HOMOLOGICAL TRANSFER BETWEEN ADDITIVE CATEGORIES AND HIGHER DIFFERENTIAL ADDITIVE CATEGORIES

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Abstract. Given an additive category $\mathcal{C}$ and an integer $n \geqslant 2$. The higher differential additive category consists of objects X in $\mathcal{C}$ equipped with an endomorphism ϵ_X satisfying $\epsilon_X^n = 0$. Let R be a finite-dimensional basic algebra over an algebraically closed field and T the augmenting functor from the category of finitely generated left R-modules to that of finitely generated left $R/(t^n)$-modules. It is proved that a finitely generated left R-module M is τ-rigid (respectively, (support) τ-tilting, almost complete τ-tilting) if and only if so is $T(M)$ as a left $R[t]/(t^n)$-module. Moreover, R is τ_m-selfinjective if and only if so is $R[t]/(t^n)$.

1. Introduction

Let R be an arbitrary associative ring with unit. A module equipped with an R-linear endomorphism of square zero is called a *differential* R-module. Since their appearance in Cartan and Eilenberg's treatise [10], differential modules has played an important role in solving some problems from commutative algebra and algebraic topology [5]. Indeed, differential R-modules are exactly modules over *the ring of dual numbers*, that is, the ring $R[\epsilon] := R[t]/(t^2)$ (the factor ring of the polynomial ring $R[t]$ in one variable t modulo the ideal generated by t^2). For a positive integer $n \geqslant 2$, Xu, Yang and Yao [32] introduced a higher analog of differential modules, called *n-th differential* modules. More precisely, an n-th differential module is such an R-module with an R-linear endomorphism of n-th power zero. Recently, Tang and Huang [28] extended the theory of n-th differential modules to additive categories and related some homological behavior of R and those of the ring $R[t]/(t^n)$. With the help of the theory of higher differential objects in additive categories, this paper is concerned with investigating the transfer of some homological properties between R and $R[t]/(t^n)$. The paper is organized as follows.

In Section 2, some terminology and notations are given. We also collect some useful general facts in higher differential additive categories, which will be frequently used in the sequel.

Let $\mathcal{C}$ be an additive category and $T : \mathcal{C} \to \mathcal{C}[\epsilon]^n$ the augmenting functor. In Section 3, we establish the relation between the (pre)covers (respectively, (pre)envelopes) in $\mathcal{C}$ and $\mathcal{C}[\epsilon]^n$, and prove that for a subcategory $\mathcal{X}$ of $\mathcal{C}$, $\mathcal{X}$ is precovering (respectively, preenveloping) in $\mathcal{C}$ if and only if $T(\mathcal{X})$ is precovering (respectively, preenveloping) in $\mathcal{C}[\epsilon]^n$ (Theorem 3.3).

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We devote the rest part of the paper to expose some applications of the obtained results. Let R be a left noetherian ring and ${}_R\omega$ a Wakamatsu tilting module with $S = \mathrm{End}({}_R\omega)$. In Section 4, we prove that an $R[t]/(t^n)$-module M is $G_{T(\omega)}$-projective if and only if M is G_ω-projective as an R-module; and that an $S[t]/(t^n)$-module N is in the Auslander class $\mathcal{A}_{T(\omega)}(S[t]/(t^n))$ if and only if N is in the Auslander class $\mathcal{A}_\omega(S)$ (Theorem 4.7). Moreover, we prove that for an artin algebra R, $R/(t^n)$ is CM-finite (respectively, CM-free) implies that so is R (Proposition 4.9); and for a finite dimensional algebra R over an algebraically closed field, if $R[t]/(t^n)$ is representation finite, then so is R (Proposition 4.12). We give examples to illustrate that neither the converses of these two propositions hold true in general.

In Section 5, we focus on the τ-tilting theory of higher differential module categories. Let R be a finite-dimensional basic algebra over an algebraically closed field. We prove that a finitely generated left R-module M is τ-rigid (respectively, (support) τ-tilting, almost complete τ-tilting) if and only if so is $T(M)$ as a left $R[t]/(t^n)$-module (Theorem 5.5). Then we apply it to study the transfer of the Bongartz complement and two-term (pre)silting complexes between R and $R/(t^n)$.

Section 6 deals with an application to m-precluster tilting subcategories of module categories. Actually, we show that an artin algebra R is τ_m-selfinjective if and only if so is $R[t]/(t^n)$ (Theorem 6.4).

2. **Preliminaries**

Throughout this paper, R is an associative ring with unit. We use $\mathrm{Mod}\, R$ (respectively, $\mathrm{mod}\, R$) to denote the class of (respectively, finitely generated) left R-modules. For a module $M \in \mathrm{Mod}\, R$, we use $\mathrm{pd}_R\, M$ to denote the projective dimension of M.

Now we start by recalling from [28] some definitions and notations. Let $\mathcal{C}$ be an additive category and $n \geqslant 2$. An *n-th differential object* of $\mathcal{C}$ is a pair (X, ϵ_X), where $X \in \mathrm{ob}\,\mathcal{C}$ and $\epsilon_X \in \mathrm{End}_{\mathcal{C}}(X)$ satisfying $\epsilon_X^n = 0$. We define the *higher differential additive category* $\mathcal{C}[\epsilon]^n$ as follows: The objects of $\mathcal{C}[\epsilon]^n$ are n-th differential objects, and the set of morphisms from (X, ϵ_X) to (Y, ϵ_Y) consists of morphisms $f : X \to Y$ of $\mathcal{C}$ satisfying the equality $f\epsilon_X = \epsilon_Y f$.

Next we introduce two functors between $\mathcal{C}$ and $\mathcal{C}[\epsilon]^n$.

(1) The *forgetful functor* $F : \mathcal{C}[\epsilon]^n \to \mathcal{C}$ is defined on the objects (X, ϵ_X) of $\mathcal{C}[\epsilon]^n$ by $F(X, \epsilon_X) = X$ and on the morphisms f in $\mathcal{C}[\epsilon]^n$ by $F(f) = f$.

(2) We define the *augmenting functor* $T : \mathcal{C} \to \mathcal{C}[\epsilon]^n$, which takes an object X of $\mathcal{C}$ to the object $T(X) = (X^{\oplus n}, \epsilon_{X^{\oplus n}})$ of $\mathcal{C}[\epsilon]^n$ with $X^{\oplus n} = \underbrace{X \oplus X \oplus \cdots \oplus X}_{n}$

and

$$\epsilon_{X^{\oplus n}} := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{n \times n},$$

and takes a morphism f in $\mathcal{C}$ to the morphism

$$\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}_{n\times n}$$

in $\mathcal{C}[\epsilon]^n$.

We state some preliminary results on $\mathcal{C}[\epsilon]^n$ as follows.

Fact 2.1. Let $\mathcal{C}$ be an additive category, and let $M, N \in \mathrm{ob}\,\mathcal{C}$ and $(X, \epsilon_X) \in \mathrm{ob}\,\mathcal{C}[\epsilon]^n$.

(1) If R is a ring and $\mathcal{C} = \mathrm{Mod}\,R$, then $(\mathrm{Mod}\,R)[\epsilon]^n \cong \mathrm{Mod}(R[t]/(t^n))$.
(2) Both (F, T) and (T, F) are adjoint pairs.
(3) $f \in \mathrm{Hom}_{\mathcal{C}[\epsilon]^n}(T(M), T(N))$ if and only if

$$f = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix} \text{ with } a_i \in \mathrm{Hom}_{\mathcal{C}}(M, N).$$

(4) If $f \in \mathrm{Hom}_{\mathcal{C}[\epsilon]^n}(T(M), (X, \epsilon_X))$, then $f = (f', \epsilon_X f', \cdots, \epsilon_X^{n-1} f')$ with $f' \in \mathrm{Hom}_{\mathcal{C}}(M, X)$.
(5) If $g \in \mathrm{Hom}_{\mathcal{C}[\epsilon]^n}((X, \epsilon_X), T(M))$, then $g = (g'\epsilon_X^{n-1}, \cdots, g'\epsilon_X, g')^T$ with $g' \in \mathrm{Hom}_{\mathcal{C}}(X, M)$.

Proof. The assertions (1), (2) and (3) follow from [28, p.130], [28, Proposition 3.1] and [28, Proposition 3.4] respectively. The assertions (4) and (5) are obvious. □

The following definition is cited from [9].

