Bull. Malays. Math. Sci. Soc. (2026) 49:136
https://doi.org/10.1007/s40840-026-02135-0

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

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ICE-Closed Subcategories, Wide τ-Tilting Modules and Recollements

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Received: 5 February 2026 / Revised: 29 April 2026 / Accepted: 11 June 2026

Abstract
Let $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be a recollement of abelian categories. We establish a bijection between certain ICE-closed subcategories of $\mathcal{A}$ and those of $\mathcal{A}''$. As an application, when Λ', Λ and Λ'' are artin algebras such that (mod Λ', mod Λ, mod Λ'') is a recollement of abelian categories, we establish a bijection between certain doubly functorially finite ICE-closed subcategories of mod Λ and those of mod Λ''. Furthermore, we provide some constructions of wide τ-tilting modules in a recollement.

Keywords ICE-closed subcategories · Wide τ-tilting modules · Support τ-tilting modules · Torsion classes · Recollements

Mathematics Subject Classification 18E10 · 16G10

1 Introduction

Wide subcategories and torsion classes play an important role in the representation theory of rings and algebras, and they are closely related to each other, see [10, 23, 27] and references therein. Recently, the notion of ICE-closed subcategories of an abelian category was introduced and studied by Enomoto [9], and such subcategories are closed under images, cokernels and extensions. This notion generalizes torsion

Communicated by Rosihan M. Ali.

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classes and wide subcategories. Later on, Enomoto and Sakai [10] introduced the notion of wide τ-tilting modules, which generalizes that of support τ-tilting modules introduced by Adachi, Iyama and Reiten [1]. It was proved in [1] that there exists a bijection between support τ-tilting modules and functorially finite torsion classes. As a generalization, Enomoto and Sakai established a bijection between wide τ-tilting modules and doubly functorially finite ICE-closed subcategories [10].

Recollements of abelian and triangulated categories were introduced by Beĭlinson, Bernstein and Deligne [4] in connection with derived categories of sheaves on topological spaces with the idea that one triangulated category may be "glued together" from two others, which play an important role in representation theory of algebras. Gluing and reduction techniques with respect to a recollement have been investigated; for instance, for a given recollement of triangulated categories or abelian categories, Chen [6] glued cotorsion pairs; Liu, Vitória and Yang presented the construction of gluing of silting objects [16]; Ma and Huang [19] glued torsion pairs and torsion classes; Ma and Zhao [21] glued tilting modules; Zhang glued wide subcategories [30] and support τ-tilting modules [31]. For more references, see [2, 5, 13, 14, 17, 22, 24–26, 28, 29]. On the other hand, the reduction of τ-tilting modules provides an effective method for constructing support τ-tilting modules [8, 12, 15, 32].

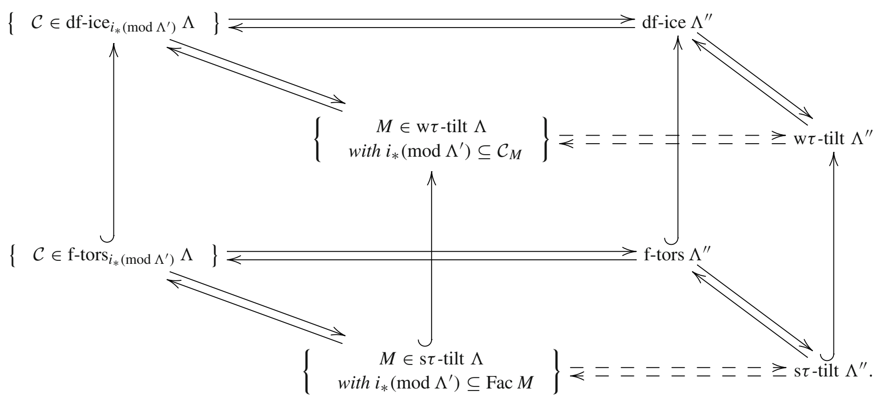
In this paper, we use $\operatorname{mod}\Lambda$ to denote the category of finitely generated left Λ-module for an artin algebra Λ. Let Λ', Λ and Λ'' be artin algebras such that

$$\operatorname{mod}\Lambda' \;\begin{smallmatrix}\xleftarrow{\;i^*\;}\\ \xrightarrow{\;i_*\;}\\ \xleftarrow{\;i^!\;}\end{smallmatrix}\; \operatorname{mod}\Lambda \;\begin{smallmatrix}\xleftarrow{\;j_!\;}\\ \xrightarrow{\;j^*\;}\\ \xleftarrow{\;j_*\;}\end{smallmatrix}\; \operatorname{mod}\Lambda''$$

is a recollement of abelian categories.

The aim of this paper is to glue and reduce ICE-closed subcategories in a recollement. As applications, we provide some constructions and reductions of wide τ-tilting modules in a recollement. Our main result is as follows.

Theorem 1.1 (Corollary 4.9) *Let* $(\operatorname{mod}\Lambda', \operatorname{mod}\Lambda, \operatorname{mod}\Lambda'')$ *be a recollement of abelian categories as above. If* $i^!$ *is exact, then we have the following commutative diagram with all horizontal maps bijective, where the dashed arrows are derived from solid ones. See Section 2 for the meaning of the notations in the following diagram.*



This paper is organized as follows. In Section 2, we recall some notions and preliminary results. In Section 3, let $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be a recollement of abelian categories. We establish a bijection between certain ICE-closed subcategories of $\mathcal{A}$ and those of $\mathcal{A}''$. In particular, we get a bijection between certain torsion classes of $\mathcal{A}$ and those of $\mathcal{A}''$. In Section 4, as applications of Section 3, we focus on wide τ-tilting modules and support τ-tilting modules over artin algebras. Let Λ', Λ and Λ'' be artin algebras such that $(\text{mod}\,\Lambda', \text{mod}\,\Lambda, \text{mod}\,\Lambda'')$ is a recollement of abelian categories. We establish a bijection between certain doubly functorially finite ICE-closed subcategories of $\text{mod}\,\Lambda$ and those of $\text{mod}\,\Lambda''$. In particular, we establish a bijection between certain functorially finite torsion classes of $\text{mod}\,\Lambda$ and those of $\text{mod}\,\Lambda''$. Furthermore, using the bijection between doubly functorially finite ICE-closed subcategories and wide τ-tilting modules and the bijection between functorially finite torsion classes and support τ-tilting modules, we provide some constructions of wide τ-tilting modules and support τ-tilting modules in a recollement. Finally, in Section 5, we give some examples to illustrate the obtained results.

2 Preliminaries

Throughout this paper, all abelian categories have enough projective and injective objects and all subcategories involved are full, additive and closed under isomorphisms.

Definition 2.1 Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a subcategory of $\mathcal{A}$.

(1) $\mathcal{C}$ is *closed under extensions* if for any exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\mathcal{A}$, $L, N \in \mathcal{C}$ implies $M \in \mathcal{C}$.

(2) $\mathcal{C}$ is *closed under quotients (resp. subobjects) in* $\mathcal{A}$ if for each object $C \in \mathcal{C}$, any quotient (resp. subobject) of C in $\mathcal{A}$ belongs to $\mathcal{C}$.

(3) $\mathcal{C}$ is a *torsion class (resp. torsion-free class) in* $\mathcal{A}$ if $\mathcal{C}$ is closed under extensions and quotients (resp. subobjects).

(4) $\mathcal{C}$ is *closed under images (resp. kernels, cokernels)* if for any map $f : C_1 \to C_2$ with $C_1, C_2 \in \mathcal{C}$, $\text{Im}\, f \in \mathcal{C}$ (resp. $\text{Ker}\, f \in \mathcal{C}$, $\text{Coker}\, f \in \mathcal{C}$).

(5) $\mathcal{C}$ is a *wide subcategory of* $\mathcal{A}$ if $\mathcal{C}$ is closed under extensions, kernels and cokernels.

(6) $\mathcal{C}$ is an *ICE-closed subcategory of* $\mathcal{A}$ if $\mathcal{C}$ is closed under images, cokernels and extensions.

(7) $\mathcal{C}$ is a *contravariantly finite subcategory* if for any object $A \in \mathcal{A}$, there exists a morphism $C \longrightarrow A$ with $C \in \mathcal{C}$ such that $\text{Hom}_{\mathcal{A}}(\mathcal{C}, C) \longrightarrow \text{Hom}_{\mathcal{A}}(\mathcal{C}, A) \longrightarrow 0$ is exact; dually, a *covariantly finite subcategory* is defined. The subcategory $\mathcal{C}$ is *functorially finite* if it is both contravariantly finite and covariantly finite.

Note that all torsion classes and wide subcategories are ICE-closed subcategories. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. We use the following notations:

- $\text{tors}\,\mathcal{A}$ denotes the set of torsion classes in $\mathcal{A}$.
- $\text{ice}\,\mathcal{A}$ denotes the set of ICE-closed subcategories of $\mathcal{A}$.

- $\operatorname{tors}_{\mathcal{C}} \mathcal{A}$ denotes the set of torsion classes in $\mathcal{A}$ containing $\mathcal{C}$.
- $\operatorname{ice}_{\mathcal{C}} \mathcal{A}$ denotes the set of ICE-closed subcategories of $\mathcal{A}$ containing $\mathcal{C}$.

Let $\mathcal{C}$ be a subcategory of an abelian category $\mathcal{A}$. We use $\operatorname{add} \mathcal{C}$ to denote the subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of objects in $\mathcal{C}$.

Definition 2.2 Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$ closed under extensions.

(1) An object $P \in \mathcal{C}$ is Ext-*projective in* $\mathcal{C}$ if $\operatorname{Ext}^1_{\mathcal{A}}(P, C) = 0$ for any $C \in \mathcal{C}$. We denote by $\mathcal{P}(\mathcal{C})$ the subcategory of $\mathcal{C}$ consisting of Ext-projective objects in $\mathcal{C}$.
(2) The subcategory $\mathcal{C}$ is said to *have enough* Ext-*projectives* if for any $C \in \mathcal{C}$, there exists an exact sequence

$$0 \longrightarrow C' \longrightarrow P \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$ such that $P \in \mathcal{P}(\mathcal{C})$ and $C' \in \mathcal{C}$.
(3) The subcategory $\mathcal{C}$ is said to *have an* Ext-*progenerator* P if $\mathcal{C}$ has enough Ext-projectives and $\mathcal{P}(\mathcal{C}) = \operatorname{add} P$ holds.

