Acta Math. Hungar., **147** (1) (2015), 135–157
DOI: 10.1007/s10474-015-0540-7
First published online July 31, 2015

WEAK INJECTIVE COVERS AND DIMENSION OF MODULES

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(Received October 15, 2014; revised April 23, 2015; accepted April 24, 2015)

Abstract. For a ring R, we prove that all left R-modules have weak injective covers and weak injective preenvelopes. Then we characterize the right weak injective dimension of modules in terms of the properties of left derived functors of Hom and right derived functors of $\otimes$. For any $n \geqq 2$, we prove that the global right weak injective dimension of ${}_R\mathcal{M}$ is at most n if and only if its global left weak injective dimension is at most $n-2$.

1. Introduction

The motivation for this paper is to extend the viewpoint stated by Stenström in [20]: many results of a homological nature may be generalized from coherent rings to arbitrary rings. In this process, finitely presented modules were in general replaced by super finitely presented modules, and the so-called weak injective and weak flat modules appeared.

For an arbitrary ring R, it is not possible to describe all R-modules in general. Unless each R-module is a direct sum of indecomposable ones, we have to study some special classes of R-modules. A successful method to overcome this obstacle is to approximate arbitrary modules by selecting some known classes of modules. This approach has been used to investigate injective envelopes, projective covers, as well as pure-injective envelopes of modules by Matlis, Bass, Warfield et al. in the 1960's. Note that (pre)envelopes and (pre)covers are dual notions in the category theoretic sense and that Lambek [13] showed that, over any ring, a module is flat if and only if its

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Key words and phrases: weak injective module, weak flat module, weak injective cover, weak injective preenvelope, weak injective dimension.
Mathematics Subject Classification: 18G25, 16E10, 16E30.

character module is injective. So Enochs [5] studied injective and flat covers, envelopes of modules. A classical result of Enochs says that a ring R is left Noetherian if and only if every left R-module has an injective cover. He also raised the well-known flat cover conjecture: every module over any ring has a flat cover. It was proved by Bican, El Bashir and Enochs [1] in two different ways.

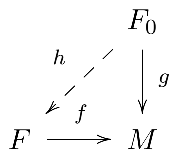
Stenström [20] introduced the notion of FP-injective modules. A left R-module M is said to be *FP-injective* (or *absolutely pure*) [16,20] if $\mathrm{Ext}^1_R(F, M) = 0$ for any finitely presented left R-module F. It is well-known that this class of modules plays an important role in characterizing coherent rings; and many results about injective modules over Noetherian rings should have a counterpart about FP-injective modules (see [6,9,17,19,20] and so on). Along the same lines, various generalizations of FP-injective modules are given and some generalized coherent rings have been studied by many authors (see e.g., [2,3,10,14,18,21]). Recently, Pinzon [19] showed that every left R-module has an FP-injective cover if R is a left coherent ring. This impels us to study the existence of weak injective covers over any ring, and investigate weak injective dimension of modules in terms of left weak injective resolutions of modules.

In this paper, we show that many results about (FP-)injective modules over coherent rings have a counterpart about weak injective modules, and some known results are obtained as corollaries. In Section 2, we give some terminology and some preliminary results. In Section 3, we first prove that every left R-module has a weak injective cover and a weak injective preenvelope. As applications, the duality properties of weak flat (resp. weak injective) preenvelopes and weak injective (resp. weak flat) precovers are discussed. We prove that the injective envelope of any weak flat left R-module is weak flat if and only if the weak flat cover of any injective left R-module is injective. In Section 4, we obtain some criteria for computing the right and left weak injective dimensions of modules in terms of the properties of left derived functors of Hom. Then we deduce that the global right weak injective dimension of a ring is at most n if and only if its global left weak injective dimension is at most $n-2$ for any $n \geqq 2$. In Section 5, we first show that $-\otimes_R-$ is right balanced on $\mathcal{M}_R \times {}_R\mathcal{M}$ by $\mathcal{WF} \times \mathcal{WI}$, and then we obtain some criteria for computing the right weak injective dimension of modules and the global right weak injective dimension of a ring R in terms of the properties of right derived functors of $\otimes$.

2. Preliminaries

Throughout this paper, R is an associative ring with identity, all modules are unitary, ${}_R\mathcal{M}$ (resp. $\mathcal{M}_R$) is the category of left (resp. right) R-modules and all subcategories of ${}_R\mathcal{M}$ (resp. $\mathcal{M}_R$) are full and closed under isomorphisms. For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers.

Definition 2.1 [6]. Let $\mathcal{F}$ be a subcategory of ${}_R\mathcal{M}$. The homomorphism $f : F \to M$ in ${}_R\mathcal{M}$ with $F \in \mathcal{F}$ is called an *$\mathcal{F}$-precover* of M if for any homomorphism $g : F_0 \to M$ in ${}_R\mathcal{M}$ with $F_0 \in \mathcal{F}$, there exists a homomorphism $h : F_0 \to F$ such that the following diagram commutes:



The homomorphism $f : F \to M$ is called *right minimal* if an endomorphism $h : F \to F$ is an automorphism whenever $f = fh$. An $\mathcal{F}$-precover $f : F \to M$ is called an *$\mathcal{F}$-cover* if f is right minimal. $\mathcal{F}$ is called a *covering subcategory* of ${}_R\mathcal{M}$ if every module in ${}_R\mathcal{M}$ has an $\mathcal{F}$-cover. Dually, the notions of an *$\mathcal{F}$-preenvelope*, a *left minimal homomorphism,* an *$\mathcal{F}$-envelope* and an *enveloping subcategory* are defined.

The following two lemmas play a crucial role in this paper.

Lemma 2.2 ([6, Proposition 5.2.2]). *If $\mathcal{F}$ is a full subcategory of ${}_R\mathcal{M}$ closed under direct sums, then a module M in ${}_R\mathcal{M}$ has an $\mathcal{F}$-precover if and only if there exists a cardinal number $\aleph_\alpha$ such that any homomorphism $D \to M$ with $D \in \mathcal{F}$ has a factorization $D \to C \to M$ with $C \in \mathcal{F}$ and $|C| \leqq \aleph_\alpha$.*

Recall that a short exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ in ${}_R\mathcal{M}$ is called *pure* if the functor $\mathrm{Hom}_R(F, -)$ preserves its exactness for every finitely presented left R-module F. In this case $f(A)$ is called a *pure submodule* of B (cf. [6,9]).

Lemma 2.3 ([1, Theorem 5]). *For each cardinal λ, there exists a cardinal κ such that for any $M \in {}_R\mathcal{M}$ and for any $L \leqq M$ such that $|M| \geqq \kappa$ and $|M/L| \leqq \lambda$, the submodule L contains a nonzero submodule that is pure in M.*

Definition 2.4 [8]. A module M in ${}_R\mathcal{M}$ (resp. N in $\mathcal{M}_R$) is said to be *weak injective* (resp. *weak flat*) if $\mathrm{Ext}^1_R(F, M) = 0$ (resp. $\mathrm{Tor}^R_1(N, F) = 0$) for any super finitely presented left R-module F, that is, for any left R-module F satisfying that there is an exact sequence: $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to F \to 0$ in ${}_R\mathcal{M}$ with each P_i finitely generated and projective. We use $\mathcal{WI}$ (resp. $\mathcal{WF}$) to denote the full subcategory of ${}_R\mathcal{M}$ (resp. $\mathcal{M}_R$) consisting of weak injective modules (resp. weak flat modules).

Remark 2.5. (1) In [15], Lee gave a definition of weak-injective modules as follows. An R-module D is called *weak-injective* if $\mathrm{Ext}^1_R(M,D)=0$ for all R-modules M of weak dimension $\leqq 1$. The notion of weak injective modules given in Definition 2.4 is different from the one in [15].

(2) It is trivial that an FP-injective module is weak injective. When R is left coherent, a module in ${}_R\mathcal{M}$ is FP-injective if and only if it is weak injective. It was showed in [11, Corollary 3.8] that, even for a left and right coherent ring R, an FP-injective left R-module does not have a decomposition as a direct sum of indecomposable FP-injective submodules in general. This also shows that, for a ring R, a weak injective left R-module does not in general have a decomposition as a direct sum of indecomposable weak injective submodules.

(3) It is trivial that a flat module is weak flat. When R is left coherent, a module in $\mathcal{M}_R$ is flat if and only if it is weak flat.

Proposition 2.6. (1) *Let $0\to A\to B\to C\to 0$ be an exact sequence in ${}_R\mathcal{M}$ with A weak injective. Then B is weak injective if and only if C is weak injective.*

(2) *Let $0\to A\to B\to C\to 0$ be an exact sequence in $\mathcal{M}_R$ with C weak flat. Then B is weak flat if and only if A is weak flat.*

Proof. (1) Let (†) $0\to A\to B\to C\to 0$ be a short exact sequence with A weak injective. For any super finitely presented left R-module F, applying $\mathrm{Hom}_R(F,-)$ to the sequence (†), we get the following exact sequence:

$$0=\mathrm{Ext}^1_R(F,A)\to \mathrm{Ext}^1_R(F,B)\to \mathrm{Ext}^1_R(F,C)\to \mathrm{Ext}^2_R(F,A)=0$$

by [8, Proposition 3.1]. Now the assertion (1) follows clearly.

