



# Extension Closure of Adjoint Cotorsionfree Modules

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## Abstract

Let  $R$  and  $S$  be rings and  ${}_R\omega_S$  a semidualizing bimodule, and let  $n \geq 1$ . We characterize the extension closure of the category of adjoint  $k$ -cotorsionfree modules with respect to  $\omega$  for any  $1 \leq k \leq n$  in terms of the (strong) cograde conditions of certain modules.

**Keywords** Semidualizing bimodules · (Strong) Ext-cograde · (Strong) Tor-cograde · Extension closed · Adjoint  $n$ -cotorsionfree modules

**Mathematics Subject Classification** 18G25 · 18G15 · 16E30

## 1 Introduction

Throughout this paper, all rings are associative rings with units. For a ring  $R$ , we use  $\text{Mod } R$  to denote the category of left  $R$ -modules. Recall that a subcategory  $\mathcal{X}$  of  $\text{Mod } R$  is called *extension closed* provided that for any exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

in  $\text{Mod } R$ , if  $A$  and  $C$  are in  $\mathcal{X}$ , then so is  $B$ . The extension closure of certain subcategories has been proved to be important in characterizing rings. We mention a well-known result about the extension closure of the category of  $n$ -syzygy modules over a Noetherian algebra  $R$  ([2, Theorem 4.7]), which shows that the category of

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finitely generated  $i$ -syzygy modules is extension closed for any  $1 \leq i \leq n$  if and only if  $R$  is quasi  $n$ -Gorenstein in the sense of [9]. Applying this theorem, under Serre's condition, Goto and Takahashi characterized a commutative Noetherian local ring in height less than  $n$  to be Gorenstein in terms of the extension closure of the category of finitely generated  $n$ -syzygy modules ([6, Theorem B]). The extension closure of various subcategories has been studied extensively, see [3, 4, 10, 11, 15] and references therein.

In particular, Huang [10] initiated the study of extension closure of the category of  $n$ -torsionfree modules with respect to a semidualizing bimodule  ${}_R\omega_S$  by using the properties of the (strong) grade of modules. In [16] and [18] we dualized the Auslander transpose and introduced the notions of  $n$ - $\omega$ -cotorsionfree modules and adjoint  $n$ - $\omega$ -cotorsionfree modules respectively. These two classes have many dual properties of relative  $n$ -torsionfree modules. It is thus natural to ask the following question:

**Question 1.1** When are the category of  $n$ - $\omega$ -cotorsionfree modules and that of adjoint  $n$ - $\omega$ -cotorsionfree modules extension closed?

This question has been partially solved by Zhao and Zhang so far, and they proved that the category of  $i$ - $\omega$ -cotorsionfree modules is extension closed for any  $1 \leq i \leq n$  if and only if the strong Tor-cograde of  $\text{Ext}_R^{i+1}(\omega, M)$  is at least  $i$  for any  $\omega$ - $i$ -syzygy module  $M$  and  $1 \leq i \leq n$  ([22, Theorem 3.10]). The purpose of this paper is to proceed with the study of Question 1.1. Indeed, we will investigate the extension closure of the category of adjoint  $n$ - $\omega$ -cotorsionfree modules.

The organization of this paper is as follows. Section 2 contains some basic definitions and preliminary results. Let  $R, S$  be arbitrary rings and let  ${}_R\omega_S$  be a semidualizing bimodule. In Section 3, we show that the categories of adjoint 1-cotorsionfree modules and adjoint 2-cotorsionfree modules are extension closed if and only if  $\text{Tor}_{k-1}^S(\omega, \text{Ext}_R^k(\omega, M)) = 0$  for any left  $R$ -module  $M$  and  $k = 1, 2$ , and if and only if the Tor-cograde of  $\text{Ext}_R^k(\omega, M)$  with respect to  $\omega$  is at least  $k$  for any left  $R$ -module  $M$  and  $k = 1, 2$  (Theorem 3.10).

Let  $\mathcal{A}_\omega(S)$  be the Auslander class with respect to  $\omega$ . In Section 4, we show that the category of adjoint  $k$ - $\omega$ -cotorsionfree modules is extension closed for any  $1 \leq k \leq n$ , if and only if the category of  $k$ - $\mathcal{A}_\omega(S)$ -syzygy modules is extension closed for any  $1 \leq k \leq n$ , if and only if the strong Ext-cograde of  $\text{Tor}_{k+1}^S(\omega, N)$  with respect to  $\omega$  is at least  $k$  for any left  $S$ -module  $N$  and  $1 \leq k \leq n$ , and if and only if the Tor-cograde of  $\text{Ext}_R^k(\omega, M)$  with respect to  $\omega$  is at least  $k$  for any left  $R$ -module  $M$  and  $1 \leq k \leq n$  (Theorem 4.6). As a consequence, we obtain some equivalent characterizations of right quasi  $n$ -Gorenstein rings (Corollary 4.8).

## 2 Preliminaries

This section is devoted to stating the definitions and basic properties of notions which are needed in the sequel.

**Definition 2.1** [1, 8]. Let  $R$  and  $S$  be rings. An  $(R, S)$ -bimodule  ${}_R\omega_S$  is called *semidualizing* if the following conditions are satisfied.

- (a1)  $R\omega$  admits a degreewise finite  $R$ -projective resolution.
- (a2)  $\omega_S$  admits a degreewise finite  $S^{op}$ -projective resolution.
- (b1) The homothety map  ${}_R R R \xrightarrow{R\gamma} \text{Hom}_{S^{op}}(\omega, \omega)$  is an isomorphism.
- (b2) The homothety map  ${}_S S S \xrightarrow{S\gamma} \text{Hom}_R(\omega, \omega)$  is an isomorphism.
- (c1)  $\text{Ext}_R^{\geq 1}(\omega, \omega) = 0$ .
- (c2)  $\text{Ext}_{S^{op}}^{\geq 1}(\omega, \omega) = 0$ .

From now on,  $R$  and  $S$  are arbitrary rings and we fix a semidualizing bimodule  ${}_R\omega_S$ . For convenience, we write

$$(-)_* := \text{Hom}(\omega, -),$$

$${}_R\omega^\perp := \{M \in \text{Mod } R \mid \text{Ext}_R^{\geq 1}(\omega, M) = 0\},$$

$${}_\omega S^\top := \{N \in \text{Mod } S \mid \text{Tor}_{\geq 1}^S(\omega, N) = 0\}.$$

Following [8], set

$$\mathcal{F}_\omega(R) := \{\omega \otimes_S F \mid F \text{ is flat in Mod } S\},$$

$$\mathcal{P}_\omega(R) := \{\omega \otimes_S P \mid P \text{ is projective in Mod } S\},$$

$$\mathcal{I}_\omega(S) := \{I_* \mid I \text{ is injective in Mod } R\}.$$

The modules in  $\mathcal{F}_\omega(R)$ ,  $\mathcal{P}_\omega(R)$  and  $\mathcal{I}_\omega(S)$  are called  $\omega$ -flat,  $\omega$ -projective and  $\omega$ -injective respectively. For a subcategory  $\mathcal{X}$  of  $\text{Mod } R$  (resp.  $\text{Mod } S$ ), we use  $\text{Add } \mathcal{X}$  (resp.  $\text{Prod } \mathcal{X}$ ) to denote the subcategory of  $\text{Mod } R$  (resp.  $\text{Mod } S$ ) consisting of modules isomorphic to direct summands of direct sums (resp. products) of modules in  $\mathcal{X}$ .

We write  $(-)^+ := \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ , where  $\mathbb{Z}$  is the additive group of integers and  $\mathbb{Q}$  is the additive group of rational numbers. By [13, Proposition 2.4], we have

$$\mathcal{P}_\omega(R) = \text{Add}_R \omega \text{ and } \mathcal{I}_\omega(S) = \text{Prod } \omega^+.$$

Let  $M \in \text{Mod } R$  and  $N \in \text{Mod } S$ . Then we have the following two canonical valuation homomorphisms

$$\theta_M : \omega \otimes_S M_* \rightarrow M$$

defined by  $\theta_M(x \otimes f) = f(x)$  for any  $x \in \omega$  and  $f \in M_*$ ; and

$$\mu_N : N \rightarrow (\omega \otimes_S N)_*$$

defined by  $\mu_N(y)(x) = x \otimes y$  for any  $y \in N$  and  $x \in \omega$ . Recall that a module  $M \in \text{Mod } R$  is called  $\omega$ -cotorsionless (resp.  $\omega$ -coreflexive) if  $\theta_M$  is an epimorphism (resp. an isomorphism) ([16]); and a module  $N \in \text{Mod } S$  is called *adjoint*  $\omega$ -cotorsionless (resp. *adjoint*  $\omega$ -coreflexive) if  $\mu_N$  is a monomorphism (resp. an isomorphism) ([19]).

**Definition 2.2** [8].

(1) The Auslander class  $\mathcal{A}_\omega(S)$  with respect to  $\omega$  consists of all left  $S$ -modules  $N$  satisfying the following conditions.

- (A1)  $N \in \omega_S^\top$ .
- (A2)  $\omega \otimes_S N \in {}_R\omega^\perp$ .
- (A3)  $N$  is adjoint  $\omega$ -coreflexive.

(2) The Bass class  $\mathcal{B}_\omega(R)$  with respect to  $\omega$  consists of all left  $R$ -modules  $M$  satisfying the following conditions.

- (B1)  $M \in {}_R\omega^\perp$ .
- (B2)  $M_* \in \omega_S^\top$ .
- (B3)  $M$  is  $\omega$ -coreflexive.

For a module  $M \in \text{Mod } R$ , we use

$$0 \rightarrow M \rightarrow I^0(M) \xrightarrow{g^0} I^1(M) \tag{2.1}$$

to denote the minimal injective copresentation of  $M$  in  $\text{Mod } R$ . For a module  $N \in \text{Mod } S$ , we use

$$F_1(N) \xrightarrow{f_0} F_0(N) \rightarrow N \rightarrow 0 \tag{2.2}$$

to denote the minimal flat presentation of  $N$  in  $\text{Mod } S$ .