If $\mathcal{C}$ has an Ext-progenerator, then we denote by $P(\mathcal{C})$ the unique basic Ext-progenerator, or equivalently, $P(\mathcal{C})$ is a direct sum of non-isomorphic indecomposable Ext-projective objects in $\mathcal{C}$.

Definition 2.3 ([10, Definition 4.8]) Let Λ be an artin algebra and let $\mathcal{C}$ be an ICE-closed subcategory of $\operatorname{mod} \Lambda$. The subcategory $\mathcal{C}$ is *doubly functorially finite* if there exists a functorially finite wide subcategory $\mathcal{W}$ of $\operatorname{mod} \Lambda$ such that $\mathcal{C}$ is a functorially finite torsion class in $\mathcal{W}$.

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod} \Lambda$. We use the following notations:

- $\operatorname{df-ice} \Lambda$: the set of doubly functorially finite ICE-closed subcategories of $\operatorname{mod} \Lambda$.
- $\operatorname{df-ice}_{\mathcal{C}} \Lambda$: the set of doubly functorially finite ICE-closed subcategories of $\operatorname{mod} \Lambda$ containing $\mathcal{C}$.
- $\operatorname{f-tors} \Lambda$: the set of functorially finite torsion classes in $\operatorname{mod} \Lambda$.
- $\operatorname{f-tors}_{\mathcal{C}} \Lambda$: the set of functorially finite torsion classes in $\operatorname{mod} \Lambda$ containing $\mathcal{C}$.

2.1 Recollements

Definition 2.4 ([11]) A recollement, denoted by $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$, of abelian categories is a diagram

$$\mathcal{A}' \; \overset{\xleftarrow{\;i^*\;}}{\underset{\xleftarrow[\;i^!\;]{}}{\xrightarrow{\;i_*\;}}} \; \mathcal{A} \; \overset{\xleftarrow{\;j_!\;}}{\underset{\xleftarrow[\;j_*\;]{}}{\xrightarrow{\;j^*\;}}} \; \mathcal{A}''$$

of abelian categories and additive functors such that

(1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs.
(2) i_*, $j_!$ and j_* are fully faithful.
(3) $\operatorname{Im} i_* = \operatorname{Ker} j^*$.

We list some properties of recollements of abelian categories in [11, 19, 20, 25], which will be used in the sequel.

Lemma 2.5 *Let $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be a recollement of abelian categories as in the Definition 2.4.*

(1) The functors i_* and j^* are exact, the functors i^* and $j_!$ are right exact, and the functors $i^!$ and j_* are left exact.

(2) All natural transformations $i^*i_* \longrightarrow \mathrm{Id}_{\mathcal{A}'}$, $\mathrm{Id}_{\mathcal{A}'} \longrightarrow i^!i_*$, $\mathrm{Id}_{\mathcal{A}''} \longrightarrow j^*j_!$ and $j^*j_* \longrightarrow \mathrm{Id}_{\mathcal{A}''}$ are natural isomorphisms.

(3) For any object $A \in \mathcal{A}$, there exist the following exact sequences

$$0 \longrightarrow i_*i^!(A) \longrightarrow A \longrightarrow j_*j^*(A) \longrightarrow i_*(A_1) \longrightarrow 0\ ,$$

$$0 \longrightarrow i_*(A_2) \longrightarrow j_!j^*(A) \longrightarrow A \longrightarrow i_*i^*(A) \longrightarrow 0$$

in $\mathcal{A}$ with $A_1, A_2 \in \mathcal{A}'$.

(4) For any object $A \in \mathcal{A}$, if $i^!$ is exact, then we have the following exact sequence

$$0 \longrightarrow i_*i^!(A) \longrightarrow A \longrightarrow j_*j^*(A) \longrightarrow 0$$

in $\mathcal{A}$.

(5) If $i^!$ is exact, then there exists an exact sequence of natural transformations

$$0 \longrightarrow i_*i^!j_! \longrightarrow j_! \longrightarrow j_* \longrightarrow 0\ .$$

2.2 τ-tiltings

Let Λ be an artin algebra and $M \in \mathrm{mod}\,\Lambda$. We use $|M|$ to denote the number of pairwise nonisomorphic indecomposable direct summands of M. In particular, $|\Lambda|$ denotes the number of pairwise nonisomorphic simple modules in $\mathrm{mod}\,\Lambda$.

Definition 2.6 ([1, Definitions 0.1 and 0.3, and Proposition 2.3]) Let Λ be an artin algebra and $M \in \mathrm{mod}\,\Lambda$.

(1) M is *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$, and M is *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$.

(2) For $P \in \mathrm{proj}\,\Lambda$, a pair (M, P) is a *support τ-tilting pair* if M is τ-rigid, $\mathrm{Hom}_\Lambda(P, M) = 0$ and $|M| + |P| = |\Lambda|$.

(3) M is *support τ-tilting* if there exists some $P \in \mathrm{proj}\,\Lambda$ such that (M, P) is a support τ-tilting pair.

We denote by $\mathrm{s}\tau\text{-tilt}\,\Lambda$ the set of isomorphism classes of basic support τ-tilting Λ-modules.

Let Λ be an artin algebra. A wide subcategory $\mathcal{W}$ of $\mathrm{mod}\,\Lambda$ is functorially finite if and only if there exists an artin algebra Γ such that $\mathcal{W}$ is equivalent to $\mathrm{mod}\,\Gamma$ [9, Proposition 4.12].

Definition 2.7 ([10, Definition 4.11]) Let Λ be an artin algebra.

(1) For a functorially finite wide subcategory $\mathcal{W}$ of $\operatorname{mod}\Lambda$ and $M \in \mathcal{W}$, fix an equivalence $F : \mathcal{W} \simeq \operatorname{mod}\Gamma$ for an artin algebra Γ. We say that M is *$\tau_{\mathcal{W}}$-tilting* if $F(M)$ is a τ-tilting Γ-module.
(2) M is a *wide τ-tilting module* if there exists some functorially finite wide subcategory $\mathcal{W}$ such that M is $\tau_{\mathcal{W}}$-tilting.

We denote by $\mathrm{w}\tau\text{-tilt}\,\Lambda$ the set of isomorphism classes of basic wide τ-tilting modules.

Following [10, Proposition 4.12], wide τ-tilting modules can be regard as a generalization of support τ-tilting modules. Enomoto and Sakai [10, Theorem 4.13] established a bijection between wide τ-tilting modules and doubly functorially finite ICE-closed subcategories in $\operatorname{mod}\Lambda$, which generalizes the bijection between support τ-tilting modules and functorially finite torsion classes [1, Theorem 2.7].

Let Λ be an artin algebra and M a Λ-module. Set

$$\operatorname{cok} M := \{\, A \in \operatorname{mod}\Lambda \mid \text{there exists an exact sequence } M_1 \longrightarrow M_2 \longrightarrow A \longrightarrow 0 \text{ in } \operatorname{mod}\Lambda \text{ with } M_1, M_2 \in \operatorname{add} M\}.$$

$$\operatorname{Fac} M := \{\, A \in \operatorname{mod}\Lambda \mid \text{there exists an exact sequence } \widetilde{M} \longrightarrow A \longrightarrow 0 \text{ in } \operatorname{mod}\Lambda \text{ with } \widetilde{M} \in \operatorname{add} M\}.$$

Theorem 2.8 ([10, Corollary 4.15]) *For an artin algebra Λ, there exists the following commutative diagram with all horizontal maps bijective.*

$$\begin{array}{ccc} \mathrm{w}\tau\text{-tilt}\,\Lambda & \underset{\operatorname{cok}}{\overset{P(-)}{\rightleftarrows}} & \text{df-ice}\,\Lambda \\ \uparrow & & \uparrow \\ \mathrm{s}\tau\text{-tilt}\,\Lambda & \underset{\operatorname{Fac}}{\overset{P(-)}{\rightleftarrows}} & \text{f-tors}\,\Lambda \end{array}$$

By this result, if M is a support τ-tilting Λ-module, then $\operatorname{Fac} M = \operatorname{cok} M$. For a wide τ-tilting module M in $\operatorname{mod}\Lambda$, we use $\mathcal{C}_M := \operatorname{cok} M$ to denote the doubly functorially finite ICE-closed subcategory of $\operatorname{mod}\Lambda$.

3 ICE-closed subcategories and recollements

In this section, we study how to construct ICE-closed subcategories in a recollement. Assume that

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{A}' & \xrightarrow{i_*} & \mathcal{A} & \xrightarrow{j^*} & \mathcal{A}'' \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

is a recollement of abelian categories.

Proposition 3.1 *If $\mathcal{C}$ is an ICE-closed subcategory of $\mathcal{A}$ with $i_*i^*(\mathcal{C}) \subseteq \mathcal{C}$ (resp. $i_*i^!(\mathcal{C}) \subseteq \mathcal{C}$), then $i^*(\mathcal{C})$ (resp. $i^!(\mathcal{C})$) is an ICE-closed subcategory of $\mathcal{A}'$.*

Proof We only prove that $i^*(\mathcal{C})$ is an ICE-closed subcategory of $\mathcal{A}'$, the other assertion follows similarly.

