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Homological Behavior of Auslander's *k*-Gorenstein Rings

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Abstract In this paper we mainly study the homological properties of dual modules over k-Gorenstein rings. For a right quasi k-Gorenstein ring Λ, we show that the right self-injective dimension of Λ is at most k if and only if each $M \in \text{mod}\ \Lambda$ satisfying the condition that $\text{Ext}^i_{\Lambda}(M, \Lambda) = 0$ for any $1 \leq i \leq k$ is reflexive. For an ∞-Gorenstein ring, we show that the big and small finitistic dimensions and the self-injective dimension of Λ are identical. In addition, we show that if Λ is a left quasi ∞-Gorenstein ring and $M \in \text{mod}\ \Lambda$ with $\text{grade}\, M$ finite, then $\text{Ext}^i_{\Lambda}(\text{Ext}^i_{\Lambda^{op}}(\text{Ext}^{\text{grade}\, M}_{\Lambda}(M, \Lambda), \Lambda), \Lambda) = 0$ if and only if $i \neq \text{grade}\, M$. For a 2-Gorenstein ring Λ, we show that a non-zero proper left ideal I of Λ is reflexive if and only if Λ/I has no non-zero pseudo-null submodule.

Keywords (Quasi) k-Gorenstein rings · (Quasi) ∞-Gorenstein rings · (Strong) grade · Reduced grade · Homological dimensions · Reflexive modules

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1 Introduction

Throughout this paper, Λ is a left and right Noetherian ring. We use mod Λ to denote the category of finitely generated left Λ-modules. Let $M \in \text{mod}\ \Lambda$. We use $\text{pd}_{\Lambda} M$, $\text{fd}_{\Lambda} M$ and $\text{id}_{\Lambda} M$ to denote the projective, flat and injective dimensions of M, respectively. For a non-negative integer i, recall from [5] that the *grade* of M,

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denoted by grade$M$, is said to be at least i if $\mathrm{Ext}^j_\Lambda(M,\Lambda)=0$ for any $0\le j<i$; and the *strong grade* of M, denoted by s.grade$M$, is said to be at least i if grade$X\ge i$ for each submodule X of M. In addition, we fix minimal injective resolutions

$$0\to {}_\Lambda\Lambda\to I_0^{'}\to I_1^{'}\to\cdots\to I_i^{'}\to\cdots$$

and

$$0\to \Lambda_\Lambda\to I_0\to I_1\to\cdots\to I_i\to\cdots$$

of ${}_\Lambda\Lambda$ and Λ_Λ, respectively.

Recall from [11] that Λ is called a *k-Gorenstein ring* if $\mathrm{fd}_\Lambda I_i^{'}\le i$ for any $0\le i\le k-1$, and Λ is called an ∞*-Gorenstein ring* if it is k-Gorenstein for all k. Auslander gave some equivalent characterizations of k-Gorenstein rings in terms of the strong grade and the flat dimension of injective modules as follows.

Auslander's Theorem [11, Theorem 3.7] *The following statements are equivalent.*

(1) Λ *is a k-Gorenstein ring.*
(2) $\mathrm{fd}_{\Lambda^{op}} I_i\le i$ *for any* $0\le i\le k-1$.
(3) $\mathrm{s.grade}\mathrm{Ext}^i_\Lambda(M,\Lambda)\ge i$ *for any* $M\in\mathrm{mod}\,\Lambda$ *and* $1\le i\le k$.
(4) $\mathrm{s.grade}\mathrm{Ext}^i_{\Lambda^{op}}(N,\Lambda)\ge i$ *for any* $N\in\mathrm{mod}\,\Lambda^{op}$ *and* $1\le i\le k$.

The Auslander's theorem shows that the notion of k-Gorenstein rings is left-right symmetric. The properties of k-Gorenstein rings and related rings have been studied by many authors (see [1, 4, 5, 8–11, 16, 17, 19, 20, 24–26, 29], and so on). In this paper we mainly investigate the homological properties of dual modules over k-Gorenstein rings and related rings.

It was showed in [21, Theorem 2.2] that if $\mathrm{id}_{\Lambda^{op}}\Lambda\le k$, then each $M\in\mathrm{mod}\,\Lambda$ satisfying the condition that $\mathrm{Ext}^i_\Lambda(M,\Lambda)=0$ for any $1\le i\le k$ is reflexive. In Section 2, we show that the converse of this result holds true for right quasi k-Gorenstein rings (see Section 2 for the definition of such rings). It was showed in [22] that the big and small finitistic dimensions of Λ are usually different even when Λ is an Artinian algebra. The other aim of Section 2 is to show that for a $(k+1)$-Gorenstein ring Λ, the small finitistic dimension of Λ is at most k if and only if $\mathrm{id}_\Lambda\Lambda\le k$, which yields that for an ∞-Gorenstein ring Λ, the big and small finitistic dimensions of Λ and $\mathrm{id}_\Lambda\Lambda$ are identical. As a consequence, we get some equivalent versions of the Nakayama conjecture.

In Section 3, we get some properties of the grade of modules over quasi ∞-Gorenstein rings. We prove that if Λ is a left quasi ∞-Gorenstein ring and $M\in\mathrm{mod}\,\Lambda$ with grade$M$ finite, then $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_{\Lambda^{op}}(\mathrm{Ext}^{\mathrm{grade}M}_\Lambda(M,\Lambda),\Lambda),\Lambda)=0$ if and only if $i\ne$grade$M$. As an application of this result, we show that if Λ is a left and right quasi Auslander-Gorenstein ring with $\mathrm{id}_\Lambda\Lambda=\mathrm{id}_{\Lambda^{op}}\Lambda=t$, then the functors $\mathrm{Ext}^t_\Lambda(\ ,\Lambda)$ and $\mathrm{Ext}^t_{\Lambda^{op}}(\ ,\Lambda)$ give a duality between $\{M\in\mathrm{mod}\,\Lambda\mid\mathrm{grade}M=t\}$ and $\{N\in\mathrm{mod}\,\Lambda^{op}\mid\mathrm{grade}N=t\}$. This generalizes a result of Iwanaga in [24].

In Section 4, we give a criterion for judging when a finitely generated torsionless module is reflexive. As a consequence of this criterion, we prove that for a 2-Gorenstein ring Λ, a non-zero proper left ideal I of Λ is reflexive if and only if Λ/I has no non-zero pseudo-null submodule. This generalizes a result of Coates, Schneider and Sujatha in [10]. We also study the following question: For a positive integer k, when is each k-torsionfree module in mod Λ projective? We prove that the answer to this question is affirmative if gl.dimΛ (the global dimension of Λ) is at most k.

2 Reflexive Modules and Homological Dimensions

Lemma 2.1 *The following statements are equivalent for a positive integer k.*

(1) $\mathrm{fd}_\Lambda I_i^{'} \leq i+1$ *for any* $0 \leq i \leq k-1$.
(2) $\mathrm{s.gradeExt}_{\Lambda^{op}}^{i+1}(N, \Lambda) \geq i$ *for any* $N \in \mathrm{mod}\ \Lambda^{op}$ *and* $1 \leq i \leq k$.
(3) $\mathrm{gradeExt}_\Lambda^i(M, \Lambda) \geq i$ *for any* $M \in \mathrm{mod}\ \Lambda$ *and* $1 \leq i \leq k$.

Proof By [5, Theorem 0.1] and [16, Theorem 4.1]. □

Definition 2.2 [19, 20] Λ is called a *left quasi k-Gorenstein ring* if one of the equivalent conditions in Lemma 2.1 is satisfied. Λ is called a *left quasi* ∞*-Gorenstein ring* if it is left quasi k-Gorenstein for all k. Dually, the notions of *right quasi k-Gorenstein rings* and *right quasi* ∞*-Gorenstein rings* are defined.

Remark A k-Gorenstein ring clearly is a left and right quasi k-Gorenstein ring. However, the interesting property of the left-right symmetry enjoyed by k-Gorenstein rings fails for quasi k-Gorenstein rings (see [17]).