**Definition 2.3** [16, 18]. Let  $M \in \text{Mod } R$  and  $N \in \text{Mod } S$ , and let  $n \geq 1$ .

- (1)  $c\text{Tr}_\omega M := \text{Coker}(g^0_*)$  is called the cotranspose of  $M$  with respect to  $\omega$ , where  $g^0$  is as in (2.1).
- (2)  $M$  is called  $n$ - $\omega$ -cotorsionfree if  $\text{Tor}_{1 \leq i \leq n}^S(\omega, c\text{Tr}_\omega M) = 0$ .
- (3)  $\text{acTr}_\omega N := \text{Ker}(1_\omega \otimes f_0)$  is called the adjoint cotranspose of  $N$  with respect to  $\omega$ , where  $f_0$  is as in (2.2).
- (4)  $N$  is called adjoint  $n$ - $\omega$ -cotorsionfree if  $\text{Ext}_R^{1 \leq i \leq n}(\omega, \text{acTr}_\omega N) = 0$ .

We use  $c\mathcal{T}_\omega^n(R)$  (resp.  $\text{ac}\mathcal{T}_\omega^n(S)$ ) to denote the subcategory of  $\text{Mod } R$  (resp.  $\text{Mod } S$ ) consisting of  $n$ - $\omega$ -cotorsionfree (resp. adjoint  $n$ - $\omega$ -cotorsionfree) modules. By [16, Proposition 3.2], we have that a module in  $\text{Mod } R$  is  $\omega$ -cotorsionless (resp.  $\omega$ -coreflexive) if and only if it is in  $c\mathcal{T}_\omega^1(R)$  (resp.  $c\mathcal{T}_\omega^2(R)$ ). In particular, we have

$$\mathcal{F}_\omega(R) \subseteq \mathcal{B}_\omega(R) \subseteq c\mathcal{T}_\omega^i(R)$$

for any  $i \geq 1$  by [8, Corollary 6.1] and [16, Theorem 3.9]. On the other hand, by [18, Proposition 3.2], we have that a module in  $\text{Mod } S$  is adjoint  $\omega$ -cotorsionless (resp. adjoint  $\omega$ -coreflexive) if and only if it is in  $\text{ac}\mathcal{T}_\omega^1(S)$  (resp.  $\text{ac}\mathcal{T}_\omega^2(S)$ ). We have

$$\mathcal{I}_\omega(S) \subseteq \mathcal{A}_\omega(S) \subseteq \text{ac}\mathcal{T}_\omega^i(S)$$

for any  $i \geq 1$  by [8, Corollary 6.1] and [18, Proposition 3.4].

**Definition 2.4** [17]

- (1) Let  $M \in \text{Mod } R$  and  $n \geq 0$ . The *Ext-cograde* of  $M$  with respect to  $\omega$  is defined as  $\text{E-cograde}_\omega M := \inf\{i \geq 0 \mid \text{Ext}_R^i(\omega, M) \neq 0\}$ ; and *the strong Ext-cograde* of  $M$  with respect to  $\omega$ , denoted by  $\text{s.E-cograde}_\omega M$ , is said to be at least  $n$  if  $\text{E-cograde}_\omega X \geq n$  for any quotient module  $X$  of  $M$ .
- (2) Let  $N \in \text{Mod } S$  and  $n \geq 0$ . The *Tor-cograde* of  $N$  with respect to  $\omega$  is defined as  $\text{T-cograde}_\omega N := \inf\{i \geq 0 \mid \text{Tor}_i^S(\omega, N) \neq 0\}$ ; and *the strong Tor-cograde* of  $N$  with respect to  $\omega$ , denoted by  $\text{s.T-cograde}_\omega N$ , is said to be at least  $n$  if  $\text{T-cograde}_\omega Y \geq n$  for any submodule  $Y$  of  $N$ .

**Definition 2.5** [20] Let  $\mathcal{X}$  be a subcategory of an abelian category  $\mathcal{E}$  and  $n \geq 1$ . If there exists an exact sequence

$$0 \rightarrow N \rightarrow X_0 \rightarrow \dots \rightarrow X_{n-1} \rightarrow M \rightarrow 0$$

in  $\mathcal{E}$  with all  $X_i$  in  $\mathcal{X}$ , then  $N$  is called an  $n$ - $\mathcal{X}$ -syzygy of  $M$  and  $M$  is called an  $n$ - $\mathcal{X}$ -cosyzygy of  $N$ .

For subcategories  $\mathcal{X}, \mathcal{Y}$  of an abelian category  $\mathcal{E}$  and  $n \geq 1$ , we write

$$\begin{aligned} \Omega_{\mathcal{X}}^n(\mathcal{Y}) &:= \{N \in \mathcal{E} \mid N \text{ is an } n\text{-}\mathcal{X}\text{-syzygy of some object in } \mathcal{Y}\}, \\ \text{co}\Omega_{\mathcal{X}}^n(\mathcal{Y}) &:= \{M \in \mathcal{E} \mid M \text{ is an } n\text{-}\mathcal{X}\text{-cosyzygy of some object in } \mathcal{Y}\}. \end{aligned}$$

In particular,  $\Omega_{\mathcal{X}}^0(\mathcal{Y}) = \mathcal{Y} = \text{co}\Omega_{\mathcal{X}}^0(\mathcal{Y})$  and  $\Omega_{\mathcal{X}}^{-1}(\mathcal{Y}) = 0 = \text{co}\Omega_{\mathcal{X}}^{-1}(\mathcal{Y})$ . For convenience, we write

$$\begin{aligned} \Omega_{\mathcal{A}}^n(S) &:= \Omega_{\mathcal{A}_\omega(S)}^n(\text{Mod } S), \quad \Omega_{\mathcal{I}_\omega}^n(S) := \Omega_{\mathcal{I}_\omega(S)}^n(\text{Mod } S), \\ \Omega_{\text{ac}\mathcal{T}_\omega^i}^n(S) &:= \Omega_{\text{ac}\mathcal{T}_\omega^i}^n(\text{Mod } S), \\ \text{co}\Omega_{\mathcal{B}}^n(R) &:= \text{co}\Omega_{\mathcal{B}_\omega(R)}^n(\text{Mod } R), \quad \text{co}\Omega_{\mathcal{F}_\omega}^n(R) := \text{co}\Omega_{\mathcal{F}_\omega(R)}^n(\text{Mod } R), \\ \text{co}\Omega_{\mathcal{P}_\omega}^n(R) &:= \text{co}\Omega_{\mathcal{P}_\omega(R)}^n(\text{Mod } R), \quad \text{co}\Omega_{\text{c}\mathcal{T}_\omega^i}^n(R) := \text{co}\Omega_{\text{c}\mathcal{T}_\omega^i}^n(\text{Mod } R). \end{aligned}$$

### 3 Tor-Cograde and Extension Closure

Our aim in this section is to show how the extension closure of the subcategories  $\text{ac}\mathcal{T}_\omega^1(S)$  and  $\text{ac}\mathcal{T}_\omega^2(S)$  is connected with the Tor-cograde of  $\text{Ext}_R^k(\omega, M)$  for any  $M \in \text{Mod } R$  and  $k = 1, 2$ .

In what follows, for any  $i \geq 1$ , we use  $\mathcal{C}_i$  (resp.  $\mathcal{D}_i$ ) to denote a subcategory of  $\text{Mod } R$  (resp.  $\text{Mod } S$ ) satisfying

$$\mathcal{F}_\omega(R) \subseteq \mathcal{C}_i \subseteq \text{c}\mathcal{T}_\omega^i(R) \quad (\text{resp. } \mathcal{I}_\omega(S) \subseteq \mathcal{D}_i \subseteq \text{ac}\mathcal{T}_\omega^i(S)).$$

We begin by proving the following lemma.

**Lemma 3.1** *For any  $i \geq 1$ , it holds that*

- (1)  $\Omega_{\mathcal{D}_i}^1(S) = \text{ac}\mathcal{T}_\omega^1(S)$ .
- (2)  $\text{co}\Omega_{\mathcal{C}_i}^1(R) = \text{c}\mathcal{T}_\omega^1(R)$ .

**Proof** (1) Since  $\mathcal{I}_\omega(S) \subseteq \mathcal{D}_i$ , we have  $\text{ac}\mathcal{T}_\omega^1(S) \subseteq \Omega_{\mathcal{D}_i}^1(S)$  by [18, Lemma 3.7(1)]. Now let  $N \in \Omega_{\mathcal{D}_i}^1(S)$ . We may assume that  $f^0 : N \rightarrow H$  is a monomorphism in  $\text{Mod } S$  with  $H \in \mathcal{D}_i$ . As  $\mathcal{D}_i \subseteq \text{ac}\mathcal{T}_\omega^i(S) \subseteq \text{ac}\mathcal{T}_\omega^1(S)$ , we have that  $\mu_H$  is a monomorphism. Then from the following commutative diagram

$$\begin{array}{ccc}
 N & \xrightarrow{f^0} & H \\
 \downarrow \mu_N & & \downarrow \mu_H \\
 (\omega \otimes_S N)_* & \xrightarrow{(1_\omega \otimes f^0)_*} & (\omega \otimes_S H)_*,
 \end{array}$$

we get that  $\mu_N$  is a monomorphism. Thus  $N \in \text{ac}\mathcal{T}_\omega^1(S)$  and  $\Omega_{\mathcal{D}_i}^1(S) \subseteq \text{ac}\mathcal{T}_\omega^1(S)$ .