Let $f : M \longrightarrow N$ be a morphism in $\mathcal{A}'$ with $M, N \in i^*(\mathcal{C})$. Consider the following commutative diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\quad f \quad} & N & \longrightarrow & \operatorname{Coker} f & \longrightarrow & 0. \\ & \searrow & \nearrow & & & & \\ & \operatorname{Im} f & & & & & \end{array}$$

Since i_* is exact, we have the following commutative diagram

$$\begin{array}{ccccccc} i_*(M) & \xrightarrow{\quad i_*(f) \quad} & i_*(N) & \longrightarrow & i_*(\operatorname{Coker} f) & \longrightarrow & 0. \\ & \searrow & \nearrow & & & & \\ & i_*(\operatorname{Im} f) & & & & & \end{array}$$

Notice that $i_*(M), i_*(N) \in i_*i^*(\mathcal{C}) \subseteq \mathcal{C}$, so $i_*(\operatorname{Coker} f), i_*(\operatorname{Im} f) \in \mathcal{C}$ by the assumption that $\mathcal{C}$ is closed under cokernels and images. Then $\operatorname{Coker} f \cong i^*i_*(\operatorname{Coker} f) \in i^*(\mathcal{C})$ and $\operatorname{Im} f \cong i^*i_*(\operatorname{Im} f) \in i^*(\mathcal{C})$, and thus $i^*(\mathcal{C})$ is closed under cokernels and images.

Let

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence in $\mathcal{A}'$ with $L, N \in i^*(\mathcal{C})$. Applying the exact functor i_* to the above exact sequence yields the following exact sequence

$$0 \longrightarrow i_*(L) \longrightarrow i_*(M) \longrightarrow i_*(N) \longrightarrow 0$$

in $\mathcal{A}$. Notice that $i_*(L), i_*(N) \in i_*i^*(\mathcal{C}) \subseteq \mathcal{C}$, so $i_*(M) \in \mathcal{C}$ by the assumption that $\mathcal{C}$ is closed under extensions. Then $M \cong i^*i_*(M) \in i^*(\mathcal{C})$, and so $i^*(\mathcal{C})$ is closed under extensions. Thus $i^*(\mathcal{C})$ is ICE-closed. □

Lemma 3.2 *If $\mathcal{C}$ is an ICE-closed subcategory of $\mathcal{A}$ with $i_*(\mathcal{A}') \subseteq \mathcal{C}$, then it holds that*

(1) $i_*i^*(\mathcal{C}) \subseteq \mathcal{C}$, $i_*i^!(\mathcal{C}) \subseteq \mathcal{C}$ and $j_*j^*(\mathcal{C}) \subseteq \mathcal{C}$.
(2) If $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$ and $C \in \mathcal{A}$ such that $j^*(C) \in j^*(\mathcal{C})$, then $C \in \mathcal{C}$.

Proof (1) It is clear that $i_*i^*(\mathcal{C}) \subseteq \mathcal{C}$ and $i_*i^!(\mathcal{C}) \subseteq \mathcal{C}$.

Let $C \in \mathcal{C}$. By Lemma 2.5, there exists an exact sequence

$$0 \longrightarrow i_*i^!(C) \longrightarrow C \longrightarrow j_*j^*(C) \longrightarrow i_*(A_1) \longrightarrow 0$$

in $\mathcal{A}$ with $A_1 \in \mathcal{A}'$. Notice that $i_*(A_1) \in \mathcal{C}$, $i_*i^!(C) \in \mathcal{C}$ and $C \in \mathcal{C}$, so $j_*j^*(C) \in \mathcal{C}$ by the assumption that $\mathcal{C}$ is an ICE-closed subcategory of $\mathcal{A}$. Thus $j_*j^*(\mathcal{C}) \subseteq \mathcal{C}$.

(2) Let $C \in \mathcal{C}$. By Lemma 2.5, there exists an exact sequence

$$0 \longrightarrow i_*(A_2) \longrightarrow j_!j^*(C) \longrightarrow C \longrightarrow i_*i^*(C) \longrightarrow 0$$

in $\mathcal{A}$ with $A_2 \in \mathcal{A}'$. Notice that $j_!j^*(C) \in j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$, $i_*(A_2) \in \mathcal{C}$ and $i_*i^*(C) \in \mathcal{C}$, so $C \in \mathcal{C}$ by the assumption that $\mathcal{C}$ is an ICE-closed subcategory of $\mathcal{A}$. □

If $\mathcal{C}$ is a wide subcategory of $\mathcal{A}$ with $i_*(\mathcal{A}') \subseteq \mathcal{C}$, then $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$ by [30, Lemma 3.1]. For an ICE-closed subcategory $\mathcal{C}$, the assertion "$j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$" does not hold true in general (see Example 5.1(3)). However, we have the following easy observation.

Proposition 3.3 *Let $\mathcal{C}$ be an ICE-closed subcategory of $\mathcal{A}$. If $i^!$ is exact and $i_*(\mathcal{A}') \subseteq \mathcal{C}$, then $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$.*

Proof Suppose $C \in \mathcal{C}$. Then $j^*(C) \in \mathcal{A}''$. By Lemma 2.5, we have an exact sequence

$$0 \longrightarrow i_*i^!j_!j^*(C) \longrightarrow j_!j^*(C) \longrightarrow j_*j^*(C) \longrightarrow 0$$

in $\mathcal{A}$. Since $j_*j^*(C) \in j_*j^*(\mathcal{C}) \subseteq \mathcal{C}$ by Lemma 3.2 and since $i_*i^!j_!j^*(C) \in i_*(\mathcal{A}') \subseteq \mathcal{C}$ by assumption, we have $j_!j^*(C) \in \mathcal{C}$ and $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$. □

Proposition 3.4 *If $\mathcal{C}$ is an ICE-closed subcategory of $\mathcal{A}$ with $i_*(\mathcal{A}') \subseteq \mathcal{C}$ and $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$, then $j^*(\mathcal{C})$ is an ICE-closed subcategory of $\mathcal{A}''$.*

Proof Let $f : M \longrightarrow N$ be a morphism in $\mathcal{A}''$ with $M, N \in j^*(\mathcal{C})$. Consider the following commutative diagram

$$\begin{array}{ccccccc} j_*(M) & \xrightarrow{\quad j_*(f) \quad} & j_*(N) & \longrightarrow & \operatorname{Coker} j_*(f) & \longrightarrow & 0. \\ & \searrow \quad \nearrow & & & & & \\ & \operatorname{Im} j_*(f) & & & & & \end{array}$$

in $\mathcal{A}$. By Lemma 3.2(1), we have $j_*(M), j_*(N) \in j_*j^*(\mathcal{C}) \subseteq \mathcal{C}$. Since $\mathcal{C}$ is closed under cokernels and images, we have $\operatorname{Coker} j_*(f)$, $\operatorname{Im} j_*(f) \in \mathcal{C}$. Applying the exact functor j^* to the above exact sequence yields the following commutative diagram

$$\begin{array}{ccccccc} j^*j_*(M) & \xrightarrow{\quad j^*j_*(f) \quad} & j^*j_*(N) & \longrightarrow & j^*(\operatorname{Coker} j_*(f)) & \longrightarrow & 0. \\ & \searrow \quad \nearrow & & & & & \\ \downarrow \cong & j^*(\operatorname{Im} j_*(f)) & \downarrow \cong & & \vdots & & \\ & \vdots & & & \downarrow & & \\ M & \xrightarrow{\quad f \quad} & N & \longrightarrow & \operatorname{Coker} f & \longrightarrow & 0. \\ & \searrow \quad \downarrow \quad \nearrow & & & & & \\ & \operatorname{Im} f & & & & & \end{array}$$

Then Coker $f \cong j^*(\text{Coker } j_*(f)) \in j^*(\mathcal{C})$ and Im $f \cong j^*(\text{Im } j_*(f)) \in j^*(\mathcal{C})$. Thus $j^*(\mathcal{C})$ is closed under cokernels and images.

Let

$$0 \longrightarrow L \xrightarrow{g} M \longrightarrow N \longrightarrow 0$$

be an exact sequence in $\mathcal{A}$ with $L, N \in j^*(\mathcal{C})$. Since j_* is left exact, one can get the following exact sequence

$$0 \longrightarrow j_*(L) \xrightarrow{j_*(g)} j_*(M) \longrightarrow \text{Coker } j_*(g) \longrightarrow 0 \tag{3.1}$$

in $\mathcal{A}$. Applying the exact functor j^* to the above exact sequence yields the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j^*j_*(L) & \longrightarrow & j^*j_*(M) & \longrightarrow & j^*(\text{Coker } j_*(g)) & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

Notice that $j^*(\text{Coker } j_*(g)) \cong N \in j^*(\mathcal{C})$, so Coker $j_*(g) \in \mathcal{C}$ by Lemma 3.2(2). Since $j_*(L) \in j_*j^*(\mathcal{C}) \subseteq \mathcal{C}$, it follows from the exact sequence (3.1) that $j_*(M) \in \mathcal{C}$. Then $M \cong j^*j_*(M) \in j^*(\mathcal{C})$ and $j^*(\mathcal{C})$ is closed under extensions. Thus $j^*(\mathcal{C})$ is an ICE-closed subcategory of $\mathcal{A}$. □

The following result provides a construction of ICE-closed subcategories of $\mathcal{A}$ from those of $\mathcal{A}''$.

Proposition 3.5 *If $\mathcal{C}''$ is an ICE-closed subcategory of $\mathcal{A}''$, then*

$$\mathcal{C} := \{A \in \mathcal{A} \mid j^*(A) \in \mathcal{C}''\}$$

is an ICE-closed subcategory of $\mathcal{A}$ with $i_*(\mathcal{A}') \subseteq \mathcal{C}$. In particular, $j_!(\mathcal{C}'') \subseteq \mathcal{C}$, $j_*(\mathcal{C}'') \subseteq \mathcal{C}$, $j^*(\mathcal{C}) = \mathcal{C}''$ and $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$.