Let $M \in \mathrm{mod}\ \Lambda$ and k a positive integer. Recall from [14] that the *reduced grade* of M, denoted by r.grade$M$, is said to be at least k if $\mathrm{Ext}_\Lambda^j(M, \Lambda) = 0$ for any $1 \leq j < k$. It is trivial that grade$M \geq k$ implies r.grade$M \geq k$. Let $\sigma_M : M \to M^{**}$ via $\sigma_M(x)(f) = f(x)$ for any $x \in M$ and $f \in M^*$ be the canonical evaluation homomorphism, where $(\ )^* = \mathrm{Hom}(\ , \Lambda)$. Recall that M is called *torsionless* if σ_M is a monomorphism; and M is called *reflexive* if σ_M is an isomorphism. It was showed in [21, Theorem 2.2] that if $\mathrm{id}_{\Lambda^{op}}\Lambda \leq k$, then each module in mod Λ with reduced grade at least $k+1$ is reflexive. However, we do not know whether the converse holds true. Jans showed in [28, Theorem 5.1] that the converse holds true when $k = 1$. One of the main results in this section is the following

Theorem 2.3 *Let* Λ *be a right quasi k-Gorenstein ring. Then the following statements are equivalent.*

(1) $\mathrm{id}_{\Lambda^{op}}\Lambda \leq k$.
(2) *Each module in* mod Λ *with reduced grade at least* $k+1$ *is reflexive.*
(3) *Each module in* mod Λ *with reduced grade at least* $k+1$ *is torsionless.*

Before proving this theorem, we recall some notions from [3] and [28]. Let $M \in \mathrm{mod}\ \Lambda$ and $\mathcal{D}$ be a full subcategory of mod Λ. A homomorphism $M \stackrel{f}{\to} D$ in mod Λ with $D \in \mathcal{D}$ is called a left $\mathcal{D}$-*approximation* of M if $\mathrm{Hom}_\Lambda(f, D')$ is epic for any $D' \in \mathcal{D}$ (see [3]). Let $X \in \mathrm{mod}\ \Lambda$ and $Y \in \mathrm{mod}\ \Lambda^{op}$. A monomorphism $X^{**} \stackrel{\rho^*}{\to} Y^*$ is called a *double dual embedding* if it is the dual of an epimorphism $Y \stackrel{\rho}{\to} X^*$. For a positive integer k, a torsionless module $T_k \in \operatorname{mod} \Lambda$ is said to be of *D-class k* if it can be fitted into a sequence of $k-1$ exact sequences of the form:

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & \longrightarrow & T_{k-1}^{**} & \longrightarrow & P_{k-1} & \longrightarrow & T_k \longrightarrow 0 \\
 & & & & & & & & \uparrow \sigma_{T_{k-1}} & & & & \\
 & & & & \cdots & \longrightarrow & P_{k-2} & \longrightarrow & T_{k-1} & \longrightarrow & 0 & & \\
 & & & & \cdots & & & & & & & & \\
 & & 0 & \longrightarrow & T_2^{**} & \longrightarrow & \cdots & & & & & & \\
 & & & & \uparrow \sigma_{T_2} & & & & & & & & \\
0 \longrightarrow T_1^{**} & \longrightarrow & P_1 & \longrightarrow & T_2 & \longrightarrow & 0 & & & & & &
\end{array}$$

where each $P_i \in \operatorname{mod} \Lambda$ is projective and the horizontal monomorphisms are double dual embeddings. Any torsionless module in mod Λ is said to be of *D-class 1* (see [28]). It follows from [28, p.335] that a torsionless module in mod Λ is of D-class k if its reduced grade is at least k. The following two results are cited from [28], which play a crucial role in proving Theorem 2.3.

Lemma 2.4 [28, Theorem 4.3] $\mathrm{id}_{\Lambda^{op}} \Lambda \leq k$ *if and only if each module of D-class k in* mod Λ *is reflexive.*

The proof of [28, Theorem 3.1] can be applied to obtain the following more general result.

Lemma 2.5 *If* Λ *is a right quasi k-Gorenstein ring, then the reduced grade of each module of D-class k in* mod Λ *is at least k.*

Proof of Theorem 2.3

(1) $\Rightarrow$ (2) follows from [21, Theorem 2.2], and (2) $\Rightarrow$ (3) is trivial.
(3) $\Rightarrow$ (1) Assume that $M \in \operatorname{mod} \Lambda$ is of D-class k. Then M is torsionless and σ_M is a monomorphism. By Lemma 2.5, we have that $\mathrm{r.grade}\, M \geq k$.

Let $P \xrightarrow{f} M^* \to 0$ be exact in mod Λ^{op} with P projective. Put $g = f^* \sigma_M$ and $X = \mathrm{Coker} g$. It is not difficult to verify that the monomorphism $g : M \to P^*$ is a left $\mathcal{P}^0(\Lambda)$-approximation of M, where $\mathcal{P}^0(\Lambda)$ denotes the full subcategory of mod Λ consisting of projective modules. Then $\mathrm{r.grade}\, X \geq 2$ and so $\mathrm{r.grade}\, X \geq k+1$. Thus by (3) we have that X is torsionless and σ_X is a monomorphism. Furthermore, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{g} & P^* & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow \sigma_M & & \cong \downarrow \sigma_{P^*} & & \downarrow \sigma_X & & \\
0 & \longrightarrow & M^{**} & \xrightarrow{g^{**}} & P^{***} & \longrightarrow & X^{**} & &
\end{array}$$

Then by the snake lemma we have that σ_M is an isomorphism and M is reflexive. So $\mathrm{id}_{\Lambda^{op}}\Lambda \leq k$ by Lemma 2.4. □

The following lemma improves [18, Corollary 3].

Proposition 2.6 *Let* Λ *be a left and right Artinian ring. If* $\mathrm{id}_{\Lambda^{op}}\Lambda = k$ *and* $\mathrm{fd}_{\Lambda^{op}} \bigoplus_{i=0}^{k-2} I_i < \infty$, *then* $\mathrm{id}_\Lambda \Lambda = k$.

Proof Assume that $\mathrm{fd}_{\Lambda^{op}} \bigoplus_{i=0}^{k-2} I_i = r(< \infty)$. By [26, 6.1(1)], we have that $\mathrm{s.grade}\mathrm{Ext}_\Lambda^{r+1}(M, \Lambda) \geq k-1$ for any $M \in \mathrm{mod}\ \Lambda$. Then $\mathrm{id}_\Lambda \Lambda \leq (r+1)+k-1 = r+k$ by [18, Theorem]. It follows from [32, Lemma A] that $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{op}}\Lambda = k$. □

By Proposition 2.6 and the left-right symmetry of k-Gorenstein rings, we immediately have the following

Corollary 2.7 *Let* Λ *be a left and right Artinian ring. If* Λ *is* $(k-1)$*-Gorenstein, then* $\mathrm{id}_{\Lambda^{op}}\Lambda \leq k$ *if and only if* $\mathrm{id}_\Lambda \Lambda \leq k$.

The following result generalizes [4, Corollary 5.5(b)].

Corollary 2.8 *Let* Λ *be a left and right Artinian ring. If* Λ *is* ∞*-Gorenstein, then* $\mathrm{id}_{\Lambda^{op}}\Lambda = \mathrm{id}_\Lambda \Lambda$.

Corollary 2.9 *Let* Λ *be a left and right Artinian ring. If* Λ *is k-Gorenstein, then the following statements are equivalent.*

(1) $\mathrm{id}_{\Lambda^{op}}\Lambda \leq k$.
(2) *Each module in* $\mathrm{mod}\ \Lambda$ *with reduced grade at least* $k+1$ *is reflexive.*
(3) *Each module in* $\mathrm{mod}\ \Lambda$ *with reduced grade at least* $k+1$ *is torsionless.*

$(1)^{op}$ $\mathrm{id}_\Lambda \Lambda \leq k$.
$(2)^{op}$ *Each module in* $\mathrm{mod}\ \Lambda^{op}$ *with reduced grade at least* $k+1$ *is reflexive.*
$(3)^{op}$ *Each module in* $\mathrm{mod}\ \Lambda^{op}$ *with reduced grade at least* $k+1$ *is torsionless.*

Proof By Theorem 2.3 and Corollary 2.7. □

Recall that the *big finitistic dimension* of Λ, denoted by $\mathrm{Fin.dim}\Lambda$, is defined to be $\sup\{\mathrm{pd}_\Lambda M | M$ is a left Λ-module with $\mathrm{pd}_\Lambda M < \infty\}$; and the *small finitistic dimension* of Λ, denoted by $\mathrm{fin.dim}\Lambda$, is defined to be $\sup\{\mathrm{pd}_\Lambda M | M \in \mathrm{mod}\ \Lambda$ with $\mathrm{pd}_\Lambda M < \infty\}$. According to [22], the big and small finitistic dimensions are not identical in general even for Artinian algebras. The other aim of this section is to show that for an ∞-Gorenstein ring Λ, these two dimensions and $\mathrm{id}_\Lambda \Lambda$ are identical. As an application of this result, we get some equivalent versions of the Nakayama conjecture.

We begin with the following easy observation.

Lemma 2.10 *Let* $M \in \mathrm{mod}\ \Lambda$ *with* $\mathrm{r.grade} M \geq k+1$. *If* $\mathrm{pd}_\Lambda M \leq k$, *then* M *is projective.*

Proof Consider the projective resolution of M in mod Λ. Then we get easily the assertion. □

Lemma 2.11 [7, Proposition 4.3] fin.dimΛ $\leq$Fin.dimΛ $\leq$id$_\Lambda \Lambda$.

Let $M \in$ mod Λ and k a positive integer. Recall that M is called a *k-syzygy module* if there exists an exact sequence $0 \to M \to Q_0 \to Q_1 \to \cdots \to Q_{k-1}$ in mod Λ with all Q_i projective. On the other hand, assume that

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

is a projective presentation of M in mod Λ. Then we get an exact sequence:

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{Tr} M \to 0$$

in mod Λ^{op}, where $\operatorname{Tr} M = \operatorname{Coker} f^*$ is the *transpose* of M. M is called a *k-torsionfree module* if r.gradeTr $M \geq k+1$; and M is called an ∞*-torsionfree module* if r.gradeTr $M \geq k$ for all k (see [1]). We remark that the transpose of M depends on the choice of the projective resolution of M, but it is unique up to projective equivalence. So the notions of k-torsionfree modules and ∞-torsionfree modules are well defined.