Definition 2.2. Let $\mathcal{C}$ be an additive category. A *kernel-cokernel pair* (i, p) in $\mathcal{C}$ is a pair of composable morphisms $A \xrightarrow{i} B \xrightarrow{p} C$ such that i is a kernel of p and p is a cokernel of i. We shall call i an *admissible monic* and p an *admissible epic.*

An *exact category* $(\mathcal{C}, \mathscr{E})$ is an additive category $\mathcal{C}$ with a class $\mathscr{E}$ of kernel-cokernel pairs which is closed under isomorphisms and satisfies the following axioms:

[E0] For all objects $C \in \mathcal{C}$, the identity morphism 1_C is an admissible monic.
[E0$^{\mathrm{op}}$] For all objects $C \in \mathcal{C}$, the identity morphism 1_C is an admissible epic.
[E1] The class of admissible monics is closed under compositions.
[E1$^{\mathrm{op}}$] The class of admissible epics is closed under compositions.
[E2] The push-out of an admissible monic along an arbitrary morphism exists and yields an admissible monic.
[E2$^{\mathrm{op}}$] The pull-back of an admissible epic along an arbitrary morphism exists and yields an admissible epic.

Elements of $\mathscr{E}$ are called *short exact sequences.*

Remark 2.3. (cf. [23, p.39]) Equivalently, an additive category $\mathcal{C}$ with a class $\mathscr{E}$ of composable morphisms $A \to B \to C$ is called *exact* if it satisfies the following axioms.

(1) An admissible monic (respectively, epic) is a kernel (respectively, cokernel) of any corresponding admissible epic (respectively, monic).

(2) Axioms [E0], [E0$^{\rm op}$], [E1], [E1$^{\rm op}$], [E2] and [E2$^{\rm op}$] hold.

According to [9, 22], an additive category $\mathcal{C}$ is called *idempotent complete* if every idempotent endomorphism $e = e^2$ of an object $X \in \operatorname{ob}\mathcal{C}$ splits, that is, there exists a factorization

$$X \xrightarrow{\pi} Y \xrightarrow{\iota} X$$

of e with $\pi\iota = 1_Y$.

Let $(\mathcal{C}, \mathscr{E})$ be an exact category, and let $\mathscr{E}_F$ be the class of pairs of composable morphisms in $\mathcal{C}[\epsilon]^n$ that become short exact sequences in $\mathcal{C}$ via the forgetful functor F. The following result characterizes projective (respectively, injective) objects of $\mathcal{C}[\epsilon]^n$ in terms of that of $\mathcal{C}$.

Lemma 2.4. ([28, Proposition 3.6]) *Let $(\mathcal{C}, \mathscr{E})$ be an idempotent complete exact category. Then we have*

(1) P is a projective object of $(\mathcal{C}[\epsilon]^n, \mathscr{E}_F)$ if and only if $P \cong T(Q)$ for some projective object Q of $\mathcal{C}$.

(2) I is an injective object of $(\mathcal{C}[\epsilon]^n, \mathscr{E}_F)$ if and only if $I \cong T(E)$ for some injective object E of $\mathcal{C}$.

Let $\mathcal{X}$ be a class of objects in an additive category $\mathcal{C}$ and $M \in \operatorname{ob}\mathcal{C}$. Recall that an *$\mathcal{X}$-precover* of M is a morphism $f : X \to M$ in $\mathcal{C}$ with $X \in \mathcal{X}$ such that any morphism $g : X' \to M$ in $\mathcal{C}$ with $X' \in \mathcal{X}$ factors through f. An $\mathcal{X}$-precover $f : X \to M$ of M is an *$\mathcal{X}$-cover* if every endomorphism $g : X \to X$ in $\mathcal{C}$ with $fg = f$ is an automorphism. We call the class $\mathcal{X}$ *precovering* in $\mathcal{C}$ if any $M \in \operatorname{ob}\mathcal{C}$ has an $\mathcal{X}$-precover. Dually, the notions of preenvelopes and preenveloping classes are defined (cf. [14]).

3. Precovering and Preenveloping Classes

From now on, we fix an exact category $(\mathcal{C}, \mathscr{E})$. This section investigates how to construct precovering and preenveloping classes in $\mathcal{C}[\epsilon]^n$ via the augmenting functor T.

A sequence (of finite or infinite length):

$$\cdots \to X_m \xrightarrow{f_m} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0$$

in $\mathcal{C}$ is called an *$\mathcal{X}$-resolution* of M if all X_i are in $\mathcal{X}$ and

$$0 \to \operatorname{Ker} f_i \to X_i \to \operatorname{Im} f_i \to 0$$

is a short exact sequence for any $i \geqslant 0$ (note: $\operatorname{Im} f_0 = M$); furthermore, such an $\mathcal{X}$-resolution is called *proper* if it remains exact after applying the functor $\operatorname{Hom}_{\mathcal{C}}(X, -)$ for any $X \in \mathcal{X}$. Dually, the notions of an *$\mathcal{X}$-coresolution* and an *$\mathcal{X}$-coproper coresolution* of M are defined.

Proposition 3.1. *Let $\mathcal{X}$ be a subcategory of $(\mathcal{C}, \mathscr{E})$ and $(M, \epsilon_M) \in \operatorname{ob}\mathcal{C}[\epsilon]^n$.*

(1) If

$$0 \to L \xrightarrow{\lambda} X \xrightarrow{\pi} M \to 0 \tag{3.1}$$

is a short exact sequence in $\mathcal{C}$ such that π is an $\mathcal{X}$-precover of M, then there is a short exact sequence

$$0 \to (L \oplus X^{\oplus(n-1)}, \epsilon) \xrightarrow{g} T(X) \xrightarrow{f} (M, \epsilon_M) \to 0 \tag{3.2}$$

in $(\mathcal{C}[\epsilon]^n, \mathscr{E}_F)$ *such that* f *is a* $T(\mathcal{X})$*-precover of* (M, ϵ_M)*, where* $f = (\pi, \epsilon_M\pi, \cdots, \epsilon_M^{n-1}\pi)$ *and*

$$g = \begin{pmatrix} \lambda & h & 0 & \cdots & 0 \\ 0 & -1 & h & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}_{n\times n} \quad \textit{with } h \in \mathrm{End}_{\mathcal{C}}(X).$$

(2) If

$$0 \to M \xrightarrow{\lambda'} X \xrightarrow{\pi'} L \to 0 \tag{3.3}$$

is a short exact sequence in $\mathcal{C}$ *such that* λ' *is an* $\mathcal{X}$*-preenvelope of* M*, then there is a short exact sequence*

$$0 \to (M, \epsilon_M) \xrightarrow{f'} T(X) \xrightarrow{g'} (L \oplus X^{\oplus(n-1)}, \epsilon) \to 0 \tag{3.4}$$

in $(\mathcal{C}[\epsilon]^n, \mathscr{E}_F)$ *such that* f' *is a* $T(\mathcal{X})$*-preenvelope of* (M, ϵ_M)*, where* $f' = (\lambda'\epsilon_M^{n-1}, \cdots, \lambda'\epsilon_M, \lambda')^T$ *and*

$$g' = \begin{pmatrix} \pi' & 0 & 0 & \cdots & 0 \\ -1 & h' & 0 & \cdots & 0 \\ 0 & -1 & h' & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h' \end{pmatrix}_{n\times n} \quad \textit{with } h' \in \mathrm{End}_{\mathcal{C}}(X).$$

Proof. (1) Since π is admissible epic, [9, Proposition 2.9] implies that $T(\pi)$ is admissible epic. Then

$$f = (\pi, \epsilon_M\pi, \cdots, \epsilon_M^{n-1}\pi) = p'_M T(\pi)$$

is also admissible epic by [28, Lemma 3.5], where $p'_M = (1, \epsilon_M, \cdots, \epsilon_M^{n-1})$. As π is an $\mathcal{X}$-precover of M, there is a morphism $h \in \mathrm{End}_{\mathcal{C}}(X)$ such that $\pi h = \epsilon_M\pi$. Thus $fg = 0$.

Now we prove that (3.2) is a short exact sequence. Let

$$t = (t_1, t_2, \cdots, t_n)^T : C \to X^{\oplus n}$$

be a morphism in $\mathcal{C}$ such that $ft = 0$. Then

$$\pi t_1 + \epsilon_M\pi t_2 + \cdots + \epsilon_M^{n-1}\pi t_n = 0,$$

and hence

$$\pi t_1 + \pi h t_2 + \cdots + \pi h^{n-1} t_n = 0.$$

Since λ is the kernel of π, there exists a unique morphism $s_1 : C \to L$ such that

$$\lambda s_1 = t_1 + h t_2 + \cdots + h^{n-1} t_n.$$

Set

$$s_i := -h^{n-i}t_n - h^{n-i-1}t_{n-1} - \cdots - h t_{i+1} - t_i \ (2 \leqslant i \leqslant n-1) \text{ and } s_n := -t_n.$$

Clearly $s = (s_1, s_2, \cdots, s_n)^T : C \to L \oplus X^{\oplus(n-1)}$ satisfies $gs = t$. We conclude that g is the kernel of f.