(2) The proof is similar to that of (1), so we omit it. □

By Proposition 2.6, we have that $\mathcal{WI}$ is coresolving in ${}_R\mathcal{M}$ and $\mathcal{WF}$ is resolving in $\mathcal{M}_R$ in the sense of [9].

Definition 2.7 [8]. For a module M in ${}_R\mathcal{M}$, the *weak injective dimension* of M, denoted by $\mathrm{wid}_R(M)$, is defined as $\inf\big\{n \mid \mathrm{Ext}^{n+1}_R(F,M)=0$ for any super finitely presented left R-module $F\big\}$. If no such n exists, set $\mathrm{wid}_R(M)=\infty$.

For a module N in $\mathcal{M}_R$, the *weak flat dimension* of N, denoted by $\mathrm{wfd}_R(N)$, is defined as $\inf\big\{n \mid \mathrm{Tor}^R_{n+1}(N,F)=0$ for any super finitely presented left R-module $F\big\}$. If no such n exists, set $\mathrm{wfd}_R(N)=\infty$.

The *left super finitely presented dimension* of R, denoted by $\mathrm{l.sp.gldim}(R)$, is defined as $\sup\big\{\mathrm{pd}_R(M) \mid M$ is a super finitely presented left R-module$\big\}$.

LEMMA 2.8 ([8, Theorem 3.8]). (1) $\mathrm{l.sp.gldim}(R) \leqq \mathrm{w.gl.dim}(R)$, *with equality* $\mathrm{l.sp.gldim}(R) = \mathrm{w.gl.dim}(R)$ *when R is a left coherent ring.*

$$\begin{aligned}\mathrm{l.sp.gldim}(R) &= \sup\left\{ \mathrm{wid}_R(M) \mid M \textit{ is a left } R\textit{-module}\right\}\\ &= \sup\left\{ \mathrm{wfd}_R(N) \mid N \textit{ is a right } R\textit{-module}\right\}.\end{aligned}$$

Let M be in ${}_R\mathcal{M}$ and $\mathcal{F}$ a subcategory of ${}_R\mathcal{M}$. Following [6], a sequence $\mathbb{E}$ in ${}_R\mathcal{M}$ is called $\mathrm{Hom}_R(\mathcal{F},-)$ (resp. $\mathrm{Hom}_R(-,\mathcal{F})$) *exact* if $\mathrm{Hom}_R(F,\mathbb{E})$ (resp. $\mathrm{Hom}_R(\mathbb{E},F)$) is exact for any $F \in \mathcal{F}$; a *left (resp. right) $\mathcal{F}$-resolution* of M is a $\mathrm{Hom}_R(\mathcal{F},-)$ (resp. $\mathrm{Hom}_R(-,\mathcal{F})$) exact complex $\cdots \to F_1 \to F_0 \to M \to 0$ (resp. $0 \to M \to F^0 \to F^1 \to \cdots$) with each F_i (resp. F^i) in $\mathcal{F}$.

For a left $\mathcal{F}$-resolution $\cdots \to F_1 \to F_0 \to M \to 0$ of M, set

$$K_0 = M, \quad K_1 = \mathrm{Ker}(F_0 \to M), \quad K_i = \mathrm{Ker}(F_{i-1} \to F_{i-2}) \ \text{ for } i \geqq 2.$$

The nth kernel K_n ($n \geqq 0$) is called the nth *$\mathcal{F}$-syzygy* of M.

For a right $\mathcal{F}$-resolution $0 \to M \to F^0 \to F^1 \to \cdots$ of M, set

$$L^0 = M, \quad L^1 = \mathrm{Coker}(M \to F^0), \quad L^i = \mathrm{Coker}\left(F^{i-2} \to F^{i-1}\right) \ \text{ for } i \geqq 2.$$

The nth cokernel L^n ($n \geqq 0$) is called the nth *$\mathcal{F}$-cosyzygy* of M.

Following [6, Definition 8.4.1], the *left $\mathcal{F}$-dimension* of M, denoted by left $\mathcal{F}$-dim M, is defined as $\inf\{n \mid$ there exists a left $\mathcal{F}$-resolution of the form $0 \to F_n \to \cdots \to F_0 \to M \to 0\}$. If there exists no such n, set left $\mathcal{F}$-dim $M = \infty$. The *global left $\mathcal{F}$-dimension* of ${}_R\mathcal{M}$, denoted by gl left $\mathcal{F}$-$\dim_R \mathcal{M}$, is defined to be $\sup\{$left $\mathcal{F}$-dim $M \mid M \in {}_R\mathcal{M}\}$ and is infinite otherwise. The right versions can be defined similarly, and they are denoted by right $\mathcal{F}$-dim M and gl right $\mathcal{F}$-dim ${}_R\mathcal{M}$ respectively.

3. Weak injective covers and preenvelopes

In this section, we show that all left R-modules have weak injective covers and weak injective preenvelopes. The duality properties of weak flat (resp. weak injective) preenvelopes and weak injective (resp. weak flat) precovers are discussed, and some known results are obtained as corollaries.

The following theorem is the main result in this section.

THEOREM 3.1. *Every module in ${}_R\mathcal{M}$ has a weak injective cover.*

PROOF. Let M be in ${}_R\mathcal{M}$ with $|M| = \lambda$. We first prove that M has a weak injective precover. Let κ be a cardinal as in Lemma 2.3. By Lemma 2.2, it suffices to show that any homomorphism $A \to M$ in ${}_R\mathcal{M}$ with A weak injective factors through a weak injective module B with $|B| \leqq \kappa$.

Consider any homomorphism $f : A \to M$ in ${}_R\mathcal{M}$ with A weak injective. If $|A| \leqq \kappa$, then the assertion holds by taking $B = A$. So we may assume that $|A| > \kappa$. Let $K = \operatorname{Ker} f$. Since A/K can be embedded in M, it follows that $|A/K| \leqq \lambda$. Thus K contains a nonzero submodule A_0 which is pure in A by Lemma 2.3. The pure exact sequence $0 \to A_0 \to A \to A/A_0 \to 0$ induces a split exact sequence $0 \to (A/A_0)^+ \to A^+ \to A_0^+ \to 0$. Hence $(A/A_0)^+$ is weak flat since A^+ is weak flat by [8, Theorem 2.10]. Therefore, A/A_0 is weak injective by [8, Theorem 2.10] again.

If $|A/A_0| \leqq \kappa$, then the desired result follows immediately since f factors through A/A_0. Now suppose that $|A/A_0| > \kappa$. Set $\mathcal{L} = \{X \mid A_0 \leqq X \leqq K$ and A/X is weak injective$\}$. It is obvious that $\mathcal{L} \neq \emptyset$ since $A_0 \in \mathcal{L}$. Let $\{X_i \in \mathcal{L} \mid i \in I\}$ be an ascending chain. Then we have that $A_0 \leqq \cup X_i \leqq K$ and $A/\cup X_i = A/\varinjlim X_i = \varinjlim (A/X_i)$ is weak injective by [8, Proposition 2.6] since A/X_i is weak injective. Thus $\cup X_i \in \mathcal{L}$. By the Zorn lemma, $\mathcal{L}$ has a maximal element B.

Next we claim that $|A/B| \leqq \kappa$. Otherwise, suppose $|A/B| > \kappa$. Because $B \subsetneqq K$, there exists $g : A/B \to M$ such that $\operatorname{Ker} g = K/B$. Notice that $\big|(A/B)/(K/B)\big| = |A/K| \leqq \lambda$, so K/B contains a nonzero submodule B'/B which is pure in A/B by Lemma 2.3. Then B'/B is weak injective by [8, Proposition 2.9]. Therefore, $A/B' \cong (A/B)/(B'/B)$ is weak injective by Proposition 2.6(1). This implies that $B' \in \mathcal{L}$, which gives a contradiction to the maximality of B.

Finally, it is clear that A/B is weak injective and f factors through A/B. It follows from [8, Proposition 2.3] that $\mathcal{WI}$ is closed under direct sums, and hence M has a weak injective precover by Lemma 2.2. In addition, since $\mathcal{WI}$ is closed under direct limits by [8, Proposition 2.6], M has a weak injective cover by [22, Theorem 2.2.8]. □

As applications of Theorem 3.1, we have

Corollary 3.2 ([22, Theorem 2.4.2]). *If R is left Noetherian, then every module M in ${}_R\mathcal{M}$ has an injective cover.*

Corollary 3.3 ([19, Corollary 2.7]). *If R is left coherent, then every module M in ${}_R\mathcal{M}$ has an FP-injective cover.*

It was shown in [6, Proposition 6.2.4] that the subcategory of ${}_R\mathcal{M}$ consisting of FP-injective modules is preenveloping. The next theorem extends this result.