Lemma 2.12 [1, Proposition 2.6] *Let* $M \in$ mod Λ. *Then we have the following exact sequences:*

$$0 \to \operatorname{Ext}^1_{\Lambda^{op}}(\operatorname{Tr} M, \Lambda) \to M \xrightarrow{\sigma_M} M^{**} \to \operatorname{Ext}^2_{\Lambda^{op}}(\operatorname{Tr} M, \Lambda) \to 0,$$

$$0 \to \operatorname{Ext}^1_{\Lambda}(M, \Lambda) \to \operatorname{Tr} M \xrightarrow{\sigma_{\operatorname{Tr} M}} (\operatorname{Tr} M)^{**} \to \operatorname{Ext}^2_{\Lambda}(M, \Lambda) \to 0.$$

It follows from Lemma 2.12 that a module in mod Λ is torsionless (resp. reflexive) if and only if it is 1-torsionfree (resp. 2-torsionfree). The following observation is well known, which is an immediate consequence of Lemma 2.12.

Proposition 2.13 *A module* M *in* mod Λ *is projective if and only if* Tr M *is projective in* mod Λ^{op}.

Proof The necessity is trivial, so it suffices to prove the sufficiency. Let $M \in$ mod Λ with Tr M projective. Then M^* is projective. So by Lemma 2.12 we have that σ_M is an isomorphism and $M(\cong M^{**})$ is projective. □

Lemma 2.14 *Consider the following conditions.*

(1) *Each module in* mod Λ^{op} *is reflexive.*
(2) *Each module in* mod Λ^{op} *is torsionless.*
(3) Λ *is self-injective.*
(4) fin.dim$\Lambda = 0$.
(5) *A module* $N \in$ mod Λ^{op} *is projective provided* N^* *is projective.*
We have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5). *If* Λ *is a 1-Gorenstein ring, then all of these conditions are equivalent.*

Proof

(1) $\Rightarrow$ (2) is trivial. (1) $\Leftrightarrow$ (3) and (3) $\Rightarrow$ (4) follow from [27, Corollary 1.2] and Lemma 2.11, respectively.
(2) $\Rightarrow$ (3) Let $M \in \text{mod}\,\Lambda$. Then $\text{Tr}\,M \in \text{mod}\,\Lambda^{op}$ is torsionless by (2). It follows from Lemma 2.12 that $\text{Ext}^1_\Lambda(M, \Lambda) \cong \sigma_{\text{Tr}\,M} = 0$ and Λ is self-injective.
(4) $\Rightarrow$ (5) Let $N \in \text{mod}\,\Lambda^{op}$ with N^* projective. Then $\text{Tr}\,N \in \text{mod}\,\Lambda$ and $\text{pd}_\Lambda \text{Tr}\,N \leq 2$. By (4), $\text{Tr}\,N$ is projective. It follows from Proposition 2.13 that N is projective.
(5) $\Rightarrow$ (4) Let $N \in \text{mod}\,\Lambda^{op}$ with $N^* = 0$, Then N is projective by (5) and so $N = 0$. It follows from [6, Corollary 5.6 and Theorem 5.4] that $\text{fin.dim}\Lambda = 0$.

Now let Λ be a 1-Gorenstein ring. We will prove (4) $\Rightarrow$ (3). Suppose $\text{fin.dim}\Lambda = 0$. Because Λ is 1-Gorenstein, for any $M \in \text{mod}\,\Lambda$ we have that $\text{s.gradeExt}^1_\Lambda(M, \Lambda) \geq 1$ and $[\text{Ext}^1_\Lambda(M, \Lambda)]^* = 0$. Then by [6, Corollary 5.6 and Theorem 5.4], $\text{Ext}^1_\Lambda(M, \Lambda) = 0$. It implies that Λ is self-injective. □

Theorem 2.15 *Let k be a non-negative integer and Λ a $(k+1)$-Gorenstein ring with* $\text{fin.dim}\Lambda = k$. *Then* $\text{id}_\Lambda \Lambda = k$.

Proof The case for $k = 0$ follows from Lemma 2.14. Now suppose that $k \geq 1$ and M is any module in $\text{mod}\,\Lambda$. Since Λ is $(k+1)$-Gorenstein, $\text{s.gradeExt}^{k+1}_\Lambda(M, \Lambda) \geq k+1$. Let

$$Q_{k+1} \to Q_k \to \cdots \to Q_1 \to Q_0 \to \text{Ext}^{k+1}_\Lambda(M, \Lambda) \to 0$$

be a projective resolution of $\text{Ext}^{k+1}_\Lambda(M, \Lambda)$ in $\text{mod}\,\Lambda^{op}$. Put $N_k = \text{Coker}(Q_{k+1} \to Q_k)$. Notice that a k-syzygy module is k-torsionfree by [5, Proposition 1.6], so N_k is k-torsionfree and $\text{r.gradeTr}\,N_k \geq k+1$. On the other hand, we have the following exact sequence:

$$0 \to Q_0^* \to Q_1^* \to \cdots \to Q_k^* \to Q_{k+1}^* \to \text{Tr}\,N_k \to 0$$

in $\text{mod}\,\Lambda$. Then $\text{pd}_\Lambda \text{Tr}\,N_k \leq k+1$. But $\text{fin.dim}\Lambda = k$, so $\text{pd}_\Lambda \text{Tr}\,N_k \leq k$ and hence $\text{Tr}\,N_k$ is projective by Lemma 2.10. It follows from Proposition 2.13 that N_k is projective and thus $\text{pd}_{\Lambda^{op}}\text{Ext}^{k+1}_\Lambda(M, \Lambda) \leq k$. Now assume that $\text{pd}_{\Lambda^{op}}\text{Ext}^{k+1}_\Lambda(M, \Lambda) = t (\leq k)$. If $\text{Ext}^{k+1}_\Lambda(M, \Lambda) \neq 0$, then it is not difficult to verify that $\text{Ext}^t_{\Lambda^{op}}(\text{Ext}^{k+1}_\Lambda(M, \Lambda), \Lambda) \neq 0$, which contradicts the assumption that Λ is $(k+1)$-Gorenstein. Thus $\text{Ext}^{k+1}_\Lambda(M, \Lambda) = 0$ and therefore $\text{id}_\Lambda \Lambda \leq k$. It follows from Lemma 2.11 that $\text{id}_\Lambda \Lambda = k$. □

Note that Jans showed in [28, Theorem 4.2] that for a positive integer k, $\text{fin.dim}\Lambda \leq k$ if and only if a module $N \in \text{mod}\,\Lambda^{op}$ of D-class k is projective provided N^* is projective. Summarizing the results obtained above, we get the following

Theorem 2.15 *Let k be a non-negative integer and Λ a $(k+1)$-Gorenstein ring. Then the following statements are equivalent.*

(1) $\text{id}_\Lambda \Lambda \leq k$.
(2) $\text{Fin.dim}\Lambda \leq k$.

(3) $\mathrm{fin.dim}\Lambda \leq k$.
(4) *Each module in* mod Λ^{op} *with reduced grade at least* $k+1$ *is reflexive.*
(5) *Each module in* mod Λ^{op} *with reduced grade at least* $k+1$ *is torsionless.*
(6) *Each module in* mod Λ^{op} *of D-class* k *is reflexive.*
(7) *A module* $N \in$ mod Λ^{op} *of D-class* k *is projective provided* N^* *is projective.*

For any positive integer n, Huisgen-Zimmermann showed in [22] that there exists a finite-dimensional monomial relation algebra Λ with $J^4 = 0$ (where J is the Jacobson radical of Λ) such that $\mathrm{fin.dim}\Lambda = n+1$ and $\mathrm{Fin.dim}\Lambda = n+2$. In [30] Smalø constructed examples of algebras Λ with arbitrarily big difference between $\mathrm{fin.dim}\Lambda$ and $\mathrm{Fin.dim}\Lambda$. But very little appears to be known when $\mathrm{fin.dim}\Lambda = \mathrm{Fin.dim}\Lambda$. As an immediate consequence of Theorem 2.16 we have

Corollary 2.17 $\mathrm{fin.dim}\Lambda = \mathrm{Fin.dim}\Lambda = \mathrm{id}_\Lambda \Lambda$ *for an* ∞*-Gorenstein ring* Λ.

Corollary 2.18 *Let* Λ *be a left and right Artinian ring. If* Λ *is* ∞*-Gorenstein, then* $\mathrm{fin.dim}\Lambda = \mathrm{fin.dim}\Lambda^{op} = \mathrm{Fin.dim}\Lambda = \mathrm{Fin.dim}\Lambda^{op} = \mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{op}} \Lambda$.

