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k-Gorenstein Modules

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Abstract Let Λ and Γ be artin algebras and ${}_\Lambda U_\Gamma$ a faithfully balanced selforthogonal bimodule. In this paper, we first introduce the notion of k-Gorenstein modules with respect to ${}_\Lambda U_\Gamma$ and then characterize it in terms of the U-resolution dimension of some special injective modules and the property of the functors $\mathrm{Ext}^i(\mathrm{Ext}^i(-,U),U)$ preserving monomorphisms, which develops a classical result of Auslander. As an application, we study the properties of dual modules relative to Gorenstein bimodules. In addition, we give some properties of ${}_\Lambda U_\Gamma$ with finite left or right injective dimension.

Keywords k-Gorenstein modules, grade of modules, injective dimension
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1 Introduction

Let Λ be a ring. We use mod Λ (resp. mod Λ^{op}) to denote the category of finitely generated left Λ-modules (resp. right Λ-modules).

Let Λ and Γ be rings. A bimodule ${}_\Lambda T_\Gamma$ is said to be faithfully balanced if the natural maps $\Lambda \to \mathrm{End}(T_\Gamma)$ and $\Gamma \to \mathrm{End}({}_\Lambda T)^{op}$ are isomorphisms; and it is said to be selforthogonal if $\mathrm{Ext}^i_\Lambda({}_\Lambda T, {}_\Lambda T) = 0$ and $\mathrm{Ext}^i_\Gamma(T_\Gamma, T_\Gamma) = 0$ for any $i \geq 1$.

Let U and A be in mod Λ (resp. mod Γ^{op}) and i be a non-negetive integer. We say that the grade of A with respect to U, written $\mathrm{grade}_U A$, is greater than or equal to i if $\mathrm{Ext}^j_\Lambda(A,U) = 0$ (resp. $\mathrm{Ext}^j_\Gamma(A,U) = 0$) for any $0 \leq j < i$. We say that the strong grade of A with respect to U, written $\mathrm{s.grade}_U A$, is greater than or equal to i if $\mathrm{grade}_U B \geq i$ for all submodules B of A (see [1]). We give the definition of (k-)Gorenstein modules in terms of the strong grade of modules as follows.

Definition 1.1 *For a non-negative integer k, a module $U \in$ mod Λ with $\Gamma = \mathrm{End}({}_\Lambda U)$ is called k-Gorenstein if $\mathrm{s.grade}_U \mathrm{Ext}^i_\Gamma(N,U) \geq i$ for any $N \in$mod Γ^{op} and $1 \leq i \leq k$. U is called Gorenstein if it is k-Gorenstein for all $k \geq 1$. Similarly, we may define the notions of k-Gorenstein modules and Gorenstein modules in* mod Γ^{op}. *A bimodule ${}_\Lambda U_\Gamma$ is called a (k-)Gorenstein bimodule if both ${}_\Lambda U$ and U_Γ are (k-)Gorenstein.*

Definition 1.2 [2] *Let U be in mod Λ (resp.* mod Γ^{op}*) and k a non-negetive integer. A module M in* mod Λ *(resp.* mod Γ^{op}*) is said to have U-dominant dimension greater than or equal to k, written U-dom.dim$({}_\Lambda M)$ (resp. U-dom.dim(M_Γ)) $\geq k$, if each of the first k terms in a minimal injective resolution of M is cogenerated by ${}_\Lambda U$ (resp. U_Γ), that is, each of these terms can be embedded into a direct product of copies of ${}_\Lambda U$ (resp. U_Γ).*

It is clear that any module in mod Λ (resp. mod Γ^{op}) is 0-Gorenstein. Let Λ and Γ be artin algebras and ${}_\Lambda U_\Gamma$ a faithfully balanced selforthogonal bimodule with ${}_\Lambda U \in$mod Λ and $U_\Gamma \in$mod Γ^{op}. If U-dom.dim$({}_\Lambda U) \geq k$, then each of the first k terms in a minimal injective resolution of ${}_\Lambda U$ is finitely cogenerated, and so each of these terms can be embedded into a

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finite direct product of copies of ${}_\Lambda U$. It follows from Lemma 2.6 below that ${}_\Lambda U$ is k-Gorenstein. It was shown in [3] that U-dom.dim$({}_\Lambda U)$ = U-dom.dim(U_Γ). So, at this moment, U_Γ is also k-Gorenstein. Recall from [4] that a module M in mod Λ (resp. mod Γ^{op}) is called a QF-3 module if $G(M)$ has a cogenerator which is a direct summand of every other cogenerator, where $G(M)$ is the subcategory of mod Λ (resp. mod Γ^{op}) consisting of all submodules of the modules generated by M. It was shown in [4, Proposition 2.2] that a finitely cogenerated Λ-module (resp. Γ^{op}-module) M is a QF-3 module if and only if M cogenerates its injective envelope. So we have that ${}_\Lambda U$ (resp. U_Γ) is 1-Gorenstein if it is a QF-3 module.

A left and right noetherian ring Λ is called k-Gorenstein if, for any $1 \leq i \leq k$, the i-th term in a minimal injective resolution of ${}_\Lambda\Lambda$ has flat dimension at most $i-1$. This notion was introduced by Auslander and Reiten in [5] as a non-commutative version of commutative Gorenstein rings. By Definition 1.1 and [6, Auslander's Theorem 3.7], Λ is a k-Gorenstein ring if it is k-Gorenstein as a Λ-module. Auslander further proved that the notion of k-Gorenstein rings is left-right symmetric (see [6, Auslander's Theorem 3.7]). Wakamatsu in [7, Theorem 7.5] generalized this result and established the left-right symmetry of the notion of k-Gorenstein modules.

In this paper, we will give some further characterizations of k-Gorenstein modules in terms of the U-resolution dimension of some special injective modules and the property of the functors $\mathrm{Ext}^i(\mathrm{Ext}^i(-,U),U)$ preserving monomorphisms, which develops the result of Auslander mentioned above. Our characterizations will lead to a better comprehension about the theory of selforthogonal bimodules and cotilting theory (Note: the class of cotilting bimodules is such a kind of faithfully balanced selforthogonal bimodules with finite left and right injective dimensions [8]).

Throughout this paper, both Λ and Γ are artin algebras (unless stated otherwise), ${}_\Lambda U_\Gamma$ is a faithfully balanced selforthogonal bimodule with ${}_\Lambda U \in$mod Λ and $U_\Gamma \in$mod Γ^{op}.

The following is an outline of this paper. In Section 2 we list some lemmas which will be used later. In Section 3 we characterize k-Gorenstein modules with respect to ${}_\Lambda U_\Gamma$ in terms of the U-resolution dimension (see Section 2 for the definition) of some special injective modules and the property of the functors $\mathrm{Ext}^i(\mathrm{Ext}^i(-,U),U)$ preserving monomorphisms. In fact, we will prove the following theorem, which extends [6, Auslander's Theorem 3.7]:

Theorem *The following statements are equivalent*:

(1) ${}_\Lambda U$ *is* k-*Gorenstein*;

(2) U-resol.dim$_\Lambda(E_i) \leq i$, *where* E_i *is the* $(i+1)$-*st term in a minimal injective of* U *as a left* Λ-*module, for any* $0 \leq i \leq k-1$;

(3) $\mathrm{Ext}^i_\Gamma(\mathrm{Ext}^i_\Lambda(-,U),U)$: mod $\Lambda \to$ mod Λ *preserves monomorphisms for any* $0 \leq i \leq k-1$;

$(1)^{op}$ U_Γ *is* k-*Gorenstein*;

$(2)^{op}$ U-resol.dim$_\Gamma(E'_i) \leq i$, *where* E'_i *is the* $(i+1)$-*st term in a minimal injective of* U *as a right* Γ-*module, for any* $0 \leq i \leq k-1$;

$(3)^{op}$ $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_\Gamma(-,U),U)$: mod $\Gamma^{op} \to$ mod Γ^{op} *preserves monomorphisms for any* $0 \leq i \leq k-1$.

As mentioned above, Wakamatsu in [7, Theorem 7.5] got the equivalence of (1) and $(1)^{op}$ for noetherian rings. However, the proof here is rather different from that in [7]. Moreover, to prove such an equivalence (Proposition 3.5), we get some other results (for example, Lemmas 3.2 and 3.3), which are of independent interest themselves. As corollaries of the Theorem above, we get a new characterization of k-Gorenstein algebras, and we in addition have that, for a faithfully balanced selforthogonal bimodule ${}_\Lambda U_\Lambda$, its left injective dimension and right injective dimension are identical provided ${}_\Lambda U$ (or U_Λ) is Gorenstein. We, in Section 4, study the dual theory relative to Gorenstein modules (Theorems 4.1 and 4.4). In the final section we give some properties of ${}_\Lambda U_\Gamma$ with finite left or right injective dimension. Some known results in [9] and [10] are obtained as corollaries.

