The Extension Dimension of Subcategories and Recollements of Abelian Categories *†‡

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Abstract

We investigate the behavior of the extension dimension of subcategories of abelian categories under recollements. Let $\Lambda', \Lambda, \Lambda''$ be artin algebras such that $(\operatorname{mod}\Lambda', \operatorname{mod}\Lambda, \operatorname{mod}\Lambda'')$ is a recollement, and let $\mathcal{D}'$ and $\mathcal{D}''$ be subcategories of $\operatorname{mod}\Lambda'$ and $\operatorname{mod}\Lambda''$ respectively. For any $n, m \geq 0$, under some conditions, we get $\dim \Omega^k(\mathcal{D}) \leq \dim \Omega^n(\mathcal{D}') + \dim \Omega^m(\mathcal{D}'') + 1$, where $k = \max\{m, n\}$ and $\mathcal{D}$ is the subcategory of $\operatorname{mod}\Lambda$ glued by $\mathcal{D}'$ and $\mathcal{D}''$; moreover, we give a sufficient condition such that the converse inequality holds true. As applications, some results for Igusa-Todorov subcategories and syzygy finite subcategories are obtained.

1 Introduction

Given a triangulated category $\mathcal{T}$, Rouquier introduced in [25, 26] the dimension of $\mathcal{T}$ under the idea of Bondal and van den Bergh in [7]. This dimension and the infimum of the Orlov spectrum of $\mathcal{T}$ coincide, see [4, 20]. This dimension plays an important role in representation theory. For example, it can be used to compute the representation dimension of artin algebras ([25, 19]). As an analogue of the dimension of triangulated categories, the extension dimension $\dim_{\mathcal{A}} \mathcal{D}$ of a subcategory $\mathcal{D}$ of an abelian category $\mathcal{A}$ was introduced by Beligiannis in [5], also see [10]. Let Λ be an artin algebra. Note that the representation dimension of Λ is at most two (that is, Λ is of finite representation type) if and only if $\dim \operatorname{mod}\Lambda (:= \dim_{\operatorname{mod}\Lambda} \operatorname{mod}\Lambda) = 0$ ([5]). So, like the representation dimension of Λ, the extension dimension $\dim \operatorname{mod}\Lambda$ is also an invariant that measures how far Λ is from of finite representation type. It was shown that the extension dimension is useful in studying the representation type of algebras and finitistic dimension conjecture ([33]).

Recollements of triangulated and abelian categories were introduced in [6, 11] in connection with derived categories of sheaves on topological spaces with the idea that one triangulated category may be "glued together" from two others. Recollements provide a useful reduction technique for some homological properties such as the finiteness of global dimension and finitistic dimension [9, 13, 21, 29], the Gorensteinness [1, 12, 17, 24] and the representation type and representation dimension of artin algebras as well as the extension dimension of abelian categories [21, 33], and so on. Following the above philosophy, we will study the behavior of the extension dimension of certain subcategories of an abelian category under recollements.

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For an artin algebra Λ, we use $\operatorname{mod}\Lambda$ to denote the category of finitely generated left Λ-modules. Let Λ', Λ and Λ'' be artin algebras such that there is a recollement of module categories:

$$\operatorname{mod}\Lambda' \begin{array}{c}\xleftarrow{\;i^*\;}\\ \xrightarrow{\;i_*\;}\\ \xleftarrow{\;i^!\;}\end{array} \operatorname{mod}\Lambda \begin{array}{c}\xleftarrow{\;j_!\;}\\ \xrightarrow{\;j^*\;}\\ \xleftarrow{\;j_*\;}\end{array} \operatorname{mod}\Lambda'' .$$

Our main results are as follows.

Theorem 1.1. (Theorem 3.8) *Let* $(\operatorname{mod}\Lambda', \operatorname{mod}\Lambda, \operatorname{mod}\Lambda'')$ *be a recollement, and let* $\mathcal{D}'$ *and* $\mathcal{D}''$ *be subcategories of* $\operatorname{mod}\Lambda'$ *and* $\operatorname{mod}\Lambda''$ *respectively. Assume that* $i^!$ *is exact. For any* $n, m \geq 0$*, if one of the following conditions holds:*

(1) $m = 0$,

(2) $m \geq 1$ *and* $i^! j_!$ *preserves projective objects,*

then

$$\dim_{\operatorname{mod}\Lambda} \Omega^k(\mathcal{D}) \leq \dim_{\operatorname{mod}\Lambda'} \Omega^n(\mathcal{D}') + \dim_{\operatorname{mod}\Lambda''} \Omega^m(\mathcal{D}'') + 1,$$

where $k = \max\{m, n\}$ *and* $\mathcal{D} = \{D \in \operatorname{mod}\Lambda \mid i^!(D) \in \mathcal{D}'$ *and* $j^*(D) \in \mathcal{D}''\}$.

Moreover, we have the following

Theorem 1.2. (Theorem 3.12) *Let* $(\operatorname{mod}\Lambda', \operatorname{mod}\Lambda, \operatorname{mod}\Lambda'')$ *be a recollement, and let* $\mathcal{D}$ *be a subcategory of* $\operatorname{mod}\Lambda$ *with* $i_* i^!(\mathcal{D}) \subseteq \mathcal{D}$ *and* $j_* j^*(\mathcal{D}) \subseteq \mathcal{D}$*. If* $i^!$ *is exact, then*

$$\max\{\dim_{\operatorname{mod}\Lambda'} \Omega^n(i^!(\mathcal{D})), \dim_{\operatorname{mod}\Lambda''} \Omega^n(j^*(\mathcal{D}))\} \leq \dim_{\operatorname{mod}\Lambda} \Omega^n(\mathcal{D})$$

for some $n \geq 0$.

Then we apply these results to Igusa-Todorov subcategories and syzygy finite subcategories. Some known results are obtained as corollaries. Finally, we give some examples to illustrate the obtained results.

Throughout this paper, all abelian categories have enough projective and injective objects and all subcategories are full, additive and closed under isomorphisms. All algebras are artin algebras. Finally, we recall the notion of upper triangular matrix artin algebras. Let Λ', Λ'' be artin algebras and ${}_{\Lambda'}M_{\Lambda''}$ an (Λ', Λ'')-bimodule such that ${}_{\Lambda'}M$ and $M_{\Lambda''}$ are finitely generated, and let $\Lambda = \left(\begin{smallmatrix} \Lambda' & M \\ 0 & \Lambda'' \end{smallmatrix}\right)$ be a triangular matrix algebra. Then Λ is an artin algebra ([3, Proposition III.2.1]). A module in $\operatorname{mod}\Lambda$ can be uniquely written as a triple $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_f$ with $X \in \operatorname{mod}\Lambda'$, $Y \in \operatorname{mod}\Lambda''$ and $f \in \operatorname{Hom}_{\Lambda'}(M \otimes_{\Lambda''} Y, X)$ ([3, p.76]).

2 Preliminaries

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{D}$ be a class of objects in $\mathcal{A}$. We use $\operatorname{add}\mathcal{D}$ to denote the subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of objects in $\mathcal{D}$.