Proof By Corollaries 2.8 and 2.17. □

Recall that Λ is said to have *dominant dimension* at least k if each $I_i^{'}$ is flat for any $0 \leq i \leq k-1$.

The following are some important homological conjectures in representation theory of algebras.

Nakayama Conjecture (**NC**) [31]: An Artinian algebra Λ is self-injective if Λ has infinite dominant dimension.

It is easy to see that the Nakayama conjecture is a special case of the following conjecture.

Auslander-Reiten Conjecture (**ARC**) [4]: An ∞-Gorenstein Artinian algebra has finite left and right self-injective dimensions.

Finitistic Dimension Conjecture (**FDC**) [6, 22]: $\mathrm{fin.dim}\Lambda$ is finite for an Artinian algebra Λ.

It follows from Corollary 2.18 that **ARC** and **FDC** are equivalent for ∞-Gorenstein Artinian algebras. So we get the following implications: **FDC**⇒**ARC**⇒ **NC**. Furthermore, we get some equivalent versions of **NC** as follows.

Corollary 2.19 *Let* Λ *be an Artinian algebra with infinite dominant dimension. Then the following statements are equivalent.*

(1) Λ *is self-injective.*
(2) $\mathrm{fin.dim}\Lambda = 0$.
(3) *The right annihilator of any proper left ideal of* Λ *is non-zero.*
(4) *The right annihilator of any maximal left ideal of* Λ *is non-zero.*
(5) $\mathrm{grade}M < \infty$ *for any non-zero module* $M \in$ mod Λ.
(6) $\mathrm{grade}S < \infty$ *for any simple module* $S \in$ mod Λ.

Proof (1) ⇔ (2) follows from Corollary 2.17. By Corollary 2.18 and [6, Corollary 5.6 and Theorem 5.4], we have that (3) ⇔ (2) ⇒ (5). Both (3) ⇒ (4) and (5) ⇒ (6) are trivial.

(4) $\Rightarrow$ (3) Assume that I is a proper left ideal of Λ. Then I is contained in a maximal left ideal L of Λ. So the right annihilator of L is contained in that of I.

(6) $\Rightarrow$ (4) Assume that $S \in \text{mod } \Lambda$ is a simple module. Then grade$S < \infty$ by (6). Put $K_i = \text{Ker}(I_i^{'} \to I_{i+1}^{'})$ for any $i \geq 0$. Then we get an exact sequence:

$$0 \to \text{Hom}_\Lambda(S, K_i) \to \text{Hom}_\Lambda(S, I_i') \to \text{Hom}_\Lambda(S, K_{i+1}) \to \text{Ext}_\Lambda^{i+1}(S, \Lambda) \to 0$$

for any $i \geq 0$. Since Λ has infinite dominant dimension, $I_i^{'}$ is projective for any $i \geq 0$.

We claim that $S^* \neq 0$. Otherwise, if $S^* = 0$, then $\text{Hom}_\Lambda(S, I_i^{'}) = 0$ and $\text{Hom}_\Lambda(S, K_i) = 0$ for any $i \geq 0$. Then by the exactness of the above sequence, we get that $\text{Ext}_\Lambda^i(S, \Lambda) = 0$ for any $i \geq 0$, which contradicts the fact that grade$S < \infty$. The claim is proved. Now assume that L is a maximal left ideal of Λ. Then $\Lambda/L \in \text{mod } \Lambda$ is a simple module. By [6, (4.1)], we have that the right annihilator of L is isomorphic to $(\Lambda/L)^*$, which is non-zero by the above claim. □

The generalized Nakayama conjecture (**GNC**), posed by Auslander and Reiten in [2], has an equivalent version as follows: Over an Artinian algebra Λ, grade$S < \infty$ for any simple module $S \in \text{mod } \Lambda$. It is well known that **GNC**$\Rightarrow$**NC** (see [2]). Corollary 2.19 not only gives another proof of this implication, but also shows that in order to verify **NC** it suffices to verify **GNC** for Artinian algebras with infinite dominant dimension.

3 Some Properties of Grade of Modules

In this section, we study the properties of grade of modules over quasi ∞-Gorenstein rings. We begin with the following lemma.

Lemma 3.1 [15, Lemma 6.2] *Let* $M \in \text{mod } \Lambda$ *and* n *a non-negative integer. If* grade$M \geq n$ *and* grade$\text{Ext}_\Lambda^n(M, \Lambda) \geq n+1$*, then* grade$M \geq n+1$.

The following is the main result in this section.

Theorem 3.2 *Let* Λ *be a left quasi* ∞*-Gorenstein ring and* $M \in \text{mod } \Lambda$ *with* grade$M$ *finite. Then* $\text{Ext}_\Lambda^i(\text{Ext}_{\Lambda^{op}}^i(\text{Ext}_\Lambda^{\text{grade}M}(M, \Lambda), \Lambda), \Lambda) = 0$ *if and only if* $i \neq$grade$M$.

Proof Because Λ is left quasi ∞-Gorenstein, s.grade$\text{Ext}_{\Lambda^{op}}^{i+1}(N, \Lambda) \geq i$ for any $N \in \text{mod } \Lambda^{op}$ and $i \geq 1$ by Lemma 2.1.

Let $M \in \text{mod } \Lambda$ and

$$\cdots \to P_i \xrightarrow{f_i} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to M \to 0 \quad (1)$$

be a projective resolution of M in mod Λ. Put $t =$grade$M$ and $X_i = \text{Coker} f_i^*$ for any $i \geq 1$.

If $t = 0$ (that is, $M^* \neq 0$), then $M^{***} \cong M^* (\neq 0)$ by [15, Lemma 1.6]. On the other hand, for any $i \geq 1$, we have $\text{Ext}_{\Lambda^{op}}^i(M^*, \Lambda) \cong \text{Ext}_{\Lambda^{op}}^{i+2}(X_1, \Lambda)$. So grade$\text{Ext}_{\Lambda^{op}}^i(M^*, \Lambda) =$ grade$\text{Ext}_{\Lambda^{op}}^{i+2}(X_1, \Lambda) \geq i+1$ and hence $\text{Ext}_\Lambda^i(\text{Ext}_{\Lambda^{op}}^i(M^*, \Lambda), \Lambda) = 0$ for any $i \geq 1$.

Now suppose $t \geq 1$. Then from the exact sequence (1), we get the following exact sequence:

$$0 \to M^*(=0) \to P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{f_2^*} \cdots \xrightarrow{f_t^*} P_t^* \to X_t \to 0$$

in mod Λ^{op}, which implies $\mathrm{pd}_{\Lambda^{op}} X_t \leq t$ and induces an exact sequence:

$$P_t^{**} \xrightarrow{f_t^{**}} \cdots \xrightarrow{f_2^{**}} P_1^{**} \xrightarrow{f_1^{**}} P_0^{**} \to \mathrm{Ext}^t_\Lambda(X_t, \Lambda) \to 0.$$

So we have $M \cong \mathrm{Ext}^t_{\Lambda^{op}}(X_t, \Lambda)$.

By [18, Lemma 2], we have an exact sequence:

$$0 \to \mathrm{Ext}^t_\Lambda(M, \Lambda) \to X_t \to P_{t+1}^* \to X_{t+1} \to 0 \tag{2}$$

Put $Y_t = \mathrm{Im}(X_t \to P_{t+1}^*)$. Notice that $\mathrm{grade}\,\mathrm{Ext}^t_\Lambda(M, \Lambda) \geq t$ by Lemma 2.1, we then get an exact sequence:

$$\begin{aligned} 0 \;=\; &\mathrm{Ext}^{t-1}_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda) \to \mathrm{Ext}^t_{\Lambda^{op}}(Y_t, \Lambda) \to \mathrm{Ext}^t_{\Lambda^{op}}(X_t, \Lambda) \\ &\to \mathrm{Ext}^t_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda) \to \mathrm{Ext}^{t+1}_{\Lambda^{op}}(Y_t, \Lambda) \to \mathrm{Ext}^{t+1}_{\Lambda^{op}}(X_t, \Lambda) = 0 \end{aligned} \tag{3}$$

and the isomorphisms:

$$\mathrm{Ext}^i_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda) \cong \mathrm{Ext}^{i+1}_{\Lambda^{op}}(Y_t, \Lambda) \text{ for any } i \geq t+1,$$

and

$$\mathrm{Ext}^{i+1}_{\Lambda^{op}}(Y_t, \Lambda) \cong \mathrm{Ext}^{i+2}_{\Lambda^{op}}(X_{t+1}, \Lambda) \text{ for any } i \geq 0.$$

So, from the exact sequence (3), we get an exact sequence:

$$0 \to \mathrm{Ext}^{t+1}_{\Lambda^{op}}(X_{t+1}, \Lambda) \to M \xrightarrow{\pi_M} \mathrm{Ext}^t_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda) \to \mathrm{Ext}^{t+2}_{\Lambda^{op}}(X_{t+1}, \Lambda) \to 0 \tag{4}$$

and an isomorphism:

$$\mathrm{Ext}^i_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda) \cong \mathrm{Ext}^{i+2}_{\Lambda^{op}}(X_{t+1}, \Lambda) \text{ for any } i \geq t+1.$$

It follows that $\mathrm{grade}\,\mathrm{Ext}^i_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda) = \mathrm{grade}\,\mathrm{Ext}^{i+2}_{\Lambda^{op}}(X_{t+1}, \Lambda) \geq i+1$ for any $i \geq t+1$. Then $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda), \Lambda) = 0$ for any $i \geq t+1$. On the other hand, since $\mathrm{grade}\,\mathrm{Ext}^t_\Lambda(M, \Lambda) \geq t$, $\mathrm{Ext}^i_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda) = 0$ for any $0 \leq i \leq t-1$. So we conclude that $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^i_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda), \Lambda) = 0$ for any $i \neq t$.

