases.*

PROOF. Assume that $\operatorname{pd}_{B^{op}} P = n$ and $X \in \operatorname{mod} B$. Then for any $m \geqq 0$, there exist $Y, Q \in \operatorname{mod} A$ with Q projective such that $\Omega_B^{n+m}(X) \cong \operatorname{Hom}_A\left(P, P \otimes_B \Omega_B^{n+m}(X)\right) \cong \operatorname{Hom}_A(P, \Omega_A^m(Y) \oplus Q)$ by [14, Lemma 3.1] and Lemma 3.5.

(1) If A is an m-syzygy-finite algebra, then there exists a module $E \in \operatorname{mod} A$ such that $\Omega_A^m(\operatorname{mod} A) \subseteqq \operatorname{add}_A E$. For any $X \in \operatorname{mod} B$, we have that $\Omega_B^{n+m}(X) \cong \operatorname{Hom}_A\left(P, \Omega_A^m(Y) \oplus Q\right) \in \operatorname{add}_B \operatorname{Hom}_A(P, E \oplus A)$, and B is $(m+n)$-syzygy-finite.

(2) If A is an m-Igusa–Todorov algebra with V an m-Igusa–Todorov module, then by Lemma 3.6, there exists an exact sequence $0 \to V_1 \to V_0 \to \Omega_A^m(Y) \to 0$ in $\operatorname{mod} A$ with $V_0, V_1 \in \operatorname{add}_A V$. So we get an exact sequence

$$0 \to V_1 \to V_0 \oplus Q \to \Omega_A^m(Y) \oplus Q \to 0 \tag{3.3}$$

in $\operatorname{mod} A$. Note that $\Omega_B^{n+m}(X) \cong \operatorname{Hom}_A(P, \Omega_A^m(Y) \oplus Q)$. Applying the functor $\operatorname{Hom}_A(P, -)$ to (3.3), we get an exact sequence $0 \to \operatorname{Hom}_A(P, V_1) \to \operatorname{Hom}_A(P, V_0 \oplus Q) \to \Omega_B^{n+m}(X) \to 0$ in $\operatorname{mod} B$. Thus B is an $(m+n)$-Igusa–Todorov algebra with $\operatorname{Hom}_A(P, V \oplus A)$ an $(m+n)$-Igusa–Todorov module. □

Recall from [3] that an ideal I of A is called a *strong idempotent* if the natural map $\operatorname{Ext}^i_{A/I}(M,N) \to \operatorname{Ext}^i_A(M,N)$ is an isomorphism for any $M, N \in \operatorname{mod} A/I$ and $i \geqq 0$.

Corollary 3.8. *Let* $\operatorname{gl.dim} A < \infty$ *and* e *be an idempotent of* A. *If the ideal* AeA *is a strong idempotent ideal of* A, *then* eAe *is a syzygy-finite algebra and an Igusa–Todorov algebra*; *in particular*, $\operatorname{fin.dim} eAe < \infty$.

Proof. Let $\operatorname{gl.dim} A < \infty$ and e be an idempotent of A. Then A is a syzygy-finite algebra and an Igusa–Todorov algebra, and $\operatorname{End}_A Ae = eAe$. If the ideal AeA is a strong idempotent ideal of A, then $\operatorname{pd}_{(eAe)^{op}} Ae < \infty$ by [3]. So eAe is a syzygy-finite algebra and an Igusa–Todorov algebra by Theorem 3.7. □

Corollary 3.9. *Let* e *be a non-zero idempotent of* A *and* $f = 1 - e$ *such that* $\operatorname{pd}_{A^{op}} fA/fJ \leqq 1$, *where* J *is the Jacobson radical of* A. *If* A *is an* m*-syzygy-finite algebra* (*resp. an* m*-Igusa–Todorov algebra*) *with* $m \geqq 0$, *then so is* eAe; *in particular*, *the finitistic dimension of* A *and* eAe *are finite.*

Proof. By the proof of [5, Proposition 2.1], we have that $fAe \in \operatorname{mod}(eAe)^{op}$ is projective and $Ae = eAe \oplus fAe$ is also projective in $\operatorname{mod}(eAe)^{op}$. Then the assertion follows from Theorem 3.7. □