Proof Let $f : M \longrightarrow N$ be a morphism in $\mathcal{A}$ with $M, N \in \mathcal{C}$. Consider the following commutative diagram

$$\begin{array}{ccccccc} M & \xrightarrow{f} & N & \longrightarrow & \text{Coker } f & \longrightarrow & 0. \\ & \searrow & \nearrow & & & & \\ & \text{Im } f & & & & & \end{array}$$

Since j^* is exact, we have the following commutative diagram

$$\begin{array}{ccccccc} j^*(M) & \xrightarrow{j^*(f)} & j^*(N) & \longrightarrow & j^*(\operatorname{Coker} f) & \longrightarrow & 0. \\ & \searrow & \nearrow & & & & \\ & j^*(\operatorname{Im} f) & & & & & \end{array}$$

Since $j^*(M), j^*(N) \in \mathcal{C}''$ and $\mathcal{C}''$ is closed under cokernels and images, we have $j^*(\operatorname{Coker} f), j^*(\operatorname{Im} f) \in \mathcal{C}''$. Then $\operatorname{Coker} f, \operatorname{Im} f \in \mathcal{C}$. Hence $\mathcal{C}$ is closed under cokernels and images.

Let

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence in $\mathcal{A}$ with $L, N \in \mathcal{C}$. Applying the exact functor j^* to the above exact sequence yields the following exact sequence

$$0 \longrightarrow j^*(L) \longrightarrow j^*(M) \longrightarrow j^*(N) \longrightarrow 0$$

in $\mathcal{A}$. Notice that $j^*(L), j^*(N) \in \mathcal{C}''$ and $\mathcal{C}''$ is closed under extensions by assumption, so $j^*(M) \in \mathcal{C}''$. Then $M \in \mathcal{C}$ and $\mathcal{C}$ is closed under extensions. Thus $\mathcal{C}$ is an ICE-closed subcategory of $\mathcal{A}$.

It is clear that $i_*(\mathcal{A}') \subseteq \mathcal{C}$ since $j^* i_* = 0$. The last assertion follows from the fact that $j^* j_! \cong \mathrm{Id}_{\mathcal{A}''}$ and $j^* j_* \cong \mathrm{Id}_{\mathcal{A}''}$. □

Theorem 3.6 *There exists a bijection*

$$\{\mathcal{C} \in \operatorname{ice}_{i_*(\mathcal{A}')} \mathcal{A} \text{ with } j_! j^*(\mathcal{C}) \subseteq \mathcal{C}\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \operatorname{ice} \mathcal{A}'' ,$$

which is given by $\Phi(\mathcal{C}) = j^*(\mathcal{C})$ *and* $\Psi(\mathcal{C}'') = \{A \in \mathcal{A} \mid j^*(A) \in \mathcal{C}''\}$ *for* $\mathcal{C} \in \operatorname{ice}_{i_*(\mathcal{A}')} \mathcal{A}$ *with* $j_! j^*(\mathcal{C}) \subseteq \mathcal{C}$*, and* $\mathcal{C}'' \in \operatorname{ice} \mathcal{A}''$*.*

Proof By Propositions 3.4 and 3.5, it suffices to prove $\Phi\Psi = \mathrm{Id}$ and $\Psi\Phi = \mathrm{Id}$.

Let $\mathcal{C}'' \in \operatorname{ice} \mathcal{A}''$. Write $\widetilde{\mathcal{C}} := \{A \in \mathcal{A} \mid j^*(A) \in \mathcal{C}''\}$. Then

$$\Phi\Psi(\mathcal{C}'') = \Phi(\widetilde{\mathcal{C}}) = j^*(\widetilde{\mathcal{C}}).$$

Clearly, we have $j^*(\widetilde{\mathcal{C}}) \subseteq \mathcal{C}''$. Since $j_!(\mathcal{C}'') \subseteq \widetilde{\mathcal{C}}$ by Proposition 3.5, we have $\mathcal{C}'' \cong j^* j_!(\mathcal{C}'') \subseteq j^*(\widetilde{\mathcal{C}})$, and so $j^*(\widetilde{\mathcal{C}}) = \mathcal{C}''$. Thus $\Phi\Psi(\mathcal{C}'') = \mathcal{C}''$ and $\Phi\Psi = \mathrm{Id}$.

Let $\mathcal{C} \in \operatorname{ice}_{i_*(\mathcal{A}')} \mathcal{A}$ with $j_! j^*(\mathcal{C}) \subseteq \mathcal{C}$. Then

$$\Psi\Phi(\mathcal{C}) = \Psi(j^*(\mathcal{C})) = \{A \in \mathcal{A} \mid j^*(A) \in j^*(\mathcal{C})\}.$$

Clearly, we have $\mathcal{C} \subseteq \{A \in \mathcal{A} \mid j^*(A) \in j^*(\mathcal{C})\}$. Now suppose $C \in \{A \in \mathcal{A} \mid j^*(A) \in j^*(\mathcal{C})\}$, that is, $j^*(C) \in j^*(\mathcal{C})$. Then $C \in \mathcal{C}$ by Lemma 3.2(2), and so $\{A \in \mathcal{A} \mid j^*(A) \in j^*(\mathcal{C})\} \subseteq \mathcal{C}$ and $\{A \in \mathcal{A} \mid j^*(A) \in j^*(\mathcal{C})\} = \mathcal{C}$. Thus $\Psi\Phi(\mathcal{C}) = \mathcal{C}$ and $\Psi\Phi = \mathrm{Id}$. □

In particular, restricting Propositions 3.4 and 3.5 and Theorem 3.6 to wide subcategories, which recover [30, Propositions 3.2 and 3.3 and Theorem 3.4], respectively. Restricting Propositions 3.4 and 3.5 to torsion classes, one can get the following two results.

Proposition 3.7 *If $\mathcal{C}$ is a torsion class in $\mathcal{A}$ with $i_*(\mathcal{A}') \subseteq \mathcal{C}$ and $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$, then $j^*(\mathcal{C})$ is a torsion class in $\mathcal{A}''$.*

Proof By Proposition 3.4, we only need to prove that $j^*(\mathcal{C})$ is closed under quotients. Let $f : M \twoheadrightarrow N$ be an epimorphism in $\mathcal{A}''$ with $M \in j^*(\mathcal{C})$. Since $j_!$ is right exact, we have that $j_!(M) \longrightarrow j_!(N) \longrightarrow 0$ is an exact sequence in $\mathcal{A}$. Since $j_!(M) \in j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$ and $\mathcal{C}$ is closed under quotients by assumption, we have $j_!(N) \in \mathcal{C}$. Then $N \cong j^*j_!(N) \in j^*(\mathcal{C})$ and $j^*(\mathcal{C})$ is closed under quotients. □

Proposition 3.8 *If $\mathcal{C}''$ is a torsion class in $\mathcal{A}''$, then*

$$\mathcal{C} := \{A \in \mathcal{A} \mid j^*(A) \in \mathcal{C}''\}$$

is a torsion class in $\mathcal{A}$ with $i_*(\mathcal{A}') \subseteq \mathcal{C}$. In particular, $j_!(\mathcal{C}'') \subseteq \mathcal{C}$, $j_*(\mathcal{C}'') \subseteq \mathcal{C}$, $j^*(\mathcal{C}) = \mathcal{C}''$ and $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$.

Proof By Proposition 3.5, we only need to prove that $\mathcal{C}$ is closed under quotients. Let $f : M \twoheadrightarrow N$ be an epimorphism in $\mathcal{A}$ with $M \in \mathcal{C}$. Since j^* is exact, we have that $j^*(M) \twoheadrightarrow j^*(N) \longrightarrow 0$ is an exact sequence in $\mathcal{A}''$. Since $j^*(M) \in \mathcal{C}''$ and $\mathcal{C}''$ is closed under quotients by assumption, we have $j^*(N) \in \mathcal{C}''$. Then $N \in \mathcal{C}$ and $\mathcal{C}$ is closed under quotients. □

Restricting Theorem 3.6 to torsion classes, one can get the following result.

Theorem 3.9 *There exists a bijection*

$$\{\mathcal{C} \in \mathrm{tors}_{i_*(\mathcal{A}')}\,\mathcal{A} \text{ with } j_!j^*(\mathcal{C}) \subseteq \mathcal{C}\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{tors}\,\mathcal{A}'' ,$$

which is given by $\Phi(\mathcal{C}) = j^*(\mathcal{C})$ and $\Psi(\mathcal{C}'') = \{A \in \mathcal{A} \mid j^*(A) \in \mathcal{C}''\}$ for $\mathcal{C} \in \mathrm{tors}_{i_*(\mathcal{A}')}\,\mathcal{A}$ with $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$, and $\mathcal{C}'' \in \mathrm{tors}\,\mathcal{A}''$.

Proof It follows from Propositions 3.7 and 3.8 and Theorem 3.6. □

4 Wide τ-tilting modules and recollements

In this section, we establish some correspondences with respect to doubly functorially finite ICE-closed subcategories and functorially finite torsion classes, and provide some constructions of wide τ-tilting modules and support τ-tilting modules in a recollement.

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$, and let $\mathcal{C}$ and $\mathcal{D}$ be subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively. If $F(\mathcal{C}) \subseteq \mathcal{D}$, we denote $\overline{F} := F|_{\mathcal{C}}$. The following two results provide certain construction of a recollement from wide subcategories.

Theorem 4.1 ([30, Theorem 3.8]) *Let $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be a recollement of abelian categories as in the Definition 2.4. If $\mathcal{C}$ is a wide subcategory of $\mathcal{A}$ with $i_*(\mathcal{A}') \subseteq \mathcal{C}$, then*

$$\mathcal{A}' \underset{\overline{i^!}}{\overset{\overline{i^*}}{\leftleftarrows}} \xrightarrow{\overline{i_*}} \mathcal{C} \underset{\overline{j_*}}{\overset{\overline{j_!}}{\leftleftarrows}} \xrightarrow{\overline{j^*}} j^*(\mathcal{C})$$

is a recollement of abelian categories.

Corollary 4.2 *Let $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be a recollement of abelian categories as in the Definition 2.4. If $\mathcal{C}''$ is a wide subcategory of $\mathcal{A}''$, then*

$$\mathcal{A}' \underset{\overline{i^!}}{\overset{\overline{i^*}}{\leftleftarrows}} \xrightarrow{\overline{i_*}} \mathcal{C} \underset{\overline{j_*}}{\overset{\overline{j_!}}{\leftleftarrows}} \xrightarrow{\overline{j^*}} \mathcal{C}''$$

is a recollement of abelian categories, where $\mathcal{C} := \{A \in \mathcal{A} \mid j^*(A) \in \mathcal{C}''\}$.