Now we show that f is the cokernel of g. Let

$$u = (u_1, u_2, \cdots, u_n) : X^{\oplus n} \to C$$

be a morphism in $\mathcal{C}$ such that $ug = 0$. Then

$$(u_1\lambda, u_1h - u_2, \cdots, u_{n-1}h - u_n) = 0.$$

Since π is the cokernel of λ, there exists a unique morphism $p : M \to C$ such that $p\pi = u_1$. Notice that $\pi h = \epsilon_M \pi$, so $\pi h^i = \epsilon_M^i \pi$ and

$$p\epsilon_M^i \pi = p\pi h^i = u_1 h^i = u_2 h^{i-1} = \cdots = u_i h = u_{i+1}$$

for any $1 \leqslant i \leqslant n-1$. It follows that $pf = u$ and f is the cokernel of g. Therefore we conclude that (3.2) is a short exact sequence by Remark 2.3. Consequently we get an endomorphism $\epsilon \in \mathrm{End}_{\mathcal{C}}(L \oplus X^{\oplus n-1})$ satisfying $g\epsilon = \epsilon_{X^{\oplus n}} g$. Then $\epsilon^n = 0$ since $\epsilon^n_{X^{\oplus n}} = 0$.

Now let $\beta \in \mathrm{Hom}_{\mathcal{C}[\epsilon]^n}(T(X'), (M, \epsilon_M))$. Then $\beta = (\beta', \varepsilon_M \beta', \cdots, \varepsilon_M^{n-1}\beta')$ with $\beta' \in \mathrm{Hom}_{\mathcal{C}}(X', M)$ by Fact 2.1(4). Since $\pi : X \to M$ is an $\mathcal{X}$-precover of M, there exists a morphism $\gamma : X' \to X$ such that $\pi\gamma = \beta'$. Thus $fT(\gamma) = \beta$ and f is a $T(\mathcal{X})$-precover of (M, ϵ_M).

(2) It is dual to (1). □

As a consequence, we get the following

Corollary 3.2. *Let $\mathcal{X}$ be an additive subcategory of $\mathcal{C}$ and $(M, \epsilon_M) \in \mathrm{ob}\,\mathcal{C}[\epsilon]^n$.*

(1) If

$$\cdots \to X_m \xrightarrow{f_m} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0$$

is a proper $\mathcal{X}$-resolution in $\mathcal{C}$, then there exists a proper $T(\mathcal{X})$-resolution

$$\cdots \to X'_m \xrightarrow{f'_m} \cdots \to X'_1 \xrightarrow{f'_1} X'_0 \xrightarrow{f'_0} (M, \epsilon_M) \to 0$$

in $\mathcal{C}[\epsilon]^n$ with $X'_i = T(X_i \oplus X_{i-1}^{(n-1)} \oplus \cdots \oplus X_0^{(n-1)^i})$.

(2) If

$$0 \to M \xrightarrow{g_0} X_0 \xrightarrow{g_1} \cdots \xrightarrow{g_m} X_m \to \cdots$$

is a coproper $\mathcal{X}$-coresolution in $\mathcal{C}$, then there exists a coproper $T(\mathcal{X})$-coresolution

$$0 \to (M, \epsilon_M) \xrightarrow{g'_0} X'_0 \xrightarrow{g'_1} \cdots \xrightarrow{g'_m} X'_m \to \cdots$$

in $\mathcal{C}[\epsilon]^n$ with $X'_i = T(X_i \oplus X_{i-1}^{(n-1)} \oplus \cdots \oplus X_0^{(n-1)^i})$.

Proof. (1) Set $M_{i+1} := \mathrm{Ker}\, f_i$ for any $i \geqslant 0$. By Proposition 3.1(1), there exists a short exact sequence

$$0 \to (M_1 \oplus X_0^{\oplus(n-1)}, \epsilon) \to T(X_0) \to (M, \epsilon_M) \to 0 \tag{3.5}$$

in $\mathcal{C}[\epsilon]^n$ such that $\mathrm{Hom}_{\mathcal{C}[\epsilon]^n}(T(X), (3.5))$ is exact for any $X \in \mathrm{ob}\,\mathcal{C}$. Note that

$$0 \to M_2 \to X_1 \oplus X_0^{\oplus(n-1)} \to M_1 \oplus X_0^{\oplus(n-1)} \to 0 \tag{3.6}$$

is a short exact sequence in $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, (3.6))$ is exact for any $X \in \mathrm{ob}\,\mathcal{C}$. Then by Proposition 3.1(1) again, we have a short exact sequence

$$0 \to (M_2 \oplus (X_1 \oplus X_0^{\oplus(n-1)})^{\oplus(n-1)}, \epsilon') \to T(X_1 \oplus X_0^{\oplus(n-1)}) \to (M_1 \oplus X_0^{\oplus(n-1)}, \epsilon) \to 0 \tag{3.7}$$

in $\mathcal{C}[\epsilon]^n$ such that $\mathrm{Hom}_{\mathcal{C}[\epsilon]^n}(T(X), (3.7))$ is exact for any $X \in \mathrm{ob}\,\mathcal{C}$. Continuing in this way, we obtain the desired sequence.

(2) It is dual to (1). □

The following result will be used frequently in the sequel.

Theorem 3.3. *Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$ and $M \in \operatorname{ob}\mathcal{C}$. Then the following statements hold.*

(1) $f : X \to M$ is an $\mathcal{X}$-(pre)cover of M if and only if $T(f) : T(X) \to T(M)$ is a $T(\mathcal{X})$-(pre)cover of $T(M)$.
(2) $g : M \to X$ is an $\mathcal{X}$-(pre)envelope of M if and only if $T(g) : T(M) \to T(X)$ is a $T(\mathcal{X})$-(pre)envelope of $T(M)$.
(3) $\mathcal{X}$ is precovering in $\mathcal{C}$ if and only if $T(\mathcal{X})$ is precovering in $\mathcal{C}[\epsilon]^n$.
(4) $\mathcal{X}$ is preenveloping in $\mathcal{C}$ if and only if $T(\mathcal{X})$ is preenveloping in $\mathcal{C}[\epsilon]^n$.

Proof. We will only prove (1) and (3). Dually, we get (2) and (4).

(1) We first prove the necessity. We use $\varepsilon : FT \to 1_{\mathcal{C}}$ (respectively, $\eta : 1_{\mathcal{C}[\epsilon]^n} \to TF$) to denote the counit (respectively, unit) of the adjoint pair (F, T). Given a morphism $f' \in \operatorname{Hom}_{\mathcal{C}[\epsilon]^n}(T(X'), T(M))$ with $X' \in \mathcal{X}$. Then we get the following commutative diagram

$$\begin{array}{ccc} T(X') & \xrightarrow{\eta_{T(X')}} & TFT(X') \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle TF(f')} \\ T(M) & \xrightarrow{\eta_{T(M)}} & TFT(M). \end{array}$$

Notice that $FT(X')$ is a finite direct sum of X', so there exists $h : FT(X') \to X$ such that $fh = \varepsilon_M F(f')$. Thus we have

$$T(f)T(h)\eta_{T(X')} = T(fh)\eta_{T(X')} = T(\varepsilon_M)TF(f')\eta_{T(X')} = T(\varepsilon_M)\eta_{T(M)}f' = f'.$$

It follows that $T(f) : T(X) \to T(M)$ is a $T(\mathcal{X})$-precover of $T(M)$. Moreover, suppose that f is an $\mathcal{X}$-cover of M. Now given an endomorphism

$$h' := \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \\ c_2 & c_1 & 0 & \cdots & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \end{pmatrix} \in \operatorname{End}_{\mathcal{C}[\epsilon]^n}(T(X)).$$

If $T(f)h' = T(f)$, then $fc_1 = f$. Thus c_1 must be an isomorphism since f is an $\mathcal{X}$-cover of M. It follows that h' is also an isomorphism and $T(f) : T(X) \to T(M)$ is a $T(\mathcal{X})$-cover of $T(M)$.