Theorem 3.4. *Every module in ${}_R\mathcal{M}$ has a weak injective preenvelope.*

Proof. Let M be in ${}_R\mathcal{M}$. By [6, Lemma 5.3.12], there exists a cardinal number $\aleph_\alpha$ such that for any R-homomorphism $f : M \to N$ with N weak injective, there exists a pure submodule S of N such that $|S| \leqq \aleph_\alpha$ and $f(M) \subset S$. Since N is weak injective, S is weak injective by [8, Proposition

2.9]. It follows from [8, Proposition 2.3] that $\mathcal{WI}$ is closed under direct products. Therefore, M has a weak injective preenvelope by [6, Proposition 6.2.1]. □

In [5], Enochs proved that R is right coherent if and only if every left R-module has a flat preenvelope ([5, Proposition 5.1]). For any ring R, we showed [8, Theorem 2.15] that every right R-module has a weak flat preenvelope. Eklof and Trlifaj proved [4, Theorem 12] that if $\mathcal{B}$ is a subcategory of ${}_R\mathcal{M}$, then every module in $\mathcal{M}_R$ has a $\operatorname{Ker}\operatorname{Tor}_1^R(-,\mathcal{B})$-cover, where $\operatorname{Ker}\operatorname{Tor}_1^R(-,\mathcal{B}) = \{ A \mid \operatorname{Tor}_1^R(A,B) = 0 \text{ for any } B \in \mathcal{B} \}$. By taking $\mathcal{B}$ as the subcategory of ${}_R\mathcal{M}$ consisting of all super finitely presented modules, one can deduce immediately that every right R-module has a weak flat cover. Here we have

PROPOSITION 3.5. (1) *If $f : C \to D$ is a weak flat preenvelope of a module C in $\mathcal{M}_R$, then $f^+ : D^+ \to C^+$ is a weak injective precover of C^+ in ${}_R\mathcal{M}$.*

(2) *If $f : C \to D$ is a weak injective preenvelope of a module C in ${}_R\mathcal{M}$, then $f^+ : D^+ \to C^+$ is a weak flat precover of C^+ in $\mathcal{M}_R$.*

PROOF. By [8, Remark 2.2 and Theorem 2.10], we have $\mathcal{WF}^+ \subseteqq \mathcal{WI}$ and $\mathcal{WI}^+ \subseteqq \mathcal{WF}$. Now both assertions follows immediately from [7, Corollary 3.2]. □

COROLLARY 3.6. *Let R be a left coherent ring.*

(1) ([7, Corollary 3.3]) *If $f : A \to F$ in $\mathcal{M}_R$ is a flat preenvelope of A, then $f^+ : F^+ \to A^+$ is an (FP-)injective precover of A^+ in ${}_R\mathcal{M}$.*

(2) *If $f : A \to E$ in ${}_R\mathcal{M}$ is an (FP-)injective preenvelope of A, then $f^+ : E^+ \to A^+$ is a flat precover of A^+ in $\mathcal{M}_R$.*

PROOF. Since weak injective left R-modules coincide with FP-injective left R-modules and weak flat right R-modules coincide with flat right R-modules over a left coherent ring R, the assertions follow from Propositions 3.5(1) and (2), respectively. □

Let $\mathcal{C}$ be a covering subcategory and $\mathcal{E}$ an enveloping subcategory of ${}_R\mathcal{M}$. For a module M in ${}_R\mathcal{M}$, the $\mathcal{C}$-cover and the $\mathcal{E}$-envelope of M are denoted by $\mathcal{C}_0(M)$ and $\mathcal{E}^0(M)$ respectively. The following proposition is of independent interest.

PROPOSITION 3.7. *Let $\mathcal{C}$ be a covering subcategory and $\mathcal{E}$ an enveloping subcategory of ${}_R\mathcal{M}$, such that both $\mathcal{C}$ and $\mathcal{E}$ are closed under direct summands. Then the following statements are equivalent.*

(1) *$\mathcal{E}^0(M) \in \mathcal{C}$ for any $M \in \mathcal{C}$.*

(2) *$\mathcal{C}_0(N) \in \mathcal{E}$ for any $N \in \mathcal{E}$.*

PROOF. (1) ⇒ (2). For any $N \in \mathcal{E}$, suppose that $\alpha : \mathcal{C}_0(N) \to N$ is the $\mathcal{C}$-cover of N and $\beta : \mathcal{C}_0(N) \to \mathcal{E}^0\big(\mathcal{C}_0(N)\big)$ is the $\mathcal{E}$-envelope of $\mathcal{C}_0(N)$. Then there exists $\theta : \mathcal{E}^0\big(\mathcal{C}_0(N)\big) \to N$ such that $\alpha = \theta\beta$. On the other hand, since $\mathcal{E}^0\big(\mathcal{C}_0(N)\big) \in \mathcal{C}$ by (1), there exists $\lambda : \mathcal{E}^0\big(\mathcal{C}_0(N)\big) \to \mathcal{C}_0(N)$ such that $\alpha\lambda = \theta$. So $\alpha = \alpha\lambda\beta$, and hence $\lambda\beta$ is an isomorphism since α is a cover. It implies that $\mathcal{C}_0(N)$ is a direct summand of $\mathcal{E}^0\big(\mathcal{C}_0(N)\big)$. Since $\mathcal{E}$ is closed under direct summands by assumption, $\mathcal{C}_0(N) \in \mathcal{E}$.

Dually we get (2) $\Rightarrow$ (1). □

We denote by $E^0(M)$ (resp. $WF_0(M)$) the injective envelope (resp. the weak flat cover) of a module M in ${}_R\mathcal{M}$. By Proposition 3.7, we get immediately the following

Corollary 3.8. *The following statements are equivalent.*
(1) $E^0(M)$ *is weak flat for any weak flat left* R*-module* M.
(2) $WF_0(N)$ *is injective for any injective left* R*-module* N.

The following result was proved in [12, Theorem 2.2] when R is a commutative Noetherian ring.

Corollary 3.9. *The following statements are equivalent.*
(1) $E^0(M)$ *is flat for any flat left* R*-module* M.
(2) $F_0(I)$ (*the flat cover of* I) *is injective for any injective left* R*-module* I.

4. Left derived functors of Hom and right $\mathcal{WI}$-dimension of modules

In this section, we investigate the right $\mathcal{WI}$-dimension of modules in terms of left derived functors of Hom and the left $\mathcal{WI}$-resolutions of modules. Some known results in [6] are developed.

By Theorem 3.1, we have that every left R-module has a weak injective cover. So every left R-module M has a *left* $\mathcal{WI}$*-resolution*, that is, there exists a $\mathrm{Hom}_R(\mathcal{WI}, -)$ exact complex $\cdots \to E_1 \to E_0 \to M \to 0$ (not necessarily exact) in ${}_R\mathcal{M}$ with each E_i weak injective. On the other hand, every left R-module M has a weak injective preenvelope by Theorem 3.4. Thus M has a *right* $\mathcal{WI}$*-resolution*, that is, there exists a $\mathrm{Hom}_R(-, \mathcal{WI})$ exact complex $0 \to M \to E^0 \to E^1 \to \cdots$ in ${}_R\mathcal{M}$ with each E^i weak injective. Clearly, this complex is exact.

The following result shows that the left super finitely presented dimension of R and the global right weak injective dimension of ${}_R\mathcal{M}$ are identical.

Proposition 4.1. (1) $\mathrm{wid}_R(M) = \mathrm{right}\,\mathcal{WI}\text{-}\mathrm{dim}\, M$ *for any* $M \in {}_R\mathcal{M}$.
(2) $\mathrm{l.sp.gldim}(R) = \mathrm{gl\,right}\,\mathcal{WI}\text{-}\mathrm{dim}_R\,\mathcal{M}$.

Proof. (1) Let M be in ${}_R\mathcal{M}$. It is obvious that $\mathrm{wid}_R(M) \leqq \mathrm{right}\,\mathcal{WI}$-$\dim M$. Conversely, suppose that $\mathrm{wid}_R(M) = n < \infty$. Take a partial right $\mathcal{WI}$-resolution of M:

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1}$$

Set $L = \operatorname{Coker}(E^{n-2} \to E^{n-1})$. Thus we get the following exact sequence:

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to L \to 0$$

in ${}_R\mathcal{M}$. Then L is weak injective by [8, Proposition 3.3]. Consequently, right $\mathcal{WI}$-dim $M \leqq n$.

(2) This result follows directly from (1) and Lemma 2.8. □

Proposition 4.2. *If* l.sp.gldim$(R) < \infty$, *then* l.sp.gldim$(R) = \operatorname{wid}({}_RR)$.

Proof. It is clear that $\operatorname{wid}({}_RR) \leqq$ l.sp.gldim(R) by Lemma 2.8. Now it suffices to show that if l.sp.gldim$(R) = n < \infty$, then $\operatorname{wid}({}_RR) \geqq n$. For any super finitely presented left R-module F, there exists some M in ${}_R\mathcal{M}$ such that $\operatorname{Ext}_R^n(F, M) \neq 0$. Let $0 \to K \to P \to M \to 0$ be an exact sequence in ${}_R\mathcal{M}$ with P free. Then we have the following exact sequence:

$$\operatorname{Ext}_R^n(F, P) \to \operatorname{Ext}_R^n(F, M) \to \operatorname{Ext}_R^{n+1}(F, K).$$

Since l.sp.gldim$(R) = n$, we have $\operatorname{Ext}_R^{n+1}(F, K) = 0$. It follows that $\operatorname{Ext}_R^n(F, P) \neq 0$ and $\operatorname{Ext}_R^n(F, {}_RR) \neq 0$. So $\operatorname{wid}({}_RR) \geqq n$. □

By Proposition 4.2, we immediately have the following

Corollary 4.3 ([20, Proposition 3.5]). *Let R be left coherent. If* w.gl.dim$(R) < \infty$, *then* w.gl.dim$(R) = $ FP-id$({}_RR)$.