Let $\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n$ be classes of objects in $\mathcal{A}$. Define

$$\begin{aligned}\mathcal{U}_1 \diamond \mathcal{U}_2 := \operatorname{add}\{A \in \mathcal{A} \mid &\text{ there exists an exact sequence} \\ &0 \longrightarrow U_1 \longrightarrow A \longrightarrow U_2 \longrightarrow 0 \ \text{ in } \mathcal{A} \text{ with } U_1 \in \mathcal{U}_1 \text{ and } U_2 \in \mathcal{U}_2\}.\end{aligned}$$

Inductively, define

$$\mathcal{U}_1 \diamond \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n := \operatorname{add}\{A \in \mathcal{A} \mid \text{there exists an exact sequence } 0 \longrightarrow U \longrightarrow A \longrightarrow V \longrightarrow 0 \\ \text{in } \mathcal{A} \text{ with } U \in \mathcal{U}_1 \text{ and } V \in \mathcal{U}_2 \diamond \cdots \diamond \mathcal{U}_n\}.$$

For a class $\mathcal{U}$ of $\mathcal{A}$, set $\langle \mathcal{U} \rangle_0 := 0$, $\langle \mathcal{U} \rangle_1 := \operatorname{add} \mathcal{U}$, $\langle \mathcal{U} \rangle_n := \langle \mathcal{U} \rangle_1 \diamond \langle \mathcal{U} \rangle_{n-1}$ for any $n \geq 2$, and $\langle \mathcal{U} \rangle_\infty := \bigcup_{n \geq 0} \langle \mathcal{U} \rangle_n$ ([5]). For subcategories $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{A}$, by [10, Proposition 2.2], we have

$$(\mathcal{U} \diamond \mathcal{V}) \diamond \mathcal{W} = \mathcal{U} \diamond (\mathcal{V} \diamond \mathcal{W}).$$

Definition 2.1. ([5, 10, 33]) For a subcategory $\mathcal{D}$ of $\mathcal{A}$, the *extension dimension* $\dim_{\mathcal{A}} \mathcal{D}$ of $\mathcal{D}$ is defined as

$$\dim_{\mathcal{A}} \mathcal{D} := \inf\{n \geq 0 \mid \mathcal{D} \subseteq \langle T \rangle_{n+1} \text{ with } T \in \mathcal{A}\}.$$

When there is no ambiguity, we write $\dim \mathcal{D} := \dim_{\mathcal{A}} \mathcal{D}$ for short.

Let $\mathcal{A}$ be an abelian category, and let $M \in \mathcal{A}$ and $m \geq 0$. We use $\Omega^m_{\mathcal{A}}(M)$ to denote the m-th *syzygy* of M; in particular, $\Omega^0_{\mathcal{A}}(M) = M$. Let $\mathcal{D}$ be a subcategory of $\mathcal{A}$. We use $\Omega^m_{\mathcal{A}}(\mathcal{D})$ to denote the full subcategory of $\mathcal{A}$ consisting of those objects in $\mathcal{A}$ that are either projective or direct summands of m-th syzygies of objects in $\mathcal{D}$. Dually, the m-th *cosyzygy* $\Omega^{-m}_{\mathcal{A}}(M)$ of M and the subcategory $\Omega^{-m}_{\mathcal{A}}(\mathcal{D})$ are defined.

Lemma 2.2. *Let $\mathcal{A}$ be an abelian category and let $X, T \in \mathcal{A}$. If $X \in \langle T \rangle_n$, then for any $n \geq 1$ and $i \geq 0$, we have*

(1) $\Omega^i_{\mathcal{A}}(X) \in \langle \Omega^i_{\mathcal{A}}(T) \rangle_n$.

(2) $\Omega^{-i}_{\mathcal{A}}(X) \in \langle \Omega^{-i}_{\mathcal{A}}(T) \rangle_n$.

Immediately, we get the following result.

Lemma 2.3. *Let $\mathcal{A}$ be an abelian category and $\mathcal{D}$ a subcategory of $\mathcal{A}$. Then for any $m \geq n \geq 0$, we have* $\dim \Omega^m_{\mathcal{A}}(\mathcal{D}) \leq \dim \Omega^n_{\mathcal{A}}(\mathcal{D})$.

Lemma 2.4. *Let $\mathcal{A}$ be an abelian category and $n \geq 1$, and let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence in $\mathcal{A}$. If there exist $T_1, T_2 \in \mathcal{A}$ such that $A \in \langle T_1 \rangle_n$ and $B \in \operatorname{add} T_2$, then $C \in \langle T_2 \oplus \Omega^{-1}_{\mathcal{A}}(T_1) \rangle_{n+1}$.

Proof. By [33, Lemma 3.2], we have the following exact sequence

$$0 \longrightarrow B \longrightarrow C \oplus I \longrightarrow \Omega^{-1}_{\mathcal{A}}(A) \longrightarrow 0$$

in $\mathcal{A}$ with I injective. Then, by Lemma 2.2 and [33, Proposition 2.2(1) and Corollary 2.3(1)], we have

$$C \in \langle T_2 \rangle_1 \diamond \langle \Omega^{-1}_{\mathcal{A}}(T_1) \rangle_n \subseteq \langle T_2 \oplus \Omega^{-1}_{\mathcal{A}}(T_1) \rangle_{n+1}.$$

□

We need the following easy and useful fact.

Lemma 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $n \geq 0$, and let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. If F preserves projective objects, then $\Omega^n_{\mathcal{B}}(F(X)) = F(\Omega^n_{\mathcal{A}}(X))$ for any $X \in \mathcal{A}$.*

Proof. For any $X \in \mathcal{A}$, consider the following exact sequence

$$0 \longrightarrow \Omega^n_{\mathcal{A}}(X) \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

in $\mathcal{A}$ with all P_i projective. Applying the functor F to it yields an exact sequence

$$0 \longrightarrow F(\Omega^n_{\mathcal{A}}(X)) \longrightarrow F(P_{n-1}) \longrightarrow F(P_{n-2}) \longrightarrow \cdots \longrightarrow F(P_0) \longrightarrow F(X) \longrightarrow 0$$

in $\mathcal{B}$ with all $F(P_i)$ projective by assumption. Thus $\Omega^n_{\mathcal{B}}(F(X)) = F(\Omega^n_{\mathcal{A}}(X))$. □

The following definition is cited from [11].

Definition 2.6. A recollement, denoted by $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, of abelian categories is a diagram

$$\begin{array}{ccccc} & \xleftarrow{\ i^*\ } & & \xleftarrow{\ j_!\ } & \\ \mathcal{A} & \xrightarrow{\ i_*\ } & \mathcal{B} & \xrightarrow{\ j^*\ } & \mathcal{C} \\ & \xleftarrow{\ i^!\ } & & \xleftarrow{\ j_*\ } & \end{array}$$

of abelian categories and additive functors such that

(1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs.

(2) i_*, $j_!$ and j_* are fully faithful.

(3) $\operatorname{Im} i_* = \operatorname{Ker} j^*$.

In the rest of this section, we assume that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a recollement of abelian categories as in Definition 2.6. We list some properties of such recollements (see [11, 16], [18]–[23] and [33]), which will be used in the sequel.

Lemma 2.7. *We have*

(1) $i^* j_! = 0 = i^! j_*$.

(2) The functors i_*, j^* are exact, and the functors i^*, $j_!$ are right exact, and the functors $i^!$, j_* are left exact.

(3) All natural transformations $i^* i_* \longrightarrow 1_{\mathcal{A}}$, $1_{\mathcal{A}} \longrightarrow i^! i_*$, $1_{\mathcal{C}} \longrightarrow j^* j_!$, and $j^* j_* \longrightarrow 1_{\mathcal{C}}$ are natural isomorphisms. Moreover, all functors i^*, $i^!$ and j^* are dense.

(4) For any object $X \in \mathcal{B}$, if i^* is exact, there is the following exact sequence

$$0 \longrightarrow j_! j^*(X) \longrightarrow X \longrightarrow i_* i^*(X) \longrightarrow 0$$

in $\mathcal{B}$; and if $i^!$ is exact, then there is the following exact sequence

$$0 \longrightarrow i_* i^!(X) \longrightarrow X \longrightarrow j_* j^*(X) \longrightarrow 0$$

in $\mathcal{B}$.

(5) If i^* is exact, then $i^! j_! = 0$ and $j_!$ is exact; and if $i^!$ is exact, then $i^* j_* = 0$ and j_* is exact.