Proof It follows from [30, Proposition 3.3], Proposition 3.5 and Theorem 4.1. □

Recall from [7] that a pair of subcategories $(\mathcal{T}, \mathcal{F})$ of an abelian category $\mathcal{A}$ is called a *torsion pair* if the following conditions are satisfied.

(1) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$; that is, $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
(2) For any object $A \in \mathcal{A}$, there exists an exact sequence

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0$$

in $\mathcal{A}$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

For a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, we have that $\mathcal{T}$ is a torsion class in $\mathcal{A}$. For a length abelian category $\mathcal{A}$, that is, $\mathcal{A}$ is an abelian category in which every object has finite length, it is well known that $\mathcal{T}$ is a torsion class in $\mathcal{A}$ if and only if $(\mathcal{T}, \mathcal{T}^{\perp})$ is a torsion pair in $\mathcal{A}$, where $\mathcal{T}^{\perp} := \{A \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(T, A) = 0 \text{ for any } T \in \mathcal{T}\}$.

Proposition 4.3 *Let $\mathcal{A}'$, $\mathcal{A}$ and $\mathcal{A}''$ be length abelian categories. If $\mathcal{C}$ is a functorially finite torsion class in $\mathcal{A}$ with $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$, then $j^*(\mathcal{C})$ is a functorially finite torsion class in $\mathcal{A}''$.*

Proof See [22, Theorem 3.7(1)]. □

Proposition 4.4 *Let $\mathcal{A}'$, $\mathcal{A}$ and $\mathcal{A}''$ be length abelian categories, and let $\mathcal{C}'$ and $\mathcal{C}''$ be functorially finite torsion classes in $\mathcal{A}'$ and $\mathcal{A}''$, respectively. Set $\mathcal{C} := \{A \in \mathcal{A} \mid i^*(A) \in \mathcal{C}' \text{ and } j^*(A) \in \mathcal{C}''\}$. If $i^!$ is exact, then the following assertions hold.*

(1) If $i_*i^!(\mathcal{C}) \subseteq \mathcal{C}$, then $\mathcal{C}$ is a functorially finite torsion class in $\mathcal{A}$.

(2) If $i_*(\mathcal{A}') \subseteq \mathcal{C}$, then $\mathcal{C} = \{A \in \mathcal{A} \mid j^*(A) \in \mathcal{C}''\}$ and $\mathcal{C}$ is a functorially finite torsion class in $\mathcal{A}$.

Proof (1) See [22, Theorem 3.3].

(2) If $i_*(\mathcal{A}') \subseteq \mathcal{C}$, then $\mathcal{A}' \cong i^*i_*(\mathcal{A}') \subseteq i^*(\mathcal{C}) \subseteq \mathcal{C}' \subseteq \mathcal{A}'$, and so $i^*(\mathcal{C}) = \mathcal{C}' = \mathcal{A}'$. Thus $i^*(\mathcal{A}) \subseteq \mathcal{A}' = \mathcal{C}'$. It is easy to check that $\mathcal{C} = \{A \in \mathcal{A} \mid j^*(A) \in \mathcal{C}''\}$. On the other hand, it is obvious that $i_*i^!(\mathcal{C}) \subseteq i_*(\mathcal{A}') \subseteq \mathcal{C}$. Thus $\mathcal{C}$ is a functorially finite torsion class in $\mathcal{A}$ by (1). $\square$

From now on, assume that Λ', Λ and Λ'' are artin algebras such that

$$\mathrm{mod}\,\Lambda' \begin{array}{c} \xleftarrow{\quad i^* \quad} \\ \xrightarrow{\quad i_* \quad} \\ \xleftarrow{\quad i^! \quad} \end{array} \mathrm{mod}\,\Lambda \begin{array}{c} \xleftarrow{\quad j_! \quad} \\ \xrightarrow{\quad j^* \quad} \\ \xleftarrow{\quad j_* \quad} \end{array} \mathrm{mod}\,\Lambda''$$

is a recollement of abelian categories.

Proposition 4.5 *If $\mathcal{C}$ is a doubly functorially finite ICE-closed subcategory of* mod Λ *with $i_*(\mathrm{mod}\,\Lambda') \subseteq \mathcal{C}$ and $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$, then $j^*(\mathcal{C})$ is a doubly functorially finite ICE-closed subcategory of* mod Λ''.

Proof By assumption, there exists a functorially finite wide subcategory $\mathcal{W}$ of mod Λ such that $\mathcal{C}$ is a functorially finite torsion class in $\mathcal{W}$. By [30, Proposition 3.2], we have that $j^*(\mathcal{W})$ is a wide subcategory of mod Λ''. It follows from [22, Lemmas 3.5 and 3.6] that $j^*(\mathcal{W})$ is functorially finite in mod Λ''. By Theorem 4.1, we have that

$$\mathrm{mod}\,\Lambda' \begin{array}{c} \xleftarrow{\quad \overline{i^*} \quad} \\ \xrightarrow{\quad \overline{i_*} \quad} \\ \xleftarrow{\quad \overline{i^!} \quad} \end{array} \mathcal{W} \begin{array}{c} \xleftarrow{\quad \overline{j_!} \quad} \\ \xrightarrow{\quad \overline{j^*} \quad} \\ \xleftarrow{\quad \overline{j_*} \quad} \end{array} j^*(\mathcal{W})$$

is a recollement of abelian categories. By Proposition 4.3, we know that $j^*(\mathcal{C})$ is a functorially finite torsion class in $j^*(\mathcal{W})$. Thus $j^*(\mathcal{C})$ is a doubly functorially finite ICE-closed subcategory of mod Λ''. $\square$

Proposition 4.6 *Assume that $i^!$ is exact. If $\mathcal{C}'' \subseteq$* mod Λ'' *is a doubly functorially finite ICE-closed subcategory, then*

$$\mathcal{C} := \{A \in \mathrm{mod}\,\Lambda \mid j^*(A) \in \mathcal{C}''\}$$

is a doubly functorially finite ICE-closed subcategory of mod Λ *with $i_*(\mathrm{mod}\,\Lambda') \subseteq \mathcal{C}$. In particular, $j_!(\mathcal{C}'') \subseteq \mathcal{C}$, $j_*(\mathcal{C}'') \subseteq \mathcal{C}$, $j^*(\mathcal{C}) = \mathcal{C}''$ and $j_!j^*(\mathcal{C}) \subseteq \mathcal{C}$.*

Proof By assumption, there exists a functorially finite wide subcategory $\mathcal{W}''$ of mod Λ'' such that $\mathcal{C}''$ is a functorially finite torsion class in $\mathcal{W}''$. By [30, Proposition 3.3], we have that $\mathcal{W} := \{A \in \mathrm{mod}\,\Lambda \mid j^*(A) \in \mathcal{W}''\}$ is a wide subcategory of mod Λ and $\mathcal{C} \subseteq \mathcal{W}$. It follows from [31, Proposition 3.1] that $\mathcal{W}$ is covariantly finite in mod Λ. Then $\mathcal{W}$ is functorially finite in mod Λ by [9, Proposition 4.12], and thus

$$\mathrm{mod}\,\Lambda' \begin{array}{c} \xleftarrow{\quad \overline{i^*} \quad} \\ \xrightarrow{\quad \overline{i_*} \quad} \\ \xleftarrow{\quad \overline{i^!} \quad} \end{array} \mathcal{W} \begin{array}{c} \xleftarrow{\quad \overline{j_!} \quad} \\ \xrightarrow{\quad \overline{j^*} \quad} \\ \xleftarrow{\quad \overline{j_*} \quad} \end{array} \mathcal{W}''$$

is a recollement of abelian categories by Corollary 4.2. Since $i^!$ is exact, we have that $\overline{i^!}$ is exact. It follows from Proposition 4.4 that $\mathcal{C}$ is a functorially finite torsion class in $\mathcal{W}$. Thus $\mathcal{C}$ is a doubly functorially finite ICE-closed subcategory of $\operatorname{mod}\Lambda$. □

Theorem 4.7 *If $i^!$ is exact, then there exists a bijection*

$$\{\mathcal{C} \in \text{df-ice}_{i_*(\operatorname{mod}\Lambda')}\Lambda\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \text{df-ice}\,\Lambda'',$$

which is given by $\Phi(\mathcal{C}) = j^*(\mathcal{C})$ and $\Psi(\mathcal{C}'') = \{A \in \operatorname{mod}\Lambda \mid j^*(A) \in \mathcal{C}''\}$ for $\mathcal{C} \in \text{df-ice}_{i_*(\operatorname{mod}\Lambda')}\Lambda$ and $\mathcal{C}'' \in \text{df-ice}\,\Lambda''$.

Proof Since $i^!$ is exact by assumption, we have that $j_! j^*(\mathcal{C}) \subseteq \mathcal{C}$ for any $\mathcal{C} \in \text{df-ice}_{i_*(\operatorname{mod}\Lambda')}\Lambda$ by Proposition 3.3. The assertion follows from Propositions 4.5 and 4.6, and Theorem 3.6. □

Clearly, a functorially finite torsion class is a doubly functorially finite ICE-closed subcategory. Restricting Theorem 4.7 to functorially finite torsion classes, we get the following result.