2 Preliminaries

In this section we give some lemmas, which are useful in the rest of this paper.

Suppose that $A \in$mod Λ (resp. mod Γ^{op}). We call $\mathrm{Hom}_\Lambda({}_\Lambda A, {}_\Lambda U_\Gamma)$ (resp. $\mathrm{Hom}_\Gamma(A_\Gamma, {}_\Lambda U_\Gamma)$) the dual module of A with respect to U, and denote either of these modules by A^*. For a homomorphism f between Λ-modules (resp. Γ^{op}-modules), we put $f^* = \mathrm{Hom}(f, {}_\Lambda U_\Gamma)$. Let $\sigma_A : A \to A^{**}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$ be the canonical evaluation homomorphism. A is called U-reflexive if σ_A is an isomorphism. Under the assumption of ${}_\Lambda U_\Gamma$ being faithfully balanced, it is easy to see that any projective module in mod Λ (resp. mod Λ^{op}) is U-reflexive.

For a Λ-module (resp. Λ^{op}-module) X, we use $l.\mathrm{fd}_\Lambda(X)$ (resp. $r.\mathrm{fd}_\Lambda(X)$) to denote the left (resp. right) flat dimension of X, and use $l.\mathrm{id}_\Lambda(X)$ (resp. $r.\mathrm{id}_\Lambda(X)$) to denote its left (resp. right) injective dimension. For a Λ-module (resp. Γ^{op}-module) Y, we denote either of $\mathrm{Hom}_\Lambda({}_\Lambda U_\Gamma, {}_\Lambda Y)$ and $\mathrm{Hom}_\Gamma({}_\Lambda U_\Gamma, Y_\Gamma)$ by ${}^*Y$.

Lemma 2.1 *Let Λ and Γ be left and right noetherian rings and n a non-negative integer. If ${}_\Lambda E$ (resp. E_Γ) is injective, then $l.\mathrm{fd}_\Gamma({}^*E)$ (resp. $r.\mathrm{fd}_\Lambda({}^*E)$) $\leq n$ if and only if $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^{n+1}(A, U), E)$ (resp. $\mathrm{Hom}_\Gamma(\mathrm{Ext}_\Lambda^{n+1}(A, U), E)) = 0$ for any $A \in$mod Γ^{op} (resp. mod Λ).*

Proof It is trivial by [11, Chapter VI, Proposition 5.3].

We use $\mathrm{add}_\Lambda U$ (resp. $\mathrm{add} U_\Gamma$) to denote the subcategory of mod Λ (resp. mod Γ^{op}) consisting of all modules isomorphic to direct summands of finite sums of copies of ${}_\Lambda U$ (resp. U_Γ). Let $A \in$mod Λ. If there is an exact sequence $\cdots \to U_n \to \cdots \to U_1 \to U_0 \to A \to 0$ in mod Λ with $U_i \in \mathrm{add}_\Lambda U$ for any $i \geq 0$, then we define the U-resolution dimension of A, denoted by U-resol.dim$_\Lambda(A)$, as $\inf\{n|$ there is an exact sequence $0 \to U_n \to \cdots \to U_1 \to U_0 \to A \to 0$ in mod Λ with $U_i \in \mathrm{add}_\Lambda U$ for any $0 \leq i \leq n\}$. We set U-resol.dim$_\Lambda(A)$ infinity if no such an integer exists. Similarly, for a module B in mod Γ^{op}, we may define U-resol.dim$_\Gamma(B)$.

Lemma 2.2 *Let Λ and Γ be left and right noetherian rings and n a non-negative integer. For a module X in mod Γ^{op}, if $\mathrm{grade}_U X \geq n$ and $\mathrm{grade}_U \mathrm{Ext}_\Gamma^n(X, U) \geq n+1$, then $\mathrm{Ext}_\Gamma^n(X, U) = 0$.*

Proof The proof of [3, Lemma 2.6] remains valid here, so we omit it.

Lemma 2.3 ([3, Lemma 2.7]) *Let $E \in$ mod Λ (resp. mod Γ^{op}) be injective. Then $l.\mathrm{fd}_\Gamma({}^*E)$ (resp. $r.\mathrm{fd}_\Lambda({}^*E)$) $\leq n$ if and only if U-resol.dim$_\Lambda(E)$ (resp. U-resol.dim$_\Gamma(E)$) $\leq n$.*

Lemma 2.4 ([3, Proposition 3.2]) *The following statements are equivalent*:

(1) U-dom.dim$({}_\Lambda U) \geq 1$;

(2) $(-)^{**}$: mod $\Lambda \to$ mod Λ *preserves monomorphisms*;

(3) $0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**}$ *is exact*;

(1) ^{op} U-dom.dim$(U_\Gamma) \geq 1$;

(2) ^{op} $(-)^{**}$: mod $\Gamma^{op} \to$ mod Γ^{op} *preserves monomorphisms*;

(3) ^{op} $0 \to (U_\Gamma)^{**} \xrightarrow{(f_0')^{**}} (E_0')^{**}$ *is exact.*

Lemma 2.5 ([1, Lemma 2.7]) *Let Λ and Γ be left and right noetherian rings. The following statements are equivalent*:

(1) *M^* is U-reflexive for any $M \in$ mod Λ;*

(2) *$[\mathrm{Ext}_\Lambda^2(M, U)]^* = 0$ for any $M \in$ mod Λ;*

(1) ^{op} *N^* is U-reflexive for any $N \in$ mod Γ^{op};*

(2) ^{op} *$[\mathrm{Ext}_\Gamma^2(N, U)]^* = 0$ for any $N \in$ mod Γ^{op}.*

From now on, assume that

$$0 \to {}_\Lambda U \to E_0 \to E_1 \cdots \to E_i \to \cdots$$

is a minimal injective resolution of ${}_\Lambda U$, and

$$0 \to U_\Gamma \to E'_0 \to E'_1 \cdots \to E'_i \to \cdots$$

is a minimal injective resolution of U_Γ.

Lemma 2.6 ([12, Corollary 3.7]) (1) U-resol.dim$_\Lambda(E_i) \leq i$ *for any* $0 \leq i \leq k-1$ *if and only if* s.grade$_U$Ext$^i_\Gamma(N,U) \geq i$ *for any* $N \in$mod Γ^{op} *and* $1 \leq i \leq k$;

(2) U-resol.dim$_\Gamma(E'_i) \leq i$ *for any* $0 \leq i \leq k-1$ *if and only if* s.grade$_U$Ext$^i_\Lambda(M,U) \geq i$ *for any* $M \in$mod Λ *and* $1 \leq i \leq k$.

Lemma 2.7 ${}_\Lambda U$ *is* 1*-Gorenstein if and only if* U_Γ *is* 1*-Gorenstein.*

Proof By [3, Corollary 2.5], ${}^*E_0$ is left Γ-flat if and only if ${}^*E'_0$ is right Λ-flat. By Lemma 2.3, we then have that E_0 is in add$_\Lambda U$ if and only if E'_0 is in addU_Γ. So, it follows from Lemma 2.6 that s.grade$_U$Ext$^1_\Gamma(N,U) \geq 1$ for any $N \in$mod Γ^{op} if and only if s.grade$_U$Ext$^1_\Lambda(M,U) \geq 1$ for any $M \in$mod Λ. Hence we conclude that ${}_\Lambda U$ is 1-Gorenstein if and only if U_Γ is 1-Gorenstein.

3 Characterizations of k-Gorenstein Modules

In this section, we characterize k-Gorenstein modules in terms of the U-resolution dimension of some special injective modules and the property of the functors Ext^i(Ext$^i(-,U),U)$ preserving monomorphisms, and also establish the left-right symmetry of the notion of k-Gorenstein modules by using different methods from that in [7]. In order to get our main theorem, we need some lemmas.

Lemma 3.1 *If* ${}_\Lambda U$ *is* k*-Gorenstein, then* Ext$^i_\Lambda$(Ext$^i_\Gamma(-,U),U)$: mod Γ^{op} →mod Γ^{op} *preserves monomorphisms for any* $0 \leq i \leq k-1$.

Proof We proceed by induction on k. The case for $k=1$ follows from Lemma 2.6 and Lemma 2.4.