By [6, Definition 8.2.13], we can easily see that $\operatorname{Hom}_R(-,-)$ is left balanced on ${}_R\mathcal{M} \times {}_R\mathcal{M}$ by $\mathcal{WI} \times \mathcal{WI}$. Let $\operatorname{Ext}_n^{\mathcal{WI}}(-,-)$ denote the nth left derived functor of $\operatorname{Hom}_R(-,-)$ with respect to $\mathcal{WI} \times \mathcal{WI}$. For any M and N in ${}_R\mathcal{M}$, $\operatorname{Ext}_n^{\mathcal{WI}}(M, N)$ can be computed by using a right $\mathcal{WI}$-resolution of M or a left $\mathcal{WI}$-resolution of N. Let

$$0 \to M \xrightarrow{f^0} E^0 \xrightarrow{f^1} E^1 \to \cdots$$

be a right $\mathcal{WI}$-resolution of M in ${}_R\mathcal{M}$. Applying $\operatorname{Hom}_R(-, N)$ to the sequence, we get the deleted complex

$$\cdots \to \operatorname{Hom}_R(E^1, N) \xrightarrow{\operatorname{Hom}_R(f^1, N)} \operatorname{Hom}_R(E^0, N) \to 0.$$

Then $\operatorname{Ext}_n^{\mathcal{WI}}(M, N)$ is exactly the nth homology of the above complex, and there exists a canonical homomorphism:

$$\sigma :\ \operatorname{Ext}_0^{\mathcal{WI}}(M, N) = \operatorname{Hom}_R(E^0, N)/\operatorname{Im}\operatorname{Hom}_R(f^1, N) \to \operatorname{Hom}_R(M, N),$$

which is defined by

$$\sigma\big(\alpha + \operatorname{Im}\operatorname{Hom}_R(f^1, N)\big) = \alpha f^0 \quad \text{for any} \quad \alpha \in \operatorname{Hom}_R(E^0, N).$$

Proposition 4.4. *The following statements are equivalent for any* $M \in {}_R\mathcal{M}$.

(1) *M is weak injective.*

(2) $\sigma : \mathrm{Ext}_0^{\mathcal{WI}}(M, N) \to \mathrm{Hom}_R(M, N)$ *is an epimorphism for any* $N \in {}_R\mathcal{M}$.

(3) $\sigma : \mathrm{Ext}_0^{\mathcal{WI}}(M, M) \to \mathrm{Hom}_R(M, M)$ *is an epimorphism.*

Proof. (1) $\Rightarrow$ (2) is clear by letting $E^0 = M$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). By assumption, there exists $\alpha \in \mathrm{Hom}_R(E^0, M)$ such that $\sigma\big(\alpha + \mathrm{Im}\,\mathrm{Hom}_R(f^1, M)\big) = \alpha f^0 = 1_M$. So M is isomorphic to a direct summand of E^0, and hence it is weak injective. $\square$

As an application of Proposition 4.4, we have the following

Corollary 4.5. *The following statements are equivalent.*

(1) *${}_RR$ is weak injective.*

(2) $\sigma : \mathrm{Ext}_0^{\mathcal{WI}}({}_RR, N) \to \mathrm{Hom}_R({}_RR, N)$ *is an epimorphism for any* $N \in {}_R\mathcal{M}$.

(3) $\sigma : \mathrm{Ext}_0^{\mathcal{WI}}({}_RR, {}_RR) \to \mathrm{Hom}_R({}_RR, {}_RR)$ *is an epimorphism.*

(4) *Every right R-module has a monic weak flat preenvelope.*

(5) *Every left R-module has an epic weak injective cover.*

(6) *Every right R-module is a submodule of a weak flat right R-module.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Proposition 4.4.

(1) $\Leftrightarrow$ (4) follows from [8, Proposition 2.17].

(1) $\Rightarrow$ (5). Let M be in ${}_R\mathcal{M}$. Then M has a weak injective cover $g : E \to M$ by Theorem 3.1. On the other hand, there exists an exact sequence $F \to M \to 0$ in ${}_R\mathcal{M}$ with F free. Notice that F is weak injective by (1) and [8, Proposition 2.3], so g is an epimorphism.

(5) $\Rightarrow$ (1). Since there exists an epic weak injective cover $f : E \to {}_RR$ by (5), ${}_RR$ is isomorphic to a direct summand of E. Thus ${}_RR$ is weak injective by [8, Proposition 2.3].

(4) $\Rightarrow$ (6) is trivial.

(6) $\Rightarrow$ (4). By [8, Theorem 2.15], every right R-module has a weak flat preenvelope. Thus the assertion holds by (6). $\square$

Proposition 4.6. *The following statements are equivalent for any* $M \in {}_R\mathcal{M}$.

(1) right $\mathcal{WI}$-dim $M \leqq 1$.

(2) $\sigma : \mathrm{Ext}_0^{\mathcal{WI}}(M, N) \to \mathrm{Hom}_R(M, N)$ *is a monomorphism for any* $N \in {}_R\mathcal{M}$.

Proof. (1) $\Rightarrow$ (2). By assumption, M has a right $\mathcal{WI}$-resolution $0 \to M \to E^0 \to E^1 \to 0$. Then we get an exact sequence $0 \to \mathrm{Hom}_R(E^1, N) \to \mathrm{Hom}_R(E^0, N) \to \mathrm{Hom}_R(M, N)$ for any $N \in {}_R\mathcal{M}$. So σ is a monomorphism.

(2) $\Rightarrow$ (1). Let $0 \to M \to E \to L \to 0$ be an exact sequence in ${}_R\mathcal{M}$ with $M \to E$ a weak injective preenvelope of M. It suffices to show that L is weak injective. By [6, Theorem 8.2.3], we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & \mathrm{Ext}_0^{\mathcal{WI}}(L,L) & \longrightarrow & \mathrm{Ext}_0^{\mathcal{WI}}(E,L) & \longrightarrow & \mathrm{Ext}_0^{\mathcal{WI}}(M,L) & \longrightarrow & 0 \\ & & \sigma_1 \downarrow & & \sigma_2 \downarrow & & \sigma_3 \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(L,L) & \longrightarrow & \mathrm{Hom}_R(E,L) & \longrightarrow & \mathrm{Hom}_R(M,L). & & \end{array}$$

Notice that σ_2 is an epimorphism by Proposition 4.4 and that σ_3 is a monomorphism by assumption, so σ_1 is an epimorphism by the snake lemma. Thus L is weak injective by Proposition 4.4. $\square$

By Proposition 4.6, we get directly the following

COROLLARY 4.7. *The following statements are equivalent.*

(1) right $\mathcal{WI}$-dim$({}_RR) \leqq 1$.

(2) $\sigma : \mathrm{Ext}_0^{\mathcal{WI}}({}_RR, N) \to \mathrm{Hom}_R({}_RR, N)$ *is a monomorphism for any* $N \in {}_R\mathcal{M}$.

THEOREM 4.8. *The following statements are equivalent.*

(1) gl right $\mathcal{WI}$-dim$_R\,\mathcal{M} \leqq 1$.

(2) $\sigma : \mathrm{Ext}_0^{\mathcal{WI}}(M, N) \to \mathrm{Hom}_R(M, N)$ *is a monomorphism for any* $M, N \in {}_R\mathcal{M}$.

(3) l.sp.gldim$(R) < \infty$ *and* $\sigma : \mathrm{Ext}_0^{\mathcal{WI}}({}_RR, N) \to \mathrm{Hom}_R({}_RR, N)$ *is a monomorphism for any* $N \in {}_R\mathcal{M}$.

(4) *Every module in* ${}_R\mathcal{M}$ *has a monic weak injective cover.*

(5) *Every module in* $\mathcal{M}_R$ *has an epic weak flat preenvelope.*

(6) *Every submodule of any module in* $\mathcal{WF}$ *is weak flat.*

(7) *Every quotient of any module in* $\mathcal{WI}$ *is weak injective.*

(8) *The kernel of any weak injective precover of a module in* ${}_R\mathcal{M}$ *is weak injective.*

(9) *The cokernel of any weak injective preenvelope of a module in* ${}_R\mathcal{M}$ *is weak injective.*

(10) *The cokernel of any weak flat preenvelope of a module in* $\mathcal{M}_R$ *is weak flat.*

PROOF. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) is clear by Propositions 4.6 and 4.1.

(3) $\Rightarrow$ (1) follows from Corollary 4.7 and Propositions 4.1 and 4.2.

(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) hold by [8, Propositions 2.18 and 3.3].

(7) $\Rightarrow$ (9) $\Rightarrow$ (1) are trivial.

(4) $\Rightarrow$ (8). Let M be in ${}_R\mathcal{M}$ and $f : E \to M$ a weak injective precover of M with $K = \mathrm{Ker}\, f$. Because there exists a monic weak injective cover

$f' : E' \to M$ of M by (4), we have $K \oplus E' \cong E$ by [6, Lemma 8.6.3]. Thus K is weak injective by [8, Proposition 2.3].