(6) The functors i^* and $j_!$ preserve projective objects. If $i^!$ is exact, then i_* and j^* preserve projective objects.

We get the following observation.

Lemma 2.8. *We have the following assertions.*

(1) If $i^!$ is exact, then there is an exact sequence of natural transformations

$$0 \longrightarrow i_* i^! j_! \longrightarrow j_! \longrightarrow j_* \longrightarrow 0\ .$$

(2) If i^* is exact, then there is an exact sequence of natural transformations

$$0 \longrightarrow j_! \longrightarrow j_* \longrightarrow i_* i^* j_* \longrightarrow 0\ .$$

Proof. (1) By [11, Proposition 4.4], there is an exact sequence of natural transformations

$$0 \longrightarrow i_* i^! j_! \longrightarrow j_! \longrightarrow j_* \longrightarrow i_* i^* j_* \longrightarrow 0\ .$$

Since $i^!$ is exact, we have $i^* j_* = 0$ by Lemma 2.7, and thus the assertion follows.

(2) It is a dual of (1). □

As a consequence of Lemmas 2.7 and 2.8, we get the following

Remark 2.9. *Let*

$$0 \longrightarrow C'' \longrightarrow C' \longrightarrow C \longrightarrow 0$$

be an exact sequence in $\mathcal{C}$. Assume that $i^!$ is exact. By Lemma 2.7, we have that j_* is exact. So

$$0 \longrightarrow j_*(C'') \longrightarrow j_*(C') \longrightarrow j_*(C) \longrightarrow 0$$

is exact in $\mathcal{B}$. By Lemma 2.8, we have the following exact sequence

$$0 \longrightarrow i_* i^! j_!(C') \longrightarrow j_!(C') \longrightarrow j_*(C') \longrightarrow 0$$

in $\mathcal{B}$. One can get the following pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & i_* i^! j_!(C') & = & i_* i^! j_!(C') & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & K & \dashrightarrow & j_!(C') & \dashrightarrow & j_*(C) & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \parallel & & \\
0 & \longrightarrow & j_*(C'') & \longrightarrow & j_*(C') & \longrightarrow & j_*(C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

The following result generalizes [8, Lemma 4.2], which is useful in the sequel.

Lemma 2.10. *For any $n \geq 1$, assume that $i^!$ is exact and $i^! j_!$ preserves projective objects. Then, for any $X \in \mathcal{B}$, there exists an exact sequence*

$$0 \longrightarrow i_* i^! j_!(P_{n-1}) \longrightarrow \Omega^n_{\mathcal{B}}(j_* j^*(X)) \longrightarrow j_*(\Omega^n_{\mathcal{C}}(j^*(X))) \longrightarrow 0$$

in $\mathcal{B}$, where P_{n-1}, a projective object in $\mathcal{C}$, lies in the exact sequence

$$0 \longrightarrow \Omega^n_{\mathcal{C}}(j^*(X)) \longrightarrow P_{n-1} \longrightarrow \Omega^{n-1}_{\mathcal{C}}(j^*(X)) \longrightarrow 0 \ .$$

Proof. Notice that $j^*(X) \in \mathcal{C}$, consider the following exact sequence

$$0 \longrightarrow \Omega^1_{\mathcal{C}}(j^*(X)) \longrightarrow P_0 \longrightarrow j^*(X) \longrightarrow 0$$

in $\mathcal{C}$ with P_0 a projective object. By Remark 2.9, we get the following pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & i_* i^! j_!(P_0) & = & i_* i^! j_!(P_0) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & K_1 & \dashrightarrow & j_!(P_0) & \dashrightarrow & j_* j^*(X) & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & j_*(\Omega^1_{\mathcal{C}}(j^*(X))) & \longrightarrow & j_*(P_0) & \longrightarrow & j_* j^*(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Since $j_!$ preserves projective objects by Lemma 2.7, $j_!(P_0)$ is a projective object in $\mathcal{B}$. So $\Omega^1_{\mathcal{B}}(j_* j^*(X)) = K_1$ and the assertion for $n = 1$ follows.

Now applying Remark 2.9 to the exact sequence

$$0 \longrightarrow \Omega^2_{\mathcal{C}}(j^*(X)) \longrightarrow P_1 \longrightarrow \Omega^1_{\mathcal{C}}(j^*(X)) \longrightarrow 0$$

in $\mathcal{C}$ with P_1 projective yields the following pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & i_* i^! j_!(P_1) & = & i_* i^! j_!(P_1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & K_2 & \dashrightarrow & j_!(P_1) & \dashrightarrow & j_*(\Omega^1_{\mathcal{C}}(j^*(X))) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & j_*(\Omega^2_{\mathcal{C}}(j^*(X))) & \longrightarrow & j_*(P_1) & \longrightarrow & j_*(\Omega^1_{\mathcal{C}}(j^*(X))) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

We also get the following pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_2 & = & K_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & i_*i^!j_!(P_0) & \to & i_*i^!j_!(P_0)\oplus j_!(P_1) & \to & j_!(P_1) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & i_*i^!j_!(P_0) & \to & K_1 & \to & j_*(\Omega^1_{\mathcal{C}}(j^*(X))) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Notice that i_* and $i^!j_!$ preserve projective objects by Lemma 2.7 and assumption, so $\Omega^2_{\mathcal{B}}(j_*j^*(X)) = K_2$. Repeating this process, we can get the desired exact sequence

$$0 \longrightarrow i_*i^!j_!(P_{n-1}) \longrightarrow \Omega^n_{\mathcal{B}}(j_*j^*(X)) \longrightarrow j_*(\Omega^n(j^*(X))) \longrightarrow 0$$

in $\mathcal{B}$, where P_{n-1}, a projective object, lies in the exact sequence

$$0 \longrightarrow \Omega^n_{\mathcal{C}}(j^*(X)) \longrightarrow P_{n-1} \longrightarrow \Omega^{n-1}_{\mathcal{C}}(j^*(X)) \longrightarrow 0\ .$$

□

3 Main results

Let Λ be an artin algebra, and $\mathcal{D}$ a subcategory of $\operatorname{mod}\Lambda$.

(1) $\mathcal{D}$ is said to be of *finite representation type*, if there is some $N \in \operatorname{mod}\Lambda$ such that $\operatorname{add}\mathcal{D} = \operatorname{add} N$; that is, the number of non-isomorphic indecomposable Λ-modules appeared in $\mathcal{D}$ is finite. In particular, if $\mathcal{D} = \operatorname{mod}\Lambda$, it is said that Λ is of *finite representation type* (see [2]).

(2) $\mathcal{D}$ is said to be *m-syzygy finite* if the subcategory $\Omega^m(\mathcal{D}) := \Omega^m_{\operatorname{mod}\Lambda}(\mathcal{D})$ is of finite representation type. In particular, if $\mathcal{D} = \operatorname{mod}\Lambda$, it is said that Λ is an *m-syzygy finite algebra* (see [27]).

Remark 3.1. *A subcategory $\mathcal{D} \subseteq \operatorname{mod}\Lambda$ is n-syzygy finite if and only if $\dim \Omega^n(\mathcal{D}) = 0$. In particular, Λ is of finite representation type if and only if $\dim \operatorname{mod}\Lambda = 0$* ([33, Corollary 3.8]).

Definition 3.2. ([28, Definition 3.1]) Let $\mathcal{D}$ be a subcategory of $\operatorname{mod}\Lambda$. Then $\mathcal{D}$ is said to be (n-)*Igusa-Todorov* provided that there exist $V \in \operatorname{mod}\Lambda$ and $n \geq 0$, such that for any $M \in \Omega^n(\mathcal{D})$, there is an exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow M \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ with $V_1, V_0 \in \operatorname{add} V$. The module V is then called an (n-)*$\mathcal{D}$-Igusa-Todorov module.*

In particular, if $\mathcal{D} = \operatorname{mod}\Lambda$, it is said that Λ is an *(n-)Igusa-Todorov algebra* and the module V is then called an (n-)-Igusa-Todorov module (see [27] and [14, Lemma 3.6]).