Next we prove the sufficiency. Let $f' : X' \to M$ be a morphism in $\mathcal{C}$. Since $T(f) : T(X) \to T(M)$ is a $T(\mathcal{X})$-(pre)cover of $T(M)$, by Fact 2.1(3) there exists a morphism

$$h := \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ h_2 & h_1 & 0 & \cdots & 0 \\ h_3 & h_2 & h_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n-1} & h_{n-2} & \cdots & h_1 \end{pmatrix} \in \operatorname{Hom}_{\mathcal{C}[\epsilon]^n}(T(X'), T(X))$$

such that $T(f)h = T(f')$. Namely,

$$\begin{pmatrix} f & 0 & 0 & \cdots & 0 \\ 0 & f & 0 & \cdots & 0 \\ 0 & 0 & f & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ h_2 & h_1 & 0 & \cdots & 0 \\ h_3 & h_2 & h_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n-1} & h_{n-2} & \cdots & h_1 \end{pmatrix} = \begin{pmatrix} f' & 0 & 0 & \cdots & 0 \\ 0 & f' & 0 & \cdots & 0 \\ 0 & 0 & f' & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f' \end{pmatrix}.$$

One can get that $fh_1 = f'$. It means that $f : X \to M$ is an $\mathcal{X}$-precover of M. Finally, it is not hard to prove that $f : X \to M$ is an $\mathcal{X}$-cover of M provided that $T(f) : T(X) \to T(M)$ is a $T(\mathcal{X})$-cover of $T(M)$.

(3) The necessity follows from the proof of Proposition 3.1(1).

In the following, we prove the sufficiency. Let $M \in \operatorname{ob}\mathcal{C}$. By assumption, there exists a $T(\mathcal{X})$-precover $f : T(X) \to T(M)$ of $T(M)$. We may assume that f has the following form

$$f = \begin{pmatrix} f_1 & 0 & 0 & \cdots & 0 \\ f_2 & f_1 & 0 & \cdots & 0 \\ f_3 & f_2 & f_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n-1} & f_{n-2} & \cdots & f_1 \end{pmatrix} \text{with } f_i \in \operatorname{Hom}_{\mathcal{C}}(X, M).$$

We will show that $f_1 : X \to M$ is an $\mathcal{X}$-precover of M. Given a morphism $g : X' \to M$ with $X' \in \mathcal{X}$. Since $f : T(X) \to T(M)$ is a $T(\mathcal{X})$-precover, there exists a morphism $h : T(X') \to T(X)$ such that $fh = T(g)$. Note that h must have the following form

$$h = \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ h_2 & h_1 & 0 & \cdots & 0 \\ h_3 & h_2 & h_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n-1} & h_{n-2} & \cdots & h_1 \end{pmatrix} \text{with } h_i \in \operatorname{Hom}_{\mathcal{C}}(X', X).$$

It implies $f_1h_1 = g$. So $\mathcal{X}$ is precovering in $\mathcal{C}$. □

4. **Wakamatsu tilting subcategories**

In this section, assume that the given exact category $(\mathcal{C}, \mathscr{E})$ has enough projectives. We will apply the established results in the previous section to study Wakamatsu tilting subcategories through the functor T.

Let $\mathcal{W}$ be a subcategory of $\mathcal{C}$. We use $\operatorname{Add}(\mathcal{W})$ (respectively, $\operatorname{add}(\mathcal{W})$) to denote the subcategory of $\mathcal{C}$ consisting of objects isomorphic to direct summands of (respectively, finite) direct sums of objects in $\mathcal{W}$.

We write ${}^{\perp}\mathcal{W} := \{X \in \mathcal{C} \mid \operatorname{Ext}_{\mathcal{C}}^{\geqslant 1}(X, W) = 0 \text{ for any } W \in \mathcal{W}\}$ and $\mathcal{X}_{\mathcal{W}} := \{X^0 \in {}^{\perp}\mathcal{W} \mid$ there exist short exact sequences

$$0 \to X^0 \to W^0 \to X^1 \to 0,\ 0 \to X^1 \to W^1 \to X^2 \to 0,\ \cdots$$

in $\mathcal{C}$ with all $W^i \in \mathcal{W}$ and $X^i \in {}^{\perp}\mathcal{W}\}$.

Definition 4.1. ([15, Definition 3.1]) Let $\mathcal{W}$ an additive subcategory of $\mathcal{C}$. We say that $\mathcal{W}$ is a *Wakamatsu tilting* subcategory of $\mathcal{C}$ if it satisfies the following conditions.

(1) $\mathcal{W}$ is *self-orthogonal*, that is, $\mathcal{W} \subseteq {}^{\perp}\mathcal{W}$.
(2) $\mathcal{X}_{\mathcal{W}}$ contains all projectives in $\mathcal{C}$.

Remark 4.2.

(1) It is trivial that the subcategory of $\mathcal{C}$ consisting of all projectives is Wakamatsu tilting in $\mathcal{C}$.
(2) Let R be a left noetherian ring and $\mathcal{C} = \operatorname{mod} R$. Recall from [15] that a module $\omega \in \operatorname{mod} R$ is called *Wakamatsu tilting* (or *semidualizing*) if $\operatorname{add}(\omega)$ is a Wakamatsu tilting subcategory of $\mathcal{C}$. This definition coincides with the usual one (cf. [4, 18, 26, 31]).
(3) Let R be a left noetherian ring and ω a Wakamatsu tilting module. If $\mathcal{C} = \operatorname{Mod} R$ and $\mathcal{W} = \operatorname{Add}(\omega)$, then $\mathcal{X}_{\mathcal{W}}$ is exactly the class of all G_ω-projective modules (see [24, Definition 2.5]).
(4) Let R be a left noetherian ring and ω a Wakamatsu tilting module with $S = \operatorname{End}_R(\omega)$. According to [18], the *Auslander class* $\mathcal{A}_\omega(S)$ with respect to ω consists of all left S-modules N satisfying the following conditions: (a) $\operatorname{Tor}^S_{\geqslant 1}(\omega, N) = 0 = \operatorname{Ext}_R^{\geqslant 1}(\omega, \omega \otimes_S N)$, and (b) $N \cong \operatorname{Hom}_R(\omega, \omega \otimes_S N)$. If $\mathcal{C} = \operatorname{Mod} R$ and $\mathcal{W} = \{\operatorname{Hom}_R(\omega, I) \mid I \text{ is injective}\}$, then $\mathcal{X}_{\mathcal{W}}$ is exactly the Auslander class $\mathcal{A}_\omega(S)$ (see [29, Theorem 3.11(1)]).

Proposition 4.3. *Let $\mathcal{C}$ be idempotent complete and $\mathcal{W}$ an additive and self-orthogonal subcategory of $\mathcal{C}$. Then the following statements hold for any $(M, \epsilon_M) \in \mathcal{C}[\epsilon]^n$.*

(1) $M \in {}^{\perp}\mathcal{W}$ if and only if $(M, \epsilon_M) \in {}^{\perp}T(\mathcal{W})$.
(2) $M \in \mathcal{X}_{\mathcal{W}}$ if and only if $(M, \epsilon_M) \in \mathcal{X}_{T(\mathcal{W})}$.

Proof. (1) Let

$$\cdots \to T(P_m) \to T(P_{m-1}) \to \cdots \to T(P_1) \to T(P_0) \to (M, \epsilon_M) \to 0$$

be a projective resolution of (M, ϵ_M) in $\mathcal{C}[\epsilon]^n$. Then by Lemma 2.4,

$$\cdots \to FT(P_m) \to FT(P_{m-1}) \to \cdots \to FT(P_1) \to FT(P_0) \to M \to 0$$

is a projective resolution of M. For any $W \in \mathcal{W}$ and $i \geqslant 1$, by Fact 2.1(2) we have

$$\operatorname{Hom}_{\mathcal{C}[\epsilon]^n}(T(P_i), T(W)) \cong \operatorname{Hom}_{\mathcal{C}}(FT(P_i), W).$$

This isomorphism gives the assertion.

(2) By (1), we have that $M \in {}^{\perp}\mathcal{W}$ if and only if $(M, \epsilon_M) \in {}^{\perp}T(\mathcal{W})$. If $M \in \mathcal{X}_{\mathcal{W}}$, then $(M, \epsilon_M) \in \mathcal{X}_{T(\mathcal{W})}$ by Corollary 3.2(2). Conversely, if $(M, \epsilon_M) \in \mathcal{X}_{T(\mathcal{W})}$, then there exist short exact sequences

$$0 \to (M, \epsilon_M) \to T(W^0) \to X^1 \to 0,\ 0 \to X^1 \to T(W^1) \to X^2 \to 0,\ \cdots$$

in $\mathcal{C}[\epsilon]^n$ with all $T(W^i) \in T(\mathcal{W})$ and $X^i \in {}^{\perp}T(\mathcal{W})$. So by (1), we get short exact sequences

$$0 \to M \to FT(W^0) \to F(X^1) \to 0,\ 0 \to F(X^1) \to FT(W^1) \to F(X^2) \to 0,\ \cdots$$

in $\mathcal{C}$ with all $FT(W^i) \in \mathcal{W}$ and $F(X^i) \in {}^{\perp}\mathcal{W}$. It follows that $M \in \mathcal{X}_{\mathcal{W}}$. □

This induces the following easy consequence.