Theorem 4.8 *If $i^!$ is exact, then there exists a bijection*

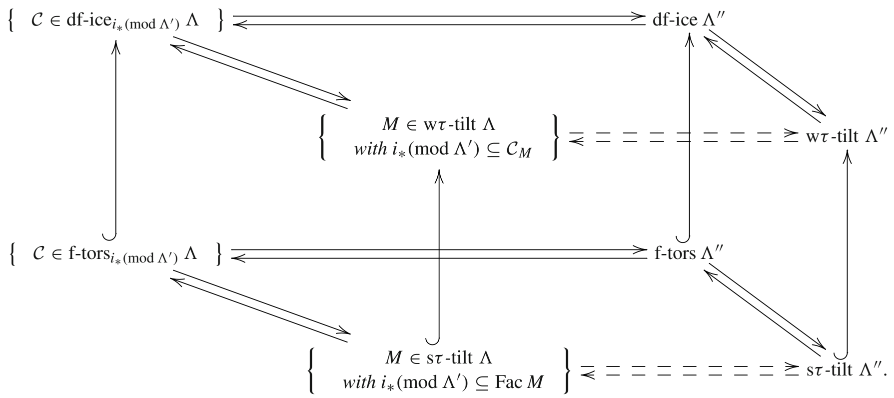
$$\{\mathcal{C} \in \text{f-tors}_{i_*(\operatorname{mod}\Lambda')}\Lambda\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \text{f-tors}\,\Lambda'' ,$$

which is given by $\Phi(\mathcal{C}) = j^*(\mathcal{C})$ and $\Psi(\mathcal{C}'') = \{A \in \operatorname{mod}\Lambda \mid j^*(A) \in \mathcal{C}''\}$ for $\mathcal{C} \in \text{f-tors}_{i_*(\operatorname{mod}\Lambda')}\Lambda$ and $\mathcal{C}'' \in \text{f-tors}\,\Lambda''$.

Proof It follows from Propositions 4.3 and 4.4 and Theorem 4.7. □

Combing Theorems 2.8, 4.7 and 4.8, we immediately get the following result.

Corollary 4.9 *If $i^!$ is exact, then we have the following commutative diagram with all horizontal maps bijective, where the dashed arrows derive from solid arrows.*



The following result shows that the converse of [10, Lemma 4.4] holds true, which is useful in the sequel.

Lemma 4.10 *Let $\mathcal{C}$ be an ICE-closed subcategory of an abelian category $\mathcal{A}$ with M an* Ext*-projective object in $\mathcal{C}$. Then $\mathcal{C} = \operatorname{cok} M$ if and only if M is an* Ext*-progenerator of $\mathcal{C}$.*

Proof The sufficiency follows from [10, Lemma 4.4]. In the following, we prove the necessity.

Let P be an indecomposable Ext-projective object in $\mathcal{C}$. By assumption, there exists an exact sequence

$$M_1 \xrightarrow{f} M_2 \longrightarrow P \longrightarrow 0$$

in $\mathcal{A}$ with $M_1, M_2 \in \operatorname{add} M$. Since $\mathcal{C}$ is ICE-closed, we have $\operatorname{Im} f \in \mathcal{C}$. Then the following exact sequence

$$0 \longrightarrow \operatorname{Im} f \longrightarrow M_2 \longrightarrow P \longrightarrow 0$$

splits, and so P is a direct summand of M_2. It follows that $P \in \operatorname{add} M$ and $\mathcal{P}(\mathcal{C}) \subseteq \operatorname{add} M$. Clearly, $\operatorname{add} M \subseteq \mathcal{P}(\mathcal{C})$. Thus $\mathcal{P}(\mathcal{C}) = \operatorname{add} M$. On the other hand, for any $C \in \mathcal{C}$, there exists an exact sequence

$$\widetilde{M}_1 \xrightarrow{g} \widetilde{M}_2 \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$ with $\widetilde{M}_1, \widetilde{M}_2 \in \operatorname{add} M$. Notice that $\operatorname{Im} g \in \mathcal{C}$ since $\mathcal{C}$ is ICE-closed, thus M is an Ext-progenerator of $\mathcal{C}$. □

As an application of Proposition 4.5, we give a construction of wide τ-tilting Λ''-modules from wide τ-tilting Λ-modules.

Proposition 4.11 *Assume that j_* is exact. If M is a wide τ-tilting Λ-module such that $i_*(\operatorname{mod} \Lambda') \subseteq \mathcal{C}_M$ and $j_! j^*(\mathcal{C}_M) \subseteq \mathcal{C}_M$, then $j^*(M)$ is a wide τ-tilting module in* $\operatorname{mod} \Lambda''$.

Proof By assumption, we have that $\mathcal{C}_M$ is a doubly functorially finite ICE-closed subcategory of $\operatorname{mod} \Lambda$. It follows from Proposition 4.5 that $j^*(\mathcal{C}_M)$ is a doubly functorially finite ICE-closed subcategory of $\operatorname{mod} \Lambda''$.

We claim that $\operatorname{cok} j^*(M) = j^*(\mathcal{C}_M)$. Clearly, $j^*(M) \in j^*(\mathcal{C}_M)$ holds. It follows that $\operatorname{cok} j^*(M) \subseteq j^*(\mathcal{C}_M)$ from the fact that $j^*(\mathcal{C}_M)$ is closed under cokernels. Now suppose $A_1 \in j^*(\mathcal{C}_M)$. Then there exists an object $C \in \mathcal{C}_M$ such that $A_1 \cong j^*(C)$. Since $\mathcal{C}_M = \operatorname{cok} M$, there exists an exact sequence

$$M_1 \longrightarrow M_2 \longrightarrow C \longrightarrow 0$$

in $\operatorname{mod} \Lambda$ with $M_1, M_2 \in \operatorname{add} M$. Since j^* is exact, we get the following exact sequence

$$j^*(M_1) \longrightarrow j^*(M_2) \longrightarrow j^*(C)(\cong A_1) \longrightarrow 0$$

in mod Λ'' with $j^*(M_1), j^*(M_2) \in \operatorname{add} j^*(M)$. Then $A_1 \in \operatorname{cok} j^*(M)$ and $j^*(\mathcal{C}_M) \subseteq \operatorname{cok} j^*(M)$. Thus $\operatorname{cok} j^*(M) = j^*(\mathcal{C}_M)$. The claim is proved.

By Theorem 2.8, we have that $M'' := P(j^*(\mathcal{C}_M))$ is a wide τ-tilting module with $\operatorname{cok} M'' = j^*(\mathcal{C}_M)$, then $\operatorname{cok} M'' = j^*(\mathcal{C}_M) = \operatorname{cok} j^*(M)$. On the other hand, since j_* is exact by assumption and $j_* j^*(\mathcal{C}_M) \subseteq \mathcal{C}_M$ by Lemma 3.2(1), we have

$$\operatorname{Ext}^1_{\Lambda''}(j^*(M), j^*(\mathcal{C}_M)) \cong \operatorname{Ext}^1_{\Lambda}(M, j_* j^*(\mathcal{C}_M)) = 0$$

by [22, Proposition 2.8]. Then $j^*(M)$ is an Ext-projective object in $j^*(\mathcal{C}_M)$. So $\operatorname{add} M'' = \operatorname{add} j^*(M)$ by Lemma 4.10, and hence $j^*(M)$ is a wide τ-tilting Λ''-module. □

In the following result, we provide a construction of wide τ-tilting modules in $\operatorname{mod} \Lambda$ from those in $\operatorname{mod} \Lambda''$.

Proposition 4.12 *Let M'' be a wide τ-tilting Λ''-module. If $i^!$ and $j_!$ are exact, then $i_*(\Lambda') \oplus j_!(M'')$ is a wide τ-tilting Λ-module.*

Proof By assumption, we have that $\mathcal{C}_{M''}$ is a doubly functorially finite ICE-closed subcategory of $\operatorname{mod} \Lambda''$. It follows from Proposition 4.6 that $\mathcal{C} := \{A \in \operatorname{mod} \Lambda \mid j^*(A) \in \mathcal{C}_{M''}\}$ is a doubly functorially finite ICE-closed subcategory of $\operatorname{mod} \Lambda$. By Theorem 2.8, there exists a wide τ-tilting module $M := P(\mathcal{C})$ such that $\operatorname{cok} M = \mathcal{C}$. Since $j^* i_* = 0$ and $j^* j_! \cong \mathrm{Id}$, we have $i_*(\Lambda') \oplus j_!(M'') \in \mathcal{C}$, and so $\operatorname{cok}(i_*(\Lambda') \oplus j_!(M'')) \subseteq \mathcal{C}$ by the fact that $\mathcal{C}$ is closed under cokernels.

On the other hand, assume $A \in \mathcal{C}$. Then $j^*(A) \in \mathcal{C}_{M''} = \operatorname{cok} M''$ and there exists the following exact sequence

$$M''_1 \xrightarrow{f} M''_2 \longrightarrow j^*(A) \longrightarrow 0$$

in $\operatorname{mod} \Lambda''$ with $M''_1, M''_2 \in \operatorname{add} M'' \subseteq \mathcal{C}_{M''}$. Thus $\operatorname{Im} f \in \mathcal{C}_{M''}$. Since $j_!$ is exact by assumption, we get the following exact sequence

$$0 \longrightarrow j_!(\operatorname{Im} f) \longrightarrow j_!(M''_2) \longrightarrow j_! j^*(A) \longrightarrow 0$$

in $\operatorname{mod} \Lambda$. Since $i^!$ is exact by assumption, we further get the following exact sequence

$$0 \longrightarrow i_* i^! j_! j^*(A) \longrightarrow j_! j^*(A) \longrightarrow j_* j^*(A) \longrightarrow 0$$

in mod Λ by Lemma 2.5. Consider the following pull-back diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & j_!(\mathrm{Im}\, f) & = & j_!(\mathrm{Im}\, f) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & K & \dashrightarrow & j_!(M_2'') & \dashrightarrow & j_* j^*(A) & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & i_* i^! j_! j^*(A) & \longrightarrow & j_! j^*(A) & \longrightarrow & j_* j^*(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Note that $i_* i^! j_! j^*(A) \in i_*(\mathrm{mod}\, \Lambda') \subseteq \mathcal{C}$ and $j_!(\mathrm{Im}\, f) \in j_!(\mathcal{C}_{M''}) \subseteq \mathcal{C}$ by Proposition 4.6. One can get $K \in \mathcal{C}$ since $\mathcal{C}$ is closed under extensions. Because $i^!(A) \in \mathrm{mod}\, \Lambda'$, there exists an exact sequence