The following result generalizes [33, Theorem 3.14], which gives an equivalent characterization of (n-)Igusa-Todorov subcategories and means that $\dim \Omega^n(\mathrm{mod}\,\Lambda)$ is an invariant for measuring how far a subcategory of $\mathrm{mod}\,\Lambda$ is from being Igusa-Todorov.

Proposition 3.3. *Let $\mathcal{D}$ be a subcategory of $\mathrm{mod}\,\Lambda$. Then for any $n \geq 0$, the following statements are equivalent.*

(1) $\mathcal{D}$ is n-Igusa-Todorov.

(2) $\dim \Omega^n(\mathcal{D}) \leq 1$.

Proof. (1) $\Rightarrow$ (2) Assume that $\mathcal{D}$ is n-Igusa-Todorov. Let $X \in \Omega^n(\mathcal{D})$. Then there exists $V \in \mathrm{mod}\,\Lambda$ such that the following sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow X \longrightarrow 0$$

in $\mathrm{mod}\,\Lambda$ with $V_1, V_0 \in \mathrm{add}\,V$ is exact. By Lemma 2.4, we have

$$X \in \langle V \oplus \Omega^{-1}(V) \rangle_2.$$

Thus $\dim \Omega^n(\mathcal{D}) \leq 1$.

(2) $\Rightarrow$ (1) Assume that $\dim \Omega^n(\mathcal{D}) \leq 1$ and $X \in \mathcal{D}$. Then there exists $V \in \mathrm{mod}\,\Lambda$ such that the following sequence

$$0 \longrightarrow V_1 \longrightarrow \Omega^n(X) \longrightarrow V_2 \longrightarrow 0,$$

in $\mathrm{mod}\,\Lambda$ with $V_1, V_2 \in \langle V \rangle_1$ is exact. By [33, Lemma 3.2], we obtain the following exact sequence

$$0 \longrightarrow \Omega^1(V_2) \longrightarrow V_1 \oplus P \longrightarrow \Omega^n(X) \longrightarrow 0$$

in $\mathrm{mod}\,\Lambda$ with P projective. Notice that both $\Omega^1(V_2)$ and $V_1 \oplus P$ are in $\mathrm{add}(\Omega^1(V) \oplus V \oplus \Lambda)$, so $\mathcal{D}$ is n-Igusa-Todorov. □

As an immediate consequence, we get the following

Corollary 3.4. ([28, Proposition 3.4]) *Let $\mathcal{D}$ be a subcategory of $\mathrm{mod}\,\Lambda$. If $\mathcal{D}$ is n-Igusa-Todorov, then $\Omega^i(\mathcal{D})$ is also n-Igusa-Todorov for any $i \geq 1$. In particular, $\mathcal{D}$ is m-Igusa-Todorov for $m \geq n$ in that case.*

Proof. It follows from Lemma 2.3 and Proposition 3.3. □

Remark 3.5. ([27, Remark 2.4]) *If Λ is an n-Igusa-Todorov algebra, then Λ is also an m-Igusa-Todorov algebra for any $m \geq n$.*

The following result generalizes [33, Proposition 3.15].

Proposition 3.6. *Let $\mathcal{D}$ be a subcategory of $\mathrm{mod}\,\Lambda$ and $m, n \geq 0$. If $\dim \Omega^n(\mathcal{D}) \leq m$, then $\dim \Omega^{n-i}(\mathcal{D}) \leq m + i$ for any $0 \leq i \leq n$. In particular, if $\mathcal{D}$ is n-Igusa-Todorov, then $\dim \mathcal{D} \leq n + 1$.*

Proof. Let $X \in \mathcal{D}$. If $n = 0$ (also $i = 0$), then $\dim \mathcal{D} \leq m$. Now suppose $n \geq 1$. Consider the following exact sequence

$$0 \longrightarrow \Omega^n(X) \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_{n-i} \longrightarrow P_{n-i-1} \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

$$P_{n-i} \searrow \Omega^{n-i}(X) \nearrow P_{n-i-1}$$

in $\operatorname{mod} \Lambda$ with all P_l projective. Using Lemma 2.4 repeatedly, we have $\Omega^{n-i}(X) \in \langle T \rangle_{m+i+1}$ for some $T \in \operatorname{mod} \Lambda$. Thus $\dim \Omega^{n-i}(\mathcal{D}) \leq m + i$. The last assertion follows from Proposition 3.3. □

We have the following

Remark 3.7. Let $\mathcal{D}$ be a subcategory of $\operatorname{mod} \Lambda$.

(1) If $\dim \mathcal{D} = n$, then $\dim \Omega^i(\mathcal{D}) \geq n - i$ for any $0 \leq i \leq n$ by Proposition 3.6. In particular, if $\dim \operatorname{mod} \Lambda = n$, then $\dim \Omega^i(\operatorname{mod} \Lambda) \geq n - i$ for any $0 \leq i \leq n$.

(2) If $\mathcal{D}$ is n-syzygy finite, then $\dim \Omega^{n-1}(\mathcal{D}) \leq 1$, and so $\mathcal{D}$ is $(n-1)$-Igusa-Todorov by Proposition 3.3. In addition, we have $\dim \mathcal{D} \leq n$ by Proposition 3.6. In particular, if Λ is n-syzygy finite, then $\dim \Omega^{n-1}(\operatorname{mod} \Lambda) \leq 1$, it follows from Proposition 3.3 that Λ is a $(n-1)$-Igusa-Todorov algebra ([27, Proposition 2.5]). In addition, we have $\dim \operatorname{mod} \Lambda \leq n$.

From now on, assume that Λ', Λ and Λ'' are artin algebras and

$$\operatorname{mod} \Lambda' \underset{\xleftarrow{\;i^!\;}}{\overset{\xleftarrow{\;i^*\;}}{\xrightarrow{\;i_*\;}}} \operatorname{mod} \Lambda \underset{\xleftarrow{\;j_*\;}}{\overset{\xleftarrow{\;j_!\;}}{\xrightarrow{\;j^*\;}}} \operatorname{mod} \Lambda''$$

is a recollement.

Theorem 3.8. *Let $\mathcal{D}'$ and $\mathcal{D}''$ be subcategories of $\operatorname{mod} \Lambda'$ and $\operatorname{mod} \Lambda''$ respectively. Assume that $i^!$ is exact. For any $n, m \geq 0$, if one of the following conditions holds:*

(1) $m = 0$,

(2) $m \geq 1$ and $i^! j_!$ preserves projective modules,

then

$$\dim \Omega^k(\mathcal{D}) \leq \dim \Omega^n(\mathcal{D}') + \dim \Omega^m(\mathcal{D}'') + 1,$$

where $k = \max\{m, n\}$ and $\mathcal{D} = \{D \in \operatorname{mod} \Lambda \mid i^!(D) \in \mathcal{D}'$ and $j^*(D) \in \mathcal{D}''\}$.

Proof. Let $X \in \mathcal{D}$. Then $i^!(X) \in \mathcal{D}'$ and $j^*(X) \in \mathcal{D}''$. Suppose $\dim \Omega^n(\mathcal{D}') = p$ and $\dim \Omega^m(\mathcal{D}'') = q$. Then there exist $V' \in \operatorname{mod} \Lambda'$ and $V'' \in \operatorname{mod} \Lambda''$ such that $\Omega^n(i^!(X)) \in \langle V' \rangle_{p+1}$ and $\Omega^m(j^*(X)) \in \langle V'' \rangle_{q+1}$.

