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ON THE FLATNESS AND INJECTIVITY OF DUAL MODULES (III)

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Abstract. For a commutative ring R and an injective cogenerator E in the category of R-modules, we characterize von Neumann regular rings and semisimple artinian rings in terms of the properties of dual modules with respect to E.

1. Introduction

Throughout this paper, R is a commutative ring with identity and all modules are unital. We use Mod$R$ to denote the category of R-modules and unless stated otherwise E to denote a certain injective cogenerator in Mod$R$. Such an R-module E is called in [9] faithfully injective.

Let $M \in$Mod$R$. We call in [7] $\mathrm{Hom}_R(M, E)$ the dual module of M with respect to E, and denote it by M^e. We use $\sigma_M : M \to M^{ee}$ to denote the canonical evaluation homomorphism which is defined as $\sigma_M(x)(f) = f(x)$ for any $x \in M$ and $f \in M^e$. It follows from [2, Proposition 20.14] that σ_M^e is an epimorphism, where $\sigma_M^e = \mathrm{Hom}_R(\sigma_M, E)$. Notice that E is a cogenerator in Mod$R$, so, by [2, Proposition 18.14], σ_M is an embedding.

We characterize in [7] and [8] several classes of rings, such as coherent rings, noetherian rings, artinian rings, quasi-Frobenius rings and IF rings, by using the flatness, injectivity, FP-injectivity and projectivity of duals with respect to E.

As the classes of rings having homological dimensions zero, both von Neumann regular rings and semisimple artinian rings are very important classes of rings in ring theory, which are characterized respectively as follows (see [10] and [4]).

Theorem A. *The following statements are equivalent.*

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(1) *R is a von Neumann regular ring.*
(2) *The weak global dimension of R is zero.*
(3) *Each R-module is flat.*
(4) *For any $a \in R$, there is an element $a' \in R$ with $aa'a = a$.*

THEOREM B. *The following statements are equivalent.*
(1) *R is a semisimple artinian ring.*
(2) *The global dimension of R is zero.*
(3) *Each (finitely generated) R-module is projective.*
(4) *Each (finitely generated) R-module is quasi-projective.*
(5) *Each R-module is injective.*
(6) *R is a direct sum of simple R-modules.*

In this paper we characterize von Neumann regular rings and semisimple artinian rings by using the quasi-injectivity and quasi-projectivity of duals with respect to E, respectively. We show that R is a von Neumann regular ring (resp. a semisimple artinian ring) if and only if the dual of each module in Mod$R$ is quasi-injective (resp. quasi-projective). As corollaries, we have that R is a von Neumann regular ring if and only if each pure-injective R-module is injective if and only if each pure-injective R-module is quasi-injective if and only if each pure-injective R-module is (quasi-)flat; and R is a semisimple artinian ring if and only if each pure-injective R-module is projective if and only if each pure-injective R-module is quasi-projective. These results generalize the classical ones mentioned above.

2. Main results

LEMMA 1. ([9, Proposition 3.6]) *An R-module M is flat if and only if M^e is injective.*

Recall that an exact sequence $0 \to L \to M \to N \to 0$ in Mod$R$ is called pure if $0 \to L \otimes_R A \to M \otimes_R A \to N \otimes_R A \to 0$ is exact for any (finitely presented) R-module A. In this case, $0 \to L \to M$ is called a pure monomorphism (see [11]).

LEMMA 2. *If E is a cogenerator (not necessarily injective) in* Mod$R$, *then an exact sequence* $0 \to L \to M \to N \to 0$ *in* Mod$R$ *is pure provided* $0 \to N^e \to M^e \to L^e \to 0$ *is exact and pure.*

Proof. The argument in proving (2) ⇒ (1) of [7, Lemma 1] remains valid in our assumption, we omit it. □

Recall that an R-module X is called quasi-injective if the induced homomorphism $\mathrm{Hom}_R(X,X) \stackrel{\mathrm{Hom}_R(f,X)}{\longrightarrow} \mathrm{Hom}_R(Y,X)$ is epic for any monomorphism $Y \stackrel{f}{\to} X$ (see [1]). We now characterize von Neumann regular rings by using the (quasi-)injectivity of duals with respect to E as follows.