Corollary 4.4. *Let $\mathcal{C}$ be idempotent complete and $\mathcal{W}$ an additive and self-orthogonal subcategory of $\mathcal{C}$. Then $\mathcal{W}$ is a Wakamatsu tilting subcategory of $\mathcal{C}$ if and only if $T(\mathcal{W})$ is a Wakamatsu tilting subcategory of $\mathcal{C}[\epsilon]^n$.*

Proof. It follows from Proposition 4.3 and [15, Proposition 3.2]. □

The following definition is cited from [6].

Definition 4.5. Let R be a ring and $m \geqslant 0$. A left R-module ω is called *m-tilting* if and only if the following conditions are satisfied.

(1) $\mathrm{pd}_R\,\omega \leqslant m$.
(2) $\omega \in {}^{\perp}\omega^{(\lambda)}$ for every cardinal λ.
(3) There exists an $\mathrm{Add}(\omega)$-coresolution

$$0 \to R \to \omega_0 \to \cdots \to \omega_m \to 0$$

in $\mathrm{Mod}\,R$.

By applying Proposition 4.3, we also get the following result.

Proposition 4.6. *Let R be a ring and $m \geqslant 0$. Then ω is an m-tilting R-module if and only if $T(\omega)$ is an m-tilting $R[t]/(t^n)$-module.*

Proof. Observe that $T(\mathrm{Add}(\omega)) = \mathrm{Add}(T(\omega))$. It is easy to see that $\mathrm{pd}_R\,\omega \leqslant m$ if and only if $\mathrm{pd}_{R[t]/(t^n)}\,T(\omega) \leqslant m$. Moreover, for every cardinal λ, the fact that $\omega \in {}^{\perp}\omega^{(\lambda)}$ if and only if $T(\omega) \in {}^{\perp}T(\omega)^{(\lambda)}$ follows from the proof of Proposition 4.3(1). If R admits an $\mathrm{Add}(\omega)$-coresolution

$$0 \to R \to \omega_0 \to \cdots \to \omega_m \to 0$$

in $\mathrm{Mod}\,R$, then applying the exact functor T to it yields an $\mathrm{Add}(T(\omega))$-coresolution

$$0 \to T(R) \to T(\omega_0) \to \cdots \to T(\omega_m) \to 0$$

of $T(R)$ in $\mathrm{Mod}\,R[t]/(t^n)$. Conversely, if $T(R)$ admits an $\mathrm{Add}(T(\omega))$-coresolution

$$0 \to T(R) \to T(\omega_0) \to \cdots \to T(\omega_m) \to 0$$

in $\mathrm{Mod}\,R[t]/(t^n)$, then it follows from [30, Lemma 4.6] that there exists an $\mathrm{Add}(\omega)$-coresolution

$$0 \to R \to \omega_0' \to \cdots \to \omega_m' \to 0$$

of R in $\mathrm{Mod}\,R$. The proof is finished. □

The main result in this section is the following theorem.

Theorem 4.7. *Let R be a left noetherian ring and ω a Wakamatsu tilting module with $S = \mathrm{End}({}_R\omega)$. Then the following statements hold.*

(1) If $M \in \mathrm{Mod}\,R[t]/(t^n)$, then M is $G_{T(\omega)}$-projective if and only if M is G_ω-projective as an R-module.
(2) If $N \in \mathrm{Mod}\,S[t]/(t^n)$, then $N \in \mathcal{A}_{T(\omega)}(S[t]/(t^n))$ if and only if $N \in \mathcal{A}_\omega(S)$.

Proof. Note that R is left noetherian if and only if so is $R[t]/(t^n)$ by [28, Corollary 3.8(1)]. Also note that $\mathrm{End}({}_{R[t]/(t^n)}T(\omega)) \cong S[t]/(t^n)$ and $T(\mathrm{Hom}_R(\omega, I)) \cong \mathrm{Hom}_{T(R)}(T(\omega), T(I))$ for any injective left R-module I. Then in view of Remark 4.2, Proposition 4.3 and Corollary 4.4, we get the assertions. □

Taking $\mathcal{W}$ to be the subcategory of $\mathcal{C}$ consisting of all projectives, objects in $\mathcal{X}_{\mathcal{W}}$ are called *Gorenstein projective* (see [15, Definition 3.7]). In our setting, Theorem 4.7(1) can be regarded as a generalisation of [32, Theorem 3.10(1)].

Let R be an artin algebra. A module $M \in \mathrm{mod}\,R$ is called *semi-Gorenstein-projective* provided that $\mathrm{Ext}_R^{\geqslant 1}(M, R) = 0$. Moreover, R is said to be *left weakly*

Gorenstein if any semi-Gorenstein-projective module is Gorenstein-projective (see [27]).

Corollary 4.8. *Let R be an artin algebra and $M \in \mathrm{mod}\, R[t]/(t^n)$. Then the following statements hold.*

(1) M is semi-Gorenstein-projective R-module if and only if M is semi-Gorenstein-projective $R[t]/(t^n)$-module.

(2) R is left weakly Gorenstein if and only if $R[t]/(t^n)$ is left weakly Gorenstein.

Proof. (1) It follows from Proposition 4.3(1).

(2) It follows from (1) and Theorem 4.7(1). □

Let R be an artin algebra. Recall from [7, 8] that R is called *Cohen-Macaulay finite* (*CM-finite*, for short) provided there are only finitely many pairwise non-isomorphic indecomposable finitely generated Gorenstein projective R-modules. Recall from [11] that R is called *CM-free* if all its finitely generated Gorenstein projective modules are projective.

Proposition 4.9. *Let R be an artin algebra. If $R[t]/(t^n)$ is CM-finite (respectively, CM-free), then so is R.*

Proof. Let $R[t]/(t^n)$ be CM-finite and $\{G_1, G_2, \cdots, G_m\}$ the set of all pairwise non-isomorphic indecomposable finitely generated Gorenstein projective $R[t]/(t^n)$-modules. For each i, since G_i is finitely generated as an R-module, G_i can be decomposed as a direct sum of finitely many indecomposable R-modules, that is, $G_i = \bigoplus_{j=1}^{i_j} G_i^j$. Because G_i is a Gorenstein projective $R[t]/(t^n)$-module, it follows that G_i is a Gorenstein projective R-module by Theorem 4.7(1). Thus each G_i^j is a Gorenstein projective R-module as well.

Now let G be an indecomposable Gorenstein projective R-module. Then $T(G)$ is an indecomposable Gorenstein projective $R[t]/(t^n)$-module by Theorem 4.7(1). So $T(G)$ is isomorphic to some G_i as an $R[t]/(t^n)$-module, which implies that $T(G)$ is also isomorphic to G_i as an R-module. Thus G is isomorphic to some G_i^j. It follows that R is CM-finite.

Assume that $R[t]/(t^n)$ is CM-free. If G is a finitely generated Gorenstein projective R-module, then $T(G)$ is a Gorenstein projective $R[t]/(t^n)$-module by Theorem 4.7(1). By assumption, there exists a projective module P such that $T(G) \cong T(P)$. Thus G is projective as an R-module, and therefore R is CM-free. □

In the following, we study the transfer of representation type between R and $R/(t^n)$.

Definition 4.10. ([12]) If R is a ring and G is an R-module. We say G is a *generic module* if it is indecomposable, of infinite length over R, but of finite length when regarded in the natural way as a module over its endomorphism ring.

We need the following observation.

Lemma 4.11. *If R is an artin algebra and $G \in \mathrm{Mod}\, R$, then G is a generic R-module if and only if $T(G)$ is a generic $R[t]/(t^n)$-module.*

Proof. By [28, Proposition 3.4], we have that G is indecomposable if and only if so is $T(G)$. Note that R is an artin algebra if and only if so is $R[t]/(t^n)$ by the proof of [28, Theorem 3.13], and note that a module over an arin algebra has finite length if and only if it is finitely generated. Thus G is of infinite length over R if only if $T(G)$ is of infinite length over $R[t]/(t^n)$. On the other hand, by Theorem 3.3(2), we have that R admits an add(G)-preenvelope if and only if $T(R)$ admits an add($T(G)$)-preenvelope. Now the assertion follows from [3, Proposition 1.2]. □

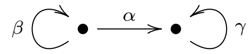
Proposition 4.12. *Let R be a finite dimensional algebra over algebraically closed field. If $R[t]/(t^n)$ is representation finite, then so is R.*

Proof. Note that a finite dimensional algebra over an algebraically closed field is representation finite if and only if it has no generic modules ([13, p.157, Corollary]). If R is representation infinite, then there exists a generic R-module G. Thus $T(G)$ is a generic $R[t]/(t^n)$-module by Lemma 4.11. It follows that $R[t]/(t^n)$ is representation infinite. □

The following example illustrates that neither the converses of Propositions 4.9 and 4.12 hold true in general.