Since $i^!$ is exact, there exists the following exact sequence

$$0 \longrightarrow i_* i^!(X) \longrightarrow X \longrightarrow j_* j^*(X) \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ by Lemma 2.7. Set $k := \max\{m, n\}$. By the horseshoe lemma, there exists an exact sequence

$$0 \longrightarrow \Omega^k(i_*i^!(X)) \longrightarrow \Omega^k(X) \longrightarrow \Omega^k(j_*j^*(X)) \longrightarrow 0 \tag{3.1}$$

in $\operatorname{mod}\Lambda$.

(1) If $m = 0$, then $j^*(X) \in \langle V''\rangle_{q+1}$. Since j_* is exact by Lemma 2.7, we have

$$j_*j^*(X) \in j_*\langle V''\rangle_{q+1} \subseteq \langle j_*(V'')\rangle_{q+1}$$

by [33, Lemma 2.4]. It follows from Lemma 2.2 that $\Omega^k(j_*j^*(X)) \in \langle \Omega^k(j_*(V''))\rangle_{q+1}$.

Note that $\Omega^k(i^!(X)) \in \langle \widetilde{V'}\rangle_{p+1}$ for some $\widetilde{V'} \in \operatorname{mod}\Lambda'$ by Lemma 2.3. Since i_* is exact and preserves projective modules by Lemma 2.7, we have

$$\begin{aligned} \Omega^k(i_*i^!(X)) &= i_*(\Omega^k(i^!(X))) && \text{(by Lemma 2.5)} \\ &\in i_*(\langle \widetilde{V'}\rangle_{p+1}) \subseteq \langle i_*(\widetilde{V'})\rangle_{p+1}. && \text{(by [33, Lemma 2.4])} \end{aligned}$$

Following the exact sequence (3.1), we have

$$\Omega^k(X) \in \langle i_*(\widetilde{V'})\rangle_{p+1} \diamond \langle \Omega^k(j_*(V''))\rangle_{q+1} \subseteq \langle i_*(\widetilde{V'}) \oplus \Omega^k(j_*(V''))\rangle_{p+q+2}$$

by [33, Proposition 2.2(1) and Corollary 2.3(1)]. Thus $\dim \Omega^k(\mathcal{D}) \leq p + q + 1$.

(2) Now let $m \geq 1$. Suppose $p \geq 1$ and $q \geq 0$. By Lemma 2.3, we have $\dim \Omega^k(\mathcal{D}') \leq p$ and $\dim \Omega^k(\mathcal{D}'') \leq q$. Without loss of generality, we still assume $\Omega^k(i^!(X)) \in \langle V'\rangle_{p+1}$ and $\Omega^k(j^*(X)) \in \langle V''\rangle_{q+1}$ for some $V' \in \operatorname{mod}\Lambda'$ and $V'' \in \operatorname{mod}\Lambda''$. Consider the following exact sequence

$$0 \longrightarrow V_1' \longrightarrow \Omega^k(i^!(X)) \longrightarrow V_2' \longrightarrow 0$$

in $\operatorname{mod}\Lambda'$ with $V_1' \in \operatorname{add} V'$ and $V_2' \in \langle V'\rangle_p$. By [31, Lemma 3.3], there is the following exact sequence

$$0 \longrightarrow \Omega^1(V_2') \longrightarrow P' \oplus V_1' \longrightarrow \Omega^k(i^!(X)) \longrightarrow 0$$

in $\operatorname{mod}\Lambda'$ with P' projective. Applying the exact functor i_* to it yields the following exact sequence

$$0 \longrightarrow i_*(\Omega^1(V_2')) \longrightarrow i_*(P' \oplus V_1') \longrightarrow i_*(\Omega^k(i^!(X))) \longrightarrow 0,$$

where $\Omega^1(V_2') \in \langle \Omega^1(V')\rangle_p$ by Lemma 2.2. Then by [33, Lemma 2.4], we have

$$i_*(\Omega^1(V_2')) \in i_*(\langle \Omega^1(V')\rangle_p) \subseteq \langle i_*(\Omega^1(V'))\rangle_p.$$

Since i_* is exact and preserves projective modules by Lemma 2.7, we have that $i_*(P')$ is projective in $\operatorname{mod}\Lambda$ and $\Omega^k(i_*i^!(X)) = i_*(\Omega^k(i^!(X)))$ by Lemma 2.5, so $\Omega^k(i_*i^!(X)) \in \langle i_*(\Omega^1(V'))\rangle_p$.

By Lemma 2.10, there exists the following exact sequence

$$0 \longrightarrow i_*i^!j_!(P_{k-1}) \longrightarrow \Omega^k(j_*j^*(X)) \longrightarrow j_*(\Omega^k(j^*(X))) \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ with P_{k-1} projective. By [31, Lemma 3.3], there exists the following exact sequence

$$0 \longrightarrow \Omega^1(j_*(\Omega^k(j^*(X)))) \longrightarrow P \oplus i_*i^!j_!(P_{k-1}) \longrightarrow \Omega^k(j_*j^*(X)) \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ with P projective. Since j_* is exact by Lemma 2.7, we have

$$j_*(\Omega^k(j^*(X))) \in j_*(\langle V''\rangle_{q+1}) \subseteq \langle j_*(V'')\rangle_{q+1}$$

by [33, Lemma 2.4]. It follows from Lemma 2.2 that

$$\Omega^1(j_*(\Omega^k(j^*(X)))) \in \langle \Omega^1(j_*(V''))\rangle_{q+1}.$$

Notice that i_* and $i^! j_!$ preserve projective modules by Lemma 2.7 and assumption, so $i_* i^! j_!(P_{k-1})$ is a projective Λ-module. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & i_*(\Omega^1(V_2')) & \dashrightarrow & V_1 & \dashrightarrow & \Omega^1(j_*(\Omega^k(j^*(X)))) & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & i_*(P' \oplus V_1') & \dashrightarrow & i_*(P' \oplus V_1') \oplus P \oplus i_* i^! j_!(P_{k-1}) & \dashrightarrow & P \oplus i_* i^! j_!(P_{k-1}) & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^k(i_* i^!(X)) & \longrightarrow & \Omega^k(X) & \longrightarrow & \Omega^k(j_* j^*(X)) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.2}$$

Then

$$\begin{aligned} V_1 &\in \langle i_*(\Omega(V'))\rangle_p \diamond \langle \Omega(j_*(V''))\rangle_{q+1} \\ &\subseteq \langle i_*(\Omega(V')) \oplus \Omega(j_*(V''))\rangle_{p+q+1} \end{aligned}$$

by [33, Proposition 2.2(1) and Corollary 2.3(1)]. Applying Lemma 2.4 to the middle column in the diagram (3.2) yields

$$\Omega^k(X) \in \langle V\rangle_{p+q+2},$$

where $V := i_*(V') \oplus \Lambda \oplus \Omega^{-1}(i_*(\Omega(V')) \oplus \Omega(j_*(V'')))$. Thus

$$\dim \Omega^k(\mathcal{D}) \leq p + q + 1.$$

If $p = 0$, then $V_2' = 0$, and the desired assertion also follows. □

Combining with Theorem 3.8 with Proposition 3.3, we have

Corollary 3.9. *Let $\mathcal{D}'$ and $\mathcal{D}''$ be subcategories of $\operatorname{mod}\Lambda'$ and $\operatorname{mod}\Lambda''$ respectively. Assume that $i^!$ is exact and $i^! j_!$ preserves projective modules. If $\mathcal{D}'$ is n-Igusa-Todorov and $\mathcal{D}''$ is m-Igusa-Todorov, then*

$$\dim \Omega^k(\mathcal{D}) \leq 3,$$

where $k = \max\{m, n\}$ and $\mathcal{D} := \{D \in \operatorname{mod}\Lambda \mid i^!(D) \in \mathcal{D}'$ and $j^*(D) \in \mathcal{D}''\}$.