(8) $\Rightarrow$ (1). Let M be a quotient of a weak injective left R-module. Note that M has a weak injective cover $f : E \to M$. So f is an epimorphism. Since $\operatorname{Ker} f$ is weak injective by (8), M is weak injective by Proposition 2.6(1). Thus any quotient of a weak injective left R-module is weak injective, and (1) follows.

(5) $\Rightarrow$ (10). Let $g : M \to F$ be a weak flat preenvelope of a module M in $\mathcal{M}_R$ and $L = \operatorname{Coker} g$. Since there exists an epic weak flat preenvelope $g' : M \to F'$ by (5), we have $L \oplus F' \cong F$ by the dual of [6, Lemma 8.6.3]. It follows from [8, Proposition 2.3] that L is weak flat, as desired.

(10) $\Rightarrow$ (1). By Lemma 2.8 and Proposition 4.1, it suffices to show that any submodule of a weak flat right R-module is weak flat. Let M be a submodule of a weak flat right R-module. Note that M has a weak flat preenvelope $f : M \to F$. It follows that f is a monomorphism. By (10), $\operatorname{Coker} f$ is weak flat. So M is weak flat by Proposition 2.6(2). $\square$

Proposition 4.9. *The following are equivalent for any $M \in {}_R\mathcal{M}$ and $n \geqq 2$.*

(1) right $\mathcal{WI}$-dim $M \leqq n$.

(2) $\operatorname{Ext}^{\mathcal{WI}}_{n+k}(M,N) = 0$ *for any* $N \in {}_R\mathcal{M}$ *and* $k \geqq -1$.

(3) $\operatorname{Ext}^{\mathcal{WI}}_{n-1}(M,N) = 0$ *for any* $N \in {}_R\mathcal{M}$.

Proof. (1) $\Rightarrow$ (2). Let $0 \to M \to E^0 \to E^1 \to \cdots \to E^n \to 0$ be a right $\mathcal{WI}$-resolution of M in ${}_R\mathcal{M}$. For any $N \in {}_R\mathcal{M}$, we have the following exact sequence:

$$0 \to \operatorname{Hom}_R(E^n, N) \to \operatorname{Hom}_R(E^{n-1}, N) \to \operatorname{Hom}_R(E^{n-2}, N).$$

Thus $\operatorname{Ext}^{\mathcal{WI}}_{n-1}(M,N) = \operatorname{Ext}^{\mathcal{WI}}_{n}(M,N) = 0$. It is clear that $\operatorname{Ext}^{\mathcal{WI}}_{n+k}(M,N) = 0$ for any $k \geqq 1$. Thus (2) follows.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Assume that $0 \to M \xrightarrow{f^0} E^0 \xrightarrow{f^1} E^1 \xrightarrow{f^2} \cdots$ is a right $\mathcal{WI}$-resolution of M in ${}_R\mathcal{M}$. Set $L^n = \operatorname{Coker} f^{n-1}$. It suffices to show that L^n is weak injective. Let $\pi : E^{n-1} \to L^n$ be the canonical projection, $\lambda : L^n \to E^n$ be a $\mathcal{WI}$-preenvelope and $f^n = \lambda\pi$. Clearly, we have the following exact

commutative diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & M & \longrightarrow & E^0 & \longrightarrow & \cdots & \longrightarrow & E^{n-2} & \overset{f^{n-1}}{\longrightarrow} & E^{n-1} & \xrightarrow{\qquad f^n \qquad} & E^n \longrightarrow \cdots . \\
 & & & & & & & & & & & \pi \searrow \quad \nearrow \lambda & \\
 & & & & & & & & & & & L^n & \\
 & & & & & & & & & & & \nearrow \quad \searrow & \\
 & & & & & & & & & & 0 & & 0
\end{array}$$

By (3), we have $\operatorname{Ext}_{n-1}^{\mathcal{WI}}(M, L^n) = 0$. So we get the following exact sequence:

$$\operatorname{Hom}_R(E^n, L^n) \overset{\operatorname{Hom}_R(f^n, L^n)}{\to} \operatorname{Hom}_R(E^{n-1}, L^n) \overset{\operatorname{Hom}_R(f^{n-1}, L^n)}{\to} \operatorname{Hom}_R(E^{n-2}, L^n).$$

Since $\operatorname{Hom}_R(f^{n-1}, L^n)(\pi) = \pi f^{n-1} = 0$,

$$\pi \in \operatorname{Ker} \operatorname{Hom}_R(f^{n-1}, L^n) = \operatorname{Im} \operatorname{Hom}_R(f^n, L^n).$$

So there exists $t \in \operatorname{Hom}_R(E^n, L^n)$ such that $\pi = \operatorname{Hom}_R(f^n, L^n)(t) = tf^n = t\lambda\pi$, and hence $t\lambda = 1$ since π is epic. Thus L^n is weak injective. $\square$

By Theorem 3.1, we have that every module N in ${}_R\mathcal{M}$ admits a minimal left $\mathcal{WI}$-resolution, that is, there exists a complex $\cdots \to E_2 \to E_1 \to E_0 \to N \to 0$ of N in ${}_R\mathcal{M}$ such that $E_0 \to N$, $E_1 \to \operatorname{Ker}(E_0 \to N)$ and $E_{i+1} \to \operatorname{Ker}(E_i \to E_{i-1})$ for $i \geqq 1$, are $\mathcal{WI}$-covers.

PROPOSITION 4.10. *The following are equivalent for any $N \in {}_R\mathcal{M}$ and $n \geqq 2$.*

(1) left $\mathcal{WI}$-$\dim N \leqq n-2$.

(2) $\operatorname{Ext}_{n+k}^{\mathcal{WI}}(M, N) = 0$ *for any $M \in {}_R\mathcal{M}$ and $k \geqq -1$.*

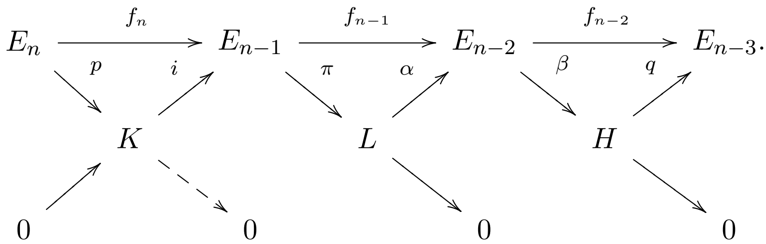
(3) $\operatorname{Ext}_{n-1}^{\mathcal{WI}}(M, N) = 0$ *for any $M \in {}_R\mathcal{M}$.*

PROOF. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (1). Let N be in ${}_R\mathcal{M}$ and let

$$\cdots \to E_n \overset{f_n}{\to} E_{n-1} \overset{f_{n-1}}{\to} E_{n-2} \overset{f_{n-2}}{\to} E_{n-3} \to \cdots \to E_1 \to E_0 \to N \to 0$$

be a minimal left $\mathcal{WI}$-resolution of N in ${}_R\mathcal{M}$. Set $K = \operatorname{Ker} f_{n-1}$, $L = E_{n-1}/K$. Let $i : K \to E_{n-1}$ be the inclusion and $\pi : E_{n-1} \to L$ the natural epimorphism. Then there exists $p : E_n \to K$ which is a $\mathcal{WI}$-cover of K such that $f_n = ip$, and there exists a homomorphism $\alpha : L \to E_{n-2}$ such that $f_{n-1} = \alpha\pi$. Put $H = E_{n-2}/\operatorname{Im}\alpha$ and let $\beta : E_{n-2} \to H$ be the natural epimorphism. Then there exists $q : H \to E_{n-3}$ such that $f_{n-2} = q\beta$. Thus we have the following commutative diagram:



By assumption, $\mathrm{Ext}^{\mathcal{WI}}_{n-1}(K, N) = 0$. Then we get the following exact sequence:

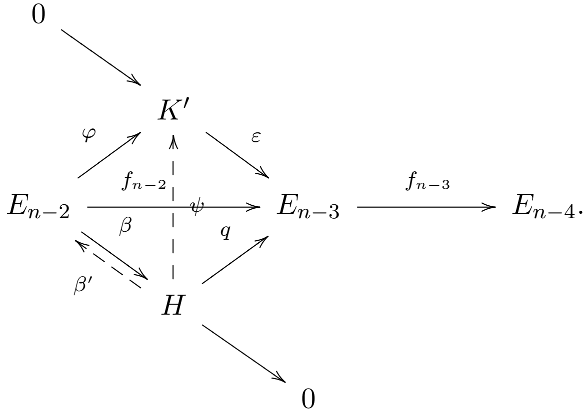
$$\mathrm{Hom}_R(K, E_n) \stackrel{\mathrm{Hom}_R(K,f_n)}{\to} \mathrm{Hom}_R(K, E_{n-1}) \stackrel{\mathrm{Hom}_R(K,f_{n-1})}{\to} \mathrm{Hom}_R(K, E_{n-2}).$$

Since $\mathrm{Hom}_R(K, f_{n-1})(i) = f_{n-1}i = 0$,

$$i \in \mathrm{Ker}\,\mathrm{Hom}_R(K, f_{n-1}) = \mathrm{Im}\,\mathrm{Hom}_R(K, f_n).$$

Then there exists $t \in \mathrm{Hom}_R(K, E_n)$ such that $i = \mathrm{Hom}_R(K, f_n)(t) = f_n t$. Note that $f_n = ip$, and so $i = ipt$. So $pt = 1$ since i is monic, and hence K is weak injective and p is an epimorphism. It follows that L and H are weak injective by Proposition 2.6(1).