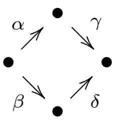
Example 4.13. Let R be a finite-dimensional algebra over an algebraically closed field.

(1) If R is hereditary of type $\mathbb{A}_2$, then $R[t]/(t^n)$ with $n > 5$ is the algebra given by the quiver

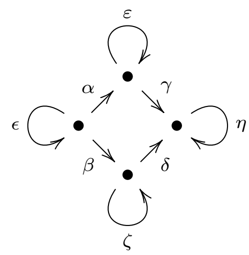


modulo the ideal generated by $\{\beta^n, \gamma^n, \alpha\beta - \gamma\alpha\}$. It is well known that R is representation finite, but $R[t]/(t^n)$ is not CM-finite by [25, Lemma 4.4], and hence not representation finite.

(2) If R is given by the quiver



modulo the ideal generated by $\{\gamma\alpha - \delta\beta\}$, then $R[t]/(t^2)$ is the algebra given by the quiver



modulo the ideal generated by $\{\gamma\alpha - \delta\beta, \epsilon^2, \varepsilon^2, \zeta^2, \eta^2, \alpha\epsilon - \varepsilon\alpha, \gamma\varepsilon - \eta\gamma, \beta\epsilon - \zeta\beta, \delta\zeta - \eta\delta\}$. Since R has finite global dimension, R is CM-free. However, $R[t]/(t^2)$ is not CM-free by [19, Example 4.10].

5. Support τ-tilting modules

In this section, R is a finite-dimensional basic algebra over an algebraically closed field k and $D := \mathrm{Hom}_k(-, k)$. We use τ_R to denote the Auslander-Reiten translation and use $\mathrm{proj}\, R$ to denote the category of finitely generated projective left R-modules. For a module M in $\mathrm{mod}\, R$, we use $\mathrm{Tr}_R(M)$ to denote the Auslander transpose of M. In fact, $R[t]/(t^n)$ is also a finite-dimensional basic algebra over k. We will study how the τ-tilting theory in $\operatorname{mod} R$ can be lifted to that in $R[t]/(t^n)$.

Firstly we need the following lemma.

Lemma 5.1. *Let $M \in \operatorname{mod} R$ and $S = R[t]/(t^n)$. Then the following statements hold.*

(1) $\tau_S(T(M)) \cong T(\tau_R(M))$.
(2) $\tau_S^{-1}(T(M)) \cong T(\tau_R^{-1}(M))$.
(3) $\operatorname{Hom}_S(T(M), \tau_S(T(M))) \cong \operatorname{Hom}_R(M^n, \tau_R(M))$.

Proof. (1) Note that $T(M) = S \otimes_R M$. For any $P \in \operatorname{proj} R$, we claim that there exists an isomorphism

$$\operatorname{Hom}_S(T(P), S) \cong \operatorname{Hom}_R(P, R) \otimes_R S.$$

Suppose $P = Re$ for some idempotent e. Then

$$\begin{aligned}\operatorname{Hom}_S(T(Re), S) &\cong \operatorname{Hom}_S(Se, S) \cong eS \cong T(eR)\\ &= \operatorname{Hom}_R(Re, R) \otimes_R S \cong \operatorname{Hom}_R(P, R) \otimes_R S.\end{aligned}$$

The claim is proved. Now let

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

be a minimal projective presentation of M. Since T is an exact functor, it follows from Theorem 3.3(1) that

$$T(P_1) \xrightarrow{T(f_1)} T(P_0) \xrightarrow{T(f_0)} T(M) \to 0$$

is a minimal projective presentation of $T(M)$. Then we get the following diagram with exact rows

$$\begin{array}{ccccccc}\operatorname{Hom}_S(T(P_0), S) & \xrightarrow{\operatorname{Hom}_S(T(f_1), S)} & \operatorname{Hom}_S(T(P_1), S) & \longrightarrow & \operatorname{Tr}_S(T(M)) & \longrightarrow & 0\\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \vdots{\scriptstyle\gamma} & & \\ \operatorname{Hom}_R(P_0, R) \otimes_R S & \xrightarrow{\operatorname{Hom}_R(f_1, R) \otimes_R S} & \operatorname{Hom}_R(P_1, R) \otimes_R S & \longrightarrow & \operatorname{Tr}_R(M) \otimes_R S & \longrightarrow & 0.\end{array}$$

By the claim above, both α and β are isomorphisms. Thus the induced map γ is also an isomorphism. Therefore we have

$$\begin{aligned}\tau_S(T(M)) &\cong D(\operatorname{Tr}_R(M) \otimes_R S)\\ &\cong \operatorname{Hom}_R(S, \tau_R(M))\\ &\cong S \otimes_R \operatorname{Hom}_R(R, \tau_R(M))\\ &\cong T(\tau_R(M)).\end{aligned}$$

(2) From the proof of (1), we have

$$\operatorname{Tr}_S(T(M)) \cong \operatorname{Tr}_R(M) \otimes_R S \cong T(\operatorname{Tr}_R(M)).$$

Thus we have

$$\begin{aligned}\tau_S^{-1}(T(M)) &\cong \operatorname{Tr}_S(D(T(M)))\\ &\cong \operatorname{Tr}_S(T(D(M)))\\ &\cong T \operatorname{Tr}_R(D(M))\\ &\cong T(\tau_R^{-1}(M)).\end{aligned}$$

(3) By(1), we have

$$\begin{aligned}\operatorname{Hom}_S(T(M),\tau_S(T(M))) &\cong \operatorname{Hom}_S(T(M),T(\tau_R(M)))\\ &\cong \operatorname{Hom}_R(FT(M),\tau_R(M))\\ &\cong \operatorname{Hom}_R(M^n,\tau_R(M)).\end{aligned}$$

□

For a module M in $\operatorname{mod} R$, $|M|$ is the number of pairwise non-isomorphic direct summands of M. The next two definitions are due to Adachi, Iyama and Reiten [2].

Definition 5.2. A module $M \in \operatorname{mod} R$ is called

(1) *τ-rigid* if $\operatorname{Hom}_R(M,\tau_R(M)) = 0$.
(2) *τ-tilting* (respectively, *almost complete τ-tilting*) if it is τ-rigid and $|M| = |R|$ (respectively, $|M| = |R| - 1$).
(3) *support τ-tilting* if there exists an idempotent e of R such that M is a τ-tilting $(R/\langle e\rangle)$-module.

Definition 5.3. Let (M,P) be a pair with $M \in \operatorname{mod} R$ and $P \in \operatorname{proj} R$.

(1) We call (M,P) a *τ-rigid* pair if M is τ-rigid and $\operatorname{Hom}_R(P,M) = 0$.
(2) We call (M,P) a *support τ-tilting* (respectively, *almost complete support τ-tilting*) pair if (M,P) is a τ-rigid pair and $|M| + |P| = |R|$ (respectively, $|M| + |P| = |R| - 1$).

The following result is crucial in proving Theorem 5.5.

Proposition 5.4. *Let (M,P) be a pair with $M \in \operatorname{mod} R$ and $P \in \operatorname{proj} R$. Then the following statements hold.*

(1) (M,P) is a τ-rigid pair if and only if $(T(M),T(P))$ is a τ-rigid pair.
(2) (M,P) is a support τ-rigid (respectively, almost complete support τ-tilting) pair if and only if $(T(M),T(P))$ is a support τ-rigid (respectively, almost complete support τ-tilting) pair.

Proof. (1) If $\operatorname{Hom}_R(P,M) = 0$, then

$$\operatorname{Hom}_{R[t]/(t^n)}(T(P),T(M)) \cong \operatorname{Hom}_R(P,FT(M)) = \operatorname{Hom}_R(P,M^n) = 0$$

by Fact 2.1(2). Conversely, it is easy to check that $\operatorname{Hom}_R(P,M) = 0$ when $\operatorname{Hom}_{R[t]/(t^n)}(T(P),T(M)) = 0$. So Lemma 5.1(3) gives the result.

The assertion (2) follows from (1) and [28, Proposition 3.4]. □

The main result in this section is stated as follows.