$$0 \longrightarrow K' \longrightarrow P' \longrightarrow i^!(A) \longrightarrow 0$$

in mod Λ' with $P' \in \mathrm{add}\, \Lambda'$. Since i_* is exact, we get the following exact sequence

$$0 \longrightarrow i_*(K') \longrightarrow i_*(P') \longrightarrow i_* i^!(A) \longrightarrow 0$$

in mod Λ. Since $j^* i_* = 0$ and $j_!$ is exact, we have

$$\mathrm{Ext}^1_\Lambda(j_!(M_2''), i_* i^!(A)) \cong \mathrm{Ext}^1_{\Lambda''}(M_2'', j^* i_* i^!(A)) = 0$$

by [22, Proposition 2.8]. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & i_*(K') & \dashrightarrow & A' & \dashrightarrow & K & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & i_*(P') & \dashrightarrow & i_*(P') \oplus j_!(M_2'') & \dashrightarrow & j_!(M_2'') & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & i_*i^!(A) & \longrightarrow & A & \longrightarrow & j_*j^*(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Notice that $i_*(K') \in \mathcal{C}$ and $K \in \mathcal{C}$, so $A' \in \mathcal{C}$. Repeating this process, we get an exact sequence

$$i_*(\widetilde{P'}) \oplus j_!(\widetilde{M_2''}) \longrightarrow i_*(P') \oplus j_!(M_2'') \longrightarrow A \longrightarrow 0$$

in mod Λ with $i_*(\widetilde{P'}) \oplus j_!(\widetilde{M_2''}), i_*(P') \oplus j_!(M_2'') \in \operatorname{add}(i_*(\Lambda') \oplus j_!(M''))$. So $A \in \operatorname{cok}(i_*(\Lambda') \oplus j_!(M''))$, and hence $\mathcal{C} \subseteq \operatorname{cok}(i_*(\Lambda') \oplus j_!(M''))$. Thus we conclude that $\operatorname{cok}(i_*(\Lambda') \oplus j_!(M'')) = \mathcal{C} = \operatorname{cok} M$.

Since $j^*(\mathcal{C}) = \mathcal{C}_{M''}$ by Proposition 4.6, we have

$$\begin{aligned}
\operatorname{Ext}^1_\Lambda(i_*(\Lambda') \oplus j_!(M''), \mathcal{C}) &\cong \operatorname{Ext}^1_\Lambda(i_*(\Lambda'), \mathcal{C}) \oplus \operatorname{Ext}^1_\Lambda(j_!(M''), \mathcal{C}) \\
&\cong \operatorname{Ext}^1_{\Lambda'}(\Lambda', i^!(\mathcal{C})) \oplus \operatorname{Ext}^1_{\Lambda''}(M'', j^*(\mathcal{C})) \quad \text{(by [22, Proposition 2.8])} \\
&= 0.
\end{aligned}$$

So $i_*(\Lambda') \oplus j_!(M'')$ is an Ext-projective object in $\mathcal{C}$. It follows from Lemma 4.10 that $\operatorname{add}(i_*(\Lambda') \oplus j_!(M'')) = \operatorname{add} M$. Thus $i_*(\Lambda') \oplus j_!(M'')$ is a wide τ-tilting module in mod Λ. ◻

If M' and M'' are wide τ-tilting modules in mod Λ' and mod Λ'', respectively, then $i_*(M') \oplus j_!(M'')$ is not necessarily a wide τ-tilting Λ-module in general (see Example 5.1(4)).

The following result provides a construction of support τ-tilting modules in a recollement.

Proposition 4.13 *Let M'' be a support τ-tilting Λ''-module. If $i^!$ and $j_!$ are exact, then $i_*(\Lambda') \oplus j_!(M'')$ is a support τ-tilting Λ-module.*

Proof Suppose that $(\Lambda', 0)$ and (M'', P'') are support τ-tilting pair in mod Λ' and mod Λ'', respectively. Then Fac Λ' = mod Λ' and Fac M'' are functorially finite torsion classes in mod Λ' and mod Λ'', respectively. It follows from Proposition 4.4 that $\mathcal{C} := \{A \in \operatorname{mod} \Lambda \mid j^*(A) \in \operatorname{Fac} M''\}$ is a functorially finite torsion class in mod Λ.

We claim that $(i_*(\Lambda') \oplus j_!(M''), j_!(P''))$ is a support τ-tilting pair in mod Λ. Note that $j_!(P'')$ is a projective Λ-module by [22, Proposition 2.5(2)].

Step 1: Since $j^*(\mathcal{C}) = \text{Fac}\, M''$ by Proposition 3.8, we have

$$\begin{aligned}\text{Ext}^1_\Lambda(i_*(\Lambda') \oplus j_!(M''), \mathcal{C}) &\cong \text{Ext}^1_\Lambda(i_*(\Lambda'), \mathcal{C}) \oplus \text{Ext}^1_\Lambda(j_!(M''), \mathcal{C})\\ &\cong \text{Ext}^1_{\Lambda'}(\Lambda', i^!(\mathcal{C})) \oplus \text{Ext}^1_{\Lambda''}(M'', j^*(\mathcal{C})) \quad \text{(by [22, Proposition 2.8])}\\ &= 0.\end{aligned}$$

Then $i_*(\Lambda') \oplus j_!(M'') \in \text{add}\, P(\mathcal{C})$ is a τ-rigid Λ-module.

Step 2: Since $j^* i_* = 0$, we have

$$\begin{aligned}&\text{Hom}_\Lambda(j_!(P''), i_*(\Lambda') \oplus j_!(M''))\\ &= \text{Hom}_\Lambda(j_!(P''), i_*(\Lambda')) \oplus \text{Hom}_\Lambda(j_!(P''), j_!(M''))\\ &\cong \text{Hom}_{\Lambda''}(P'', j^* i_*(\Lambda')) \oplus \text{Hom}_{\Lambda''}(P'', M'')\\ &= 0.\end{aligned}$$

Step 3: Since $\text{Hom}_\Lambda(j_!(M''), i_*(\Lambda')) \cong \text{Hom}_{\Lambda''}(M'', j^* i_*(\Lambda')) = 0$, we have that $i_*(\Lambda')$ and $j_!(M'')$ have no common direct summands. Thus we have

$$\begin{aligned}&|i_*(\Lambda') \oplus j_!(M'')| + |j_!(P'')|\\ &= |i_*(\Lambda')| + |j_!(M'')| + |j_!(P'')|\\ &= |\Lambda'| + |M''| + |P''| \qquad \text{(by [22, Lemma 2.12])}\\ &= |\Lambda'| + |\Lambda''|\\ &= |\Lambda|. \qquad \text{(by [22, Lemma 2.13])}\end{aligned}$$

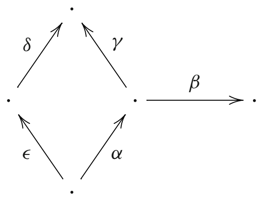
Thus $(i_*(\Lambda') \oplus j_!(M''), j_!(P''))$ is a support τ-tilting pair in mod Λ, and therefore $i_*(\Lambda') \oplus j_!(M'')$ is a support τ-tilting module in mod Λ. □

If M' and M'' are support τ-tilting modules in mod Λ' and mod Λ'', respectively, then $i_*(M') \oplus j_!(M'')$ is not necessarily a support τ-tilting Λ-module in general (see Example 5.1(5)).

5 Examples

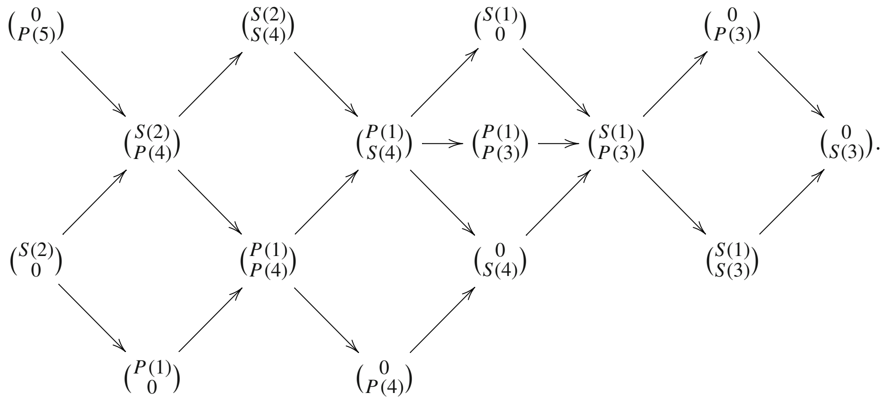
In this section, we give some examples to illustrate the obtained results. Let Λ', Λ'' be artin algebras and let ${}_{\Lambda'}M_{\Lambda''}$ be a (Λ', Λ'')-bimodule such that ${}_{\Lambda'}M$ and $M_{\Lambda''}$ are finitely generated, and let $\Lambda = \begin{pmatrix} \Lambda' & M \\ 0 & \Lambda'' \end{pmatrix}$ be the triangular matrix algebra. Then Λ is an artin algebra [3, Proposition III.2.1]. A module in mod Λ can be uniquely written as a triple $\binom{X}{Y}_f$ with $X \in \text{mod}\, \Lambda'$, $Y \in \text{mod}\, \Lambda''$ and $f \in \text{Hom}_{\Lambda'}(M \otimes_{\Lambda''} Y, X)$ [3, p.76].

Example 5.1 Let Λ' be a finite dimensional algebra given by the quiver $1 \longrightarrow 2$ and let Λ'' be a finite dimensional algebra given by the quiver $3 \xrightarrow{\alpha} 4 \xrightarrow{\beta} 5$ with the relation $\beta\alpha = 0$. Define a triangular matrix algebra $\Lambda = \begin{pmatrix} \Lambda' & M \\ 0 & \Lambda'' \end{pmatrix}$, where $M = \Lambda'$ and the right Λ''-module structure on Λ' is induced by the unique algebra surjective homomorphism $\Lambda'' \xrightarrow{\phi} \Lambda'$ satisfying $\phi(e_3) = e_1, \phi(e_4) = e_2$ and $\phi(e_5) = 0$. Then Λ is a finite dimensional algebra given by the quiver



with the relations $\gamma\alpha = \delta\epsilon$ and $\beta\alpha = 0$.