(2) Since  $\mathcal{P}_\omega(R) \subseteq \mathcal{C}_i$ , we have  $\text{c}\mathcal{T}_\omega^1(R) \subseteq \text{co}\Omega_{\mathcal{C}_i}^1(R)$  by [16, Lemma 3.6(1)]. Now let  $M \in \text{co}\Omega_{\mathcal{C}_i}^1(R)$ . We may assume that  $f_0 : L \twoheadrightarrow M$  is an epimorphism in  $\text{Mod } R$  with  $L \in \mathcal{C}_i$ . As  $\mathcal{C}_i \subseteq \text{c}\mathcal{T}_\omega^i(R) \subseteq \text{c}\mathcal{T}_\omega^1(R)$ , we have that  $\theta_L$  is an epimorphism. Then from the following commutative diagram

$$\begin{array}{ccc}
 \omega \otimes_S L_* & \xrightarrow{1_\omega \otimes f_{0*}} & \omega \otimes_S M_* \\
 \downarrow \theta_L & & \downarrow \theta_M \\
 L & \twoheadrightarrow & M,
 \end{array}$$

we get that  $\theta_M$  is an epimorphism. Thus  $M \in \text{c}\mathcal{T}_\omega^1(R)$  and  $\text{co}\Omega_{\mathcal{C}_i}^1(R) \subseteq \text{c}\mathcal{T}_\omega^1(R)$ .  $\square$

**Lemma 3.2** *The following statements are equivalent for any  $i \geq 2$ .*

- (1)  $M \in \text{co}\Omega_{\mathcal{P}_\omega}^2(R)$ .
- (2)  $M \in \text{co}\Omega_{\mathcal{C}_i}^2(R)$ .
- (3) *There is a module  $N \in \text{Mod } S$  such that  $M \cong \omega \otimes_S N$ .*

**Proof** (1)  $\Rightarrow$  (2) It is obvious.  
 (2)  $\Rightarrow$  (3) Let  $M \in \text{co}\Omega_{\mathcal{C}_i}^2(R)$  and let

$$L^0 \xrightarrow{f} L^1 \rightarrow M \rightarrow 0$$

be an exact sequence in  $\text{Mod } R$  with  $L^0, L^1 \in \mathcal{C}_i \subseteq \text{c}\mathcal{T}_\omega^i(R)$ . As  $\text{c}\mathcal{T}_\omega^i(R) \subseteq \text{c}\mathcal{T}_\omega^2(R)$ , we have that  $\theta_{L^0}$  and  $\theta_{L^1}$  are isomorphisms. Then from the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 \omega \otimes_S L^0_* & \xrightarrow{1_{\omega} \otimes f_*} & \omega \otimes_S L^1_* & \longrightarrow & \omega \otimes_S \text{Coker } f_* & \longrightarrow & 0 \\
 \downarrow \theta_{L^0} & & \downarrow \theta_{L^1} & & \downarrow h & & \\
 L^0 & \xrightarrow{f} & L^1 & \longrightarrow & M & \longrightarrow & 0,
 \end{array}$$

we get that the induced homomorphism  $h$  is an isomorphism, and thus  $M \cong \omega \otimes_S \text{Coker } f_*$ .

(3)  $\Rightarrow$  (1) Suppose  $M \cong \omega \otimes_S N$  for some  $N \in \text{Mod } S$ , and let

$$Q_1 \rightarrow Q_0 \rightarrow N \rightarrow 0$$

be a projective presentation of  $N$ . Applying the functor  $\omega \otimes_S -$  to it yields an exact sequence

$$\omega \otimes_S Q_1 \rightarrow \omega \otimes_S Q_0 \rightarrow \omega \otimes_S N \rightarrow 0.$$

Since  $\omega \otimes_S Q_1, \omega \otimes_S Q_0 \in \mathcal{P}_\omega(R)$ , we have  $M \in \text{co}\Omega_{\mathcal{P}_\omega(R)}^2$ . □

We give an analogue of Lemma 3.2.

**Lemma 3.3** *The following statements are equivalent for any  $i \geq 2$ .*

- (1)  $N \in \Omega_{\mathcal{I}_\omega}^2(S)$ .
- (2)  $N \in \Omega_{\mathcal{D}_i}^2(S)$ .
- (3) *There is a module  $M \in \text{Mod } R$  such that  $N \cong M_*$ .*

**Proof** (1)  $\Rightarrow$  (2) It is obvious.

(2)  $\Rightarrow$  (3) Let  $N \in \Omega_{\mathcal{D}_i}^2(R)$  and let

$$0 \rightarrow N \rightarrow H^0 \xrightarrow{g} H^1$$

be an exact sequence in  $\text{Mod } S$  with  $H^0, H^1 \in \mathcal{D}_i \subseteq \text{acT}_\omega^i(S)$ . As  $\text{acT}_\omega^i(S) \subseteq \text{acT}_\omega^2(S)$ , we have that  $\mu_{H^0}$  and  $\mu_{H^1}$  are isomorphisms. Then from the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & N & \longrightarrow & H^0 & \xrightarrow{g} & H^1 \\
 & & \downarrow h & & \downarrow \mu_{H^0} & & \downarrow \mu_{H^1} \\
 0 & \longrightarrow & (\text{Ker}(1_\omega \otimes g))_* & \longrightarrow & (\omega \otimes_S H^0)_* & \xrightarrow{(1_\omega \otimes g)_*} & (\omega \otimes_S H^1)_*,
 \end{array}$$

we get that the induced homomorphism  $h$  is an isomorphism, and thus  $N \cong (\text{Ker}(1_\omega \otimes g))_*$ .

(3)  $\Rightarrow$  (1) Suppose  $N \cong M_*$  for some  $M \in \text{Mod } R$ , and let

$$0 \rightarrow M \rightarrow I^0 \rightarrow I^1$$

be an injective copresentation of  $M$ . Applying the functor  $(-)_*$  to it yields an exact sequence

$$0 \rightarrow M_* \rightarrow I_*^0 \rightarrow I_*^1.$$

Since  $I_*^0, I_*^1 \in \mathcal{I}_\omega(S)$ , we have  $N \in \Omega_{\mathcal{I}_\omega(S)}^2$ . □

**Proposition 3.4** *The following statements are equivalent for any  $i \geq 2$ .*

- (1)  $M_* \in \text{ac}\mathcal{T}_\omega^2(S)$  for any  $M \in \text{Mod } R$ .
- (2)  $\omega \otimes_S N \in \text{c}\mathcal{T}_\omega^2(R)$  for any  $N \in \text{Mod } S$ .
- (3)  $\text{c}\mathcal{T}_\omega^2(R) = \text{co}\Omega_{\mathcal{C}_i}^2(R)$ .
- (4)  $\text{ac}\mathcal{T}_\omega^2(S) = \Omega_{\mathcal{D}_i}^2(S)$ .

**Proof** (1)  $\Rightarrow$  (4) Let  $N \in \Omega_{\mathcal{D}_i}^2(S)$ . Then by Lemma 3.3 and (1) there is a module  $M \in \text{Mod } R$  such that  $N \cong M_* \in \text{ac}\mathcal{T}_\omega^2(S)$ , and so  $\Omega_{\mathcal{D}_i}^2(S) \subseteq \text{ac}\mathcal{T}_\omega^2(S)$ . The inclusion  $\text{ac}\mathcal{T}_\omega^2(S) \subseteq \Omega_{\mathcal{D}_i}^2(S)$  follows from [18, Lemma 3.7(2)].

(4)  $\Rightarrow$  (1) Let  $M \in \text{Mod } R$ . Then by Lemma 3.3 and (4), we have  $M_* \in \Omega_{\mathcal{D}_i}^2(S) = \text{ac}\mathcal{T}_\omega^2(S)$ .

Similarly, we get (2)  $\Leftrightarrow$  (3) by Lemma 3.2 and [16, Lemma 3.6(2)].

(1)  $\Leftrightarrow$  (2) It follows from [20, Lemma 4.18]. □

**Proposition 3.5** *For any  $n \geq 1$ , the following statements are equivalent.*

- (1)  $T\text{-cograde}_\omega \text{Ext}_R^k(\omega, M) \geq k - 1$  for any  $M \in \text{Mod } R$  and  $1 \leq k \leq n$ .
- (2)  $T\text{-cograde}_\omega \text{Ext}_R^k(\omega, M) \geq k - 1$  for any  $M \in \Omega_{\mathcal{C}_n}^k(R)$  and  $1 \leq k \leq n$ .
- (3)  $E\text{-cograde}_\omega \text{Tor}_k^S(\omega, N) \geq k - 1$  for any  $N \in \text{Mod } S$  and  $1 \leq k \leq n$ .
- (4)  $E\text{-cograde}_\omega \text{Tor}_k^S(\omega, N) \geq k - 1$  for any  $N \in \text{co}\Omega_{\mathcal{D}_n}^k(S)$  and  $1 \leq k \leq n$ .
- (5)  $\text{c}\mathcal{T}_\omega^k(R) = \text{co}\Omega_{\mathcal{C}_n}^k(R)$  for any  $1 \leq k \leq n$ .
- (6)  $\text{ac}\mathcal{T}_\omega^k(S) = \Omega_{\mathcal{D}_n}^k(S)$  for any  $1 \leq k \leq n$ .

**Proof** (2)  $\Rightarrow$  (5) By [16, Proposition 3.7], it suffices to prove  $\text{co}\Omega_{\mathcal{C}_n}^k(R) \subseteq \text{c}\mathcal{T}_\omega^k(R)$  for any  $1 \leq k \leq n$ . We proceed by induction on  $n$ . The case for  $n = 1$  follows from Lemma 3.1.