Because $\mathrm{grade}\,\mathrm{Ext}^{t+1}_{\Lambda^{op}}(X_{t+1}, \Lambda) \geq t$ and $\mathrm{grade}\,\mathrm{Ext}^{t+2}_{\Lambda^{op}}(X_{t+1}, \Lambda) \geq t+1$, from the exact sequence (4) we get that $\mathrm{Ext}^i_\Lambda(\mathrm{Ext}^t_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda), \Lambda) \cong \mathrm{Ext}^i_\Lambda(\mathrm{Im}\pi_M, \Lambda) \cong \mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for any $0 \leq i \leq t-1$ (note: $\mathrm{grade}\,M = t$). We claim that $\mathrm{Ext}^t_\Lambda(\mathrm{Ext}^t_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda), \Lambda) \neq 0$. Otherwise, we have that $\mathrm{grade}\,\mathrm{Ext}^t_{\Lambda^{op}}(\mathrm{Ext}^t_\Lambda(M, \Lambda), \Lambda) \geq t+1$ by the above argument. Since $\mathrm{grade}\,\mathrm{Ext}^t_\Lambda(M, \Lambda) \geq t$, $\mathrm{grade}\,\mathrm{Ext}^t_\Lambda(M, \Lambda) \geq t+1$ and $\mathrm{grade}\,M \geq t+1$ by Lemma 3.1, which contradicts the fact that $\mathrm{grade}\,M = t$. The proof is finished. □

Recall from [8] that an ∞-Gorenstein ring is called *Auslander-Gorenstein* if it has finite left and right self-injective dimensions. The following result was proved by Björk in [8, Proposition 1.6] when Λ is Auslander-Gorenstein.

Corollary 3.3 *Let Λ be a left quasi ∞-Gorenstein ring and $M \in \mathrm{mod}\, \Lambda$ with* $\mathrm{grade}\, M$ *finite. Then* $\mathrm{grade}\mathrm{Ext}_{\Lambda}^{\mathrm{grade}\, M}(M, \Lambda) = \mathrm{grade}\, M$.

Proof Suppose $\mathrm{grade}\, M = k(< \infty)$. Since Λ is a left quasi ∞-Gorenstein ring, $\mathrm{grade}\mathrm{Ext}_{\Lambda}^{k}(M, \Lambda) \geq k$ by Lemma 2.1. On the other hand, $\mathrm{Ext}_{\Lambda}^{k}(\mathrm{Ext}_{\Lambda^{op}}^{k}(\mathrm{Ext}_{\Lambda}^{k}(M, \Lambda), \Lambda), \Lambda) \neq 0$ by Theorem 3.2. So $\mathrm{Ext}_{\Lambda^{op}}^{k}(\mathrm{Ext}_{\Lambda}^{k}(M, \Lambda), \Lambda) \neq 0$ and hence $\mathrm{grade}\mathrm{Ext}_{\Lambda}^{k}(M, \Lambda) \leq k$. The proof is finished. □

In viewing of the proof of Theorem 3.2, we get the following

Corollary 3.4 *Let Λ be a left quasi ∞-Gorenstein ring with $\mathrm{id}_{\Lambda^{op}}\Lambda = t(< \infty)$. If $M \in$* $\mathrm{mod}\, \Lambda$ *with* $\mathrm{grade}\, M = t$, *then* $M \cong \mathrm{Ext}_{\Lambda^{op}}^{t}(\mathrm{Ext}_{\Lambda}^{t}(M, \Lambda), \Lambda)$.

Proof Consider the exact sequence (4) in the proof of Theorem 3.2. If $\mathrm{id}_{\Lambda^{op}}\Lambda = t$, then $\mathrm{Ext}_{\Lambda^{op}}^{t+1}(X_{t+1}, \Lambda) = 0 = \mathrm{Ext}_{\Lambda^{op}}^{t+2}(X_{t+1}, \Lambda)$. So $M \cong \mathrm{Ext}_{\Lambda^{op}}^{t}(\mathrm{Ext}_{\Lambda}^{t}(M, \Lambda), \Lambda)$. □

A left (resp. right) quasi ∞-Gorenstein ring is called *left (resp. right) quasi Auslander-Gorenstein* if it has finite left and right self-injective dimensions. We denote $\mathcal{G}_t = \{M \in \mathrm{mod}\ \Lambda | \mathrm{grade}\, M = t\}$. The following corollary generalizes [24, Theorem 4].

Corollary 3.5 *Let Λ be a left and right quasi Auslander-Gorenstein ring with* $\mathrm{id}_{\Lambda}\Lambda = \mathrm{id}_{\Lambda^{op}}\Lambda = t$. *If $M \in \mathrm{mod}\, \Lambda$ with* $\mathrm{grade}\, M = t$, *then* $M \cong \mathrm{Ext}_{\Lambda^{op}}^{t}(\mathrm{Ext}_{\Lambda}^{t}(M, \Lambda), \Lambda)$. *Moreover, the functors* $\mathrm{Ext}_{\Lambda}^{t}(\ , \Lambda)$ *and* $\mathrm{Ext}_{\Lambda^{op}}^{t}(\ , \Lambda)$ *give a duality between $\mathcal{G}_t$ and $\mathcal{G}_t^{op}$.*

Proof By Corollaries 3.3 and 3.4. □

Example 3.6 There exist rings which are left and right quasi Auslander-Gorenstein, but not Auslander-Gorenstein. For example, let Λ be the path algebra given by the quiver $2 \leftarrow 1 \rightarrow 3$. Then $\mathrm{id}_{\Lambda}\Lambda = \mathrm{id}_{\Lambda^{op}}\Lambda = 1$ and $\mathrm{fd}_{\Lambda} I_0^{'} = \mathrm{fd}_{\Lambda^{op}} I_0 = 1$. So Λ is left and right quasi Auslander-Gorenstein, but not Auslander-Gorenstein.

A module $M \in \mathrm{mod}\ \Lambda$ is called *pure* if $\mathrm{grade}\, X = \mathrm{grade}\, M$ for any non-zero submodule X of M (see [8]). We use $\mathcal{C}_{\Lambda}^{n}$ to denote the full subcategory of $\mathrm{mod}\, \Lambda$ consisting of the modules M with $\mathrm{Hom}_{\Lambda}(M, \bigoplus_{i=0}^{n} I_i^{'}) = 0$ (see [10]).

Lemma 3.7 *Let Λ be an Auslander-Gorenstein ring and k a positive integer. Then the following statements are equivalent for a module $M \in \mathrm{mod}\, \Lambda$ with* $\mathrm{grade}\, M = k$.

(1) *M is pure.*
(2) $M \in \mathcal{C}_{\Lambda}^{k-1} \setminus \mathcal{C}_{\Lambda}^{k}$.
(3) $\mathrm{Ext}_{\Lambda^{op}}^{i}(\mathrm{Ext}_{\Lambda}^{i}(M, \Lambda), \Lambda) = 0$ *for every $i \neq k$.*

Proof (1) $\Leftrightarrow$ (3) See [8, Proposition 1.9].

By [10, Lemma 1.1], for a non-negative integer n, we have that a module $M \in \mathrm{mod}\, \Lambda$ is in $\mathcal{C}_{\Lambda}^{n}$ if and only if $\mathrm{s.grade}\, M \geq n + 1$. From this fact it is easy to get (1) $\Leftrightarrow$ (2). □

Björk raised in [8, p.144] a question as follows: For an Auslander-Gorenstein ring Λ, is it true that $\mathrm{Ext}_{\Lambda}^{\mathrm{grade}M}(M,\Lambda)$ is pure for any $M \in \mathrm{mod}\ \Lambda$? It was answered affirmatively by Björk and Ekström in [9, Proposition 2.11]. As an application of Theorem 3.2, we give a different proof of this result.

Proposition 3.8 *Let* Λ *be an Auslander-Gorenstein ring. Then* $\mathrm{Ext}_{\Lambda}^{\mathrm{grade}M}(M,\Lambda)$ *is pure for any* $M \in \mathrm{mod}\ \Lambda$.