Now suppose $k \geq 2$ and $0 \to X \to Y \to Z \to 0$ is an exact sequence in mod Γ^{op}. Then we have in mod Λ the following commutative diagram with the row exact:

$$\begin{array}{ccccccc} \mathrm{Ext}^{k-1}_\Gamma(Z,U) & \xrightarrow{\alpha} & \mathrm{Ext}^{k-1}_\Gamma(Y,U) & \xrightarrow{\beta} & \mathrm{Ext}^{k-1}_\Gamma(X,U) & \xrightarrow{\gamma} & \mathrm{Ext}^{k}_\Gamma(Z,U), \\ & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & \\ & A & & B & & C & \end{array}$$

where $A = \mathrm{Im}\alpha$, $B = \mathrm{Im}\beta$ and $C = \mathrm{Im}\gamma$, and each triangle in the above diagram is an epic-monic resolution. Since ${}_\Lambda U$ is k-Gorenstein, s.grade$_U$Ext$^i_\Gamma(N,U) \geq i$ for any $N \in$mod Γ^{op} and $1 \leq i \leq k$. So grade$_U A \geq k-1$, grade$_U B \geq k-1$, grade$_U C \geq k$ and we have the exact sequences:

$$0 = \mathrm{Ext}^{k-1}_\Lambda(C,U) \longrightarrow \mathrm{Ext}^{k-1}_\Lambda(\mathrm{Ext}^{k-1}_\Gamma(X,U),U) \longrightarrow \mathrm{Ext}^{k-1}_\Lambda(B,U),$$
$$0 = \mathrm{Ext}^{k-2}_\Lambda(A,U) \longrightarrow \mathrm{Ext}^{k-1}_\Lambda(B,U) \longrightarrow \mathrm{Ext}^{k-1}_\Lambda(\mathrm{Ext}^{k-1}_\Gamma(Y,U),U)$$

and we then get a composition of monomorphisms:

$$\mathrm{Ext}^{k-1}_\Lambda(\mathrm{Ext}^{k-1}_\Gamma(X,U),U) \hookrightarrow \mathrm{Ext}^{k-1}_\Lambda(B,U) \hookrightarrow \mathrm{Ext}^{k-1}_\Lambda(\mathrm{Ext}^{k-1}_\Gamma(Y,U),U),$$

which is also a monomorphism.

Let M be in mod Λ (resp. mod Γ^{op}) and $P_1 \xrightarrow{f} P_0 \to M \to 0$ a projective resolution of M in mod Λ (resp. mod Γ^{op}). Then we have an exact sequence $0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to X \to 0$, where $X = \mathrm{Coker}f^*$. For a positive integer k, recall from [13] that M is called U-k-torsionfree if Ext$^i_\Gamma(X,U) = 0$ (resp. Ext$^i_\Lambda(X,U) = 0$) for any $1 \leq i \leq k$. M is called U-k-syzygy if there is an exact sequence $0 \to M \to X_0 \to X_1 \to \cdots \to X_{k-1}$ with all X_i in add$_\Lambda U$ (resp. addU_Γ). Putting ${}_\Lambda U_\Gamma = {}_\Lambda\Lambda_\Lambda$, then, in this case, the notions of U-k-torsionfree modules and U-k-syzygy modules are just that of k-torsionfree modules and k-syzygy modules, respectively (see [14] for the definitions of k-torsionfree modules and k-syzygy modules). We use $\mathscr{T}_U^k(\text{mod } \Lambda)$ (resp. $\mathscr{T}_U^k(\text{mod } \Gamma^{op})$) and $\Omega_U^k(\text{mod } \Lambda)$ (resp. $\Omega_U^k(\text{mod } \Gamma^{op})$) to denote the full subcategory of mod Λ (resp. mod Γ^{op}) consisting of U-k-torsionfree modules and U-k-syzygy modules, respectively. In [13] it was pointed out that $\mathscr{T}_U^k(\text{mod } \Lambda) \subseteq \Omega_U^k(\text{mod } \Lambda)$ and $\mathscr{T}_U^k(\text{mod } \Gamma^{op}) \subseteq \Omega_U^k(\text{mod } \Gamma^{op})$.

The following two lemmas are of independent interest themselves:

Lemma 3.2 *Let Λ and Γ be left and right noetherian rings. If* $\text{grade}_U \text{Ext}_\Lambda^{i+1}(M, U) \geq i$ *for any* $M \in$mod Λ *and* $1 \leq i \leq k-1$, *then each k-syzygy module in* mod Λ *is in* $\Omega_U^k(\text{mod } \Lambda)$.

Proof Suppose that $\text{grade}_U \text{Ext}_\Lambda^{i+1}(M, U) \geq i$ for any $M \in$mod Λ and $1 \leq i \leq k-1$. Then by [1, Theorem 3.1], $\Omega_U^k(\text{mod } \Lambda) = \mathscr{T}_U^k(\text{mod } \Lambda)$. So it suffices to show that each k-syzygy module in mod Λ is in $\mathscr{T}_U^k(\text{mod } \Lambda)$. The following proof is similar to that of (1) $\Rightarrow$ (2) in [1, Theorem 3.1]. For the sake of completeness, we give the proof here.

We proceed by induction on k.

Notice that Λ is U-reflexive, it follows easily that each 1-syzygy module in mod Λ is in $\Omega_U^1(\text{mod } \Lambda)(= \mathscr{T}_U^1(\text{mod } \Lambda))$.

Assume that $k = 2$ and M is a 2-syzygy module in mod Λ. Then there is an exact sequence $0 \to M \to P_1 \xrightarrow{f} P_0$ in mod Λ with P_0 and P_1 projective. By [1, Lemma 2.4], $M \cong (\text{Coker} f^*)^*$. It follows from Lemma 2.5 and [13, Lemma 4] that M is U-reflexive and U-2-torsionfree. The case for $k = 2$ follows.

Now suppose that $k \geq 3$ and M is a k-syzygy module in mod Λ. Then there is an exact sequence:

$$P_{k+1} \xrightarrow{f_{k+1}} P_k \xrightarrow{f_k} P_{k-1} \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to X \to 0$$

in mod Λ such that $M = \text{Coker} f_{k+1}$, where P_i is projective for any $0 \leq i \leq k+1$. By the induction assumption, $M \in \mathscr{T}_U^{k-1}(\text{mod } \Lambda)$. We will show that $M \in \mathscr{T}_U^k(\text{mod } \Lambda)$. Notice that $k \geq 3$, so M is U-reflexive and hence it suffices to show that $\text{Ext}_\Gamma^i(M^*, U) = 0$ for any $1 \leq i \leq k-2$ by [1, Lemma 2.9].

Put $N =\text{Coker} f_{k-1}^*$. Then, by [1, Lemma 2.4], $M \cong N^*$ and $M^* \cong N^{**}$. We claim that $\text{Ext}_\Gamma^i(N, U) = 0$ for any $1 \leq i \leq k-2$. If $k = 3$, then $\text{Coker} f_{k-1}$ is a submodule of P_0. But P_0 is U-reflexive, so $\text{Coker} f_{k-1}$ is U-torsionless. By [15, Lemma 2.1], $\text{Ext}_\Gamma^1(N, U) \cong \text{Ker}\sigma_{\text{Coker} f_{k-1}} = 0$. If $k = 4$, then $\text{Coker} f_{k-1}$ is a 2-syzygy module in mod Λ and so $\text{Coker} f_{k-1}$ is U-reflexive by the above argument. Thus by [15, Lemma 2.1], $\text{Ext}_\Gamma^1(N, U) \cong \text{Ker}\sigma_{\text{Coker} f_{k-1}} = 0$ and $\text{Ext}_\Gamma^2(N, U) \cong \text{Coker}\sigma_{\text{Coker} f_{k-1}} = 0$, and the case for $k = 4$ follows. If $k \geq 5$, then $\text{Coker} f_{k-1}$ is a $(k-2)$-syzygy module in mod Λ and so $\text{Coker} f_{k-1} \in \mathscr{T}_U^{k-2}(\text{mod } \Lambda)$ by the induction assumption. It is clear that $\text{Coker} f_{k-1}$ is U-reflexive. Then by using an argument similar to that in the proof of the case for $k = 4$, we have that $\text{Ext}_\Gamma^1(N, U) = 0 =\text{Ext}_\Gamma^2(N, U) = 0$. On the other hand, by [1, Lemma 2.9], we have that $\text{Ext}_\Gamma^i((\text{Coker} f_{k-1})^*, U) = 0$ for any $1 \leq i \leq k-4$. It follows from the exact sequence $0 \to (\text{Coker} f_{k-1})^* \to P_{k-2}^* \xrightarrow{f_{k-1}^*} P_{k-1}^* \to N \to 0$ that $\text{Ext}_\Gamma^i(N, U) = 0$ for any $3 \leq i \leq k-2$. So $\text{Ext}_\Gamma^i(N, U) = 0$ for any $1 \leq i \leq k-2$.