Now let  $M \in \text{co}\Omega_{\mathcal{C}_n}^n(R)$  with  $n \geq 2$  and let

$$W_{n-1} \xrightarrow{f_{n-1}} \dots \rightarrow W_1 \xrightarrow{f_1} W_0 \xrightarrow{f_0} M \rightarrow 0 \tag{3.1}$$

be an exact sequence in  $\text{Mod } R$  with all  $W_i$  in  $\mathcal{C}_n$ . By the induction hypothesis, we have  $\text{Im } f_1 \in \text{c}\mathcal{T}_\omega^{n-1}(R)$  and there is an exact sequence

$$V_{n-1} \xrightarrow{g_{n-1}} \dots \rightarrow V_1 \xrightarrow{g_1} W_0 \xrightarrow{f_0} M \rightarrow 0 \tag{3.2}$$



in  $\text{Mod } R$  with all  $V_i$  in  $\mathcal{P}_\omega(R)$  by [16, Proposition 3.7]. Applying the functor  $(-)_*$  to (3.2) gives an exact sequence

$$0 \rightarrow (\text{Im } g_1)_* \rightarrow W_{0*} \xrightarrow{f_{0*}} M_* \rightarrow \text{Ext}_R^n(\omega, \text{Ker } g_{n-1}) \rightarrow 0. \tag{3.3}$$

Set  $N := \text{Im}(f_{0*})$  and let  $f_{0*} := \alpha\pi$  (where  $\pi : W_{0*} \twoheadrightarrow N$  and  $\alpha : N \hookrightarrow M_*$ ) be the natural epic-monic decomposition of  $f_{0*}$ . Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} \text{Tor}_1^S(\omega, N) & \xrightarrow{h} & \omega \otimes_S (\text{Im } g_1)_* & \longrightarrow & \omega \otimes_S W_{0*} & \xrightarrow{1_\omega \otimes \pi} & \omega \otimes_S N & \longrightarrow & 0 \\ & & \downarrow \theta_{\text{Im } g_1} & & \downarrow \theta_{W_0} & & \downarrow g & & \\ 0 & \longrightarrow & \text{Im } g_1 & \longrightarrow & W_0 & \xrightarrow{f_0} & M & \longrightarrow & 0. \end{array}$$

Diagram (3.1)

So we have

$$\theta_M(1_\omega \otimes \alpha)(1_\omega \otimes \pi) = \theta_M(1_\omega \otimes f_{0*}) = f_0\theta_{W_0} = g(1_\omega \otimes \pi).$$

Because  $1_\omega \otimes \pi$  is epic, we have  $\theta_M \cdot (1_\omega \otimes \alpha) = g$  and the following commutative diagram with exact rows

$$\begin{array}{ccccc} \omega \otimes_S N & \xrightarrow{1_\omega \otimes \alpha} & \omega \otimes_S M_* & \longrightarrow & \omega \otimes_S \text{Ext}_R^n(\omega, \text{Ker } g_{n-1}) \longrightarrow 0 \\ \downarrow g & & \downarrow \theta_M & & \\ M & \xlongequal{\quad} & M & & \end{array}$$

Diagram (3.2)

Since  $\text{Im } g_1 = \text{Im } f_1 \in \mathcal{cT}_\omega^{n-1}(R)$ , we have that  $\theta_{\text{Im } g_1}$  is an epimorphism. So  $g$  is an isomorphism by the snake lemma, and hence  $1_\omega \otimes \alpha$  is a monomorphism. Since  $\omega \otimes_S \text{Ext}_R^n(\omega, \text{Ker } g_{n-1}) = 0$  by assumption, we see that  $\theta_M$  is an isomorphism and  $M \in \mathcal{cT}_\omega^2(R)$  by Diagram (3.2). This shows that the assertion holds true for  $n = 2$ .

If  $n > 2$ , then  $\theta_{\text{Im } g_1}$  is an isomorphism as  $\text{Im } g_1 \in \mathcal{cT}_\omega^{n-1}(R)$ , we also have  $\text{Tor}_1^S(\omega, W_{0*}) = 0$  by [16, Corollary 3.4(3)]. So  $h$  is monic and  $\text{Tor}_1^S(\omega, N) = 0$  by Diagram (3.1). Moreover, it is clear that  $\text{Tor}_{1 \leq k \leq n-3}^S(\omega, (\text{Im } g_1)_*) = 0$  by [16, Corollary 3.4(3)]. Because  $\text{T-cograde}_\omega \text{Ext}_R^n(\omega, \text{Ker } f_{n-1}) \geq n - 1$  by assumption, applying the dimension shifting to (3.3) yields  $\text{Tor}_{1 \leq k \leq n-2}^S(\omega, M_*) = 0$ . Therefore  $M \in \mathcal{cT}_\omega^n(R)$  by [16, Corollary 3.4(3)] again.

(5)  $\Rightarrow$  (1) For any  $1 \leq k \leq n$ , since  $c\mathcal{T}_\omega^k(R) \subseteq \text{co}\Omega_{\mathcal{P}_\omega}^k(R) \subseteq \text{co}\Omega_{\mathcal{C}_n}^k(R)$ , we have  $c\mathcal{T}_\omega^k(R) = \text{co}\Omega_{\mathcal{P}_\omega}^k(R)$  by (5), and hence  $c\mathcal{T}_\omega^k(R) = \text{co}\Omega_{\mathcal{B}}^k(R)$  by [20, Proposition 4.17]. Now (1) follows from [20, Theorem 4.19].

The implications (1)  $\Rightarrow$  (2) and (3)  $\Rightarrow$  (4) are obvious.

(1)  $\Leftrightarrow$  (3) It follows from [20, Theorem 4.19].

The proofs of (4)  $\Rightarrow$  (6) and (6)  $\Rightarrow$  (2) are similar to that of (2)  $\Rightarrow$  (5) and (5)  $\Rightarrow$  (1) respectively. □

**Lemma 3.6** *For any  $M \in \text{Mod } R$ , there are two exact sequences*

$$\begin{aligned} 0 &\rightarrow \text{Ext}_R^1(\omega, M) \rightarrow c\text{Tr}_\omega M \xrightarrow{\pi} H \rightarrow 0, \\ 0 &\rightarrow H \xrightarrow{\lambda} (\omega \otimes_S c\text{Tr}_\omega M)_* \rightarrow \text{Ext}_R^2(\omega, M) \rightarrow 0 \end{aligned}$$

in  $\text{Mod } S$  such that  $\text{Hom}_S(\pi, \omega^+)$  is an isomorphism.

**Proof** By [17, Corollary 6.8], there is an exact sequence

$$0 \rightarrow \text{Ext}_R^1(\omega, M) \rightarrow c\text{Tr}_\omega M \xrightarrow{\mu_{c\text{Tr}_\omega M}} (\omega \otimes_S c\text{Tr}_\omega M)_* \rightarrow \text{Ext}_R^2(\omega, M) \rightarrow 0$$

in  $\text{Mod } S$ . Put  $H := \text{Im } \mu_{c\text{Tr}_\omega M}$  and assume that  $\mu_{c\text{Tr}_\omega M} = \lambda\pi$ , where  $\pi : c\text{Tr}_\omega M \rightarrow H$  is an epimorphism and  $\lambda : H \rightarrow (\omega \otimes_S c\text{Tr}_\omega M)_*$  is a monomorphism. Then we have the following exact sequences

$$\begin{aligned} 0 &\rightarrow \text{Ext}_R^1(\omega, M) \rightarrow c\text{Tr}_\omega M \xrightarrow{\pi} H \rightarrow 0, \\ 0 &\rightarrow H \xrightarrow{\lambda} (\omega \otimes_S c\text{Tr}_\omega M)_* \rightarrow \text{Ext}_R^2(\omega, M) \rightarrow 0. \end{aligned}$$

In view of [17, Lemma 6.1(2)],  $1_\omega \otimes \mu_{c\text{Tr}_\omega M}$  is a monomorphism, and so  $1_\omega \otimes \pi$  is an isomorphism. It follows from the adjoint isomorphism theorem that  $\text{Hom}_S(\pi, \omega^+) \cong (1_\omega \otimes \pi)^+$  is also an isomorphism. □

**Lemma 3.7** *The following statements are equivalent for any  $i \geq 1$ .*

- (1)  $\text{ac}\mathcal{T}_\omega^1(S)$  is extension closed.
- (2)  $T\text{-cograde}_\omega \text{Ext}_R^1(\omega, M) \geq 1$  for any  $M \in \Omega_{\mathcal{C}_i}^2(R)$ .
- (3)  $T\text{-cograde}_\omega \text{Ext}_R^1(\omega, M) \geq 1$  for any  $M \in \text{Mod } R$ .

**Proof** (2)  $\Rightarrow$  (1) Let

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

be an exact sequence in  $\text{Mod } S$  with  $A, C \in \text{ac}\mathcal{T}_\omega^1(S)$ . By [18, Proposition 3.2],  $\text{Ker } \mu_B \cong \text{Ext}_R^1(\omega, \text{ac}\text{Tr}_\omega B)$ . Notice that  $\text{ac}\text{Tr}_\omega B \in \Omega_{\mathcal{C}_i}^2(R)$ , so

$$\omega \otimes_S \text{Ext}_R^1(\omega, \text{ac}\text{Tr}_\omega B) = 0$$

by (2), and hence  $\omega \otimes_S \text{Ker } \mu_B = 0$ . Moreover, since  $\mu_C g = (1_\omega \otimes g)_* \mu_B$  and  $\mu_C$  is a monomorphism, we get  $\text{Ker } \mu_B \subseteq \text{Ker } g \cong A$ . Note that  $A \in \text{acT}_\omega^1(S)$  and

$$\text{Hom}_S(\text{Ker } \mu_B, \omega^+) \cong (\omega \otimes_S \text{Ker } \mu_B)^+ = 0.$$

It follows from [18, Lemma 3.7] and [13, Proposition 2.4] that  $\text{Hom}_S(\text{Ker } \mu_B, A) = 0$ , which implies  $\text{Ker } \mu_B = 0$ , and thus  $B \in \text{acT}_\omega^1(S)$ .