The following result provides a sufficient condition for a subcategory $\mathcal{D} \subseteq \operatorname{mod}\Lambda$ being n-Igusa-Todorov.

Corollary 3.10. *Let $\mathcal{D}'$ and $\mathcal{D}''$ be subcategories of $\operatorname{mod}\Lambda'$ and $\operatorname{mod}\Lambda''$ respectively. Assume that $i^!$ is exact and $i^!j_!$ preserves projective modules. If $\mathcal{D}'$ is n-syzygy finite and $\mathcal{D}''$ is m-syzygy finite, then $\mathcal{D}$ is a k-Igusa-Todorov subcategory, where $k = \max\{m, n\}$ and $\mathcal{D} = \{D \in \operatorname{mod}\Lambda \mid i^!(D) \in \mathcal{D}'$ and $j^*(D) \in \mathcal{D}''\}$.*

Proof. It follows from Theorem 3.8, Remark 3.1 and Proposition 3.3. □

Taking $\mathcal{D}' = \operatorname{mod}\Lambda'$ and $\mathcal{D}'' = \operatorname{mod}\Lambda''$ in Theorem 3.8, it is easy to check that $\mathcal{D} = \{D \in \operatorname{mod}\Lambda \mid i^!(D) \in \mathcal{D}'$ and $j^*(D) \in \mathcal{D}''\} = \operatorname{mod}\Lambda$. Then we have

Remark 3.11. Assume that $i^!$ is exact and $n, m \geq 0$. Set $k := \max\{m, n\}$. If one of the following conditions holds:

(1) $m = 0$,

(2) $m \geq 1$ and $i^!j_!$ preserves projective objects,

then

$$\dim \Omega^k(\operatorname{mod}\Lambda) \leq \dim \Omega^n(\operatorname{mod}\Lambda') + \dim \Omega^m(\operatorname{mod}\Lambda'') + 1.$$

In particular, we have

(a) (see [33, Theorem 5.5]) If $n = 0 = m$, then $\dim \operatorname{mod}\Lambda \leq \dim \operatorname{mod}\Lambda' + \dim \operatorname{mod}\Lambda'' + 1$.

(b) If Λ' is n-Igusa-Todorov and Λ'' is m-Igusa-Todorov, then $\dim \Omega^k(\operatorname{mod}\Lambda) \leq 3$.

(c) If Λ' is n-syzygy finite and Λ'' is m-syzygy finite, then Λ is k-Igusa-Todorov.

The following result shows that the converse inequality in Theorem 3.8 holds true under certain conditions.

Theorem 3.12. *Let $\mathcal{D}$ be a subcategory of $\operatorname{mod}\Lambda$ with $i_*i^!(\mathcal{D}) \subseteq \mathcal{D}$ and $j_*j^*(\mathcal{D}) \subseteq \mathcal{D}$. If $i^!$ is exact, then*

$$\max\{\dim \Omega^n(i^!(\mathcal{D})), \dim \Omega^n(j^*(\mathcal{D}))\} \leq \dim \Omega^n(\mathcal{D})$$

for some $n \geq 0$.

Proof. Suppose $\dim \Omega^n(\mathcal{D}) = p$. Then for any $X \in \mathcal{D}$, there exists $V \in \operatorname{mod}\Lambda$ such that $\Omega^n(X) \in \langle V \rangle_{p+1}$.

Let $X' \in i^!(\mathcal{D})$. Consider the following exact sequence

$$0 \longrightarrow \Omega^n(X') \longrightarrow P'_{n-1} \longrightarrow P'_{n-2} \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow X' \longrightarrow 0$$

in $\operatorname{mod}\Lambda'$ with all P'_i projective. Since i_* is exact and preserves projective modules by Lemma 2.7, we have $\Omega^n(i_*(X')) = i_*(\Omega^n(X'))$ by Lemma 2.5. Notice that $i_*(X') \in \mathcal{D}$, so $i_*(\Omega^n(X')) \in \langle V \rangle_{p+1}$ by assumption. Since $i^!$ is exact again by assumption, we have

$$\Omega^n(X') \cong i^!i_*(\Omega^n(X')) \in i^!(\langle V \rangle_{p+1}) \subseteq \langle i^!(V) \rangle_{p+1}$$

by [33, Lemma 2.4]. Thus $\dim \Omega^n(i^!(\mathcal{D})) \le p$.

Let $X'' \in j^*(\mathcal{D})$. Notice that $j_*(X'') \in \mathcal{D}$, so $\Omega^n(j_*(X'')) \in \langle V\rangle_{p+1}$ by assumption. Since j^* is exact and preserves projective modules by Lemma 2.7, we have

$$\begin{aligned}\Omega^n(X'') \cong \Omega^n(j^*j_*(X'')) &= j^*(\Omega^n(j_*(X''))) && \text{(by Lemma 2.5)}\\ &\in j^*(\langle V\rangle_{p+1}) \subseteq \langle j^*(V)\rangle_{p+1}. && \text{(by [33, Lemma 2.4])}\end{aligned}$$

Thus $\dim \Omega^n(j^*(\mathcal{D})) \le p$. □

By Theorem 3.12, Proposition 3.3 and Remark 3.1, we have

Corollary 3.13. *Let $\mathcal{D}$ be a subcategory of $\operatorname{mod}\Lambda$ with $i_*i^!(\mathcal{D}) \subseteq \mathcal{D}$ and $j_*j^*(\mathcal{D}) \subseteq \mathcal{D}$ and $n \ge 0$. Assume that $i^!$ is exact. Then we have*

(1) If $\mathcal{D}$ is n-Igusa-Todorov, then both $i^!(\mathcal{D})$ and $j^*(\mathcal{D})$ are n-Igusa-Todorov.

(2) If $\mathcal{D}$ is n-syzygy finite, then both $i^!(\mathcal{D})$ and $j^*(\mathcal{D})$ are n-syzygy finite.

Take $\mathcal{D} = \operatorname{mod}\Lambda$ in Theorem 3.12. It is clear that $i_*i^!(\mathcal{D}) \subseteq \mathcal{D}$ and $j_*j^*(\mathcal{D}) \subseteq \mathcal{D}$ and that $i^!(\mathcal{D}) = \operatorname{mod}\Lambda'$ and $j^*(\mathcal{D}) = \operatorname{mod}\Lambda''$. Then we have

Remark 3.14. If $i^!$ is exact, then

$$\max\{\dim \Omega^n(\operatorname{mod}\Lambda'), \dim \Omega^n(\operatorname{mod}\Lambda'')\} \le \dim \Omega^n(\operatorname{mod}\Lambda)$$

for some $n \ge 0$.

In particular, we have

(1) (see [33, Theorem 5.5]) If $n = 0$, then $\max\{\dim \operatorname{mod}\Lambda', \dim \operatorname{mod}\Lambda''\} \le \dim \operatorname{mod}\Lambda$.

(2) If Λ is n-Igusa-Todorov, then both Λ' and Λ'' are n-Igusa-Todorov.

(3) If Λ is n-syzygy finite, then both Λ' and Λ'' are n-syzygy finite.