By [15, Lemma 2.1], we have an exact sequence:

$$0 \to \text{Ext}_\Lambda^1(\text{Coker} f_{k-1}, U) \to N \xrightarrow{\sigma_N} N^{**} \to \text{Ext}_\Lambda^2(\text{Coker} f_{k-1}, U) \to 0.$$

Then $\text{Ker}\sigma_N \cong \text{Ext}_\Lambda^1(\text{Coker} f_{k-1}, U) \cong \text{Ext}_\Lambda^{k-1}(X, U)$ and $\text{Coker}\sigma_N \cong \text{Ext}_\Lambda^2(\text{Coker} f_{k-1}, U) \cong \text{Ext}_\Lambda^k(X, U)$. So we get the following exact sequences:

$$0 \to \text{Ext}_\Lambda^{k-1}(X, U) \to N \xrightarrow{\pi} \text{Im}\sigma_N \to 0, \tag{1}$$

$$0 \to \text{Im}\sigma_N \xrightarrow{\mu} N^{**} \to \text{Ext}_\Lambda^k(X, U) \to 0, \tag{2}$$

where $\sigma_N = \mu\pi$. Since $\text{Ext}_\Gamma^i(N, U) = 0$ for any $1 \leq i \leq k-2$ and $\text{grade}_U \text{Ext}_\Lambda^{k-1}(X, U) \geq k-2$, from the exact sequence (1) we have $\text{Ext}_\Gamma^i(\text{Im}\sigma_N, U) = 0$ for any $1 \leq i \leq k-2$. Moreover, since

$\mathrm{grade}_U \mathrm{Ext}^k_\Lambda(X, U) \geq k-1$, from the exact sequence (2) we get that $\mathrm{Ext}^i_\Gamma(N^{**}, U) = 0$ for any $1 \leq i \leq k-2$, which yields $\mathrm{Ext}^i_\Gamma(M^*, U) = 0$ for any $1 \leq i \leq k-2$. We are done.

Lemma 3.3 *Let Λ and Γ be left and right noetherian rings. For a positive integer k, the following statements are equivalent*:

(1) $\mathrm{grade}_U \mathrm{Ext}^{i+1}_\Lambda(M, U) \geq i$ *for any* $M \in \mathrm{mod}\ \Lambda$ *and* $1 \leq i \leq k-1$;
(2) $\mathrm{Ext}^{i-1}_\Gamma(\mathrm{Ext}^{i+1}_\Lambda(M, U), U) = 0$ *for any* $M \in \mathrm{mod}\ \Lambda$ *and* $1 \leq i \leq k-1$;
(1) ^{op} $\mathrm{grade}_U \mathrm{Ext}^{i+1}_\Gamma(N, U) \geq i$ *for any* $N \in \mathrm{mod}\ \Gamma^{op}$ *and* $1 \leq i \leq k-1$;
(2) ^{op} $\mathrm{Ext}^{i-1}_\Lambda(\mathrm{Ext}^{i+1}_\Gamma(N, U), U) = 0$ *for any* $N \in \mathrm{mod}\ \Gamma^{op}$ *and* $1 \leq i \leq k-1$.

Proof The implications that (1) $\Rightarrow$ (2) and $(1)^{op} \Rightarrow (2)^{op}$ are trivial.

(2) $\Rightarrow$ (1) We proceed by induction on k. It is trivial when $k = 1$ or $k = 2$. Now suppose $k \geq 3$. By the induction assumption, for any $M \in \mathrm{mod}\ \Lambda$, we have $\mathrm{grade}_U \mathrm{Ext}^{i+1}_\Lambda(M, U) \geq i$ for any $1 \leq i \leq k-2$ and $\mathrm{grade}_U \mathrm{Ext}^k_\Lambda(M, U) \geq k-2$. In addition, by (2), we have $\mathrm{Ext}^{k-2}_\Gamma(\mathrm{Ext}^k_\Lambda(M, U), U) = 0$. So $\mathrm{grade}_U \mathrm{Ext}^k_\Lambda(M, U) \geq k-1$.

$(1)^{op} \Rightarrow (1)$ We also proceed by induction on k. The case $k = 1$ is trivial. The case $k = 2$ follows from Lemma 2.5. Now suppose $k \geq 3$.

Let $M \in \mathrm{mod}\ \Lambda$ and

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

a projective resolution of M in mod Λ. Put $M_i = \mathrm{Coker}(P_i \to P_{i-1})$ (where $M_1 = M$) and $X_i = \mathrm{Coker}(P^*_{i-1} \to P^*_i)$ for any $i \geq 1$. By the induction assumption, we have $\mathrm{grade}_U \mathrm{Ext}^{i+1}_\Lambda(M, U) \geq i$ for any $1 \leq i \leq k-2$ and $\mathrm{grade}_U \mathrm{Ext}^k_\Lambda(M, U) \geq k-2$. So it suffices to prove $\mathrm{Ext}^{k-2}_\Gamma(\mathrm{Ext}^k_\Lambda(M, U), U) = 0$.

By [1, Theorem 3.1], $\Omega^i_U(\mathrm{mod}\ \Lambda) = \mathscr{T}^i_U(\mathrm{mod}\ \Lambda)$ for any $1 \leq i \leq k-1$. For any $t \geq k$, since $M_t \in \Omega^{k-1}_U(\mathrm{mod}\ \Lambda)$ by Lemma 3.2, $M_t \in \mathscr{T}^{k-1}_U(\mathrm{mod}\ \Lambda)$. It follows that $\mathrm{Ext}^j_\Gamma(X_t, U) = 0$ for any $1 \leq j \leq k-1$ and $t \geq k$.

On the other hand, by [16, Lemma 2], we have an exact sequence:

$$0 \to \mathrm{Ext}^k_\Lambda(M, U) \to X_k \to P^*_{k+1} \to X_{k+1} \to 0.$$

Put $K = \mathrm{Im}(X_k \to P^*_{k+1})$. From the exactness of $0 \to K \to P^*_{k+1} \to X_{k+1} \to 0$ we know that $\mathrm{Ext}^j_\Gamma(K, U) = 0$ for any $1 \leq j \leq k-2$ and $\mathrm{Ext}^k_\Gamma(X_{k+1}, U) \cong \mathrm{Ext}^{k-1}_\Gamma(K, U)$. Moreover, from the exactness of $0 \to \mathrm{Ext}^k_\Lambda(M, U) \to X_k \to K \to 0$ we know that $\mathrm{Ext}^{k-1}_\Gamma(K, U) \cong \mathrm{Ext}^{k-2}_\Gamma(\mathrm{Ext}^k_\Lambda(M, U), U)$. So $\mathrm{Ext}^{k-2}_\Gamma(\mathrm{Ext}^k_\Lambda(M, U), U) \cong \mathrm{Ext}^k_\Gamma(X_{k+1}, U)$. By $(1)^{op}$, we then have that $\mathrm{grade}_U \mathrm{Ext}^{k-2}_\Gamma(\mathrm{Ext}^k_\Lambda(M, U), U) = \mathrm{grade}_U \mathrm{Ext}^k_\Gamma(X_{k+1}, U) \geq k-1$. It follows from Lemma 2.2 that $\mathrm{Ext}^{k-2}_\Lambda(\mathrm{Ext}^k_\Lambda(M, U), U) = 0$.

By symmetry, we have the implications of $(2)^{op} \Rightarrow (1)^{op}$ and $(1) \Rightarrow (1)^{op}$. We are done.

The following result not only generalizes [14, Proposition 2.26], but also means that the statements in this proposition are left-right symmetric.

Corollary 3.4 *Let Λ be a left and right noetherian ring. For a positive integer k, the following statements are equivalent*:

(1) $\mathrm{grade}_\Lambda \mathrm{Ext}^{i+1}_\Lambda(M, \Lambda) \geq i$ *for any* $M \in \mathrm{mod}\ \Lambda$ *and* $1 \leq i \leq k-1$;
(2) $\mathrm{Ext}^{i-1}_\Lambda(\mathrm{Ext}^{i+1}_\Lambda(M, \Lambda), \Lambda) = 0$ *for any* $M \in \mathrm{mod}\ \Lambda$ *and* $1 \leq i \leq k-1$;
(3) $\Omega^i_\Lambda(\mathrm{mod}\ \Lambda) = \mathscr{T}^i_\Lambda(\mathrm{mod}\ \Lambda)$ *for any* $1 \leq i \leq k$;
(1) ^{op} $\mathrm{grade}_\Lambda \mathrm{Ext}^{i+1}_\Lambda(N, \Lambda) \geq i$ *for any* $N \in \mathrm{mod}\ \Lambda^{op}$ *and* $1 \leq i \leq k-1$;
(2) ^{op} $\mathrm{Ext}^{i-1}_\Lambda(\mathrm{Ext}^{i+1}_\Lambda(N, \Lambda), \Lambda) = 0$ *for any* $N \in \mathrm{mod}\ \Lambda^{op}$ *and* $1 \leq i \leq k-1$;
(3) ^{op} $\Omega^i_\Lambda(\mathrm{mod}\ \Lambda^{op}) = \mathscr{T}^i_\Lambda(\mathrm{mod}\ \Lambda^{op})$ *for any* $1 \leq i \leq k$.