Theorem 5.5. *Let $M \in \operatorname{mod} R$. Then the following statements hold.*

(1) M is a τ-rigid R-module if and only if $T(M)$ is a τ-rigid $R[t]/(t^n)$-module.
(2) M is a τ-tilting R-module if and only if $T(M)$ is a τ-tilting $R[t]/(t^n)$-module.
(3) M is an almost complete τ-tilting R-module if and only if $T(M)$ is an almost complete support τ-tilting $R[t]/(t^n)$-module.
(4) M is a support τ-tilting R-module if and only if $T(M)$ is a support τ-tilting $R[t]/(t^n)$-module.

Proof. Using [2, Proposition 2.3], we deduce that (M, P) is a τ-rigid (respectively, support τ-tilting, almost complete support τ-tilting) pair if and only if M is a τ-rigid (respectively, τ-tilting, almost complete τ-tilting) $(R/\langle e\rangle)$-module, where $Re \cong P$ with e an idempotent. Hence we get (4) immediately by Proposition 5.4(2). On the other hand, when we take $P = 0$, it is true that $(M, 0)$ is a τ-rigid (respectively, support τ-tilting, almost complete support τ-tilting) pair if and only if M is a τ-rigid (respectively, τ-tilting, almost complete τ-tilting) R-module. So the assertions (1)–(3) follow from Proposition 5.4 again. □

Given a τ-rigid module M, we use $P({}^{\perp}\tau_R(M))$ to denote the direct sum of one copy of each indecomposable Ext-projective module in ${}^{\perp}\tau_R(M)$ up to isomorphism, where ${}^{\perp}\tau_R(M) = \{X \in \operatorname{mod} R \mid \operatorname{Hom}_R(X, \tau_R(M))\} = 0$, and use U to denote the direct sum of one copy of each indecomposable Ext-projective module in ${}^{\perp}\tau_R(M)$ up to isomorphism that does not belong to $\operatorname{add}(M)$. Then $M \oplus U$ is τ-tilting and U is called the *Bongartz τ-complement* of M (see [2]). For a module $M \in \operatorname{mod} R$, we use $\operatorname{Fac} M$ to denote the category of factor modules of finite direct sums of copies of M.

The following result describes that the functor T preserves and reflects the Bongartz τ_R-complement of a τ-rigid module.

Corollary 5.6. *Let $M, U \in \operatorname{mod} R$. Then U is the Bongartz τ_R-complement of M if and only if $T(U)$ is the Bongartz $\tau_{R[t]/(t^n)}$-complement of $T(M)$.*

Proof. It follows from Theorem 5.5 that M is a τ-rigid R-module if and only if $T(M)$ is a τ-rigid $R[t]/(t^n)$-module.

We first prove the necessity. Since $M \oplus U$ is τ-tilting by assumption, $\operatorname{Hom}_R(M \oplus U, \tau_R(M \oplus U)) = 0$ implies that $U \in {}^{\perp}\tau_R(M)$ and U is a τ-rigid R-module. Hence

$$\begin{aligned}\operatorname{Hom}_{R[t]/(t^n)}(T(U), \tau_{R[t]/(t^n)}(T(M)) &\cong \operatorname{Hom}_{R[t]/(t^n)}(T(U), T(\tau_R(M)))\\ &\cong \operatorname{Hom}_R(U, FT(\tau_R(M))) = 0.\end{aligned}$$

Thus $T(U) \in {}^{\perp}\tau_{R[t]/(t^n)}(T(M))$ and $\operatorname{Fac} T(U) \subseteq {}^{\perp}\tau_{R[t]/(t^n)}(T(M))$. Note that ${}^{\perp}\tau_R(M) \subseteq {}^{\perp}\tau_R(U)$ by [2, Proposition 2.9 and Lemma 2.11]. If there exists an $R[t]/(t^n)$-module X such that $\operatorname{Hom}_{R[t]/(t^n)}(X, \tau_{R[t]/(t^n)}(T(M))) = 0$, then

$$\operatorname{Hom}_{R[t]/(t^n)}(X, T(\tau_R(M))) \cong \operatorname{Hom}_R(FX, \tau_R(M)) = 0,$$

and so

$$\operatorname{Hom}_{R[t]/(t^n)}(X, \tau_{R[t]/(t^n)}(T(U))) \cong \operatorname{Hom}_R(FX, \tau_R(U)) = 0.$$

It follows that

$${}^{\perp}\tau_{R[t]/(t^n)}(T(M)) \subseteq {}^{\perp}\tau_{R[t]/(t^n)}(T(U)).$$

Therefore, in view of [2, Proposition 2.9] again, we have

$$T(U) \in \operatorname{add}(P({}^{\perp}\tau_{R[t]/(t^n)}(T(M)))).$$

Since

$$|T(M) \oplus T(U)| = |M \oplus U| = |R| = |R[t]/(t^n)|,$$

$T(U)$ comprises all the indecomposable Ext-projective modules in ${}^{\perp}\tau_{R[t]/(t^n)}(T(M))$ up to isomorphism not in $\operatorname{add}(T(M))$. Consequently $T(U)$ is the Bongartz $\tau_{R[t]/(t^n)}$-complement of $T(M)$.

Next we prove the sufficiency. Since $T(M\oplus U)$ is τ-tilting by assumption, $M\oplus U$ is τ-tilting by Theorem 5.5 and $\mathrm{Hom}_{R[t]/(t^n)}(T(M\oplus U),\tau_{R[t]/(t^n)}(T(M\oplus U))=0$. It follows that

$$\mathrm{Hom}_R(FT(U),\tau_R(M))\cong \mathrm{Hom}_{R[t]/(t^n)}(T(U),\tau_{R[t]/(t^n)}(T(M)))=0.$$

Thus $\mathrm{Hom}_R(U,\tau_R(M))=0$ and $\mathrm{Fac}\,U\subseteq {}^{\perp}\tau_R(M)$. New let $X\in \mathrm{mod}\,R$ such that $\mathrm{Hom}_R(X,\tau_R(M))=0$, then $\mathrm{Hom}_{R[t]/(t^n)}(TX,T(\tau_R(M))=0$ by Fact 2.1(3). Because ${}^{\perp}\tau_{R[t]/(t^n)}(T(M))\subseteq {}^{\perp}\tau_{R[t]/(t^n)}(T(U))$ by [2, Proposition 2.9] and assumption, we have

$$\mathrm{Hom}_R(FT(X),\tau_R(U))\cong \mathrm{Hom}_{R[t]/(t^n)}(T(X),T(\tau_R(U))=0.$$

So $\mathrm{Hom}_R(X,\tau_R(U))=0$, which implies ${}^{\perp}\tau_R(M)\subseteq {}^{\perp}\tau_R(U)$. It follows from [2, Proposition 2.9] again that $U\in \mathrm{add}(P({}^{\perp}\tau_R(M)))$. The fact that

$$|M\oplus U|=|T(M)\oplus T(U)|=|R[t]/(t^n)|=|R|$$

gives the result. □

Definition 5.7. ([2, Definition 1.5]) Let $P\in K^b(\mathrm{proj}\,R)$, where $K^b(\mathrm{proj}\,R)$ is the homotopy category of bounded complexes of finitely generated projective left R-modules.

(1) We call P *presilting* if $\mathrm{Hom}_{K^b(\mathrm{proj}\,R)}(P,P[i])=0$ for any $i\geqslant 1$.

(2) We call P *silting* if it is presilting and satisfies $\mathrm{thick}(P)=K^b(\mathrm{proj}\,R)$, where $\mathrm{thick}(P)$ is the smallest full triangulated subcategory of $K^b(\mathrm{proj}\,R)$ containing P and being closed under direct summands.

Our next corollary concerns two-term (pre)silting complexes.

Corollary 5.8. *Let*

$$P_1\xrightarrow{f}P_0\xrightarrow{g}M\to 0$$

be a minimal projective presentation of M in $\mathrm{mod}\,R$. *Then $P=(P_1\xrightarrow{f}P_0)$ is (pre)silting if and only if $T(P)=(T(P_1)\xrightarrow{T(f)}T(P_0))$ is (pre)silting.*

Proof. By Theorem 3.3(1) and assumption,

$$T(P_1)\xrightarrow{T(f)}T(P_0)\xrightarrow{T(g)}T(M)\to 0$$

is a minimal projective presentation of $T(M)$. We have that $P=(P_1\xrightarrow{f}P_0)$ is presilting if and only if $\mathrm{Coker}\,f$ is a τ-rigid R-module by [2, Lemma 3.4], and if and only if $T(\mathrm{Coker}\,f)$ is a τ-rigid $R[t]/(t^n)$-module by Theorem 5.5. So $P=(P_1\xrightarrow{f}P_0)$ is presilting if and only if $T(P)=(T(P_1)\xrightarrow{T(f)}T(P_0))$ is presilting.