The Auslander–Reiten quiver of Λ is



Following [25, Example 2.12], we have that

$$\operatorname{mod}\Lambda' \; \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*\;} \\ \xleftarrow{\;i^!\;} \end{array} \; \operatorname{mod}\Lambda \; \begin{array}{c} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^*\;} \\ \xleftarrow{\;j_*\;} \end{array} \; \operatorname{mod}\Lambda''$$

is a recollement of module categories, where

$$i^*(\begin{pmatrix} X \\ Y \end{pmatrix}_f) = \operatorname{Coker} f, \qquad i_*(X) = \begin{pmatrix} X \\ 0 \end{pmatrix}, \qquad i^!(\begin{pmatrix} X \\ Y \end{pmatrix}_f) = X,$$
$$j_!(Y) = \begin{pmatrix} M \otimes_{\Lambda''} Y \\ Y \end{pmatrix}_1, \qquad j^*(\begin{pmatrix} X \\ Y \end{pmatrix}_f) = Y, \qquad j_*(Y) = \begin{pmatrix} 0 \\ Y \end{pmatrix}.$$

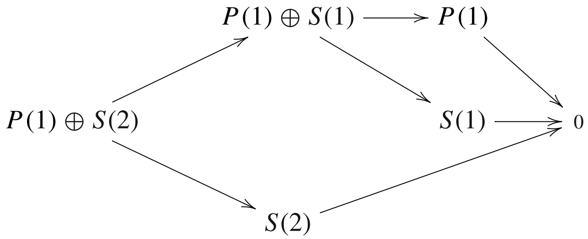
By [18, Lemma 3.2], we know that $i^!$ admits a right adjoint $\widetilde{i^!}$ and j_* admits a right adjoint $\widetilde{j_*}$, where

$$\widetilde{i^!}(X) = \begin{pmatrix} X \\ \mathrm{Hom}_{\Lambda'}(M, X) \end{pmatrix}, \qquad \widetilde{j_*}(\begin{pmatrix} X \\ Y \end{pmatrix}_f) = \mathrm{Ker}(Y \to \mathrm{Hom}_{\Lambda'}(M, X)).$$

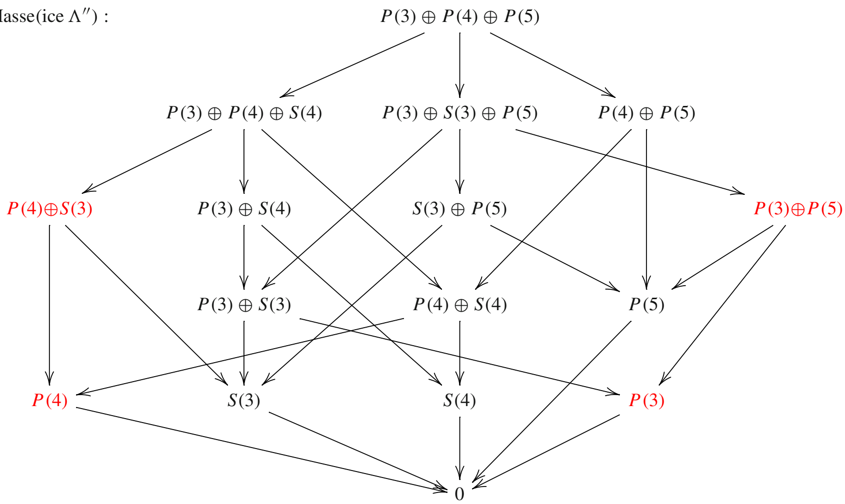
Thus $i^!$ and j_* are exact. Notice that $M_{\Lambda''}$ is projective, so $j_!$ is exact.

By [10, Examples 1.1 and 6.4], the Hasse quiver of ice Λ' and ice Λ'' are as follows, where the vertices are the corresponding wide τ-tilting modules, and the red vertices are wide τ-tilting modules which are not support τ-tilting modules.

Hasse(ice Λ') :



Hasse(ice Λ'') :



(1) By Propositions 4.12 and 4.13, we have the following table. It shows some wide τ-tilting modules (resp. support τ-tilting modules) in mod Λ constructed from those in mod Λ'', where the black marks are both wide τ-tilting and support τ-tilting modules and the red ones are wide τ-tilting modules but not support τ-tilting.

(2) The condition "$i_* i^!(\mathcal{C}) \subseteq \mathcal{C}$" in Proposition 3.1 is necessary. Taking an ICE-closed subcategory $\mathcal{C} = \mathrm{add}(\binom{S(2)}{0} \oplus \binom{S(1)}{S(3)})$ of mod Λ, we have that $i^!(\mathcal{C}) = \mathrm{add}(S(1) \oplus S(2))$ is not an ICE-closed subcategory of mod Λ' and $i_* i^!(\mathcal{C}) = \mathrm{add}(\binom{S(1)}{0} \oplus \binom{S(2)}{0}) \not\subseteq \mathcal{C}$.

$M'' \in \mathrm{w}\tau\text{-tilt}\,\Lambda''$	$i_*(\Lambda') \oplus j_!(M'') \in \mathrm{w}\tau\text{-tilt}\,\Lambda$
$P(3) \oplus P(4) \oplus P(5)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{P(1)}{P(3)} \oplus \binom{S(2)}{P(4)} \oplus \binom{0}{P(5)}$
$P(3) \oplus P(4) \oplus S(4)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{P(1)}{P(3)} \oplus \binom{S(2)}{P(4)} \oplus \binom{S(2)}{S(4)}$
$P(3) \oplus S(3) \oplus P(5)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{P(1)}{P(3)} \oplus \binom{S(1)}{S(3)} \oplus \binom{0}{P(5)}$
$P(4) \oplus P(5)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{S(2)}{P(4)} \oplus \binom{0}{P(5)}$
$P(4) \oplus S(3)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{S(2)}{P(4)} \oplus \binom{S(1)}{S(3)}$
$P(3) \oplus S(4)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{P(1)}{P(3)} \oplus \binom{S(2)}{S(4)}$
$S(3) \oplus P(5)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{S(1)}{S(3)} \oplus \binom{0}{P(5)}$
$P(3) \oplus P(5)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{P(1)}{P(3)} \oplus \binom{0}{P(5)}$
$P(3) \oplus S(3)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{P(1)}{P(3)} \oplus \binom{S(1)}{S(3)}$
$P(4) \oplus S(4)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{S(2)}{P(4)} \oplus \binom{S(2)}{S(4)}$
$P(5)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{0}{P(5)}$
$P(3)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{P(1)}{P(3)}$
$P(4)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{S(2)}{P(4)}$
$S(3)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{S(1)}{S(3)}$
$S(4)$	$\binom{S(2)}{0} \oplus \binom{P(1)}{0} \oplus \binom{S(2)}{S(4)}$
0	$\binom{S(2)}{0} \oplus \binom{P(1)}{0}$

(3) For an ICE-closed subcategory $\mathcal{C}$ of mod Λ, the assertion "$j_! j^*(\mathcal{C}) \subseteq \mathcal{C}$" does not hold true in general. In fact, $\mathcal{C} = \mathrm{add}(\binom{S(1)}{0} \oplus \binom{S(1)}{P(3)} \oplus \binom{0}{P(3)} \oplus \binom{S(1)}{S(3)} \oplus \binom{0}{S(3)})$ is an ICE-closed subcategory of mod Λ with $j_! j^*(\mathcal{C}) = \mathrm{add}(\binom{P(1)}{P(3)} \oplus \binom{S(1)}{S(3)}) \not\subseteq \mathcal{C}$.

(4) If M' and M'' are wide τ-tilting modules in mod Λ' and mod Λ'', respectively, then $i_*(M') \oplus j_!(M'')$ is not necessarily a wide τ-tilting module in general. Take wide τ-tilting modules $M' = P(1)$ and $M'' = S(3) \oplus P(4)$ in mod Λ' and mod Λ'', respectively. We have that $i_*(M') \oplus j_!(M'') = \binom{P(1)}{0} \oplus \binom{S(2)}{P(4)} \oplus \binom{S(1)}{S(3)}$ is not a wide τ-tilting Λ-module. In fact, $\mathrm{cok}(i_*(M') \oplus j_!(M'')) = \mathrm{add}(\binom{P(1)}{0} \oplus \binom{S(2)}{P(4)} \oplus \binom{S(1)}{S(3)} \oplus \binom{0}{S(3)})$ is not an ICE-closed subcategory of mod Λ.

(5) If M' and M'' are support τ-tilting modules in mod Λ' and mod Λ'', respectively, then $i_*(M') \oplus j_!(M'')$ is not necessarily a support τ-tilting module in general. Take support τ-tilting modules $M' = S(1)$ and $M'' = P(3) \oplus P(4) \oplus P(5)$ in mod Λ' and mod Λ'', respectively. We have that $M := i_*(M') \oplus j_!(M'') = \binom{S(1)}{0} \oplus \binom{P(1)}{P(3)} \oplus \binom{S(2)}{P(4)} \oplus \binom{0}{P(5)}$ is not a support τ-tilting Λ-module. In fact, there does not exist a projective Λ-module P such that (M, P) is a support τ-tilting pair.

Acknowledgements The authors thank the referees for useful suggestions.

Funding This work was supported partially by National Natural Science Foundation of China (Grant No. 12371038), Central Plains Science and Technology Innovation Youth Top-notch Talent (Grant No. 2024ZYBJRC002), and Natural Science Foundation of Henan Province (Grant No. 262300421832).

Declarations

Competing interests The authors have no relevant competing interests to declare.

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