Proof By Lemma 3.3 we have $(1) \Leftrightarrow (2) \Leftrightarrow (1)^{op} \Leftrightarrow (2)^{op}$, and by [14, Proposition 2.26] we have $(1) \Leftrightarrow (3)$ and $(1)^{op} \Leftrightarrow (3)^{op}$.

The following proposition, obtained by Wakamatsu in [7, Theorem 7.5] for noetherian rings, shows the left-right symmetry of the notion of k-Gorenstein modules. However, the proof here is rather different from that in [7].

Proposition 3.5 *For a positive integer k, the following statements are equivalent:*

(1) $\text{s.grade}_U \text{Ext}^i_\Lambda(M,U) \geq i$ *for any* $M \in \text{mod } \Lambda$ *and* $1 \leq i \leq k$;

(2) $\text{s.grade}_U \text{Ext}^i_\Gamma(N,U) \geq i$ *for any* $N \in \text{mod } \Gamma^{op}$ *and* $1 \leq i \leq k$.

Proof By symmetry, we need to prove only that (2) implies (1).

We proceed by induction on k. The case $k = 1$ follows from Lemma 2.7. Now suppose that $k \geq 2$ and $\text{s.grade}_U \text{Ext}^i_\Gamma(N,U) \geq i$ for any $N \in \text{mod } \Gamma^{op}$ and $1 \leq i \leq k$. By Lemma 3.3 and [1, Theorem 3.1], we have that $\mathscr{T}^i_U(\text{mod } \Lambda) = \Omega^i_U(\text{mod } \Lambda)$ for any $1 \leq i \leq k$. By the induction assumption, for any $M \in \text{mod } \Lambda$, we have that $\text{s.grade}_U \text{Ext}^i_\Lambda(M,U) \geq i$ for any $1 \leq i \leq k-1$ and $\text{s.grade}_U \text{Ext}^k_\Lambda(M,U) \geq k-1$.

Assume that

$$\cdots \xrightarrow{f_k} P_{k-1} \xrightarrow{f_{k-1}} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \longrightarrow M \longrightarrow 0$$

is a (minimal) projective resolution of M in mod Λ. For any $t \geq k$, since $\text{Coker} f_t$ is $(k-1)$-syzygy, by Lemma 3.2 we have that $\text{Coker} f_t \in \Omega^{k-1}_U(\text{mod } \Lambda)$ and $\text{Coker} f_t \in \mathscr{T}^{k-1}_U(\text{mod } \Lambda)$, that is, $\text{Coker} f_t$ is U-$(k-1)$-torsionfree, which implies that $\text{Ext}^j_\Gamma(\text{Coker} f^*_t, U) = 0$ for any $1 \leq j \leq k-1$ and $t \geq k$.

Let X be a submodule of $\text{Ext}^k_\Lambda(M,U)$. Then $\text{grade}_U X \geq k-1$. On the other hand, by [15, Lemma 2.1] there is an exact sequence $0 \to \text{Ext}^k_\Lambda(M,U) \to \text{Coker} f^*_k \overset{\sigma_{\text{Coker} f^*_k}}{\longrightarrow} (\text{Coker} f^*_k)^{**} \to \text{Ext}^{k+1}_\Lambda(M,U) \to 0$ and then there is a composition of monomorphisms: $X \hookrightarrow \text{Ext}^k_\Lambda(M,U) \hookrightarrow \text{Coker} f^*_k$. Put $Y = \text{Coker}(X \hookrightarrow \text{Coker} f^*_k)$. Notice that $\text{Ext}^j_\Gamma(\text{Coker} f^*_k, U) = 0$ for any $1 \leq j \leq k-1$, we then have an embedding $\text{Ext}^{k-1}_\Gamma(X,U) \hookrightarrow \text{Ext}^k_\Gamma(Y,U)$. By assumption, $\text{s.grade}_U \text{Ext}^k_\Gamma(Y,U) \geq k$. So $\text{grade}_U \text{Ext}^{k-1}_\Gamma(X,U) \geq k$ and hence $\text{Ext}^{k-1}_\Gamma(X,U) = 0$ by Lemma 2.2. It follows that $\text{grade}_U X \geq k$ and $\text{s.grade}_U \text{Ext}^k_\Lambda(M,U) \geq k$. We are done.

Lemma 3.6 *If* $\text{Ext}^i_\Gamma(\text{Ext}^i_\Lambda(-,U),U)$: mod $\Lambda \to$mod Λ *preserves monomorphisms for any* $0 \leq i \leq k-1$, *then* ${}_\Lambda U$ *is k-Gorenstein.*

Proof We proceed by induction on k.

Assume that $(-)^{**}$: mod $\Lambda \to$ mod Λ preserves monomorphisms, then, by Lemma 2.4, U-dom.dim$({}_\Lambda U) \geq 1$ and E_0 is cogenerated by ${}_\Lambda U$. But E_0 is finitely cogenerated, so $E_0 \in \text{add}_\Lambda U$. By Lemma 2.6, we then have that $\text{s.grade}_U \text{Ext}^1_\Gamma(N,U) \geq 1$ for any $N \in \text{mod } \Gamma^{op}$ and ${}_\Lambda U$ is 1-Gorenstein. The case for $k = 1$ is proved.

Now suppose $k \geq 2$ and

$$\cdots \xrightarrow{g_k} Q_{k-1} \xrightarrow{g_{k-1}} \cdots \longrightarrow Q_1 \xrightarrow{g_1} Q_0 \longrightarrow N \longrightarrow 0$$

is a (minimal) projective resolution of a module N in mod Γ^{op}. By the induction hypothesis, ${}_\Lambda U$ is $(k-1)$-Gorenstein and $\text{s.grade}_U \text{Ext}^i_\Gamma(B,U) \geq i$ for any $B \in \text{mod } \Gamma^{op}$ and $1 \leq i \leq k-1$ (and certainly, $\text{s.grade}_U \text{Ext}^{i+1}_\Gamma(B,U) \geq i$ for any $B \in \text{mod } \Gamma^{op}$ and $1 \leq i \leq k-1$). So $\mathscr{T}^i_U(\text{mod } \Gamma^{op}) = \Omega^i_U(\text{mod } \Gamma^{op})$ for any $1 \leq i \leq k$ by the dual statements of [1, Theorem 3.1]. By using a similar argument to that for (2) $\Rightarrow$ (1) in Proposition 3.5, we have that $\text{Ext}^j_\Lambda(\text{Coker} g^*_k, U) = 0$ for any $1 \leq j \leq k-1$.

Let X be a submodule of $\text{Ext}^k_\Gamma(N,U)$. Then $\text{grade}_U X \geq k-1$. By using a similar argument to that for (2) $\Rightarrow$ (1) in Proposition 3.5, we have a monomorphism $X \hookrightarrow \text{Coker} g^*_k$. By assumption, $0 \to \text{Ext}^{k-1}_\Gamma(\text{Ext}^{k-1}_\Lambda(X,U),U) \to \text{Ext}^{k-1}_\Gamma(\text{Ext}^{k-1}_\Lambda(\text{Coker} g^*_k,U),U) (= 0)$ is exact, so $\text{Ext}^{k-1}_\Gamma(\text{Ext}^{k-1}_\Lambda(X,U),U) = 0$. On the other hand, by Proposition 3.5, we have that $\text{s.grade}_U \text{Ext}^{k-1}_\Lambda(X,U) \geq k-1$. So we conclude that $\text{grade}_U \text{Ext}^{k-1}_\Lambda(X,U) \geq k$ and hence $\text{Ext}^{k-1}_\Lambda(X,U) = 0$ by Lemma 2.2. It follows that $\text{grade}_U X \geq k$ and $\text{s.grade}_U \text{Ext}^k_\Gamma(N,U) \geq k$. We are done.

We are now in a position to state the main result in this paper.

Theorem 3.7 *The following statements are equivalent*:

(1) ${}_\Lambda U$ *is* k-*Gorenstein*;

(2) U-resol.dim$_\Lambda(E_i) \leq i$ *for any* $0 \leq i \leq k-1$;

(3) $\mathrm{Ext}^i_\Gamma(\mathrm{Ext}^i_\Lambda(-,U),U)$: mod $\Lambda \to$ mod Λ *preserves monomorphisms for any* $0 \leq i \leq k-1$;

(1) ^{op} U_Γ *is* k-*Gorenstein*;

(2) ^{op} U-resol.dim$_\Gamma(E_i^{'}) \leq i$ *for any* $0 \leq i \leq k-1$;

(3) ^{op} $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_\Gamma(-,U),U)$: mod $\Gamma^{op} \to$ mod Γ^{op} *preserves monomorphisms for any* $0 \leq i \leq k-1$.