(1)  $\Rightarrow$  (3) By Lemma 3.6, there is an exact sequence

$$0 \rightarrow \text{Ext}_R^1(\omega, M) \rightarrow \text{cTr}_\omega M \xrightarrow{\pi} H \rightarrow 0$$

in  $\text{Mod } S$  such that  $\text{Hom}_S(\pi, \omega^+)$  is an isomorphism. Then

$$\text{Ker Ext}_S^1(\pi, \omega^+) \cong \text{Hom}_S(\text{Ext}_R^1(\omega, M), \omega^+) \cong (\omega \otimes_S \text{Ext}_R^1(\omega, M))^+.$$

Suppose

$$\alpha : 0 \rightarrow \omega^+ \rightarrow X \xrightarrow{f} H \rightarrow 0$$

is an element in  $\text{Ker Ext}_S^1(\pi, \omega^+)$ , that is,  $\text{Ext}_S^1(\pi, \omega^+)(\alpha) = 0$ . Then we have the following pull-back diagram with the first row splitting:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \omega^+ & \longrightarrow & Y & \xrightarrow{u} & \text{cTr}_\omega M \longrightarrow 0 \\ & & \parallel & & \downarrow t & & \downarrow \pi \\ 0 & \longrightarrow & \omega^+ & \longrightarrow & X & \xrightarrow{f} & H \longrightarrow 0. \end{array}$$

So there is a homomorphism  $u' : \text{cTr}_\omega M \rightarrow Y$  such that  $uu' = 1_{\text{cTr}_\omega M}$ . Since  $\pi u = ft$ , we have  $\pi = ftu'$ . Note that  $(\omega \otimes_S \text{cTr}_\omega M)_* \in \text{acT}_\omega^1(S)$  by [17, Lemma 6.1(1)]. Thus  $H \in \text{acT}_\omega^1(S)$  since  $H$  is a submodule of  $(\omega \otimes_S \text{cTr}_\omega M)_*$  by Lemma 3.6. So  $X \in \text{acT}_\omega^1(S)$  by (1), and hence there is a monomorphism  $0 \rightarrow X \rightarrow U^0$  in  $\text{Mod } S$  with  $U^0 \in \mathcal{I}_\omega(S) = \text{Prod } \omega^+$ . As  $\text{Hom}_S(\pi, \omega^+)$  is an isomorphism, we have that  $\text{Hom}_S(\pi, U^0)$ , and hence  $\text{Hom}_S(\pi, X)$ , is an isomorphism by [11, Lemma 2.1]. Then there is a homomorphism  $f' : H \rightarrow X$  such  $f'\pi = tu'$ , and so  $\pi = ff'\pi$ . But  $\pi$  is an epimorphism, thus  $ff' = 1_H$ , which implies that  $\alpha$  splits, and thus  $\omega \otimes_S \text{Ext}_R^1(\omega, M) = 0$ .

(3)  $\Rightarrow$  (2) It is trivial. □

**Lemma 3.8** *For any  $N \in \text{Mod } S$ , the following statements are equivalent.*

- (1)  $\omega \otimes_S N \in \text{cT}_\omega^2(R)$ .
- (2)  $\omega \otimes_S \text{Coker } \mu_N = 0$ .

**Proof** By [17, Lemma 6.1(2)], we have  $\theta_{\omega \otimes_S N}(1_\omega \otimes \mu_N) = 1_{\omega \otimes_S N}$ . It follows that  $\theta_{\omega \otimes_S N}$  is a split epimorphism and

$$\text{Ker } \theta_{\omega \otimes_S N} \cong \text{Coker}(1_\omega \otimes \mu_N) \cong \omega \otimes_S \text{Coker } \mu_N.$$

Now the assertion follows easily. □

**Lemma 3.9** *If  $\text{acT}_\omega^1(S)$  is extension closed, then  $\omega \otimes_S N \in \text{cT}_\omega^2(R)$  for any  $N \in \text{Mod } S$ .*

**Proof** By the definition of adjoint cotranspose, there is an exact sequence

$$0 \rightarrow \text{acTr}_\omega N \rightarrow \omega \otimes_S F_1 \rightarrow \omega \otimes_S F_0 \rightarrow \omega \otimes_S N \rightarrow 0$$

in  $\text{Mod } R$  with  $F_0, F_1$  flat. Let  $K := \text{Im}(\omega \otimes_S F_1 \rightarrow \omega \otimes_S F_0)$ . We have

$$\text{Coker } \mu_N \cong \text{Ext}_R^2(\omega, \text{acTr}_\omega N) \cong \text{Ext}_R^1(\omega, K)$$

by [18, Proposition 3.2]. By assumption and Lemma 3.7, we have  $\omega \otimes_S \text{Coker } \mu_N = 0$ . Thus  $\omega \otimes_S N \in \text{cT}_\omega^2(R)$  by Lemma 3.8. □

We are now in a position to prove the main result of this section.

**Theorem 3.10** *The following statements are equivalent for any  $i \geq 1$ .*

- (1)  $\text{acT}_\omega^k(S)$  is extension closed for  $k = 1, 2$ .
- (2)  $\text{Tor}_{k-1}^S(\omega, \text{Ext}_R^k(\omega, M)) = 0$  for any  $M \in \Omega_{C_i}^2(R)$  and  $k = 1, 2$ .
- (3)  $T\text{-cograde}_\omega \text{Ext}_R^k(\omega, M) \geq k$  for any  $M \in \Omega_{C_i}^2(R)$  and  $k = 1, 2$ .
- (4)  $\text{Tor}_{k-1}^S(\omega, \text{Ext}_R^k(\omega, M)) = 0$  for any  $M \in \text{Mod } R$  and  $k = 1, 2$ .
- (5)  $T\text{-cograde}_\omega \text{Ext}_R^k(\omega, M) \geq k$  for any  $M \in \text{Mod } R$  and  $k = 1, 2$ .

**Proof** (1)  $\Rightarrow$  (5) By Lemma 3.6, there are two exact sequences

$$0 \rightarrow \text{Ext}_R^1(\omega, M) \rightarrow \text{cTr}_\omega M \xrightarrow{\pi} H \rightarrow 0, \tag{3.4}$$

$$0 \rightarrow H \xrightarrow{\lambda} (\omega \otimes_S \text{cTr}_\omega M)_* \xrightarrow{\beta} \text{Ext}_R^2(\omega, M) \rightarrow 0 \tag{3.5}$$

in  $\text{Mod } S$  such that  $\text{Hom}_S(\pi, \omega^+)$  is an isomorphism and  $\lambda\pi = \mu_{\text{cTr}_\omega M}$ . Since  $\text{Hom}_S(\pi, \omega^+) \cong (\omega \otimes \pi)^+$  by the adjoint isomorphism theorem, it follows that  $(\omega \otimes \pi)^+$  and  $\omega \otimes \pi$  are isomorphisms.

By Lemma 3.7, it is easy to see that  $T\text{-cograde}_\omega \text{Ext}_R^1(\omega, M) \geq 1$  and  $\omega \otimes_S \text{Ext}_R^1(\omega, M) = 0$ . Then by the adjoint isomorphism theorem, we have that

$$\text{Hom}_S(\text{Ext}_R^1(\omega, M), \omega^+) \cong (\omega \otimes_S \text{Ext}_R^1(\omega, M))^+ = 0$$

and  $\text{Ext}_S^1(\pi, \omega^+)$  is a monomorphism. We know from [17, Lemma 6.1(2)] that  $\text{Hom}_S(\mu_{\text{cTr}_\omega M}, \omega^+)$  is an epimorphism. Then the fact that

$$\text{Hom}_S(\mu_{\text{cTr}_\omega M}, \omega^+) = \text{Hom}_S(\pi, \omega^+) \text{Hom}_S(\lambda, \omega^+)$$

in which  $\text{Hom}_S(\pi, \omega^+)$  is an isomorphism (by Lemma 3.6) implies that  $\text{Hom}_S(\lambda, \omega^+)$  is also an epimorphism. On the other hand, note that

$$\text{Ext}_S^1(\mu_{\text{cTr}_\omega M}, \omega^+) = \text{Ext}_S^1(\pi, \omega^+) \text{Ext}_S^1(\lambda, \omega^+)$$

and  $\text{Ext}_S^1(\pi, \omega^+)$  is a monomorphism by the above argument. Applying the functor  $\text{Hom}_S(-, \omega^+)$  to (3.5) gives

$$\text{Ker Ext}_S^1(\mu_{\text{cTr}_\omega M}, \omega^+) \cong \text{Ker Ext}_S^1(\lambda, \omega^+) \cong \text{Ext}_S^1(\text{Ext}_R^2(\omega, M), \omega^+).$$

Let

$$\alpha : 0 \rightarrow \omega^+ \rightarrow X \xrightarrow{f} (\omega \otimes_S \text{cTr}_\omega M)_* \rightarrow 0$$

be an element in  $\text{Ker Ext}_S^1(\mu_{\text{cTr}_\omega M}, \omega^+)$ , that is,  $\text{Ext}_S^1(\mu_{\text{cTr}_\omega M}, \omega^+)(\alpha) = 0$ . Then we have the following pull-back diagram with the first row splitting:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \omega^+ & \longrightarrow & Y & \xrightarrow{u} & \text{cTr}_\omega M \longrightarrow 0 \\ & & \parallel & & \downarrow t & & \downarrow \mu_{\text{cTr}_\omega M} \\ 0 & \longrightarrow & \omega^+ & \longrightarrow & X & \longrightarrow & (\omega \otimes_S \text{cTr}_\omega M)_* \longrightarrow 0 \end{array}$$

So there is a homomorphism  $u' : \text{cTr}_\omega M \rightarrow Y$  such that  $uu' = 1_{\text{cTr}_\omega M}$ . Since  $\mu_{\text{cTr}_\omega M} u = ft$ , we have

$$\mu_{\text{cTr}_\omega M} = ftu'.$$

By Lemma 3.9, we have  $\omega \otimes_S \text{cTr}_\omega M \in \text{cT}_\omega^2(R)$ . It follows from [19, Proposition 6.4] that  $(\omega \otimes_S \text{cTr}_\omega M)_* \in \text{acT}_\omega^2(S)$ . Since  $\text{acT}_\omega^2(S)$  is extension closed by (1), we have  $X \in \text{acT}_\omega^2(S)$ . As  $\mu_X tu' = (1_\omega \otimes tu')_* \mu_{\text{cTr}_\omega M}$ , we have

$$\begin{aligned} \mu_{\text{cTr}_\omega M} &= ftu' = f\mu_X^{-1}(1_\omega \otimes tu')_* \mu_{\text{cTr}_\omega M} \text{ and} \\ (1_{(\omega \otimes_S \text{cTr}_\omega M)_*} - f\mu_X^{-1}(1_\omega \otimes tu')_*) \mu_{\text{cTr}_\omega M} &= 0, \end{aligned}$$

and hence

$$(1_{(\omega \otimes_S \text{cTr}_\omega M)_*} - f\mu_X^{-1}(1_\omega \otimes tu')_*)\lambda = 0.$$