Proof Suppose $M \in \mathrm{mod}\ \Lambda$. Since Λ has finite self-injective dimensions, $\mathrm{grade}M$ is finite. By Corollary 3.3, $\mathrm{grade}\mathrm{Ext}_{\Lambda}^{\mathrm{grade}M}(M,\Lambda)$=$\mathrm{grade}M$. On the other hand, by Theorem 3.2, we have that $\mathrm{Ext}_{\Lambda}^{i}(\mathrm{Ext}_{\Lambda^{op}}^{i}(\mathrm{Ext}_{\Lambda}^{\mathrm{grade}M}(M,\Lambda),\Lambda),\Lambda)=0$ if and only if $i \neq \mathrm{grade}M$. It follows from Lemma 3.7 that $\mathrm{Ext}_{\Lambda}^{\mathrm{grade}M}(M,\Lambda)$ is pure. □

In view of the results obtained above, it is natural to ask the following question.

Question Does Proposition 3.8 hold true for left (and right) quasi Auslander-Gorenstein rings? That is, for a left (and right) quasi Auslander-Gorenstein ring Λ, is it true that $\mathrm{Ext}_{\Lambda}^{\mathrm{grade}M}(M,\Lambda)$ is pure for any $M \in \mathrm{mod}\ \Lambda$?

Remark This question is a generalized version of the Björk's question above. By the proof of Proposition 3.8, it is easy to see that the answer to this question is affirmative if the implication (3) $\Rightarrow$ (1) in Lemma 3.7 also holds true for left (and right) quasi Auslander-Gorenstein rings.

In the following, we give some further properties of grade of modules.

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in mod Λ. In general, we have $\mathrm{grade}M_2 \geq \min\{\mathrm{grade}M_1, \mathrm{grade}M_3\}$. Björk showed in [8, Proposition 1.8] that the equality holds true if Λ is an Auslander-Gorenstein ring. The following result shows that the assumption "Λ is Gorenstein" is not necessary for this Björk's result.

Proposition 3.9 *Let* Λ *be an* ∞*-Gorenstein ring and* $0 \to M_1 \to M_2 \to M_3 \to 0$ *an exact sequence in* mod Λ. *Then* $\mathrm{grade}M_2 = \min\{\mathrm{grade}M_1, \mathrm{grade}M_3\}$.

Proof It suffices to prove $\mathrm{grade}M_2 \leq \min\{\mathrm{grade}M_1, \mathrm{grade}M_3\}$. Put $n = \min\{\mathrm{grade}M_1, \mathrm{grade}M_3\}$. Without loss of generality, suppose $n < \infty$. We proceed in three cases.

Case I Assume that $n = \mathrm{grade}M_1 = \mathrm{grade}M_3$.

Consider the following exact sequence:

$$0 = \mathrm{Ext}_{\Lambda}^{n-1}(M_1,\Lambda) \to \mathrm{Ext}_{\Lambda}^{n}(M_3,\Lambda) \to \mathrm{Ext}_{\Lambda}^{n}(M_2,\Lambda).$$

If $\mathrm{Ext}_{\Lambda}^{n}(M_2,\Lambda)=0$, then $\mathrm{Ext}_{\Lambda}^{n}(M_3,\Lambda)=0$ and $\mathrm{grade}M_3 \geq n+1$, which is a contradiction. So $\mathrm{Ext}_{\Lambda}^{n}(M_2,\Lambda) \neq 0$ and $\mathrm{grade}M_2 \leq n$.

Case II Assume that $n = \mathrm{grade}M_3 < \mathrm{grade}M_1$.

Consider the following exact sequence:

$$0 = \mathrm{Ext}_{\Lambda}^{n-1}(M_1, \Lambda) \rightarrow \mathrm{Ext}_{\Lambda}^{n}(M_3, \Lambda) \rightarrow \mathrm{Ext}_{\Lambda}^{n}(M_2, \Lambda) \rightarrow \mathrm{Ext}_{\Lambda}^{n}(M_1, \Lambda) = 0.$$

So $\mathrm{Ext}_{\Lambda}^{n}(M_2, \Lambda) \cong \mathrm{Ext}_{\Lambda}^{n}(M_3, \Lambda) \neq 0$ and hence $\mathrm{grade}\, M_2 \leq n$.

Case III Assume that $n = \mathrm{grade}\, M_1 < \mathrm{grade}\, M_3$.
Consider the following exact sequence:

$$0 = \mathrm{Ext}_{\Lambda}^{n}(M_3, \Lambda) \rightarrow \mathrm{Ext}_{\Lambda}^{n}(M_2, \Lambda) \rightarrow \mathrm{Ext}_{\Lambda}^{n}(M_1, \Lambda) \rightarrow \mathrm{Ext}_{\Lambda}^{n+1}(M_3, \Lambda).$$

If $\mathrm{Ext}_{\Lambda}^{n}(M_2, \Lambda) = 0$, then $\mathrm{Ext}_{\Lambda}^{n}(M_1, \Lambda)$ is isomorphic to a submodule of $\mathrm{Ext}_{\Lambda}^{n+1}(M_3, \Lambda)$. Since Λ is an ∞-Gorenstein ring, $\mathrm{grade}\mathrm{Ext}_{\Lambda}^{n}(M_1, \Lambda) \geq n + 1$. By Lemma 3.1, we have that $\mathrm{grade}\, M_1 \geq n + 1$, which is a contradiction. So $\mathrm{Ext}_{\Lambda}^{n}(M_2, \Lambda) \neq 0$ and $\mathrm{grade}\, M_2 \leq n$. □

For a positive integer k, recall again that a module $M \in \mathrm{mod}\ \Lambda$ is called *k-torsionfree* if $\mathrm{r.gradeTr}\, M \geq k + 1$. We use $\mathcal{T}^k(\mathrm{mod}\ \Lambda)$ to denote the full subcategory of $\mathrm{mod}\ \Lambda$ consisting of k-torsionfree modules. Recall that a full subcategory $\mathcal{X}$ of $\mathrm{mod}\ \Lambda$ is said to be *closed under extensions* if the middle term B of any short sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is in $\mathcal{X}$ provided that the end terms A and C are in $\mathcal{X}$. By Lemma 2.1 and the proof of [17, Lemma 3.2], we have that Λ is a right quasi k-Gorenstein ring if and only if $\mathcal{T}^i(\mathrm{mod}\ \Lambda)$ is closed under extensions for any $1 \leq i \leq k$. The assertion (2) in the following proposition can be regarded as a generalization of this result.

Proposition 3.10 *Let* $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ *be an exact sequence in* $\mathrm{mod}\ \Lambda$. *For a positive integer k, if* $\mathrm{grade} C \geq k$ *where* $C = \mathrm{Coker}(M_2^* \rightarrow M_1^*)$*, then we have*

(1) *If* $M_2 \in \mathcal{T}^{k+1}(\mathrm{mod}\ \Lambda)$ *and* $M_3 \in \mathcal{T}^{k}(\mathrm{mod}\ \Lambda)$*, then* $M_1 \in \mathcal{T}^{k+1}(\mathrm{mod}\ \Lambda)$.
(2) *If* $M_1, M_3 \in \mathcal{T}^{k}(\mathrm{mod}\ \Lambda)$*, then* $M_2 \in \mathcal{T}^{k}(\mathrm{mod}\ \Lambda)$.
(3) *If* $M_1 \in \mathcal{T}^{k}(\mathrm{mod}\ \Lambda)$ *and* $M_2 \in \mathcal{T}^{k-1}(\mathrm{mod}\ \Lambda)$*, then* $M_3 \in \mathcal{T}^{k-1}(\mathrm{mod}\ \Lambda)$.

Proof Consider the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F_0 & \longrightarrow & F_0 \oplus G_0 & \longrightarrow & G_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F_1 \oplus G_1 & \longrightarrow & G_1 & \longrightarrow & 0
\end{array}$$

where all F_i and G_i are projective in mod Λ. Then we get the following commutative diagram with exact columns and rows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_3^* & \longrightarrow & M_2^* & \longrightarrow & M_1^* & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_0^* & \longrightarrow & G_0^* \oplus F_0^* & \longrightarrow & F_0^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_1^* & \longrightarrow & G_1^* \oplus F_1^* & \longrightarrow & F_1^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Tr} M_3 & & \operatorname{Tr} M_2 & & \operatorname{Tr} M_1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

It follows from the snake lemma that $0 \to M_3^* \to M_2^* \to M_1^* \to \operatorname{Tr} M_3 \to \operatorname{Tr} M_2 \to \operatorname{Tr} M_1 \to 0$ is exact. Then we get two short exact sequences: $0 \to C \to \operatorname{Tr} M_3 \to K \to 0$ and $0 \to K \to \operatorname{Tr} M_2 \to \operatorname{Tr} M_1 \to 0$, where $C = \operatorname{Ker}(\operatorname{Tr} M_3 \to \operatorname{Tr} M_2)$ and $K = \operatorname{Im}(\operatorname{Tr} M_3 \to \operatorname{Tr} M_2)$, which yield two long exact sequences:

$$
\begin{aligned}
\operatorname{Ext}^i_{\Lambda^{op}}(K, \Lambda) &\to \operatorname{Ext}^i_{\Lambda^{op}}(\operatorname{Tr} M_3, \Lambda) \to \operatorname{Ext}^i_{\Lambda^{op}}(C, \Lambda) \to \operatorname{Ext}^{i+1}_{\Lambda^{op}}(K, \Lambda) \\
&\to \operatorname{Ext}^{i+1}_{\Lambda^{op}}(\operatorname{Tr} M_3, \Lambda) \to \operatorname{Ext}^{i+1}_{\Lambda^{op}}(C, \Lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{Ext}^i_{\Lambda^{op}}(\operatorname{Tr} M_1, \Lambda) &\to \operatorname{Ext}^i_{\Lambda^{op}}(\operatorname{Tr} M_2, \Lambda) \to \operatorname{Ext}^i_{\Lambda^{op}}(K, \Lambda) \to \operatorname{Ext}^{i+1}_{\Lambda^{op}}(\operatorname{Tr} M_1, \Lambda) \\
&\to \operatorname{Ext}^{i+1}_{\Lambda^{op}}(\operatorname{Tr} M_2, \Lambda) \to \operatorname{Ext}^{i+1}_{\Lambda^{op}}(K, \Lambda)
\end{aligned}
$$

for any $i \geq 0$.