Proof (2) $\Leftrightarrow$ (1) $\Leftrightarrow$ $(1)^{op}$ See Lemma 2.6 and Proposition 3.5.

(1) $\Rightarrow$ $(3)^{op}$ By Lemma 3.1.

(3) $\Rightarrow$ (1) By Lemma 3.6.

Symmetrically we have that $(2)^{op} \Leftrightarrow (1)^{op}$, $(1)^{op} \Rightarrow (3)$ and $(3)^{op} \Rightarrow (1)^{op}$. The proof is finished.

Put ${}_\Lambda U_\Gamma = {}_\Lambda \Lambda_\Lambda$. By Theorem 3.7, we then immediately have the following corollary, which extends [6, Auslander's Theorem 3.7]:

Corollary 3.8 *The following statements are equivalent*:

(1) s.grade$_\Lambda \mathrm{Ext}^i_\Lambda(M,\Lambda) \geq i$ *for any* $M \in$mod Λ *and* $1 \leq i \leq k$;

(2) *The left flat dimension of the* i-th *term in a minimal injective resolution of* ${}_\Lambda\Lambda$ *is at most* $i-1$ *for any* $1 \leq i \leq k$;

(3) $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_\Lambda(-,\Lambda),\Lambda)$: mod $\Lambda \to$ mod Λ *preserves monomorphisms for any* $0 \leq i \leq k-1$;

(1) ^{op} s.grade$_\Lambda \mathrm{Ext}^i_\Lambda(N,\Lambda) \geq i$ *for any* $N \in$mod Λ^{op} *and* $1 \leq i \leq k$;

(2) ^{op} *The right flat dimension of the* i-th *term in a minimal injective resolution of* Λ_Λ *is at most* $i-1$ *for any* $1 \leq i \leq k$;

(3) ^{op} $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_\Lambda(-,\Lambda),\Lambda)$: mod $\Lambda^{op} \to$ mod Λ^{op} *preserves monomorphisms for any* $0 \leq i \leq k-1$.

${}_\Lambda U_\Gamma$ is called a cotilting bimodule if ${}_\Lambda U$ and U_Γ are cotilting, that is, l.id$_\Lambda(U)$ and r.id$_\Gamma(U)$ are finite[8]. If ${}_\Lambda U_\Gamma$ is a cotilting bimodule, then l.id$_\Lambda(U)$ $=r$.id$_\Gamma(U)$ (see [17, Lemma 1.7). However, in general, we don't know whether l.id$_\Lambda(U) < \infty$ implies that r.id$_\Gamma(U) < \infty$. In fact, Auslander and Reiten in [18, p. 150] posed an important question which remains open: For an artin algebra Λ, does l.id$_\Lambda(\Lambda) < \infty$ imply r.id$_\Lambda(\Lambda) < \infty$? Putting ${}_\Lambda U_\Gamma = {}_\Lambda U_\Lambda$, as applications of the results obtained above we have the following corollaries:

Corollary 3.9 *For a positive integer* k, *if* r.id$_\Lambda(U) = k$ *and* U_Λ *is* $(k-1)$-*Gorenstein, then* l.id$_\Lambda(U) = k$.

Proof The case for $k=1$ follows from [16, Corollary 1]. Now assume that r.id$_\Lambda(U) = k(\geq 2)$ and U_Λ is $(k-1)$-Gorenstein. Then, by Lemma 2.6, we have s.grade$_U \mathrm{Ext}^{k-1}_\Lambda(M,U) \geq k-1$ for any $M \in$mod Λ. It follows from [16, Theorem] that l.id$_\Lambda(U) \leq 2k-2$. So l.id$_\Lambda(U) = k$ by [17, Lemma 1.7].

Corollary 3.10 l.id$_\Lambda(U)$=r.id$_\Lambda(U)$ *if* ${}_\Lambda U$ *(or* U_Λ*) is Gorenstein.*

Proof By Theorem 3.7, Corollary 3.9 and its dual result.

Corollary 3.11 ([5, Corollary 5.5]) *Let* Λ *be a* k-*Gorenstein algebra for all* k. *Then* l.id$_\Lambda(\Lambda) =$ r.id$_\Lambda(\Lambda)$.

4 Dual Theory

In this section we study the dual theory relative to Gorenstein bimodules.

For a non-negative integer g, we use $\mathscr{G}_g$(mod Λ) (resp. $\mathscr{G}_g$(mod Γ^{op})) to denote the subcategory of mod Λ (resp. mod Γ^{op}) consisting of the modules M with grade$_U M = g$, and $\mathscr{H}_g$(mod Λ) (resp. $\mathscr{H}_g$(mod Γ^{op})) to denote the subcategory of $\mathscr{G}_g$(mod Λ) (resp. $\mathscr{G}_g$(mod Γ^{op})) consisting of the modules M with $\mathrm{Ext}^i_\Lambda(M,U) = 0$ (resp. $\mathrm{Ext}^i_\Gamma(M,U) = 0$) for any $i \neq$ grade$_U M (= g)$.

Theorem 4.1 *Let* Λ *and* Γ *be left and right noetherian rings and* ${}_\Lambda U_\Gamma$ *a Gorenstein bimodule.*

(1) *If* $r.\mathrm{id}_\Gamma(U) = g$, *then, for any* $0 \neq M \in \mathscr{G}_g(\mathrm{mod}\ \Lambda)$, $M \cong \mathrm{Ext}^g_\Gamma(\mathrm{Ext}^g_\Lambda(M,U),U)$;

(2) *If* $r.\mathrm{id}_\Gamma(U) = l.\mathrm{id}_\Lambda(U) = g$, *then there is a duality between* $\mathscr{G}_g(\mathrm{mod}\ \Lambda)$ *and* $\mathscr{G}_g(\mathrm{mod}\ \Gamma^{op})$ *given by* $M \to \mathrm{Ext}^g_\Lambda(M,U)$.

Proof (1) Let M be a non-zero module in $\mathscr{G}_g(\mathrm{mod}\ \Lambda)$ and

$$\cdots \to P_i \xrightarrow{f_i} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to M \to 0$$

a projective resolution of M in mod Λ. If $g = 0$, it is easy to see from [15, Lemma 2.1] that $M \cong M^{**}$. Now suppose $g \geq 1$. Then we get an exact sequence:

$$0 \to P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{f_2^*} \cdots \xrightarrow{f_{g-1}^*} P_{g-1}^* \to (\mathrm{Im} f_g)^* \to \mathrm{Ext}^g_\Lambda(M,U) \to 0.$$

Since ${}_\Lambda U_\Gamma$ is a Gorenstein bimodule, $\mathrm{s.grade}_U \mathrm{Ext}^g_\Lambda(M,U) \geq g$. For any $i \geq 1$, put $K_i = \mathrm{Coker} f_i^*$. We then have an exact sequence:

$$P_1^{**} \xrightarrow{f_1^{**}} P_0^{**} \to \mathrm{Ext}^1_\Gamma(K_1,U)(\cong \mathrm{Ext}^{g-1}_\Gamma(K_{g-1},U)) \to 0.$$

On the other hand, we have an exact sequence $0 \to (\mathrm{Im} f_g)^* \to P_g^* \xrightarrow{f_{g+1}^*} P_{g+1}^* \to K_{g+1} \to 0$. Since $r.\mathrm{id}_\Gamma(U) = g$, for any $i \geq g-1$ we have $\mathrm{Ext}^i_\Gamma((\mathrm{Im} f_g)^*,U) \cong \mathrm{Ext}^{i+2}_\Gamma(K_{g+1},U) = 0$. Moreover, the exact sequence $0 \to K_{g-1} \to (\mathrm{Im} f_g)^* \to \mathrm{Ext}^g_\Lambda(M,U) \to 0$ yields an exact sequence:

$$\mathrm{Ext}^{g-1}_\Gamma((\mathrm{Im} f_g)^*,U) \to \mathrm{Ext}^{g-1}_\Gamma(K_{g-1},U) \to \mathrm{Ext}^g_\Gamma(\mathrm{Ext}^g_\Lambda(M,U),U) \to \mathrm{Ext}^g_\Gamma((\mathrm{Im} f_g)^*,U).$$

So $\mathrm{Ext}^g_\Gamma(\mathrm{Ext}^g_\Lambda(M,U),U) \cong \mathrm{Ext}^{g-1}_\Gamma(K_{g-1},U) \cong \mathrm{Ext}^1_\Gamma(K_1,U)$ and we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} P_1 & \xrightarrow{f_1} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \sigma_{P_1}} & & \downarrow{\scriptstyle \sigma_{P_0}} & & \downarrow{\scriptstyle h} & & \\ P_1^{**} & \xrightarrow{f_1^{**}} & P_0^{**} & \longrightarrow & \mathrm{Ext}^g_\Gamma(\mathrm{Ext}^g_\Lambda(M,U),U) & \longrightarrow & 0, \end{array}$$

where σ_{P_1} and σ_{P_0} are isomorphisms. Hence h is also an isomorphism and $\mathrm{Ext}^g_\Gamma(\mathrm{Ext}^g_\Lambda(M,U), U) \cong M(\neq 0)$. By assumption, ${}_\Lambda U_\Gamma$ is a Gorenstein bimodule, so $\mathrm{grade}_U \mathrm{Ext}^g_\Lambda(M,U) \geq g$ and hence $\mathrm{grade}_U \mathrm{Ext}^g_\Lambda(M,U) = g$.