THEOREM 3. *The following statements are equivalent.*

(1) *R is a von Neumann regular ring.*

(2) *M^e is flat for each $M \in$*Mod*R.*

(3) *M^e is injective for each $M \in$*Mod*R.*

(4) *M^e is quasi-injective for each $M \in$*Mod*R.*

(5) *N^{ee} is injective for each $N \in$*Mod*R.*

(6) *N^{ee} is quasi-injective for each $N \in$*Mod*R.*

Proof. The implications that (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) $\Rightarrow$ (6) and (3) $\Rightarrow$ (5) $\Rightarrow$ (6) are trivial. From Lemma 1 we get the equivalence between (2) and (5). So it suffices to prove (6) $\Rightarrow$ (1).

Assume that the condition (6) is satisfied. We claim that any R-module is flat. Otherwise, if there is a non-flat R-module M, then there is a monomorphism $f: A \to B$ in Mod$R$ such that $f \otimes_R 1_M : A \otimes_R M \to B \otimes_R M$ is not monic. Put $X =$Cokerσ_M. It follows from [2, Proposition 20.14] that the sequence $0 \to X^e \to M^{eee} \stackrel{\sigma_M^e}{\longrightarrow} M^e \to 0$ is exact and split and hence is certainly pure. Then, by Lemma 2, σ_M is pure.

Put $N = B \oplus M^e$ and let g be the composition: $B \stackrel{\sigma_B}{\longrightarrow} B^{ee} \to B^{ee} \oplus M^{eee} (\cong N^{ee})$ and h the composition: $M \stackrel{\sigma_M}{\longrightarrow} M^{ee} \to B^e \oplus M^{ee}$ $(\cong N^e)$, where both $B^{ee} \to B^{ee} \oplus M^{eee}$ and $M^{ee} \to B^e \oplus M^{ee}$ are natural embeddings. Then g and h are monomorphisms.

Suppose $0 \neq \sum_{i=1}^n (a_i \otimes m_i) \in$Ker$(f \otimes_R 1_M) (\subset A \otimes_R M)$, where $a_i \in A$ and $m_i \in M$ for any $1 \leq i \leq n$. Since σ_M is a pure monomorphism, $\sum_{i=1}^n (a_i \otimes \sigma_M(m_i)) (\in A \otimes_R M^{ee})$ is non-zero. Notice that $M^{ee} \to B^e \oplus M^{ee} (\cong N^e)$ is a natural embedding, so $\sum_{i=1}^n (a_i \otimes h(m_i)) (\in A \otimes_R N^e)$ is non-zero. On the other hand, $(gf \otimes_R 1_{N^e})[\sum_{i=1}^n (a_i \otimes h(m_i))] = (g \otimes_R h)(f \otimes_R 1_M)[\sum_{i=1}^n (a_i \otimes m_i)] = 0$, which implies that $gf \otimes_R 1_{N^e}$ is not a monomorphism. Since E is cogenerator in Mod$R$, $(gf \otimes_R 1_{N^e})^e : (N^{ee} \otimes_R N^e)^e \to (A \otimes_R N^e)^e$ is not an epimorphism by [2, Proposition 18.14]. It follows from [10, Theorem 2.11] (Adjoint Isomorphism) that $\mathrm{Hom}_R(gf, N^{ee}) : \mathrm{Hom}_R(N^{ee}, N^{ee}) \to \mathrm{Hom}_R(A, N^{ee})$ is not an epimorphism. We then conclude that N^{ee} is not quasi-injective, which induces a contradiction. So R is von Neumann regular. $\square$

Definition 4. Let E be a cogenerator (not necessarily injective) in Mod$R$. An R-module M is called quasi-flat (with respect E) if for any monomorphism $f : N \to M^e$ in Mod$R$ the induced map $f \otimes_R 1_M : N \otimes_R M \to M^e \otimes_R M$ is a monomorphism.