Proposition 3.10. *Let* $P \in \operatorname{mod} A$ *be projective and* $B = \operatorname{End}_A P$. *If there exists* $n \geqq 2$ *such that*

$$\operatorname{pd}_A P \otimes_B \Omega_B^{n-2}(X) < \infty \quad \text{and} \quad \operatorname{id}_A \Omega_A^4\big(P \otimes_B \Omega_B^{n-2}(X)\big) \leqq 3$$

for any $X \in \operatorname{mod} B$, *then* B *is an* n*-syzygy-finite algebra and hence an* $(n-1)$*-Igusa–Todorov algebra*; *in particular*, $\operatorname{fin.dim} B < \infty$.

Proof. Let $X \in \operatorname{mod} B$ and $Y = P \otimes_B \Omega_B^{n-2}(X)$. We have an exact sequence

$$0 \to \Omega_A^4(Y) \to P_3(Y) \to \Omega_A^3(Y) \to 0. \tag{3.4}$$

Applying $\operatorname{Hom}_A\left(\Omega_A^2(Y), -\right)$ to (3.4), we get the following exact sequence:

$$\operatorname{Ext}^1_A\left(\Omega_A^2(Y), P_3(Y)\right) \to \operatorname{Ext}^1_A\left(\Omega_A^2(Y), \Omega_A^3(Y)\right) \to \operatorname{Ext}^2_A\left(\Omega_A^2(Y), \Omega_A^4(Y)\right).$$

Since $\operatorname{id}_A \Omega_A^4(Y) \leqq 3$ by assumption,

$$\operatorname{Ext}_A^2\left(\Omega_A^2(Y), \Omega_A^4(Y)\right) \cong \operatorname{Ext}_A^4\left(Y, \Omega_A^4(Y)\right) = 0.$$

Thus there exists an exact sequence $0 \to P_3(Y) \to Z \to \Omega_A^2(Y) \to 0$ in $\operatorname{mod} A$ such that the following commutative diagram has exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega_A^4(Y) & = & \Omega_A^4(Y) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & P_3(Y) & \longrightarrow & Z & \longrightarrow & \Omega_A^2(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_A^3(Y) & \longrightarrow & P_2(Y) & \longrightarrow & \Omega_A^2(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then $Z \cong P_2(Y) \oplus \Omega_A^4(Y)$ and we get an exact sequence $0 \to P_3(Y) \to P_2(Y) \oplus \Omega_A^4(Y) \to \Omega_A^2(Y) \to 0$ in $\operatorname{mod} A$. Since $\operatorname{pd}_A Y < \infty$ by assumption, $\operatorname{pd}_A \Omega_A^2(Y) = \operatorname{pd}_A \Omega_A^4(Y) < \infty$, which implies that $\Omega_A^2(Y)$ is projective. By Lemma 3.1(2), there exists a projective module $Q \in \operatorname{mod} A$ such that $\Omega_B^n(X) \cong \operatorname{Hom}_A\left(P, \Omega_A^2\left(P \otimes_B \Omega_B^{n-2}(X)\right) \oplus Q\right) = \operatorname{Hom}_A(P, \Omega_A^2(Y) \oplus Q)$. Thus $\Omega_B^n(X) \in \operatorname{add}_B \operatorname{Hom}_A(P, A)$. It follows that B is an n-syzygy finite algebra, and hence an $(n-1)$-Igusa–Todorov algebra by Lemma 2.3(2). □

The following result is an immediate consequence of Proposition 3.10, which generalizes [15, Proposition 3.10].

Corollary 3.11. *Let* $\operatorname{gl.dim} A < \infty$, *and let* $P \in \operatorname{mod} A$ *be projective and* $B = \operatorname{End}_A P$. *If there exists* $n \geqq 2$ *such that* $\operatorname{id}_A \Omega_A^4\left(P \otimes_B \Omega_B^{n-2}(X)\right) \leqq 3$ *for any* $X \in \operatorname{mod} B$, *then* B *is an* n*-syzygy finite algebra, and hence an* $(n-1)$*-Igusa–Todorov algebra; in particular,* $\operatorname{fin.dim} B < \infty$.

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