(2) It follows from (1) and its dual result.

The following two corollaries are immediate consequences of Theorem 4.1:

Corollary 4.2 ([8, Proposition 3.1]) *Let* Λ *and* Γ *be left and right noetherian rings. If* U_Γ *is injective, then* $M \cong M^{**}$ *for any* M *in* mod Λ.

Corollary 4.3 *Under the assumptions of Theorem* 4.1(2), *there is a duality between* $\mathscr{H}_g(\mathrm{mod}\ \Lambda)$ *and* $\mathscr{H}_g(\mathrm{mod}\ \Gamma^{op})$ *given by* $M \to \mathrm{Ext}^g_\Lambda(M,U)$ (*where* $M \in \mathscr{H}_g(\mathrm{mod}\ \Lambda)$).

The following result is a generalization of [10, Theorem 6], which gives some characterizations of the modules in $\mathscr{H}_g(\mathrm{mod}\ \Lambda)$.

Theorem 4.4 *Let* ${}_\Lambda U_\Gamma$ *be a Gorenstein bimodule with* $r.\mathrm{id}_\Gamma(U) = l.\mathrm{id}_\Lambda(U) = g$. *Then, for any* $0 \neq M \in$ mod Λ, *the following statements are equivalent*:

(1) $M \in \mathscr{H}_g(\mathrm{mod}\ \Lambda)$;

(2) $M \cong \mathrm{Ext}^g_\Gamma(\mathrm{Ext}^g_\Lambda(M,U),U)$;

(3) $M \cong \mathrm{Ext}^g_\Gamma(N,U)$ *for some* $N \in$ mod Γ^{op};

(4) $\mathrm{Hom}_\Lambda(M, \bigoplus_{i=0}^{g-1} E_i) = 0$.

Proof (1) $\Rightarrow$ (2) follows from Corollary 4.3, and (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4) Since ${}_\Lambda U_\Gamma$ is a Gorenstein bimodule, U-resol.dim$_\Lambda(E_i) \leq i$ for any $0 \leq i \leq g-1$ by Lemma 2.6. Then we get our conclusion by Lemma 2.3 and Lemma 2.1.

(4) $\Rightarrow$ (1) Since $l.\mathrm{id}_\Lambda(U) = g$, $\mathrm{Ext}^i_\Lambda(M, U) = 0$ for any $i \geq g + 1$. On the other hand, we know that $M^* = 0$ because $\mathrm{Hom}_\Lambda(M, E_0) = 0$. In addition, we have an exact sequence:

$$0 \to K_{i-1} \to E_{i-1} \to K_i \to 0,$$

for any $1 \leq i \leq g - 1$, where $K_{i-1} = \mathrm{Ker}(E_{i-1} \to E_i)$. From $\mathrm{Hom}_\Lambda(M, E_i) = 0$ we know that $\mathrm{Hom}_\Lambda(M, K_i) = 0$. But $\mathrm{Hom}_\Lambda(M, K_i) \to \mathrm{Ext}^i_\Lambda(M, U)(\cong \mathrm{Ext}^1_\Lambda(M, K_{i-1})) \to 0$ is exact, so $\mathrm{Ext}^i_\Lambda(M, U) = 0$ for any $1 \leq i \leq g - 1$ and $\mathrm{grade}_U M \geq g$. We claim that $\mathrm{grade}_U M = g$. Otherwise, if $\mathrm{grade}_U M > g$, we then have that $\mathrm{Ext}^g_\Lambda(M, U) = 0$ and $\mathrm{Ext}^i_\Lambda(M, U) = 0$ for any $i \geq 1$. It follows from [15, Corollary 2.5] that $M = 0$, which is a contradiction.

Let $\mathscr{A}$ be an abelian category and $\mathscr{B}$ a full subcategory of $\mathscr{A}$. An object $X \in \mathscr{A}$ is called an embedding cogenerator for $\mathscr{B}$ if every object in $\mathscr{B}$ admits an injection to some direct product of copies of X in $\mathscr{A}$ [15]. For any M in mod Λ we use $E(M)$ to denote the injective envelope of M.

Corollary 4.5 *Under the assumptions of Theorem* 4.4, E_g *is an injective embedding cogenerator for* $\mathscr{H}_g(\mathrm{mod}\ \Lambda)$.

Proof Let M be in $\mathscr{H}_g(\mathrm{mod}\ \Lambda)$. Notice that M is finitely cogenerated, so, by [19, Proposition 18.18], $E(M) \cong E(S_1) \bigoplus \cdots \bigoplus E(S_t)$, where S_i is isomorphic to a simple submodule of M for any $1 \leq i \leq t$. Since $M \in \mathscr{H}_g(\mathrm{mod}\ \Lambda)$, each $S_i \in \mathscr{H}_g(\mathrm{mod}\ \Lambda)$ by Theorem 4.4.

Because $r.\mathrm{id}_\Gamma(U) = g$, $\bigoplus_{i=0}^g E_i$ is an injective embedding cogenerator for mod Λ by [15, Proposition 2.8]. So $\mathrm{Hom}_\Lambda(S_i, \bigoplus_{i=0}^g E_i) \neq 0$ and hence $\mathrm{Hom}_\Lambda(S_i, E_g) \neq 0$ by Theorem 4.4, which implies that each S_i can be embedded into E_g. Therefore $M \hookrightarrow E(M) \cong \bigoplus_{i=0}^t E(S_i) \hookrightarrow E_g^{(t)}$ and E_g is an injective embedding cogenerator for $\mathscr{H}_g(\mathrm{mod}\ \Lambda)$.

5 Finite Injective Dimension

In this section we discuss the properties of ${}_\Lambda U_\Gamma$ with finite left or right injective dimension. We first have the following:

Proposition 5.1 *If* $l.\mathrm{id}_\Lambda(U) = k$ *and* E_k *is in* $\mathrm{add}_\Lambda U$ (*equivalently*, ${}^*E_k$ *is flat*), *then* ${}_\Lambda U$ *is injective.*

Proof Assume that $l.\mathrm{id}_\Lambda(U) = k \neq 0$. Then there is a simple Λ-module S such that $\mathrm{Ext}^k_\Lambda(S, U) \neq 0$. It is easy to see that $\mathrm{Hom}_\Lambda(S, E_k) \cong \mathrm{Ext}^k_\Lambda(S, U)$, so $\mathrm{Hom}_\Lambda(S, E_k) \neq 0$ and hence there is an exact sequence $0 \to S \xrightarrow{f} E_k \to \mathrm{Coker} f \to 0$, which yields an exact sequence $\mathrm{Ext}^k_\Lambda(E_k, U) \to \mathrm{Ext}^k_\Lambda(S, U) \to \mathrm{Ext}^{k+1}_\Lambda(\mathrm{Coker} f, U)$. Since $E_k \in \mathrm{add}_\Lambda U$, $\mathrm{Ext}^k_\Lambda(E_k, U) = 0$. On the other hand, $l.\mathrm{id}_\Lambda(U) = k$, so $\mathrm{Ext}^{k+1}_\Lambda(\mathrm{Coker} f, U) = 0$. Hence $\mathrm{Ext}^k_\Lambda(S, U) = 0$, which is a contradiction.