If $M_2 \in \mathcal{T}^{k+1}(\text{mod } \Lambda)$ and $M_3 \in \mathcal{T}^{k}(\text{mod } \Lambda)$, then $\text{r.grade}\operatorname{Tr} M_2 \geq k+2$ and $\text{r.grade}\operatorname{Tr} M_3 \geq k+1$. Because $\text{grade}C \geq k$ by assumption, from the above two long exact sequences we get that $\text{r.grade}K \geq k+1$ and $\operatorname{Ext}^i_{\Lambda^{op}}(\operatorname{Tr} M_1, \Lambda) = 0$ for any $2 \leq i \leq k+1$. But M_1 is clearly torsionless, so $\operatorname{Ext}^1_{\Lambda^{op}}(\operatorname{Tr} M_1, \Lambda) = 0$ and $\text{r.grade}\operatorname{Tr} M_1 \geq k+2$, which implies that M_1 is $(k+1)$-torsionfree. This finishes the proof of (1). Similarly, we get (2) and (3). □

We end this section by giving some examples of rings satisfying the grade condition "$\text{grade}C \geq k$" in Proposition 3.10.

Example 3.11

(1) From the proof of [17, Theorem 2.3], we know that if $\mathrm{fd}_{\Lambda^{op}} \bigoplus_{i=0}^{k-1} I_i \leq k$, then the grade condition in Proposition 3.10 is satisfied. In particular, if Λ is a right quasi k-Gorenstein ring, then this grade condition is also satisfied.

(2) By [23, Proposition 1], we have that $\mathrm{id}_\Lambda \Lambda = \sup\{\mathrm{fd}_{\Lambda^{op}} I \mid I$ is an injective right Λ-module$\}$. Then by (1), the grade condition in Proposition 3.10 is satisfied if $\mathrm{id}_\Lambda \Lambda \leq k$. Thus, if $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{op}} \Lambda \leq k$, then by Proposition 3.10, $\mathcal{T}^k(\mathrm{mod}\ \Lambda)$ is a resolving subcategory of mod Λ in the sense of Auslander and Reiten [3].

4 Torsionless and Reflexive Modules

Let $A \in \mathrm{mod}\ \Lambda$ be a torsionless module. Then A can be embedded into a finitely generated free Λ-module G. We use $\mathcal{E}_A$ to denote the subcategory of mod Λ consisting of the non-zero modules C such that there exists an exact sequence $0 \to A \to G \to C \to 0$ in mod Λ with G free.

Proposition 4.1 *If Λ is a right quasi k-Gorenstein ring, then, for any t-torsionfree module $A \in \mathrm{mod}\ \Lambda$ (where $1 \leq t \leq k$) and $C \in \mathcal{E}_A$, there exists an exact sequence $0 \to F \to T \to C \to 0$ in mod Λ with F free and T $(t-1)$-torsionfree.*

Proof Because Λ is a right quasi k-Gorenstein ring, we have that for any $1 \leq t \leq k$, a module in mod Λ is t-torsionfree if and only if it is t-syzygy by [5, Proposition 1.6 and Theorem 1.7]. So for a t-torsionfree module $A \in \mathrm{mod}\ \Lambda$, there exists an exact sequence $0 \to A \to F \to K \to 0$ in mod Λ with F free and K $(t-1)$-torsionfree. Let $C \in \mathcal{E}_A$. Then there exists an exact sequence $0 \to A \to G \to C \to 0$ in mod Λ with G free. Consider the following push-out diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & F & \longrightarrow & T & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & K & = & K & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

By Lemma 2.1 and the proof of [17, Lemma 3.2], $\mathcal{T}^t(\mathrm{mod}\ \Lambda)$ is closed under extensions for any $1 \leq t \leq k$. So from the exactness of the middle column in the above diagram, we get that T is $(t-1)$-torsionfree. Hence the middle row in the above diagram is as desired. □

Recall from [10] that a module $M \in \text{mod } \Lambda$ is said to be *pseudo-null* if $M \in \mathcal{C}_\Lambda^1$ (that is, $\text{Hom}_\Lambda(M, I_0' \bigoplus I_1') = 0$).

Proposition 4.2 *Let Λ be a 2-Gorenstein ring, and let $A \in \text{mod } \Lambda$ be a torsionless module and $0 \to A \to G \to C \to 0$ an exact sequence in* mod Λ *with G free and $C \in \mathcal{E}_A$. Then we have*

(1) $\text{Coker}\sigma_A$ *is pseudo-null.*
(2) A^{**} is isomorphic to a submodule of G.
(3) If C has no non-zero pseudo-null submodule, then A is reflexive.

Proof

(1) By Lemma 2.12, $\text{Coker}\sigma_A \cong \text{Ext}^2_{\Lambda^{op}}(\text{Tr } A, \Lambda)$. Since Λ is 2-Gorenstein, $\text{Hom}_\Lambda(\text{Ext}^2_{\Lambda^{op}}(\text{Tr } A, \Lambda), I_0' \bigoplus I_1') = 0$ by [25, Proposition 3].

(2) From the exact sequence $0 \to A \to G \to C \to 0$ we get an exact sequence:

$$0 \to C^* \to G^* \to A^* \to \text{Ext}^1_\Lambda(C, \Lambda) \to 0.$$

Put $K = \text{Im}(G^* \to A^*)$. Then we get two exact sequences $0 \to K^* \to G^{**} (\cong G)$ and $[\text{Ext}^1_\Lambda(C, \Lambda)]^* \to A^{**} \to K^*$. Because Λ is 2-Gorenstein, $\text{s.grade}\text{Ext}^1_\Lambda(C, \Lambda) \geq 1$ and $[\text{Ext}^1_\Lambda(C, \Lambda)]^* = 0$. Then the assertion follows.

(3) Note that $A^{**}/A \cong \text{Coker}\sigma_A$ is pseudo-null by (1) and that A^{**}/A is isomorphic to a submodule of C by (2). So $A^{**}/A = 0$ by assumption and hence $A \cong A^{**}$ and A is reflexive. □

We now give a criterion for judging when a torsionless module is reflexive.

Theorem 4.3 *Let Λ be a 2-Gorenstein ring and $A \in \text{mod } \Lambda$ a torsionless module. Then the following statements are equivalent.*

(1) A is reflexive.
(2) C has no non-zero pseudo-null submodule for any $C \in \mathcal{E}_A$.
(3) C has no non-zero pseudo-null submodule for some $C \in \mathcal{E}_A$.

Proof (2) $\Rightarrow$ (3) is trivial, and (3) $\Rightarrow$ (1) follows from Proposition 4.2(3).

(1) $\Rightarrow$ (2) Assume that $A \in \text{mod } \Lambda$ is reflexive. Then, by Proposition 4.1, for any $C \in \mathcal{E}_A$, there exists an exact sequence $0 \to F \to T \to C \to 0$ in mod Λ with F free and T torsionless. By [29, Corollary 1.3], it is easy to see that C can be embedded into a finite direct sum of $I_0' \bigoplus I_1'$ and so C has no non-zero pseudo-null submodule. □

Let Λ be an Auslander-Gorenstein ring and I a non-zero proper left ideal of Λ. Then, by Lemma 3.7, we have that Λ/I has no non-zero pseudo-null submodule if and only if Λ/I is pure of grade 1. It was showed in [10, Lemma 4.12] that if Λ is an Auslander-regular ring (that is, Λ is an ∞-Gorenstein ring with finite global dimension) without non-zero zero divisors, then I is reflexive if and only if Λ/I is pure of grade 1. The following corollary generalizes this result, which is an immediate consequence of Theorem 4.3.