Next we will show that the complex $0 \to E_{n-2} \stackrel{f_{n-2}}{\to} E_{n-3} \stackrel{f_{n-3}}{\to} \cdots \to E_1 \stackrel{f_1}{\to} E_0 \to N \to 0$ is a left $\mathcal{WI}$-resolution of N in ${}_R\mathcal{M}$. To do this, set $K' = \mathrm{Ker}\, f_{n-3}$ and let $\varepsilon : K' \to E_{n-3}$ be the inclusion. Then there exists $\varphi : E_{n-2} \to K'$ which is a $\mathcal{WI}$-cover of K' such that $f_{n-2} = \varepsilon\varphi$. So we have the following commutative diagram:



Since $f_{n-2} = q\beta = \varepsilon\varphi$ and β is epic, $\operatorname{Im} q = \operatorname{Im} f_{n-2} \subseteqq \operatorname{Ker} f_{n-3} = K'$. Then there exists $\psi : H \to K'$ such that $q = \varepsilon\psi$. So $\varepsilon\varphi = f_{n-2} = q\beta = \varepsilon\psi\beta$, and hence $\varphi = \psi\beta$ since ε is monic. Notice that $\varphi : E_{n-2} \to K'$ is a $\mathcal{WI}$-cover of K', so there exists $\beta' : H \to E_{n-2}$ such that $\psi = \varphi\beta'$. It follows that $\varphi = \psi\beta = \varphi\beta'\beta$ and that $\beta'\beta$ is an isomorphism since φ is a cover. Consequently, $\beta : E_{n-2} \to H$ is a monomorphism. But clearly $\operatorname{Im}\alpha = \operatorname{Ker}\beta = 0$, so $f_{n-1} = \alpha\pi = 0$. Let E be a weak injective left R-module. From the exactness of the following sequence

$$\operatorname{Hom}_R(E, E_{n-1}) \stackrel{\operatorname{Hom}_R(E,f_{n-1})}{\longrightarrow} \operatorname{Hom}_R(E, E_{n-2}) \stackrel{\operatorname{Hom}_R(E,f_{n-2})}{\longrightarrow} \operatorname{Hom}_R(E, E_{n-3}),$$

we get that $\operatorname{Ker}\operatorname{Hom}_R(E, f_{n-2}) = \operatorname{Im}\operatorname{Hom}_R(E, f_{n-1})$. It implies that

$$\operatorname{Ker}\operatorname{Hom}_R(E, f_{n-2}) = 0$$

since $f_{n-1} = 0$. Thus $0 \to E_{n-2} \stackrel{f_{n-2}}{\to} E_{n-3} \stackrel{f_{n-3}}{\to} \cdots \to E_1 \stackrel{f_1}{\to} E_0 \to N \to 0$ is a left $\mathcal{WI}$-resolution of N, and the assertion follows. □

By Propositions 4.9 and 4.10, we immediately have the following

THEOREM 4.11. *The following are equivalent for any* $n \geqq 2$.
(1) gl right $\mathcal{WI}$-$\dim_R \mathcal{M} \leqq n$.
(2) gl left $\mathcal{WI}$-$\dim_R \mathcal{M} \leqq n-2$.
(3) $\operatorname{Ext}^{\mathcal{WI}}_{n+k}(M, N) = 0$ *for any* $M, N \in {}_R\mathcal{M}$ *and* $k \geqq -1$.
(4) $\operatorname{Ext}^{\mathcal{WI}}_{n-1}(M, N) = 0$ *for any* $M, N \in {}_R\mathcal{M}$.

The following result is a generalization of [6, Lemma 8.4.34].

THEOREM 4.12. *The following statements are equivalent for any* $M \in {}_R\mathcal{M}$ *and* $n \geqq 0$.

(1) right $\mathcal{WI}$-dim $M \leqq n$.

(2) *For any left* $\mathcal{WI}$*-resolution* $\cdots \to E_n \to E_{n-1} \to \cdots \to E_0 \to N \to 0$ *of each* N *in* ${}_R\mathcal{M}$, $\operatorname{Hom}_R(M, E_n) \to \operatorname{Hom}_R(M, K_n) \to 0$ *is exact, where* K_n *is the nth* $\mathcal{WI}$*-syzygy of* N.

PROOF. We proceed by induction on n. Let $n = 0$. If M is weak injective, then it is clear that $\operatorname{Hom}_R(M, E_0) \to \operatorname{Hom}_R(M, K_0) \to 0$ is exact. Conversely, putting $N = M$, we have that $\operatorname{Hom}_R(M, E_0) \to \operatorname{Hom}_R(M, M)$ is surjective. So M is isomorphic to a direct summand of E_0, and hence M is weak injective.

Let $n \geqq 1$. Then, by [9, Theorem 3.2.1], there exists an exact sequence $0 \to M \to E \to L \to 0$ in ${}_R\mathcal{M}$ with E weak injective and $\operatorname{Ext}^1_R(L, G) = 0$

for any weak injective left R-module G. Thus we have the following exact commutative diagrams:

$$\begin{array}{ccccc}
\mathrm{Hom}_R(E,E_n) & \longrightarrow & \mathrm{Hom}_R(E,K_n) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}_R(M,E_n) & \longrightarrow & \mathrm{Hom}_R(M,K_n) & & \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

and

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(L,K_n) & \longrightarrow & \mathrm{Hom}_R(L,E_{n-1}) & \longrightarrow & \mathrm{Hom}_R(L,K_{n-1}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(E,K_n) & \longrightarrow & \mathrm{Hom}_R(E,E_{n-1}) & \longrightarrow & \mathrm{Hom}_R(E,K_{n-1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(M,K_n) & \longrightarrow & \mathrm{Hom}_R(M,E_{n-1}) & \longrightarrow & \mathrm{Hom}_R(M,K_{n-1}) & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

where $K_{n-1} = \mathrm{Im}(E_{n-1} \to E_{n-2})$. Then right $\mathcal{WI}$-dim $M \leqq n$ if and only if right $\mathcal{WI}$-dim $L \leqq n-1$ by Proposition 4.1(1) and [8, Proposition 3.3], if and only if $\mathrm{Hom}_R(L, E_{n-1}) \to \mathrm{Hom}_R(L, K_{n-1}) \to 0$ is exact by the induction hypothesis, if and only if $\mathrm{Hom}_R(E, K_n) \to \mathrm{Hom}_R(M, K_n)$ is surjective by the snake lemma, and if and only if $\mathrm{Hom}_R(M, E_n) \to \mathrm{Hom}_R(M, K_n)$ is surjective by the first diagram. □

In the following, we give some applications of Theorem 4.12.

COROLLARY 4.13. *The following statements are equivalent for any* $n \geqq 0$.

(1) right $\mathcal{WI}$-dim$({}_RR) \leqq n$.

(2) *Every left* $\mathcal{WI}$*-resolution* $\cdots \to E_n \to E_{n-1} \to \cdots \to E_0 \to N \to 0$ *of each* N *in* ${}_R\mathcal{M}$ *is exact at* E_i *for* $i \geqq n-1$, *where* $E_{-1} = N$.

PROOF. (1) ⇒ (2). If $n = 0$, then ${}_RR$ is weak injective. By Corollary 4.5, we have that every weak injective precover is surjective, and so $\cdots \to E_1 \to E_0 \to N \to 0$ is exact. Now suppose $n \geqq 1$. By Theorem 4.12, $\mathrm{Hom}_R({}_RR, E_n) \to \mathrm{Hom}_R({}_RR, K_n) \to 0$ is exact. So $E_n \to K_n$ is surjective, and hence $E_n \to E_{n-1} \to E_{n-2}$ is exact. Note that right $\mathcal{WI}$-$\dim({}_RR) \leqq k$ for $k \geqq n+1$. So $E_k \to E_{k-1} \to E_{k-2}$ is exact, as desired.

(2) ⇒ (1) is obvious by Theorem 4.12.

Recall that a left R-module M is called *pure-injective* if the functor $\mathrm{Hom}_R(-, M)$ preserves the exactness of any pure exact sequence in ${}_R\mathcal{M}$ (cf. [6,9]).

PROPOSITION 4.14. *The following statements are equivalent for any* $n \geqq 1$.

(1) gl right $\mathcal{WI}$-$\dim_R \mathcal{M} \leqq n$.

(2) *For every left* $\mathcal{WI}$*-resolution* $\cdots \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to N \to 0$ *of each* N *in* ${}_R\mathcal{M}$, $\mathrm{Hom}_R(M, E_n) \to \mathrm{Hom}_R(M, K_n) \to 0$ *is exact for any pure-injective left* R*-module* M, *where* K_n *is the* n*th* $\mathcal{WI}$*-syzygy of* N.

(3) l.sp.gldim$(R) < \infty$ *and every left* $\mathcal{WI}$*-resolution* $\cdots \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to N \to 0$ *of each* N *in* ${}_R\mathcal{M}$ *is exact at* E_i *for any* $i \geqq n-1$, *where* $E_{-1} = N$.