Corollary 5.2 *If* $l.\mathrm{id}_\Lambda(\Lambda) = k$ *and the* $(k+1)$-st *term* (*that is, the last term*) *in a minimal injective resolution of* ${}_\Lambda\Lambda$ *is flat, then* Λ *is self-injective.*

Corollary 5.3 *If* $l.\mathrm{id}_\Lambda(U) = k < U$-$\mathrm{dom.dim}({}_\Lambda U)$, *then* ${}_\Lambda U$ *is injective.*

Corollary 5.4 ([20, Proposition 8]) *If* $l.\mathrm{id}_\Lambda(\Lambda) = k < \Lambda$-$\mathrm{dom.dim}({}_\Lambda\Lambda)$, *then* Λ *is self-injective.*

Proposition 5.5 *If* ${}_\Lambda U_\Gamma$ *is* k-*Gorenstein and* $r.\mathrm{id}_\Gamma(U) = l.\mathrm{id}_\Lambda(U) = k$, *then* U-$\mathrm{resol.dim}_\Lambda(E_k) = U$-$\mathrm{resol.dim}_\Gamma(E'_k) = k$ *and* ${}_\Lambda U_\Gamma$ *is Gorenstein.*

Proof Assume that ${}_\Lambda U$ is k-Gorenstein. By Lemma 2.6 and Lemma 2.3, $l.\mathrm{fd}_\Gamma({}^*E_i) \leq i$ for any $0 \leq i \leq k - 1$. Since $r.\mathrm{id}_\Gamma(U) = k$, there is a module X in mod Γ^{op} such that $\mathrm{Ext}^k_\Gamma(X, U) \neq 0$. Since $\bigoplus_{i=0}^k E_i$ is an injective embedding cogenerator for mod Λ by [15, Proposition 2.8], it then follows from [11, Chapter VI, Proposition 5.3] that $0 \neq \mathrm{Hom}_\Lambda(\mathrm{Ext}^k_\Gamma(X, U), \bigoplus_{i=0}^k E_i) \cong \mathrm{Tor}^\Gamma_k(X, {}^*(\bigoplus_{i=0}^k E_i)) \cong \bigoplus_{i=0}^k \mathrm{Tor}^\Gamma_k(X, {}^*E_i) \cong \mathrm{Tor}^\Gamma_k(X, {}^*E_k)$. So $l.\mathrm{fd}_\Gamma({}^*E_k) \geq k$. On the other hand, by [12, Lemma 2.2], we have $r.\mathrm{id}_\Gamma(U) = \sup\{l.\mathrm{fd}_\Gamma({}^*E) | {}_\Lambda E$ is injective$\}$, so $l.\mathrm{fd}_\Gamma({}^*E_k) \leq k$

and hence $l.\mathrm{fd}_\Gamma(^*E_k) = k$. By Lemma 2.3, U-resol.dim$_\Lambda(E_k) = k$. It then follows from Theorem 3.7 that $_\Lambda U$ is $(k+1)$-Gorenstein. In addition, $l.\mathrm{id}_\Lambda(U) = k$ by assumption, so $_\Lambda U$ is Gorenstein. Similarly, we have that U-resol.dim$_\Gamma(E'_k) = k$ and U_Γ is Gorenstein.

Recall that an artin algebra is called an Auslander algebra if it is k-Gorenstein for all k. By Proposition 5.5, we immediately have the following

Corollary 5.6 ([9, Proposition 1.1]) *If Λ is a k-Gorenstein algebra with right and left self-injective dimensions k, then the flat dimension of the* $(k+1)$-st *term in a minimal injective resolution of $_\Lambda\Lambda$ (resp. Λ_Λ) is equal to k and Λ is an Auslander algebra.*

Compare Corollary 5.3 with the following

Proposition 5.7 *If $l.\mathrm{id}_\Lambda(U) = k \leq U$-dom.dim$(_\Lambda U)$, then $\bigoplus_{i=0}^{k} E_i$ is an injective embedding cogenerator for* mod Λ *if and only if $r.\mathrm{id}_\Gamma(U) = k$.*

Proof The sufficiency follows from [15, Proposition 2.8]. Now we prove the necessity. Since U-dom.dim$(_\Lambda U) \geq k$, $E_i \in \mathrm{add}_\Lambda U$ for any $0 \leq i \leq k-1$. On the other hand, $l.\mathrm{id}_\Lambda(U) = k$ implies that $E_i = 0$ for any $i \geq k+1$. So U-resol.dim$_\Lambda(E_k) \leq k$. Then, by Lemma 2.3, $l.\mathrm{fd}_\Gamma(^*E_k) \leq k$ and $l.\mathrm{fd}_\Gamma(^*E_i) = 0$ for any $0 \leq i \leq k-1$. It follows that $l.\mathrm{fd}_\Gamma(^*(\bigoplus_{i=0}^{k} E_i)) \leq k$. So, by Lemma 2.1, we have $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^{k+1}(X, U), \bigoplus_{i=0}^{k} E_i) = 0$ for any $X \in \mathrm{mod}\ \Gamma^{op}$. However, $\bigoplus_{i=0}^{k} E_i$ is an injective embedding cogenerator for mod Λ, so $\mathrm{Ext}_\Gamma^{k+1}(X, U) = 0$ and $r.\mathrm{id}_\Gamma(U) \leq k$. Hence we conclude that $r.\mathrm{id}_\Gamma(U) = k$ by [17, Lemma 1.7].

Finally we conjecture the following, which is a generalization of the Auslander and Reitens question mentioned in Section 3: A Gorenstein bimodule $_\Lambda U_\Gamma$ is cotilting, that is, $l.\mathrm{id}_\Lambda(U) < \infty$ and $r.\mathrm{id}_\Gamma(U) < \infty$.

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References

[1] Huang, Z. Y.: Extension closure of relative syzygy modules. *Science in China* (*Series A*), **46**, 611–620 (2003)

[2] Kato, T.: Rings of U-dominant dimension≥ 1. *Tôhoku Math. J.*, **21**, 321–327 (1969)

[3] Huang, Z. Y.: On U-dominant dimension. *J. Algebra*, **285**, 669–681 (2005)

[4] Wisbauer, R.: Decomposition properties in module categories. *Acta Univ. Carolinae–Mathematica et Physica*, **26**, 57–68 (1985)

[5] Auslander, M., Reiten, I.: k-Gorenstein algebras and syzygy modules. *J. Pure Appl. Algebra*, **92**, 1–27 (1994)

[6] Fossum, R. M., Griffith, P. A., Reiten, I.: Trivial Extensions of Abelian Categories, Lecture Notes in Mathematics 456, Springer-Verlag, Berlin-Heidelberg-New York, 1975

[7] Wakamatsu, T.: Tilting modules and Auslander's Gorenstein property. *J. Algebra*, **275**, 3–39 (2004)

[8] Huang, Z. Y.: On a generalization of the Auslander–Bridger transpose. *Comm. Algebra*, **27**, 5791–5812 (1999)

[9] Fuller, K. R., Iwanaga, Y.: On n-Gorenstein rings and Auslander rings of low injective dimension, Representations of Algebras (Ottawa, ON, 1992), Canad. Math. Soc. Conf. Proc., 14, Amer. Math. Soc., Providence, RI, 175–183, 1993

[10] Iwanaga, Y.: Duality over Auslander–Gorenstein rings. *Math. Scand.*, **81**, 184–190 (1997)

[11] Cartan, H., Eilenberg, S.: Homological Algebra, Reprint of the 1956 original, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, 1999

[12] Huang, Z. Y.: Extension closure of relative k-torsionfree modules. *Comm. Algebra*, **34**, 3585–3592 (2006)

[13] Huang, Z. Y.: ω-k-torsionfree modules and ω-left approximation dimension. *Science in China* (*Series A*), **44**, 184–192 (2001)

[14] Auslander, M., Bridger, M.: Stable Module Theory, Memoirs Amer. Math. Soc., 94, American Mathematical Society, Providence, Rhode Island, 1969

[15] Huang, Z. Y., Tang, G. H.: Self-orthogonal modules over coherent rings. *J. Pure Appl. Algebra*, **161**, 167–176 (2001)

[16] Huang, Z. Y.: Selforthogonal modules with finite injective dimension II. *J. Algebra*, **264**, 262–268 (2003)

[17] Auslander, M., Reiten, I.: Cohen–Macaulay and Gorenstein artin algebras, in: G. O. Michler and C. M. Ringel, eds. Representation Theory of Finite Groups and Finite Dimensional Algebras, Bielefeld, 1991, Progress in Mathematics 95, Birkhauser, Basel, 221–245, 1991

[18] Auslander, M., Reiten, I.: Applications of contravariantly finite subcategories. *Adv. Math.*, **86**, 111–152 (1991)

[19] Anderson, F. W., Fuller, K. R.: Rings and Categories of modules, 2nd ed, Graduate Texts in Mathematics 13, Springer-Verlag, Berlin-Heidelberg-New York, 1992

[20] Iwanaga, Y., Sato, H.: Minimal injective resolution of Gorenstein rings. *Comm. Algebra*, **18**, 3835–3856 (1990)