By the universal property of cokernels, there is a homomorphism  $g : \text{Ext}_R^2(\omega, M) \rightarrow (\omega \otimes_S \text{cTr}_\omega M)_*$  such that

$$1_{(\omega \otimes_S \text{cTr}_\omega M)_*} - f\mu_X^{-1}(1_\omega \otimes tu')_* = g\beta.$$

In addition, since  $\text{Ext}_R^2(\omega, M) \cong \text{Coker } \mu_{\text{cTr}_\omega M}$ , we have

$$\omega \otimes_S \text{Ext}_R^2(\omega, M) = 0$$

by Lemma 3.8. It follows from the adjoint isomorphism theorem that

$$\text{Hom}_S(\text{Ext}_R^2(\omega, M), \omega^+) \cong (\omega \otimes_S \text{Ext}_R^2(\omega, M))^+ = 0.$$

Moreover, since  $(\omega \otimes_S \text{cTr}_\omega M)_* \in \text{acT}_\omega^2(S)$ , we have that  $(\omega \otimes_S \text{cTr}_\omega M)_*$  is isomorphic to a submodule of some module in  $\text{Prod } \omega^+$  by [18, Lemma 3.7]. Then

$$\text{Hom}_S(\text{Ext}_R^2(\omega, M), (\omega \otimes_S \text{cTr}_\omega M)_*) = 0.$$

So  $g = 0$  and

$$1_{(\omega \otimes_S \text{cTr}_\omega M)_*} = f\mu_X^{-1}(1_\omega \otimes tu')_*,$$

which means  $\alpha = 0$ . It follows from [5, Theorem 3.2.1] that  $\text{Tor}_1^S(\omega, \text{Ext}_R^2(\omega, M)) = 0$ , and thus  $\text{T-cograde}_\omega \text{Ext}_R^2(\omega, M) \geq 2$ .

The implications (3)  $\Rightarrow$  (2), (4)  $\Rightarrow$  (2) and (5)  $\Rightarrow$  (3) + (4) are trivial.

(2)  $\Rightarrow$  (1) Let

$$0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$$

be an exact sequence in  $\text{Mod } S$  with  $A, C \in \text{acT}_\omega^2(S)$ . Then  $B \in \text{acT}_\omega^1(S)$  by (2) and Lemma 3.7. It follows from [18, Proposition 3.2] that

$$\text{Coker } \mu_B \cong \text{Ext}_R^2(\omega, \text{acTr}_\omega B).$$

Consider the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{f} & C & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow \mu_B & & \downarrow \mu_C & & \\ 0 & \longrightarrow & A' & \longrightarrow & (\omega \otimes_S B)_* & \xrightarrow{(1_\omega \otimes f)_*} & (\omega \otimes_S C)_* & & \end{array}$$

where  $A' \cong \text{Ker}(1_\omega \otimes f)_*$  and  $h$  is an induced homomorphism. By the snake lemma, we get an exact sequence

$$0 \rightarrow A \xrightarrow{h} A' \rightarrow \text{Ext}_R^2(\omega, \text{acTr}_\omega B) \rightarrow 0. \tag{3.6}$$

Since  $\text{acTr}_\omega B \in \Omega_{C_i}^2(R)$ , we have

$$\omega \otimes_S \text{Coker } \mu_B \cong \omega \otimes_S \text{Ext}_R^2(\omega, \text{acTr}_\omega B) = 0$$

by Lemmas 3.8 and 3.9. Also we have  $\text{Tor}_1^S(\omega, \text{Ext}_R^2(\omega, \text{acTr}_\omega B)) = 0$  by (2). Applying the functor  $\omega \otimes -$  to (3.6) yields that  $1_\omega \otimes h$  is an isomorphism. Since  $A'$  is a submodule of  $(\omega \otimes_S B)_*$ , we have  $A' \in \text{acT}_\omega^1(S)$ , and thus  $\mu_{A'}$  is a monomorphism. On the other hand, since  $\mu_A$  is an isomorphism and

$$(1_\omega \otimes h)_* \mu_A = \mu_{A'} h,$$

we get that  $\mu_{A'}$  is an epimorphism, and hence an isomorphism, which implies that  $h$  is also an isomorphism. Thus  $\mu_B$  is an isomorphism and  $B \in \text{ac}\mathcal{T}_\omega^2(S)$ .  $\square$

### 4 Strong Ext-Cograde and Extension Closure

In this section, we characterize the extension closure of the class  $\text{ac}\mathcal{T}_\omega^k(S)$  for any  $1 \leq k \leq n$  in terms of the strong cograde of  $\text{Tor}_{k+1}^S(\omega, C)$  for any  $C \in \text{co}\Omega_{\mathcal{D}_i}^k(S)$  and  $1 \leq k \leq n$ .

Let

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

be an exact sequence in  $\text{Mod } R$ . Then one gets two exact sequences

$$0 \rightarrow \text{Im}(1_\omega \otimes f) \rightarrow \omega \otimes_S B \xrightarrow{1_\omega \otimes g} \omega \otimes_S C \rightarrow 0, \tag{4.1}$$

$$0 \rightarrow \text{Ker}(1_\omega \otimes f) \rightarrow \omega \otimes_S A \rightarrow \text{Im}(1_\omega \otimes f) \rightarrow 0. \tag{4.2}$$

Applying the functor  $(-)_*$  to (4.1) yields the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \mu_B & & \downarrow \mu_C \\ 0 & \longrightarrow & (\text{Im}(1_\omega \otimes f))_* & \longrightarrow & (\omega \otimes_S B)_* & \longrightarrow & (\omega \otimes_S C)_* \end{array}$$

Diagram (4.1)

where  $\alpha$  is an induced homomorphism. It is easy to check that the following diagram

$$\begin{array}{ccccccc} & & A & \xlongequal{\quad} & A & & \\ & & \downarrow \mu_A & & \downarrow \alpha & & \\ 0 & \longrightarrow & (\text{Ker}(1_\omega \otimes f))_* & \longrightarrow & (\omega \otimes_S A)_* & \longrightarrow & (\text{Im}(1_\omega \otimes f))_* \longrightarrow \text{Ext}_R^1(\omega, \text{Ker}(1_\omega \otimes f)) \end{array}$$

Diagram (4.2)

is commutative with the bottom row exact.

The following two lemmas are useful in this section.

**Lemma 4.1** *The following statements are equivalent for any  $C \in \text{ac}\mathcal{T}_\omega^1(S)$ .*

- (1) *s.E-cograde $_\omega \text{Tor}_1^S(\omega, C) \geq 1$ .*

(2) If

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

is an exact sequence in  $\text{Mod } R$  with  $A \in \text{acT}_\omega^1(S)$ , then  $B \in \text{acT}_\omega^1(S)$ .

**Proof** (1)  $\Rightarrow$  (2) Since  $\text{Ker}(1_\omega \otimes f)$  is a quotient module of  $\text{Tor}_1^S(\omega, C)$ , we have  $(\text{Ker}(1_\omega \otimes f))_* = 0$  by (1). Then it follows from Diagram (4.2) that  $\alpha$  is a monomorphism. Applying the snake lemma to Diagram (4.1) gives that  $\mu_B$  is also a monomorphism, and so  $B \in \text{acT}_\omega^1(S)$ .

(2)  $\Rightarrow$  (1) Let  $L$  be a quotient module of  $\text{Tor}_1^S(\omega, C)$ . Then  $L^+$  is a submodule of  $\text{Ext}_S^1(C, \omega^+) (\cong [\text{Tor}_1^S(\omega, C)]^+)$ . Since  $\omega \in \mathcal{B}_\omega(S^{op})$ , for any cardinal  $\zeta$ , we have  $\omega^{+\zeta} \in \mathcal{A}_\omega(S)$  by [12, Theorem 3.3]. Thus

$$\text{Ext}_S^1(\omega^{+\zeta}, \omega^+) \cong [\text{Tor}_S^1(\omega, \omega^{+\zeta})]^+ = 0.$$

It follows from the proof of [21, Lemma 6.9] or the dual result of [22, Lemma 3.4] that there is a cardinal  $\lambda$  such that there is an exact sequence

$$0 \rightarrow \omega^{+\lambda} \xrightarrow{f} D \rightarrow C \rightarrow 0 \tag{4.3}$$

in  $\text{Mod } R$  with  $\text{Coker } \text{Hom}_S(f, \omega^+) \cong L^+$ . Then we get the following commutative diagram with the top and bottom rows exact:

$$\begin{array}{ccccccc}
 \text{Hom}_S(D, \omega^+) \otimes_R \omega & \xrightarrow{g} & \text{Hom}_S(\omega^{+\lambda}, \omega^+) \otimes_R \omega & \longrightarrow & L^+ \otimes_R \omega & \longrightarrow & 0 \\
 \downarrow \cong & & \downarrow \cong & & & & \\
 [(\omega \otimes_S D)_*]^+ & \longrightarrow & [(\omega \otimes_S \omega^{+\lambda})_*]^+ & & & & \\
 \downarrow (\mu_D)^+ & & \downarrow (\mu_{\omega^{+\lambda}})^+ & & & & \\
 D^+ & \longrightarrow & (\omega^{+\lambda})^+ & \longrightarrow & 0 & & 
 \end{array}$$

Diagram (4.3)

Since  $\omega^{+\lambda} \in \text{acT}_\omega^2(S)$ , we have  $D \in \text{acT}_\omega^1(S)$  by (2). It follows that  $g$  is an epimorphism and  $L^+ \otimes_R \omega = 0$ . Since  $\omega$  admits a degreewise finite projective resolution, we have

$$L^+ \otimes_R \omega \cong \text{Hom}_R(\omega, L)^+ = 0 \tag{4.4}$$

by [5, Theorem 3.2.11]. Thus  $\text{Hom}_R(\omega, L) = 0$  and  $\text{s.E-cograde}_\omega \text{Tor}_1^S(\omega, C) \geq 1$ .  $\square$



**Lemma 4.2** *The following statements are equivalent for any  $C \in \text{acT}_\omega^2(S)$ .*

- (1) *s.E-cograde $_\omega \text{Tor}_1^S(\omega, C) \geq 2$ .*
- (2) *If*

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

*is an exact sequence in  $\text{Mod } R$  with  $A \in \text{acT}_\omega^2(S)$ , then  $B \in \text{acT}_\omega^2(S)$ .*

**Proof** (1)  $\Rightarrow$  (2) Since  $\text{Ker}(1_\omega \otimes f)$  is a quotient module of  $\text{Tor}_1^S(\omega, C)$ , we have

$$[\text{Ker}(1_\omega \otimes f)]_* = 0 = \text{Ext}_R^1(\omega, \text{Ker}(1_\omega \otimes f))$$

by (1). Since  $A \in \text{acT}_\omega^2(S)$  by assumption, from Diagram (4.2) we know that  $\alpha$  is an isomorphism. Notice that  $C \in \text{acT}_\omega^2(S)$ , so  $B \in \text{acT}_\omega^2(S)$  by Diagram (4.1).