Next, we have that

$$A\to B\to C\to A[1]$$

is a triangle in $K^b(\mathrm{proj}\,R)$ if and only if

$$T(A)\to T(B)\to T(C)\to T(A)[1]$$

is a triangle in $K^b(\mathrm{proj}\,R[t]/(t^n))$. Thus it follows from [1, Lemma 2.15] that $P=(P_1\xrightarrow{f}P_0)$ is silting if and only if $T(P)=(T(P_1)\xrightarrow{T(f)}T(P_0))$ is silting. □

Following [17], R is called a *tilted algebra* if R is an algebra of the form $\mathrm{End}_H(T)$, where H is a hereditary artin algebra and T is a 1-tilting module in $\mathrm{mod}\, H$. Recall from [2] that a module $M \in \mathrm{mod}\, R$ is *sincere* if every simple R-module appears as a composition factor in M. This is equivalent to the fact that $\mathrm{Hom}_R(P, M) \neq 0$ for every indecomposable summand P of R.

Proposition 5.9. *If $R[t]/(t^n)$ is a tilted algebra, then so is R.*

Proof. Observe that an algebra R is tilted if and only if there exists a sincere module $M \in \mathrm{mod}\, R$ such that either $\mathrm{Hom}_R(X, M) = 0$ or $\mathrm{Hom}_R(M, \tau_R(X)) = 0$ for any indecomposable module $X \in \mathrm{mod}\, R$ ([21, Theorem]).

If $R[t]/(t^n)$ is a tilted algebra, then there exists a sincere module $M \in \mathrm{mod}\, R[t]/(t^n)$ such that either $\mathrm{Hom}_{R[t]/(t^n)}(X, M) = 0$ or $\mathrm{Hom}_{R[t]/(t^n)}(M, \tau_{R[t]/(t^n)}(X)) = 0$ for any indecomposable module $X \in \mathrm{mod}\, R[t]/(t^n)$. For any indecomposable projective R-module P, we have

$$\mathrm{Hom}_R(P, F(M)) \cong \mathrm{Hom}_{R[t]/(t^n)}(T(P), M) \neq 0,$$

which implies that $F(M)$ is a sincere R-module. Given an indecomposable R-module X. Since $\mathrm{Hom}_R(X, F(M) \cong \mathrm{Hom}_{R[t]/(t^n)}(T(X), M)$ and

$$\begin{aligned}\mathrm{Hom}_R(F(M), \tau_R(X)) &\cong \mathrm{Hom}_{R[t]/(t^n)}(M, T(\tau_R(X))) \\ &\cong \mathrm{Hom}_{R[t]/(t^n)}(M, \tau_{R[t]/(t^n)}(T(X))) \text{ (by Lemma 5.1)},\end{aligned}$$

it follows that R is a tilted algebra. □

However, the converse of Proposition 5.9 does not hold true in general.

Example 5.10. Let R be semisimple. It is obvious that R is a tilted algebra and the global dimension of $R[t]/(t^n)$ is infinite. If $R[t]/(t^n)$ is tilted, then the global dimension must be finite by [16, Proposition 2.1], which is a contradiction. So $R[t]/(t^n)$ is not a tilted algebra.

6. m-precluster tilting subcategories

Throughout this section, R is an artin algebra and $m \geqslant 1$. A subcategory $\mathcal{C}$ of $\mathrm{mod}\, R$ is called a *generator* (respectively, *cogenerator*) if $R \in \mathcal{C}$ (respectively, $D(R) \in \mathcal{C}$), where D is the usual duality between $\mathrm{mod}\, R$ and $\mathrm{mod}\, R^{op}$.

Definition 6.1. ([20])

(1) A subcategory $\mathcal{C}$ of $\mathrm{mod}\, R$ is called *m-cluster tilting* if $\mathcal{C}$ is precovering and preenveloping and

$$\begin{aligned}\mathcal{C} &= \{M \in \mathrm{mod}\, R \mid \mathrm{Ext}_R^{1\leqslant i<m}(M, \mathcal{C}) = 0\} \\ &= \{M \in \mathrm{mod}\, R \mid \mathrm{Ext}_R^{1\leqslant i<m}(\mathcal{C}, M) = 0\}.\end{aligned}$$

(2) $\mathcal{C}$ is called an *m-precluster tilting* subcategory if it satisfies the following conditions.
   - (i) $\mathcal{C}$ is a generator–cogenerator for $\mathrm{mod}\, R$.
   - (ii) $\tau_m(\mathcal{C}) := \tau_R\Omega^{m-1}(\mathcal{C}) \subseteq \mathcal{C}$ and $\tau_m^{-1}(\mathcal{C}) := \tau_R^{-1}\Omega^{-(m-1)}(\mathcal{C}) \subseteq \mathcal{C}$, where Ω^{m-1} and $\Omega^{-(m-1)}$ are the $(m-1)$-th syzygy and cosyzygy functors respectively.
   - (iii) $\mathrm{Ext}_R^{1\leqslant i<m}(\mathcal{C}, \mathcal{C}) = 0$.
   - (iv) $\mathcal{C}$ is a precovering and preenveloping subcategory of $\mathrm{mod}\, R$.

If moreover $\mathcal{C}$ admits an additive generator M, then we say that M is an *m-precluster tilting module*.

(3) R is called *τ_m-selfinjective* if R admits an m-precluster tilting module.

Proposition 6.2. *Let $\mathcal{C}$ be an additive subcategory of* $\operatorname{mod} R$ *closed under direct summands. Then $\mathcal{C}$ is m-precluster tilting in* $\operatorname{mod} R$ *if and only if $T(\mathcal{C})$ is m-precluster tilting in* $\operatorname{mod} R[t]/(t^n)$.

Proof. It is trivial that $\mathcal{C}$ is a generator–cogenerator for $\operatorname{mod} R$ if and only if $T(\mathcal{C})$ is a generator–cogenerator for $\operatorname{mod} R[t]/(t^n)$. By Theorem 3.3 and Lemma 5.1, we have that $\tau_m(\mathcal{C}) \subseteq \mathcal{C}$ (respectively, $\tau_m^{-1}(\mathcal{C}) \subseteq \mathcal{C}$) if and only if $\tau_m(T(C)) \subseteq T(C)$ (respectively, $\tau_m^{-1}(T(C)) \subseteq \mathcal{C}$). Using [32, Theorem 3.9], we get that $\operatorname{Ext}_R^{1\leqslant i<m}(\mathcal{C},\mathcal{C}) = 0$ if and only if $\operatorname{Ext}_{R[t]/(t^n)}^{1\leqslant i<m}(T(\mathcal{C}),T(\mathcal{C})) = 0$. Finally, it follows from Theorem 3.3 that $\mathcal{C}$ is precovering and preenveloping in $\operatorname{mod} R$ if and only if $T(\mathcal{C})$ is precovering and preenveloping in $\operatorname{mod} R[t]/(t^n)$. Consequently, the assertion holds true. □

However, Proposition 6.2 is not true for m-cluster tilting subcategories in general, as illustrated in the following example.

Example 6.3. Let $R = k$ be a algebraically closed field and $\mathcal{C} = \operatorname{mod} k$. It is obvious that $\mathcal{C}$ is m-cluster tilting. But $T(\mathcal{C}) = \operatorname{proj} k[t]/(t^n)$ is not m-cluster tilting, since $k[t]/(t^n)$ is not semisimple.

Now we can state the following result.

Theorem 6.4. *R is τ_m-selfinjective if and only if $R[t]/(t^n)$ is τ_m-selfinjective.*

Proof. The necessity follows from Proposition 6.2 directly.

In the following, we prove the sufficiency. In view of [20, Propositon 3.5], R is τ_m-selfinjective if and only if $R \in \mathcal{I}_m$ and $\operatorname{Ext}_R^{1\leqslant i<m}(\mathcal{I}_m,\mathcal{I}_m) = 0$ with $\mathcal{I}_m = \operatorname{add}\{\tau_m^i(D(R))\}_{i=0}^{\infty}$. Since

$$T(\operatorname{add}\{\tau_m^i(D(R))\}_{i=0}^{\infty}) = \operatorname{add}\{\tau_m^i(D(T(R)))\}_{i=0}^{\infty}$$

by Lemma 5.1, we have that R is τ_m-selfinjective by [32, Theorem 3.9(1)]. □

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