Remark. A flat R-module is clearly quasi-flat. However, the converse doesn't hold in general. For example, let $\mathbb{Z}$ be integers and $\mathbb{Q}$ its quotient field. Then $E = \mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in Mod$\mathbb{Z}$. Put $\bar{\mathbb{Z}} = \mathbb{Z}/(0 :_{\mathbb{Z}} \mathbb{Z}_2)$. It is clear that $\bar{\mathbb{Z}} = \mathbb{Z}/(2) \cong \mathbb{Z}_2$, so $\mathbb{Z}_2$ is a projective (and hence a flat) $\bar{\mathbb{Z}}$-module. Because $0 :_{\mathbb{Z}} \mathbb{Z}_2 = 0 :_{\mathbb{Z}} \mathbb{Z}_2^e$, both $\mathbb{Z}_2$ and $\mathbb{Z}_2^e$ are $\bar{\mathbb{Z}}$-modules. Now let $0 \to M \to \mathbb{Z}_2^e$ be an exact sequence in Mod$\mathbb{Z}$. Then it is also an exact sequence in Mod$\bar{\mathbb{Z}}$ and we get an exact sequence $0 \to M \otimes_{\bar{\mathbb{Z}}} \mathbb{Z}_2 \to \mathbb{Z}_2^e \otimes_{\bar{\mathbb{Z}}} \mathbb{Z}_2$. It is not difficult to verify that $M \otimes_{\bar{\mathbb{Z}}} \mathbb{Z}_2 \cong M \otimes_{\mathbb{Z}} \mathbb{Z}_2$ and $\mathbb{Z}_2^e \otimes_{\bar{\mathbb{Z}}} \mathbb{Z}_2 \cong \mathbb{Z}_2^e \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Thus we get an exact sequence $0 \to M \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to \mathbb{Z}_2^e \otimes_{\mathbb{Z}} \mathbb{Z}_2$, which implies that $\mathbb{Z}_2$ is a quasi-flat $\mathbb{Z}$-module. However, $\mathbb{Z}_2$ is not torsionfree over $\mathbb{Z}$, it then follows from [10, Theorem 4.33] that $\mathbb{Z}_2$ is not a flat $\mathbb{Z}$-module.

Lemma 5. *Let M be in* Mod$R$. *If E is a cogenerator in* Mod$R$ *and M^e is quasi-injective, then M is quasi-flat. The converse holds when E is injective.*

Proof. Let $N \to M^e$ be a monomorphism in Mod$R$. If M^e is quasi-injective, then $\mathrm{Hom}_R(M^e, M^e) \to \mathrm{Hom}_R(N, M^e) \to 0$ is exact. By [10, Theorem 2.11] we have that $(M^e \otimes_R M)^e \to (N \otimes_R M)^e \to 0$ is exact. Since E is a cogenerator, $0 \to N \otimes_R M \to M^e \otimes_R M$ is also exact by [2, Proposition 18.14]. Thus M is quasi-flat. Conversely, if M is quasi-flat, then $N \otimes_R M \to M^e \otimes_R M$ is a monomorphism. Under the assumption that E is injective, we then have that $(M^e \otimes_R M)^e \to (N \otimes_R M)^e \to 0$ is exact. Again by [10, Theorem 2.11], $\mathrm{Hom}_R(M^e, M^e) \to \mathrm{Hom}_R(N, M^e) \to 0$ is also exact and M^e is quasi-injective. □

As an immediate consequence of Theorem 3 and Lemma 5, we have the following

Theorem 6. *If E is a cogenerator in* Mod$R$ *and N^{ee} is quasi-injective for each $N \in$*Mod$R$ (*that is, the condition* (6) *in Theorem 3 is satisfied*), *then we have that*

(7) *M^e is quasi-flat for each $M \in$*Mod$R$.