(2)  $\Rightarrow$  (1) By [12, Theorem 3.3], we have that  $\omega^{+\lambda} \in \mathcal{A}_\omega(S)$  with  $\lambda$  same as that in the proof of (2)  $\Rightarrow$  (1) in Lemma 4.1. Then the middle term  $D$  in the exact sequence (4.3) is in  $\text{acT}_\omega^2(S)$  by (2). It follows from Diagram (4.3) that  $L^+ \otimes_R \omega = 0$ . Thus

$$\text{Hom}_R(\omega, L) = 0$$

by the exact sequence (4.4).

Applying the functor  $\text{Hom}_S(-, \omega^+)$  to the exact sequence (4.3) gives an exact sequence

$$0 \rightarrow \text{Hom}_S(C, \omega^+) \xrightarrow{\delta} \text{Hom}_S(D, \omega^+) \xrightarrow{\theta} \text{Hom}_S(\omega^{+\lambda}, \omega^+) \rightarrow L^+ \rightarrow 0 \tag{4.5}$$

in  $\text{Mod } R^{op}$ . Then we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 \text{Hom}_S(C, \omega^+) \otimes_R \omega & \xrightarrow{\delta \otimes 1_\omega} & \text{Hom}_S(D, \omega^+) \otimes_R \omega & \longrightarrow & \text{Im } \theta \otimes_R \omega & \longrightarrow & 0 \\
 \downarrow \cong & & \downarrow \cong & & \parallel & & \\
 ((\omega \otimes_S C)_*)^+ & \longrightarrow & ((\omega \otimes_S D)_*)^+ & \longrightarrow & \text{Im } \theta \otimes_R \omega & \longrightarrow & 0 \\
 \downarrow (\mu_C)^+ & & \downarrow (\mu_D)^+ & & \downarrow h & & \\
 0 & \longrightarrow & C^+ & \longrightarrow & D^+ & \longrightarrow & \omega^{+\lambda^+} \longrightarrow 0,
 \end{array}$$

where  $h$  is an induced homomorphism. As  $C, D \in \text{acT}_\omega^2(S)$ , we have that both  $(\mu_C)^+$  and  $(\mu_D)^+$  are isomorphisms, and so  $h$  is also an isomorphism. On the other hand,

from the exact sequence (4.5) we get the following commutative diagram with the top row exact:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Tor}_1^S(L^+, \omega) & \longrightarrow & \text{Im } \theta \otimes_R \omega & \longrightarrow & ((\omega \otimes_S \omega^{+\lambda})_*)^+ (\cong \text{Hom}_S(\omega^{+\lambda}, \omega^+) \otimes_R \omega) \\
 & & & & \downarrow h & & \downarrow (\mu_{\omega^{+\lambda}})^+ \\
 & & & & (\omega^{+\lambda})^+ & \xlongequal{\hspace{2cm}} & (\omega^{+\lambda})^+.
 \end{array}$$

Because both  $h$  and  $(\mu_{\omega^{+\lambda}})^+$  are isomorphisms, we have  $\text{Tor}_1^S(L^+, \omega) = 0$ , and thus  $\text{Ext}_R^1(\omega, L) = 0$ . The proof is finished.  $\square$

**Theorem 4.3** *The following statements are equivalent for any  $C \in \text{acT}_\omega^n(S)$  and  $n \geq 1$ .*

- (1) *s.E-cograde $_\omega$   $\text{Tor}_1^S(\omega, C) \geq n$ .*
- (2) *If*

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

*is an exact sequence in  $\text{Mod } S$  with  $A \in \text{acT}_\omega^n(S)$ , then  $B \in \text{acT}_\omega^n(S)$ .*

**Proof** The cases for  $n = 1$  and  $n = 2$  follow from Lemmas 4.1 and 4.2 respectively. Now suppose  $n \geq 3$ .

(1)  $\Rightarrow$  (2) By Lemma 4.2, we have  $B \in \text{acT}_\omega^2(S)$ . Since  $C \in \text{acT}_\omega^n(S)$ , we have  $\text{Ext}_R^i(\omega, \omega \otimes_S C) = 0$  for any  $1 \leq i \leq n - 2$  by [18, Corollary 3.3(3)]. On the other hand, since  $\text{Ker}(1_\omega \otimes f)$  is a quotient module of  $\text{Tor}_1^S(\omega, C)$ , we have  $\text{Ext}_R^i(\omega, \text{Ker}(1_\omega \otimes f)) = 0$  for any  $0 \leq i \leq n - 1$  by (1). Then it is induced from the exact sequence (4.2) that  $\text{Ext}_R^i(\omega, \text{Im}(1_\omega \otimes f)) = 0$  for any  $1 \leq i \leq n - 2$ . Thus it follows from the exact sequence (4.1) that  $\text{Ext}_R^i(\omega, \omega \otimes_S B) = 0$  for any  $1 \leq i \leq n - 2$ , and so  $B \in \text{acT}_\omega^n(S)$ .

(2)  $\Rightarrow$  (1) It follows from the exact sequence (4.4) that there are two exact sequences

$$\begin{aligned}
 0 \rightarrow \text{Hom}_S(C, \omega^+) &\xrightarrow{\delta} \text{Hom}_S(D, \omega^+) \rightarrow \text{Im } \theta \rightarrow 0, \\
 0 \rightarrow \text{Im } \theta &\rightarrow \text{Hom}_S(\omega^{+\lambda}, \omega^+) \rightarrow L^+ \rightarrow 0.
 \end{aligned}$$

By (2), we have  $D \in \text{acT}_\omega^n(S)$ . From the proof of Lemma 4.2, we know that  $\delta \otimes_R 1_\omega$  is a monomorphism and

$$\text{Tor}_1^S(L^+, \omega) = 0 = L^+ \otimes_R \omega.$$

Moreover, it is easy to verify that  $\text{Tor}_{2 \leq i \leq n-1}^S(L^+, \omega) = 0$ . Since

$$[\text{Ext}_R^i(\omega, L)]^+ \cong \text{Tor}_i^S(L^+, \omega)$$

for any  $i \geq 0$  by [5, Theorem 3.2.1], it follows that  $[\text{Ext}_R^{0 \leq i \leq n-1}(\omega, L)]^+ = 0$ , and hence  $\text{Ext}_R^{0 \leq i \leq n-1}(\omega, L) = 0$ . The proof is finished.  $\square$

**Proposition 4.4** *Suppose  $n \geq 1$  and  $\mathcal{D}_n \subseteq \omega_S^\top$ . Then the following assertions hold.*

- (1) *If  $\Omega_{\mathcal{D}_n}^k(S)$  is extension closed for any  $1 \leq k < n$ , then  $\text{ac}\mathcal{T}_\omega^k(S) = \Omega_{\mathcal{D}_n}^k(S)$  for any  $1 \leq k \leq n$ , and hence  $\text{Add } \Omega_{\mathcal{D}_n}^k(S) = \Omega_{\mathcal{D}_n}^k(S)$  for any  $1 \leq k \leq n$ .*
- (2) *If  $\text{Add } \Omega_{\mathcal{D}_n}^k(S)$  is extension closed for any  $1 \leq k \leq n$ , then  $\text{Add } \Omega_{\mathcal{D}_n}^k(S) = \Omega_{\mathcal{D}_n}^k(S)$  for any  $1 \leq k \leq n$ .*

**Proof** (1) We proceed by induction on  $n$ . When  $n = 1$ , the assertion follows from Lemma 3.1.

Now suppose  $n \geq 2$  and  $\text{ac}\mathcal{T}_\omega^k(S) = \Omega_{\mathcal{D}_n}^k(S)$  for any  $1 \leq k \leq n - 1$ . Let  $N \in \text{co}\Omega_{\mathcal{D}_n}^n(S)$  and let

$$A_{n-1} \rightarrow \cdots \rightarrow A_0 \rightarrow N \rightarrow 0$$

be an exact sequence in  $\text{Mod } S$  with all  $A_j \in \mathcal{D}_n$ . Set  $L := \text{Im}(A_{n-1} \rightarrow A_{n-2})$ . Then  $L \in \Omega_{\mathcal{D}_n}^{n-1}(S) = \text{ac}\mathcal{T}_\omega^{n-1}(S)$ . By Theorem 4.3, we have

$$\text{s.E-cograde}_\omega \text{Tor}_n^S(\omega, N) = \text{s.E-cograde}_\omega \text{Tor}_1^S(\omega, L) \geq n - 1.$$

Then  $\text{ac}\mathcal{T}_\omega^k(S) = \Omega_{\mathcal{D}_n}^k(S)$  for any  $1 \leq k \leq n$  by Proposition 3.5. Because  $\text{ac}\mathcal{T}_\omega^k(S)$  is closed under direct sums and direct summands for any  $k \geq 1$ , we have  $\text{Add } \Omega_{\mathcal{D}_n}^k(S) = \Omega_{\mathcal{D}_n}^k(S)$  for any  $1 \leq k \leq n$ .