(4) l.sp.gldim$(R) < \infty$ *and the* n*th* $\mathcal{WI}$*-syzygy of each* N *in* ${}_R\mathcal{M}$ *has an epic weak injective cover.*

PROOF. (1) ⇒ (2). Let M be a pure-injective left R-module. Then right $\mathcal{WI}$-$\dim M \leqq n$ by (1). So the assertion follows directly from Theorem 4.12.

(2) ⇒ (1). For any $M \in \mathcal{M}_R$, M^+ is pure-injective by [6, Proposition 5.3.7]. So right $\mathcal{WI}$-$\dim M^+ \leqq n$ by (2) and Theorem 4.12. It follows from Proposition 4.1(1) that $\mathrm{wid}_R(M^+) \leqq n$. Let $\mathcal{F}$ be the class of super finitely presented left R-modules. Then

$$\begin{aligned} \mathrm{wfd}_R(M) \leqq n &\Leftrightarrow \mathrm{Tor}^R_{n+1}(M, F) = 0 \quad \text{for all } F \in \mathcal{F} \text{ by [8, Proposition 3.4]} \\ &\Leftrightarrow [\mathrm{Tor}^R_{n+1}(M, F)]^+ = 0 \quad \text{for all } F \in \mathcal{F} \\ &\Leftrightarrow \mathrm{Ext}^{n+1}_R(F, M^+) = 0 \quad \text{for all } F \in \mathcal{F} \\ &\Leftrightarrow \mathrm{wid}_R(M^+) \leqq n \qquad \text{by [8, Proposition 3.3].} \end{aligned}$$

So $\mathrm{wfd}_R(M) \leqq n$, and hence l.sp.gldim$(R) \leqq n$ by Lemma 2.8. Consequently, gl right $\mathcal{WI}$-$\dim_R \mathcal{M} \leqq n$ by Proposition 4.1(2).

(1) ⇒ (3). By (1) and Proposition 4.1(2), we have l.sp.gldim$(R) < \infty$. It is clear that right $\mathcal{WI}$-$\dim({}_RR) \leqq n$ by (1). Thus the assertion holds by Corollary 4.13.

(3) $\Rightarrow$ (4). By (3), we have that $E_k \to E_{k-1} \to E_{k-2}$ is exact for any $k \geqq n$. So the assertion follows immediately from Theorem 3.1.

(4) $\Rightarrow$ (1). Let $\cdots \to E_n \to E_{n-1} \to \cdots \to E_0 \to N \to 0$ be a left $\mathcal{WI}$-resolution of N in ${}_R\mathcal{M}$ and K_n the nth $\mathcal{WI}$-syzygy of N. Then $E_n \to K_n$ is surjective by assumption. It follows from Theorem 4.12 that right $\mathcal{WI}$-$\dim({}_RR) \leqq n$. So $\mathrm{wid}({}_RR) \leqq n$ by Proposition 4.1(1), and hence l.sp.gldim$(R) \leqq n$ by Proposition 4.2 since l.sp.gldim$(R) < \infty$. Therefore, gl right $\mathcal{WI}$-$\dim_R \mathcal{M} \leqq n$ by Proposition 4.1(2). $\square$

5. Right derived functors of ⊗ and right $\mathcal{WI}$-dimension of modules

In this section, we show that $- \otimes_R -$ is right balanced on $\mathcal{M}_R \times {}_R\mathcal{M}$ by $\mathcal{WF} \times \mathcal{WI}$, and characterize the right $\mathcal{WI}$-dimension of modules in terms of right derived functors of $\otimes$.

Since every module in $\mathcal{M}_R$ has a $\mathcal{WF}$-preenvelope by [8, Theorem 2.15], every module M in $\mathcal{M}_R$ has a right $\mathcal{WF}$-resolution, that is, there exists a $\mathrm{Hom}_R(-, \mathcal{WF})$ exact complex $0 \to M \to F^0 \to F^1 \to \cdots$ (not necessarily exact) with each F^i weak flat. On the other hand, every module N in ${}_R\mathcal{M}$ has a $\mathcal{WI}$-preenvelope by Theorem 3.4. So N has a right $\mathcal{WI}$-resolution, that is, there exists a $\mathrm{Hom}_R(-, \mathcal{WI})$ exact complex $0 \to N \to E^0 \to E^1 \to \cdots$ with each E^i weak injective. Clearly, this complex is exact.

Proposition 5.1. *$- \otimes_R -$ is right balanced on $\mathcal{M}_R \times {}_R\mathcal{M}$ by $\mathcal{WF} \times \mathcal{WI}$.*

Proof. Assume that $M \in \mathcal{M}_R$ and $0 \to M \to F^0 \to F^1 \to \cdots$ is a right $\mathcal{WF}$-resolution of M in $\mathcal{M}_R$. Let E be a weak injective left R-module. Then E^+ is a weak flat right R-module by [8, Theorem 2.10]. So we get the exact sequence:

$$\cdots \to \mathrm{Hom}_R\left(F^1, E^+\right) \to \mathrm{Hom}_R\left(F^0, E^+\right) \to \mathrm{Hom}_R(M, E^+) \to 0,$$

which gives the exact sequence:

$$\cdots \to \left(F^1 \otimes_R E\right)^+ \to \left(F^0 \otimes_R E\right)^+ \to (M \otimes_R E)^+ \to 0.$$

Thus we get the exact sequence $0 \to M \otimes_R E \to F^0 \otimes_R E \to F^1 \otimes_R E \to \cdots$.

On the other hand, let N be in ${}_R\mathcal{M}$ and let $0 \to N \to E^0 \to E^1 \to \cdots$ be a right $\mathcal{WI}$-resolution of N. Let F be a weak flat right R-module. Then F^+ is a weak injective left R-module by [8, Remark 2.2]. Similar to the proof above, we obtain that the sequence $0 \to F \otimes_R N \to F \otimes_R E^0 \to F \otimes_R E^1 \to \cdots$ is exact, as desired. $\square$

We denote by $\mathrm{Tor}^n_{\mathcal{W}}(-,-)$ the nth right derived functor of $-\otimes_R -$ with respect to $\mathcal{WF}\times\mathcal{WI}$. Then, for any $M\in\mathcal{M}_R$ and $N\in {}_R\mathcal{M}$, $\mathrm{Tor}^n_{\mathcal{W}}(M,N)$ can be computed by using a right $\mathcal{WF}$-resolution of M in $\mathcal{M}_R$ or a right $\mathcal{WI}$-resolution of N in ${}_R\mathcal{M}$. Let $0\to N\overset{g^0}{\to}E^0\overset{g^1}{\to}E^1\to\cdots$ be a right $\mathcal{WI}$-resolution of N in ${}_R\mathcal{M}$. Applying $M\otimes_R -$ to the sequence, we obtain the deleted complex:

$$0\to M\otimes_R E^0\overset{1_M\otimes_R g^1}{\longrightarrow} M\otimes_R E^1\to M\otimes_R E^2\to\cdots.$$

Then $\mathrm{Tor}^n_{\mathcal{W}}(M,N)$ is exactly the nth homology of the above complex. There exists a canonical homomorphism:

$$\tau:\ M\otimes_R N\to \mathrm{Tor}^0_{\mathcal{W}}(M,N)=\mathrm{Ker}\left(1_M\otimes_R g^1\right),$$

which is defined by $\tau\big(\Sigma(m_i\otimes_R n_i)\big)=\Sigma\big(m_i\otimes_R g^0(n_i)\big)$ for $\Sigma(m_i\otimes_R n_i)\in M\otimes_R N$.

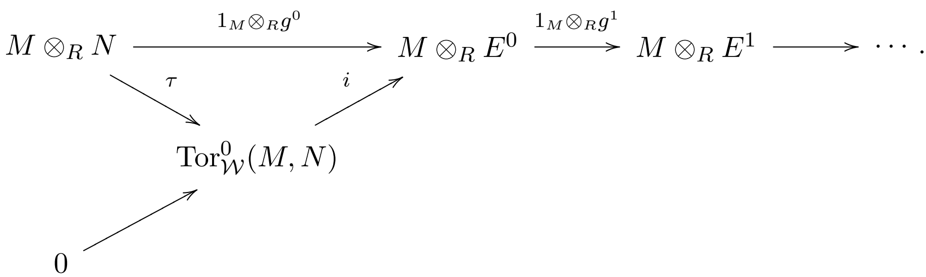
Proposition 5.2. *The following statements are equivalent for any $N\in{}_R\mathcal{M}$.*

(1) *N is weak injective.*

(2) *$\tau:\ M\otimes_R N\to\mathrm{Tor}^0_{\mathcal{W}}(M,N)$ is a monomorphism for any $M\in\mathcal{M}_R$.*

Proof. (1) $\Rightarrow$ (2) is clear by taking $E^0=N$.