Moreover, if E is injective, then the conditions $(1) - (6)$ *in Theorem 3 and the condition* (7) *above are equivalent.*

Recall that an R-module Q is called pure-injective if for any pure exact sequence $0 \to A \to B \to C \to 0$ in Mod$R$ the induced sequence

$0 \to \mathrm{Hom}_R(C,Q) \to \mathrm{Hom}_R(B,Q) \to \mathrm{Hom}_R(A,Q) \to 0$ is exact (see [11]). It is trivial that an injective R-module is pure-injective. But the converse doesn't hold in general. From Corollary 8 below, which says that each pure-injective R-module is injective if and only if R is a von Neumann regular ring, we may give a counter-example easily.

From now on, E always denotes a certain injective cogenerator in $\mathrm{Mod}R$.

LEMMA 7. *M^e is pure-injective for each $M \in \mathrm{Mod}R$.*

Proof. Let M be in $\mathrm{Mod}R$ and $0 \to A \to B \to C \to 0$ a pure exact sequence in $\mathrm{Mod}R$. Then we have that $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is exact and hence $0 \to (C \otimes_R M)^e \to (B \otimes_R M)^e \to (A \otimes_R M)^e \to 0$ is also exact by the injectivity of E. By [10, Theorem 2.11], we get an exact sequence $0 \to \mathrm{Hom}_R(C, M^e) \to \mathrm{Hom}_R(B, M^e) \to \mathrm{Hom}_R(A, M^e) \to 0$, which implies that M^e is pure-injective. □

The following result generalizes the classical characterizations of von Neumann regular rings mentioned in Section 1.

COROLLARY 8. *The following statements are equivalent.*

(1) *R is a von Neumann regular ring.*

(2) *Each pure-injective R-module is flat.*

(3) *Each pure-injective R-module is quasi-flat.*

(4) *Each pure-injective R-module is injective.*

(5) *Each pure-injective R-module is quasi-injective.*

(6) *M^e is injective for each pure-injective R-module M.*

(7) *M^e is quasi-injective for each pure-injective R-module M.*

Proof. The implications that (1) ⇒ (2) ⇒ (3), (4) ⇒ (5) and (6) ⇒ (7) are trivial.

(3) ⇒ (1) Let M be in $\mathrm{Mod}R$. Then M^e is pure-injective by Lemma 7, and by (3) it is quasi-flat. From Theorems 6 and 3 we know that R is a von Neumann regular ring. Similarly, we get (5) ⇒ (1) and (7) ⇒ (1).

(1) ⇒ (4) Let M be a pure-injective R-module. By (1) and Theorem 3, M^e is flat. Then, by Lemma 1, M^{ee} is injective. Notice that σ_M is a pure monomorphism (see the proof of Theorem 3) and M is pure-injective, so σ_M splits and M is isomorphic to a direct summand of M^{ee}. It turns out that M is injective.

(1) ⇒ (6) It follows from Theorem A and Lemma 1. □

Recall that an R-module X is called quasi-projective if the induced homomorphism $\mathrm{Hom}_R(X,X) \stackrel{\mathrm{Hom}_R(X,g)}{\longrightarrow} \mathrm{Hom}_R(X,Y)$ is epic for any epimorphism $X \stackrel{g}{\to} Y$ (see [1]). We now characterize semisimple artinian rings by using the (quasi-)projectivity of duals with respect to E as follows, where E is a certain injective cogenerator in Mod$R$.

THEOREM 9. *The following statements are equivalent.*
(1) *R is a semisimple artinian ring.*
(2) *M^e is projective for each $M \in$Mod$R$.*
(3) *M^e is quasi-projective for each $M \in$Mod$R$.*
(4) *N^{ee} is projective for each $N \in$Mod$R$.*
(5) *N^{ee} is quasi-projective for each $N \in$Mod$R$.*

To prove this theorem, we need two lemmas.