Let $\Lambda = \left(\begin{smallmatrix}\Lambda' & M\\ 0 & \Lambda''\end{smallmatrix}\right)$ be a triangular matrix artin algebra. By [21, Example 2.12], we know that

$$\operatorname{mod}\Lambda' \;\underset{\overset{\longleftarrow i^! \longleftarrow}{}}{\overset{\overset{\longleftarrow i^* \longleftarrow}{}}{\longrightarrow i_* \longrightarrow}}\; \operatorname{mod}\Lambda \;\underset{\overset{\longleftarrow j_* \longleftarrow}{}}{\overset{\overset{\longleftarrow j_! \longleftarrow}{}}{\longrightarrow j^* \longrightarrow}}\; \operatorname{mod}\Lambda''$$

is a recollement of module categories, where

$$i^*(\begin{pmatrix}X\\Y\end{pmatrix}_f) = \operatorname{Coker} f, \qquad i_*(X) = \begin{pmatrix}X\\0\end{pmatrix}, \qquad i^!(\begin{pmatrix}X\\Y\end{pmatrix}_f) = X,$$

$$j_!(Y) = \begin{pmatrix}M\otimes_{\Lambda''}Y\\Y\end{pmatrix}_1, \qquad j^*(\begin{pmatrix}X\\Y\end{pmatrix}_f) = Y, \qquad j_*(Y) = \begin{pmatrix}0\\Y\end{pmatrix}.$$

By [15, Lemma 3.2], $i^!$ admits a right adjoint, then $i^!$ is exact. If $M_{\Lambda''}$ is projective, then $j_!$ is exact. If ${}_{\Lambda'}M$ is projective, notice that $i^!j_!(Y) = i^!(\left(\begin{smallmatrix}M\otimes_{\Lambda''}Y\\Y\end{smallmatrix}\right)_1) = M \otimes_{\Lambda''} Y$, then $i^!j_!$ preserves projective objects. By Remark 3.14, we get immediately the following corollary, which generalizes some results in [8].

Corollary 3.15. *Let $\Lambda = \left(\begin{smallmatrix}\Lambda' & M\\ 0 & \Lambda''\end{smallmatrix}\right)$. Then we have*

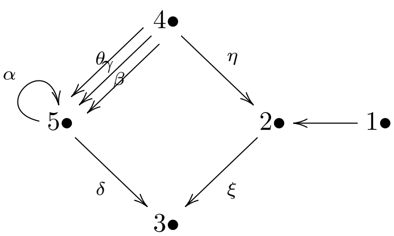
(1) (cf. [8, Theorem 4.5]) *If Λ is n-Igusa-Todorov, then both Λ' and Λ'' are n-Igusa-Todorov.*

(2) (cf. [8, Theorem 4.3]) *If Λ is n-syzygy finite, then both Λ' and Λ'' are n-syzygy finite.*

4 Examples

In this section, all algebras are finite dimensional algebras over an algebraically closed field. For a quiver Q, we use e_i to denote the idempotent corresponding to the vertex i.

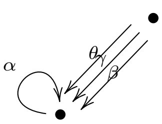
Example 4.1. Let Λ be a finite dimensional algebra given by



with $\eta\xi = \gamma\delta,\ \alpha^3 = \beta\delta = \alpha\delta = 0.$

By [27, Example 1], Λ is a 1-Igusa-Todorov algebra.

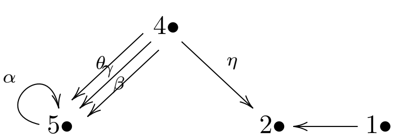
(a) Put $e = e_1 + e_2 + e_3$, $\Lambda' := e\Lambda e$ and $\Lambda'' := (1-e)\Lambda(1-e)$. It follows that $\Lambda = \begin{pmatrix}\Lambda' & M\\ 0 & \Lambda''\end{pmatrix}$ with $M = (1-e)\Lambda e$, where Λ' is a finite dimensional algebra given by $\bullet \longleftarrow \bullet \longleftarrow \bullet$ and Λ'' is a finite dimensional algebra given by



with $\alpha^3 = 0$.

By Corollary 3.15, Λ' and Λ'' are 1-Igusa-Todorov algebras. In fact, Λ' (it is of finite representation type) and Λ'' (it is a matrix algebra formed by two representation-finite algebras [27, Corollary 3.3]) are 0-Igusa-Todorov algebras.

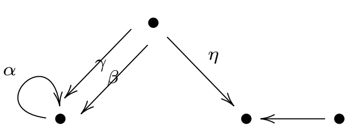
(b) Put $e := e_3$, $\Lambda' := e\Lambda e$ and $\Lambda'' := (1-e)\Lambda(1-e)$. It follows that $\Lambda = \begin{pmatrix}\Lambda' & M\\ 0 & \Lambda''\end{pmatrix}$ with $M = (1-e)\Lambda e$, where Λ' is a finite dimensional algebra given by $\bullet$ and Λ'' is a finite dimensional algebra given by



with $\alpha^3 = 0$.

By Corollary 3.15, Λ' and Λ'' are 1-Igusa-Todorov algebras. In fact, Λ' (it is of finite representation type) is a 0-Igusa-Todorov algebra.

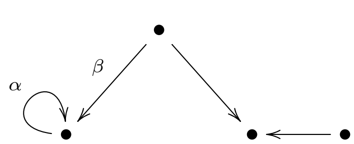
We claim that Λ'' is a 1-Igusa-Todorov algebra. Let $\widetilde{\Lambda''}$ be a finite dimensional algebra given by



with $\alpha^3 = 0$.

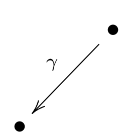
Then $\widetilde{\Lambda''}$ is a trivially twisted extension (see [30] for details) of two representation-finite algebras:

Λ_1 :



with $\alpha^3 = 0$,

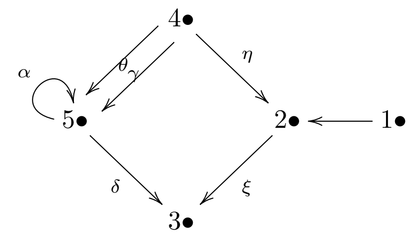
Λ_2 :



.

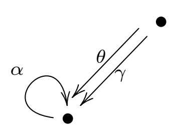
By [27, Corollary 3.3], we have that $\widetilde{\Lambda''}$ is a 0-Igusa Todorov algebra. Let I be the ideal of Λ'' generated by θ. Then $\widetilde{\Lambda''}$ is the quotient algebra of Λ by I. Since $I \operatorname{rad} \Lambda'' = 0$, it follows from [27, Theorem 3.4] that Λ'' is a 1-Igusa-Todorov algebra.

Example 4.2. Let Λ be a finite dimensional algebra given by



with $\eta\xi = \gamma\delta,\ \alpha^3 = \alpha\delta = 0.$

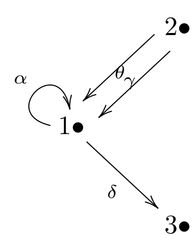
By [27, Example 2], Λ is a 2-Igusa-Todorov algebra. Put $e := e_1 + e_2 + e_3$, $\Lambda' := e\Lambda e$ and $\Lambda'' := (1-e)\Lambda(1-e)$. It follows that $\Lambda = \left(\begin{smallmatrix} \Lambda' & M \\ 0 & \Lambda'' \end{smallmatrix}\right)$ with $M = (1-e)\Lambda e$, where Λ' is a finite dimensional algebra given by $\bullet \longleftarrow \bullet \longleftarrow \bullet$ and Λ'' is a finite dimensional algebra given by



with $\alpha^3 = 0.$

By Corollary 3.15, Λ' and Λ'' are 2-Igusa-Todorov algebras. In fact, Λ' (it is of finite representation type) and Λ'' (it is a matrix algebra formed by two representation-finite algebras) are 0-Igusa-Todorov algebras.

Example 4.3. Let Λ be a finite dimensional algebra given by



with $\alpha^3 = \alpha\delta = 0.$

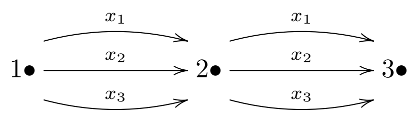
Put $e := e_1 + e_3$, $\Lambda' := e\Lambda e$ and $\Lambda'' := (1-e)\Lambda(1-e)$. It follows that $\Lambda = \left(\begin{smallmatrix} \Lambda' & M \\ 0 & \Lambda'' \end{smallmatrix}\right)$ with $M = (1-e)\Lambda e$, where Λ' is a finite dimensional algebra given by

with $\alpha^3 = 0.$

and Λ'' is a finite dimensional algebra given by $\bullet$.