(2) We proceed by induction on  $n$ . By Lemma 3.1, we have  $\Omega_{\mathcal{D}_n}^1(S) = \text{ac}\mathcal{T}_\omega^1(S)$ . Since  $\text{Add } \text{ac}\mathcal{T}_\omega^1(S) = \text{ac}\mathcal{T}_\omega^1(S)$ , the case for  $n = 1$  follows. Let  $n \geq 2$ . By the induction hypothesis, we have that  $\text{Add } \Omega_{\mathcal{D}_n}^k(S) = \Omega_{\mathcal{D}_n}^k(S)$  is extension closed for any  $1 \leq k < n$ , it follows from (1) that  $\text{ac}\mathcal{T}_\omega^n(S) = \Omega_{\mathcal{D}_n}^n(S)$ . Thus  $\Omega_{\mathcal{D}_n}^n(S)$  is closed under direct sums and direct summands and  $\text{Add } \Omega_{\mathcal{D}_n}^n(S) = \Omega_{\mathcal{D}_n}^n(S)$ .  $\square$

**Theorem 4.5** *Suppose  $n \geq 1$  and  $\mathcal{D}_n \subseteq \omega_S^\top$ . Then the following statements are equivalent.*

- (1)  *$\text{s.E-cograde}_\omega \text{Tor}_{k+1}^S(\omega, N) \geq k$  for any  $N \in \text{co}\Omega_{\mathcal{D}_n}^k(S)$  and  $1 \leq k \leq n$ .*
- (2)  *$\Omega_{\mathcal{D}_n}^k(S)$  is extension closed for any  $1 \leq k \leq n$ .*
- (3)  *$\Omega_{\mathcal{D}_n}^k(S)$  is extension closed and  $\text{ac}\mathcal{T}_\omega^k(S) = \Omega_{\mathcal{D}_n}^k(S)$  for any  $1 \leq k \leq n$ .*
- (4)  *$\text{ac}\mathcal{T}_\omega^k(S)$  is extension closed for any  $1 \leq k \leq n$ .*
- (5)  *$\text{Add } \Omega_{\mathcal{D}_n}^k(S)$  is extension closed for any  $1 \leq k \leq n$ .*

**Proof** (1)  $\Rightarrow$  (2) By Proposition 3.5, we have  $\text{ac}\mathcal{T}_\omega^k(S) = \Omega_{\mathcal{D}_n}^k(S)$  for any  $1 \leq k \leq n$ . Let  $N \in \text{ac}\mathcal{T}_\omega^k(S)$ . Then there is an exact sequence

$$0 \rightarrow N \rightarrow U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_{k-1} \rightarrow L \rightarrow 0$$

in  $\text{Mod } S$  with all  $U_i$  in  $\mathcal{D}_n$  by [18, Proposition 3.8]. Notice that  $L \in \text{co}\Omega_{\mathcal{D}_n}^k(S)$ , so

$$\text{s.E-cograde}_\omega \text{Tor}_1^S(\omega, N) = \text{s.E-cograde}_\omega \text{Tor}_{k+1}^S(\omega, L) \geq k.$$

It follows from Theorem 4.3 that  $\Omega_{\mathcal{D}_n}^k(S)$  is extension closed for any  $1 \leq k \leq n$ .

(5)  $\Leftrightarrow$  (2)  $\Rightarrow$  (3) It follows from Proposition 4.4.

(3)  $\Rightarrow$  (4) It is obvious.

(4)  $\Rightarrow$  (1) We proceed by induction on  $n$ . Let  $N \in \text{co}\Omega_{\mathcal{D}_n}^1(S)$  and let

$$0 \rightarrow L \rightarrow D \rightarrow N \rightarrow 0$$

be an exact sequence in  $\text{Mod } S$  with  $D \in \mathcal{D}_n$ . By Lemma 3.1, we have  $L \in \text{ac}\mathcal{T}_\omega^1(S)$ . Thus

$$\text{s.E-cograde}_\omega \text{Tor}_2^S(\omega, N) = \text{s.E-cograde}_\omega \text{Tor}_1^S(\omega, L) \geq 1$$

by Theorem 4.3. The case for  $n = 1$  follows.

Suppose  $n \geq 2$ . By the induction hypothesis, we have  $\text{s.E-cograde}_\omega \text{Tor}_{k+1}^S(\omega, N) \geq k$  for any  $N \in \text{co}\Omega_{\mathcal{D}_n}^k(S)$  and  $1 \leq k \leq n - 1$ . From Proposition 3.5, we know that  $\text{ac}\mathcal{T}_\omega^k(S) = \Omega_{\mathcal{D}_n}^k(S)$  for any  $1 \leq k \leq n$ . Now let  $N \in \text{co}\Omega_{\mathcal{D}_n}^n(S)$  and let

$$0 \rightarrow L \rightarrow D_0 \rightarrow D_1 \rightarrow \dots \rightarrow D_{n-1} \rightarrow N \rightarrow 0$$

be an exact sequence in  $\text{Mod } S$  with all  $D_i$  in  $\mathcal{D}_n$ . Then  $L \in \text{ac}\mathcal{T}_\omega^n(S)$ , and hence

$$\text{s.E-cograde}_\omega \text{Tor}_{n+1}^S(\omega, N) = \text{s.E-cograde}_\omega \text{Tor}_1^S(\omega, L) \geq n$$

by Theorem 4.3. □

Since  $\mathcal{A}_\omega(S)$  contains all projective left  $S$ -modules by [8, Lemma 4.1], any left  $S$ -module is in  $\text{co}\Omega_{\mathcal{A}}^k(S)$  for any  $k \geq 1$ . Moreover, we have  $\mathcal{A}_\omega(S) = \text{ac}\mathcal{T}(S) \cap \omega_S^\top$  by [18, Theorem 3.11].

**Theorem 4.6** *The following statements are equivalent.*

- (1)  $\text{s.E-cograde}_\omega \text{Tor}_{k+1}^S(\omega, N) \geq k$  for any  $N \in \text{Mod } S$  and  $1 \leq k \leq n$ .
- (2)  $T\text{-cograde}_\omega \text{Ext}_R^k(\omega, M) \geq k$  for any  $M \in \text{Mod } R$  for any  $1 \leq k \leq n$ .
- (3)  $\Omega_{\mathcal{A}}^k(S)$  is extension closed for any  $1 \leq k \leq n$ .
- (4)  $\Omega_{\mathcal{A}}^k(S)$  is extension closed and  $\text{ac}\mathcal{T}_\omega^k(S) = \Omega_{\mathcal{A}}^k(S)$  for any  $1 \leq k \leq n$ .
- (5)  $\text{ac}\mathcal{T}_\omega^k(S)$  is extension closed for any  $1 \leq k \leq n$ .
- (6)  $\text{Add } \Omega_{\mathcal{A}}^k(S)$  is extension closed for any  $1 \leq k \leq n$ .

**Proof** By [20, Proposition 4.12], we have (1)  $\Leftrightarrow$  (2). The other implications follow from Theorem 4.5 by replacing  $\mathcal{D}_n$  with  $\mathcal{A}_\omega(S)$ . □

Recall from [14] that a ring  $R$  is called *semiregular* if  $R/J(R)$  is von Neumann regular and idempotents can be lifted modulo  $J(R)$ , where  $J(R)$  is the Jacobson radical of  $R$ . The class of semiregular rings includes: (i) von Neumann regular rings; (ii) semiperfect rings; (iii) left cotorsion rings; and (iv) right cotorsion rings. See [7] for the definitions of left cotorsion rings and right cotorsion rings.

**Corollary 4.7** *Let  $R$  be semiregular and  $n \geq 1$ . Then the following statements are equivalent.*

- (1) *s.E-cograde $_{\omega} \text{Tor}_{k+1}^S(\omega, N) \geq k$  for any  $N \in \text{Mod } S$  and  $1 \leq k \leq n$ .*
- (2) *s.T-cograde $_{\omega} \text{Ext}_{S^{op}}^{k+1}(\omega, N') \geq k$  for any  $N' \in \text{Mod } S^{op}$  and  $1 \leq k \leq n$ .*
- (3)  *$cT_{\omega}^k(S^{op})$  is extension closed for any  $1 \leq k \leq n$ .*
- (4)  *$acT_{\omega}^k(S)$  is extension closed for any  $1 \leq k \leq n$ .*

**Proof** By the dual proof of Theorem 4.6, we get (2)  $\Leftrightarrow$  (3). The assertions (1)  $\Leftrightarrow$  (2) and (1)  $\Leftrightarrow$  (4) follow from [20, Theorem 4.14] and Theorem 4.6 respectively.  $\square$

Recall that an artin algebra  $R$  is called *right quasi  $n$ -Gorenstein* if the projective dimension of the  $i$ -term in a minimal injective resolution of  $R_R$  is at most  $i$  for any  $1 \leq i \leq n$  ([9]). Let  $D$  be the ordinary duality between  $\text{mod } R$  and  $\text{mod } R^{op}$ . Then  $D(R)$  is a semidualizing  $(R, R)$ -bimodule. It is induced from [20, Example 4.20] that  $R$  is right quasi  $n$ -Gorenstein if and only if s.E-cograde $_{\omega} \text{Tor}_{i+1}^R(D(R), N) \geq i$  for any  $N \in \text{Mod } R$  and  $1 \leq i \leq n$ .

**Corollary 4.8** *Let  $R$  be an artin algebra and  $n \geq 1$ . Then the following statements are equivalent.*

- (1)  *$R$  is right quasi  $n$ -Gorenstein.*
- (2)  *$cT_{D(R)}^k(R^{op})$  is extension closed for any  $1 \leq k \leq n$ .*
- (3)  *$acT_{D(R)}^k(R)$  is extension closed for any  $1 \leq k \leq n$ .*

**Proof** It is a consequence of Corollary 4.7.  $\square$

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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