LEMMA 10. *Let M be a finitely presented R-module. If M^e is quasi-injective, then M is quasi-projective.*

Proof. Assume that $M \to N$ is an epimorphism in Mod$R$ with M finitely presented. Then $N^e \to M^e$ is monic. If M^e is quasi-injective, then $\mathrm{Hom}_R(M^e, M^e) \to \mathrm{Hom}_R(N^e, M^e) \to 0$ is exact. By [10, Theorem 2.11] $(M^e \otimes_R M)^e \to (N^e \otimes_R M)^e \to 0$ is exact and then by [10, Lemma 3.60] we have an exact sequence $[\mathrm{Hom}_R(M, M)]^{ee} \to [\mathrm{Hom}_R(N, M)]^{ee} \to 0$. Thus we conclude that $\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R (N, M) \to 0$ is also an exact sequence and M is quasi-projective. □

LEMMA 11. ([3, Corollary 2.3]) *R is a quasi-Frobenius ring if and only if each injective R-module is quasi-projective.*

Proof of Theorem 9. The implications that (1) ⇒ (2) ⇒ (3) ⇒ (5) and (2) ⇒ (4) ⇒ (5) are trivial. So we only need to prove (5) ⇒ (1).

Assume that (5) holds. Let N be an injective R-module. Since σ_N is an embedding, N is isomorphic to a direct summand of N^{ee}. By (5), N^{ee} is quasi-projective. So N is also quasi-projective by [1, Proposition 2.2] and hence R is a quasi-Frobenius ring by Lemma 11.

Let M be a finitely generated R-module. Then M is finitely presented for R which is a quasi-Frobenius ring (and it is certainly noetherian). Assume that $B \to M^e$ is a monomorphism in Mod$R$. Then the induced homomorphism $M^{eeee} \to B^{eee}$ is epic. By (5), $M^{eeee} \oplus B^{eee} \cong (M^{ee} \oplus B^e)^{ee}$ is quasi-projective. It follows from [5, Lemma 2.1] that $M^{eeee} \to B^{eee}$ splits and we then yield an exact sequence $\mathrm{Hom}_R(M, M^{eeee}) \to \mathrm{Hom}_R(M, B^{eee}) \to 0$. Applying [10, Theorem 2.11 and Lemma 3.60] to this exact sequence, we get successively the exact sequences $0 \to M \otimes_R B^{ee} \to M \otimes_R M^{eee}$, $\mathrm{Hom}_R(M, M^{ee}) \to \mathrm{Hom}_R(M, B^e) \to 0$, $(M \otimes_R M^e)^e \to (M \otimes_R B)^e \to 0$ and $\mathrm{Hom}_R(M^e, M^e) \to \mathrm{Hom}_R(B, M^e) \to 0$. The exactness of the last sequence implies that M^e is quasi-injective.

Then, by Lemma 10, M is quasi-projective. Therefore R is a semisimple artinian ring by Theorem B. □

The following result contains [6, Theorem 3.4] and generalizes [4, Theorem 1.3] and the classical characterizations of semisimple artinian rings mentioned in Section 1.

COROLLARY 12. *The following statements are equivalent.*
(1) *R is a semisimple artinian ring.*
(2) *Each pure-injective R-module is projective.*
(3) *Each pure-injective R-module is quasi-projective.*
(4) *M^e is projective for each pure-injective R-module M.*
(5) *M^e is quasi-projective for each pure-injective R-module M.*

Proof. The implications that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are trivial, and both (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) follow from Lemma 7.

Assume that (5) holds. Let N be in Mod$R$. Then N^e is pure-injective by Lemma 7 and so N^{ee} is quasi-projective by (5). It follows from Theorem 9 that R is a semisimple artinian ring. This shows (5) $\Rightarrow$ (1). □

For any M in Mod$R$, $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+ (which is called the character module of M). By [11, Chapter I, Proposition 9.3], we have that R^+ is an injective cogenerator in Mod$R$. On the other hand, from [10, Theorem 2.11] it follows easily that $\mathrm{Hom}_R(M, R^+) \cong M^+$ for any M in Mod$R$. Consequently, the results obtained above remains true when the notation of $(\ )^e$ is replaced by that of $(\ )^+$.

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