(2) $\Rightarrow$ (1). Let $0\to N\overset{g^0}{\to}E^0\overset{g^1}{\to}E^1\to\cdots$ be a right $\mathcal{WI}$-resolution of N in ${}_R\mathcal{M}$. For any $M\in\mathcal{M}_R$, we have the following commutative diagram:



By assumption, $1_M\otimes_R g^0:\ M\otimes_R N\to M\otimes_R E^0$ is monic. It follows from [6, Definition 5.3.6] that N is a pure submodule of E^0. Thus N is weak injective by [8, Proposition 2.9]. $\square$

As a special case of Proposition 5.2, we have the following

Corollary 5.3. *The following statements are equivalent.*

(1) *${}_RR$ is weak injective.*

(2) *$\tau:\ M\to\mathrm{Tor}^0_{\mathcal{W}}(M,{}_RR)$ is a monomorphism for any $M\in\mathcal{M}_R$.*

Let $0 \to M \xrightarrow{h^0} F^0 \xrightarrow{h^1} F^1 \to \cdots$ be a right $\mathcal{WF}$-resolution of M in $\mathcal{M}_R$. Applying $- \otimes_R N$ to the sequence, we obtain the deleted complex:

$$0 \to F^0 \otimes_R N \xrightarrow{h^1 \otimes_R 1_N} F^1 \otimes_R N \to F^2 \otimes_R N \to \cdots .$$

Then $\mathrm{Tor}^n_{\mathcal{W}}(M,N)$ is exactly the nth homology of the above complex. There exists a canonical homomorphism:

$$\nu : \, M \otimes_R N \to \mathrm{Tor}^0_{\mathcal{W}}(M,N) = \mathrm{Ker}(h^1 \otimes_R 1_N),$$

which is defined by $\nu\big(\Sigma(m_i \otimes_R n_i)\big) = \Sigma\big(h^0(m_i) \otimes_R n_i\big)$ for $\Sigma(m_i \otimes_R n_i) \in M \otimes_R N$.

Proposition 5.4. *The following statements are equivalent for any $M \in \mathcal{M}_R$.*

(1) *M is weak flat.*

(2) *$\nu : \, M \otimes_R N \to \mathrm{Tor}^0_{\mathcal{W}}(M,N)$ is a monomorphism for any $N \in {}_R\mathcal{M}$.*

Proof. (1) $\Rightarrow$ (2) is obvious by taking $F^0 = M$.

(2) $\Rightarrow$ (1). Let $0 \to M \to F^0 \to F^1 \to \cdots$ be a right $\mathcal{WF}$-resolution of M in $\mathcal{M}_R$. Similar to the proof of (2) $\Rightarrow$ (1) in Proposition 5.2, we get that $0 \to M \to F^0$ is pure exact. Therefore, M is weak flat by [8, Proposition 2.9]. $\square$

Proposition 5.5. *The following statements are equivalent for any $N \in {}_R\mathcal{M}$.*

(1) right $\mathcal{WI}$-dim $N \leqq 1$.

(2) *$\tau : \, M \otimes_R N \to \mathrm{Tor}^0_{\mathcal{W}}(M,N)$ is an epimorphism for any $M \in \mathcal{M}_R$.*

Proof. (1) $\Rightarrow$ (2). Let $0 \to N \to E^0 \to E^1 \to 0$ be a right $\mathcal{WI}$-resolution of N in ${}_R\mathcal{M}$. For any $M \in \mathcal{M}_R$, $M \otimes_R N \to M \otimes_R E^0 \to M \otimes_R E^1 \to 0$ is exact. It follows that $\tau : \, M \otimes_R N \to \mathrm{Tor}^0_{\mathcal{W}}(M,N)$ is an epimorphism, as desired.

(2) $\Rightarrow$ (1). Let $0 \to N \to E \to L \to 0$ be an exact sequence in ${}_R\mathcal{M}$ with $N \to E$ a weak injective preenvelope of N. It suffices to show that L is weak injective. By [6, Theorem 8.2.5], we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & M \otimes_R N & \longrightarrow & M \otimes_R E & \longrightarrow & M \otimes_R L & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_3} & \\
0 \longrightarrow & \mathrm{Tor}^0_{\mathcal{W}}(M,N) & \longrightarrow & \mathrm{Tor}^0_{\mathcal{W}}(M,E) & \longrightarrow & \mathrm{Tor}^0_{\mathcal{W}}(M,L). &
\end{array}$$

Since τ_2 is a monomorphism by Proposition 5.2 and τ_1 is an epimorphism by assumption, τ_3 is a monomorphism by the snake lemma. It follows from Proposition 5.2 that L is weak injective. $\square$

LEMMA 5.6. *Let $M_1 \to M_2 \to M_3 \to M_4$ be an exact sequence in ${}_R\mathcal{M}$. If $G \otimes_R M_1 \to G \otimes_R M_2 \to G \otimes_R M_3$ is exact for any $G \in \mathcal{M}_R$, then $K = \mathrm{Ker}(M_3 \to M_4)$ is a pure submodule of M_3.*

PROOF. Let $M_1 \overset{f_1}{\to} M_2 \overset{f_2}{\to} M_3 \overset{f_3}{\to} M_4$ be an exact sequence in ${}_R\mathcal{M}$ and $K = \mathrm{Ker}\, f_3$. Then there exist an epimorphism $\pi : M_2 \to K$ and a monomorphism $i : K \to M_3$ such that $f_2 = i\pi$. Let G be in $\mathcal{M}_R$. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
G \otimes_R M_1 & \xrightarrow{1_G \otimes_R f_1} & G \otimes_R M_2 & \xrightarrow{1_G \otimes_R f_2} & G \otimes_R M_3. \\
 & & {\scriptstyle 1_G \otimes_R \pi} \searrow & & \nearrow {\scriptstyle 1_G \otimes_R i} \\
 & & & G \otimes_R K & \\
 & & & & \searrow \\
 & & & & 0
\end{array}$$

Let $x \in \mathrm{Ker}(1_G \otimes_R i)$. There exists $y \in G \otimes_R M_2$ such that $(1_G \otimes_R \pi)(y) = x$. Then $y \in \mathrm{Ker}(1_G \otimes_R f_2)$. Note that $\mathrm{Ker}(1_G \otimes_R f_2) = \mathrm{Im}(1_G \otimes_R f_1) = \mathrm{Ker}(1_G \otimes_R \pi)$. So $x = (1_G \otimes_R \pi)(y) = 0$, and hence $0 \to G \otimes_R K \to G \otimes_R M_3$ is exact. It follows that K is a pure submodule of M_3. $\square$

Now we are in a position to prove the following

THEOREM 5.7. *The following are equivalent for any $N \in {}_R\mathcal{M}$ and $n \geqq 2$.*
(1) right $\mathcal{WI}$-dim $N \leqq n$.
(2) $\mathrm{Tor}_{\mathcal{W}}^{n+k}(M, N) = 0$ *for any* $M \in \mathcal{M}_R$ *and* $k \geqq -1$.
(3) $\mathrm{Tor}_{\mathcal{W}}^{n}(M, N) = \mathrm{Tor}_{\mathcal{W}}^{n-1}(M, N) = 0$ *for any* $M \in \mathcal{M}_R$.
(4) $\mathrm{Tor}_{\mathcal{W}}^{n-1}(M, N) = 0$ *for any* $M \in \mathcal{M}_R$.

PROOF. (1) $\Rightarrow$ (2). Let $0 \to N \to E^0 \to \cdots \to E^{n-1} \to E^n \to 0$ be a right $\mathcal{WI}$-resolution of N in ${}_R\mathcal{M}$. Then the sequence $M \otimes_R E^{n-2} \to M \otimes_R E^{n-1} \to M \otimes_R E^n \to 0$ is exact for any $M \in \mathcal{M}_R$. It follows that $\mathrm{Tor}_{\mathcal{W}}^{n-1}(M, N) = \mathrm{Tor}_{\mathcal{W}}^{n}(M, N) = 0$. It is obvious that $\mathrm{Tor}_{\mathcal{W}}^{n+k}(M, N) = 0$ for any $k \geqq 1$, and the assertion follows.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1). Let $0 \to N \to E^0 \to E^1 \to \cdots$ be a right $\mathcal{WI}$-resolution of N in ${}_R\mathcal{M}$ and $L = \mathrm{Ker}(E^n \to E^{n+1})$. By assumption, $\mathrm{Tor}_{\mathcal{W}}^{n-1}(M, N) = 0$ for

any $M \in \mathcal{M}_R$. Then we get the following exact sequence:

$$M \otimes_R E^{n-2} \to M \otimes_R E^{n-1} \to M \otimes_R E^n.$$

So L is a pure submodule of E^n by Lemma 5.6, and hence L is weak injective by [8, Proposition 2.9]. Therefore, $0 \to N \to E^0 \to E^1 \to \cdots \to E^{n-1} \to L \to 0$ is a right $\mathcal{WI}$-resolution of N. The proof is finished. □

By Theorem 5.7, we immediately get the following

Corollary 5.8. *The following are equivalent for any* $n \geqq 2$.

(1) $\operatorname{gl\,right} \mathcal{WI}\text{-}\dim_R \mathcal{M} \leqq n$.

(2) $\operatorname{Tor}_{\mathcal{W}}^{n+k}(M, N) = 0$ *for any* $M \in \mathcal{M}_R$, $N \in {}_R\mathcal{M}$ *and* $k \geqq -1$.

(3) $\operatorname{Tor}_{\mathcal{W}}^{n-1}(M, N) = 0$ *for any* $M \in \mathcal{M}_R$ *and* $N \in {}_R\mathcal{M}$.

Acknowledgements. This research was partially supported by National Natural Science Foundation of China (11301042 and 11171142), China Postdoctoral Science Foundation (No. 2014M550279) and by the Scientific Research Foundation of CUIT (J201217). The authors thank the referee for the useful suggestions.

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