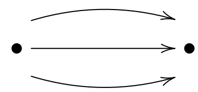
Clearly, Λ' and Λ'' are of finite representation type, and so $\dim \operatorname{mod} \Lambda' = 0 = \dim \operatorname{mod} \Lambda''$ by Remark 3.1. By Theorem 3.8 (or [33, Theorem 5.5]), we have $\dim \operatorname{mod} \Lambda \leq 1$. In fact, $\dim \operatorname{mod} \Lambda = 1$. We know that Λ is a 0-Igusa-Todorov algebra from Proposition 3.3 (or [33, Theorem 3.14]). Note that Λ is viewed as a 2-Igusa-Todorov algebra in [27, Example 3].

Example 4.4. Let Λ be a finite dimensional algebra given by



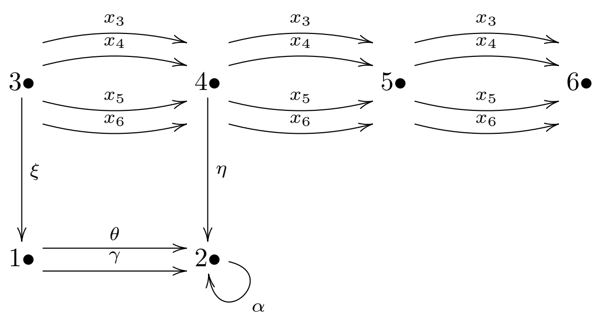
with $x_i x_j = x_j x_i,\ 1 \le i, j \le 3.$

Put $e := e_3$, $\Lambda' := e\Lambda e$ and $\Lambda'' := (1-e)\Lambda(1-e)$. It follows that $\Lambda = \left(\begin{smallmatrix} \Lambda' & M \\ 0 & \Lambda'' \end{smallmatrix}\right)$ with $M = (1-e)\Lambda e$, where Λ' is a finite dimensional algebra given by $\bullet$ and Λ'' is a finite dimensional algebra given by



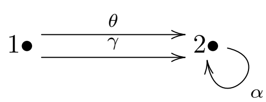
Clearly, $\dim \operatorname{mod} \Lambda' = 0$ and ${}_{\Lambda'}M$ is projective. Notice that Λ'' is a 0-Igusa-Todorov algebra (it is a matrix algebra formed by two representation-finite algebras) and is of representation infinite type, so $\dim \operatorname{mod} \Lambda'' = 1$. Notice that ${}_{\Lambda'}M$ is projective, by Theorem 3.8 and Remark 3.14 (see [33, Theorem 5.5]), we have $1 \le \dim \operatorname{mod} \Lambda \le 2$. In fact, $\dim \operatorname{mod} \Lambda = 2$ by [32, Example 3.4].

Example 4.5. Let Λ be a finite dimensional algebra given by



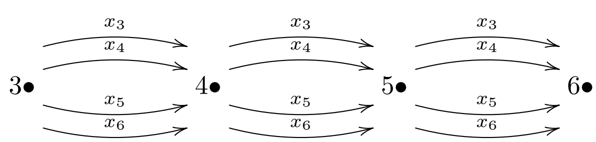
with $\alpha^3 = 0, x_i x_j = x_j x_i, x_i \eta = \xi \theta,\ 3 \le i, j \le 6.$

Put $e = e_1 + e_2$, $\Lambda' := e\Lambda e$ and $\Lambda'' := (1-e)\Lambda(1-e)$. It follows that $\Lambda = \left(\begin{smallmatrix} \Lambda' & M \\ 0 & \Lambda'' \end{smallmatrix}\right)$ with $M = (1-e)\Lambda e$, where Λ' is a finite dimensional algebra given by



with $\alpha^3 = 0$

and Λ'' is a finite dimensional algebra given by



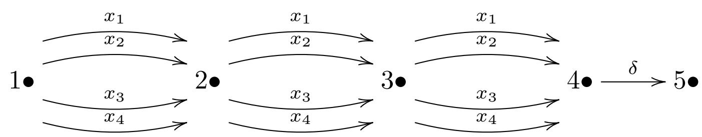
with $x_i x_j = x_j x_i,\ 3 \le i, j \le 6.$

Notice that ${}_{\Lambda'}M$ is projective. As in Example 4.2, Λ' is a 0-Igusa-Todorov algebra, so $\dim \operatorname{mod} \Lambda' \le 1$ (in fact, $\dim \operatorname{mod} \Lambda' = 1$). By [32, Example 3.4], we have $\dim \operatorname{mod} \Lambda'' = 3$. By Remark 3.7, we have $\dim \Omega^1(\operatorname{mod} \Lambda'') \ge 2$. Notice that Λ'' is 3-syzygy finite, so $\dim \Omega^1(\operatorname{mod} \Lambda'') \le 2$ by Proposition 3.6, and thus $\dim \Omega^1(\operatorname{mod} \Lambda'') = 2$. Similarly, we have $\dim \Omega^1(\operatorname{mod} \Lambda'') = 1$. On the other hand, Λ' is 2-syzygy finite (Λ' is a monomial algebra), so $\dim \operatorname{mod} \Lambda' = 0$. Then

(a) By Remarks 3.11 and 3.14 (or [33, Theorem 5.5]), $3 \le \dim \operatorname{mod} \Lambda \le 5$. By Lemma 2.3 and Remark 3.7, we have $2 \le \dim \Omega^1(\operatorname{mod} \Lambda) \le 5$, and $1 \le \dim \Omega^2(\operatorname{mod} \Lambda) \le 5$.

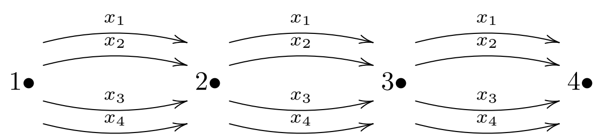
(b) By Remark 3.11 and Remark 3.14, we have $2 \le \dim \Omega^1(\operatorname{mod} \Lambda) \le 4$, and $1 \le \dim \Omega^2(\operatorname{mod} \Lambda) \le 2$. The upper bound here is better than that in (a).

Example 4.6. Let Λ be a finite dimensional algebra given by



with $x_i x_j = x_j x_i$, $\delta x_i = \delta x_j$ $1 \le i, j \le 4$.

Put $e = e_5$, $\Lambda' := e\Lambda e$ and $\Lambda'' := (1-e)\Lambda(1-e)$. It follows that $\Lambda = \left(\begin{smallmatrix} \Lambda' & M \\ 0 & \Lambda'' \end{smallmatrix}\right)$ with $M = (1-e)\Lambda e$, where Λ' is a finite dimensional algebra given by $\bullet$ and Λ'' is a finite dimensional algebra given by



with $x_i x_j = x_j x_i$, $1 \le i, j \le 4$.

Notice that ${}_{\Lambda'}M$ is projective and $\dim \operatorname{mod} \Lambda' = 0$. By [32, Example 3.4], we have $\dim \operatorname{mod} \Lambda'' = 3$. By Example 4.5, we have $\dim \Omega^1(\operatorname{mod} \Lambda'') = 2$. Then

(a) By Remarks 3.11 and 3.14 (or [33, Theorem 5.5]), we have $3 \le \dim \operatorname{mod} \Lambda \le 4$. Then by Lemma 2.3 and Remark 3.7, we have $2 \le \dim \Omega^1(\operatorname{mod} \Lambda) \le 4$.

(b) By Remarks 3.11 and 3.14, we have $2 \le \dim \Omega^1(\operatorname{mod} \Lambda) \le 3$. The upper bound here is better than that in (a).

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