Corollary 4.4 *Let Λ be a 2-Gorenstein ring. Then a non-zero proper left ideal I of Λ is reflexive if and only if Λ/I has no non-zero pseudo-null submodule.*

Recall from [12] that a right Λ-module M is said to *have an injective resolution with a redundant image* from a positive integer n if the n-th cosyzygy Ω_n has a decomposition $\Omega_n = \bigoplus_{i\in I} A_i$ such that each A_i is a direct summand of a cosyzygy Ω_{α_i} for some $\alpha_i \neq n$. It was showed in [12, Theorem 1] that if Λ_Λ has an injective resolution with a redundant image from n, then $\bigoplus_{i=0}^{n} I_i'$ is an injective cogenerator for the category of left Λ-modules. Assume that Λ is a 2-Gorenstein ring such that Λ_Λ has an injective resolution with a redundant image from 1. Then $I_0' \bigoplus I_1'$ is an injective cogenerator for the category of left Λ-modules. So every module in mod Λ has no non-zero pseudo-null submodule and hence each torsionless module in mod Λ is reflexive by Theorem 4.3. It then follows from [28, Theorem 5.1] that $\mathrm{id}_{\Lambda^{op}} \Lambda \leq 1$. Therefore we have established the following result.

Corollary 4.5 *Let Λ be a 2-Gorenstein ring. If Λ_Λ has an injective resolution with a redundant image from 1, then* $\mathrm{id}_{\Lambda^{op}} \Lambda \leq 1$.

Ramras raised in [13, p.380] an open question: When is each reflexive module in mod Λ projective? A generalized version of this question is: For a positive integer k, when is each k-torsionfree module in mod Λ projective? In the following, we will deal with these two questions and give some partial answers to them.

Proposition 4.6 *For any positive integer k, the following statements are equivalent.*

(1) *Each k-torsionfree module in* mod Λ *is projective.*
(2) *Each module in* mod Λ^{op} *with reduced grade at least $k+1$ is projective.*

Proof (1) $\Rightarrow$ (2) Let $N \in \mathrm{mod}\ \Lambda^{op}$ with $\mathrm{r.grade} N \geq k+1$ and $Q_1 \xrightarrow{g} Q_0 \to N \to 0$ be a projective presentation of N in mod Λ^{op}. Then we get an exact sequence:

$$0 \to N^* \to Q_0^* \xrightarrow{g^*} Q_1^* \to \mathrm{Tr}\, N \to 0.$$

It is easy to see that $N \cong \mathrm{Coker} g^{**} = \mathrm{Tr}\,\mathrm{Tr}\, N$. So $\mathrm{Tr}\, N \in \mathrm{mod}\ \Lambda$ is k-torsionfree and hence $\mathrm{Tr}\, N$ is projective by (1). It follows from Proposition 2.13 that N is projective.

(2) $\Rightarrow$ (1) Let $M \in$ mod Λ be a k-torsionfree module. Then $\mathrm{r.grade Tr}\, M \geq k+1$ and so $\mathrm{Tr}\, M$ is projective by (2). It follows from Proposition 2.13 that M is projective. □

It is well known that Λ is a hereditary ring (that is, $\mathrm{gl.dim}\Lambda \leq 1$) if and only if each torsionless module in mod Λ (or mod Λ^{op}) is projective. By Proposition 4.6, we give a new characterization of hereditary rings as follows.

Corollary 4.7 *Λ is hereditary if and only if each module in* mod Λ *(or* mod Λ^{op}*) with reduced grade at least 2 is projective.*

Theorem 4.8 *Let k be a positive integer or infinite. Then the following statements are equivalent.*

(1) *Each k-torsionfree module in* mod Λ *is projective.*
(2) *Each module in* mod Λ^{op} *with reduced grade at least* $k+1$ *is projective.*

Proof When k is a positive integer, it has been proved in Proposition 4.6. When k is infinite, the proof is similar to that of Proposition 4.6, so we omit it. □

Corollary 4.9 *If* gl.dim$\Lambda \leq k$*, then each k-torsionfree module in* mod Λ *is projective.*

Proof Let $N \in$ mod Λ^{op} with r.grade$N \geq k+1$. Because gl.dim$\Lambda \leq k$, $\text{pd}_{\Lambda^{op}} N \leq k$. Then by Lemma 2.10, N is projective. Thus the assertion follows from Theorem 4.8. □

In particular, if putting $k = 2$ in Corollary 4.9, then we get the following

Corollary 4.10 *If* gl.dim$\Lambda \leq 2$*, then each reflexive module in* mod Λ *is projective.*

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References

1. Auslander, M., Bridger, M.: Stable module theory. In: Memoirs Amer. Math. Soc. vol. 94. Amer. Math. Soc., Providence, RI (1969)
2. Auslander, M., Reiten, I.: On a generalized version of the Nakayama conjecture. Proc. Am. Math. Soc. **52**, 69–74 (1975)
3. Auslander, M., Reiten, I.: Applications to contravariantly finite subcategories. Adv. Math. **86**, 111–152 (1991)
4. Auslander, M., Reiten, I.: *k*-Gorenstein algebras and syzygy. J. Pure Appl. Algebra **92**, 1–27 (1994)
5. Auslander, M., Reiten, I.: Syzygy modules for Noetherian rings. J. Algebra **183**, 167–185 (1996)
6. Bass, H.: Finitistic dimension and a homological generalization of semi-primary rings. Trans. Am. Math. Soc. **95**, 466–488 (1960)
7. Bass, H.: Injective dimension in Noetherian rings. Trans. Am. Math. Soc. **102**, 18–29 (1962)
8. Björk, J.E.: The Auslander condition on Noetherian rings. In: Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année, Paris, 1987/1988, Lect. Notes in Math. vol. 1404, pp. 137–173. Springer, Berlin (1989)
9. Björk, J.E., Ekström, E.K.: Filtered Auslander-Gorenstein rings. In: Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Paris, 1989, Progr. Math. vol. 92, pp. 425–448. Birkhäuser, Boston (1990)
10. Coates, J., Schneider, P., Sujatha, R.. Modules over Iwasawa algebras. Institute J. Math. Jussieu **2**, 73–108 (2003)
11. Fossum, R.M., Griffith, P.A., Reiten, I.: Trivial Extensions of Abelian Categories, Lect. Notes in Math. vol. 456. Springer, Berlin (1975)
12. Fuller, K.R., Wang, Y.: Redundancy in resolutions and finitistic dimensions of Noetherian rings. Commun. Algebra **21**, 2983–2994 (1993)
13. Gordon, R. (ed.): Ring Theory, Proceedings of a Conference on Ring Theory held in Park City from March 2–6, 1971. Academic Press, New York (1972)
14. Hoshino, M.: Syzygies and Gorenstein rings. Arch. Math. **55**, 355–360 (1990)

15. Hoshino, M.: Noetherian rins of self-injective dimension two. Commun. Algebra **21**, 1071–1094 (1993)
16. Hoshino, M., Nishida, K.: A generalization of the Auslander formula. In: Representations of Algebras and Related Topics, Fields Instit. Comm. vol. 45, pp. 175–186. Amer. Math. Soc., Providence, RI (2005)
17. Huang, Z.Y.: Extension closure of k-torsionfree modules. Commun. Algebra **27**, 1457–1464 (1999)
18. Huang, Z.Y.: Selforthogonal modules with finite injective dimension II. J. Algebra **264**, 262–268 (2003)
19. Huang, Z.Y.: Syzygy modules for quasi k-Gorenstein rings. J. Algebra **299**, 21–32 (2006)
20. Huang, Z.Y.: Approximation presentations of modules and homological conjectures. Commun. Algebra **36**, 546–563 (2008)
21. Huang, Z.Y.,Tang, G.H.: Self-orthogonal modules over coherent rings. J. Pure Appl. Algebra **161**, 167–176 (2001)
22. Huisgen-Zimmermann, B.: Homological domino effects and its first finitistic dimension conjecture. Invent. Math. **108**, 369–383 (1992)
23. Iwanaga, Y.: On rings with finite self-injective dimension II. Tsukuba J. Math. **4**, 107–113 (1980)
24. Iwanaga, Y.: Duality over Auslander-Gorenstein rings. Math. Scand. **81**, 184–190 (1997)
25. Iwanaga, Y., Sato, H.: On Auslander's n-Gorenstein rings. J. Pure Appl. Algebra **106**, 61–76 (1996)
26. Iyama, O.: τ-categories III, Auslander orders and Auslander-Reiten quivers. Algebr. Represent. Theory **8**, 601–619 (2005)
27. Jans, J.P.: Duality in Noetherian rings. Proc. Am. Math. Soc. **12**, 829–835 (1961)
28. Jans, J.P.: On finitely generated modules over Noetherian rings. Trans. Am. Math. Soc. **106**, 330–340 (1963)
29. Miyachi, J.: Injective resolutions of Noetherian rings and cogenerators. Proc. Am. Math. Soc. **128**, 2233–2242 (2000)
30. Smalø, S.O.: The supremum of the difference between the big and little finitistic dimensions is infinite. Proc. Am. Math. Soc. **126**, 2619–2622 (1998)
31. Yamagata, K.: Frobenius algebras. In: Hazewinkel, M. (ed.) Handbook of Algebra vol. 1, pp. 841–887. North-Holland Publishing Co., Amsterdam (1996)
32. Zaks, A.: Injective dimension of semiprimary rings. J. Algebra **13**, 73–86 (1969)