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Duality of Preenvelopes and Pure Injective Modules

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Abstract. Let R be an arbitrary ring and let $(-)^+ = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the ring of integers and $\mathbb{Q}$ is the ring of rational numbers. Let $\mathcal{C}$ be a subcategory of left R-modules and $\mathcal{D}$ a subcategory of right R-modules such that $X^+ \in \mathcal{D}$ for any $X \in \mathcal{C}$ and all modules in $\mathcal{C}$ are pure injective. Then a homomorphism $f\colon A \to C$ of left R-modules with $C \in \mathcal{C}$ is a $\mathcal{C}$-(pre)envelope of A provided $f^+\colon C^+ \to A^+$ is a $\mathcal{D}$-(pre)cover of A^+. Some applications of this result are given.

1 Introduction

Throughout this paper, all rings are associative with identity. For a ring R, we use $\mathrm{Mod}\, R$ (resp. $\mathrm{Mod}\, R^{\mathrm{op}}$) to denote the category of left (resp. right) R-modules.

(Pre)envelopes and (pre)covers of modules were introduced by Enochs in [E] and are fundamental and important in relative homological algebra. Following Auslander and Smalø's terminology in [AuS], for a finitely generated module over an artinian algebra, a (pre)envelope and a (pre)cover are called a *(minimal) left approximation* and a *(minimal) right approximation*, respectively. Notice that (pre)envelopes and (pre)covers of modules are dual notions, so the dual properties between them are natural research topics. It is known that most of their properties are indeed dual (see [AuS, E, EH, EJ2, GT] and the references therein).

We write $(-)^+ = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the ring of integers and $\mathbb{Q}$ is the ring of rational numbers. For a ring R and a subcategory $\mathcal{X}$ of $\mathrm{Mod}\, R$ (or $\mathrm{Mod}\, R^{\mathrm{op}}$), we write $\mathcal{X}^+ = \{X^+ \mid X \in \mathcal{X}\}$. Enochs and Huang proved the following result, which played a crucial role in [EH].

Theorem 1.1 ([EH, Corollary 3.2]) *Let R be a ring, let $\mathcal{C}$ be a subcategory of* $\mathrm{Mod}\, R$*, and let $\mathcal{D}$ be a subcategory of* $\mathrm{Mod}\, R^{\mathrm{op}}$ *such that $\mathcal{C}^+ \subseteq \mathcal{D}$ and $\mathcal{D}^+ \subseteq \mathcal{C}$. If $f\colon A \to C$ is a $\mathcal{C}$-preenvelope of a module A in* $\mathrm{Mod}\, R$*, then $f^+\colon C^+ \to A^+$ is a $\mathcal{D}$-precover of A^+ in* $\mathrm{Mod}\, R^{\mathrm{op}}$.

However, the converse of Theorem 1.1 does not hold true in general (see [EH, Example 3.6]). So a natural question is: when does the converse of Theorem 1.1 hold true? In this paper, we will give a partial answer to this question and prove the following theorem.

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Theorem 1.2 *Let R be a ring, and let $\mathcal{C}$ be a subcategory of* $\operatorname{Mod} R$ *and $\mathcal{D}$ a subcategory of* $\operatorname{Mod} R^{\mathrm{op}}$ *such that $\mathcal{C}^+ \subseteq \mathcal{D}$ and all modules in $\mathcal{C}$ are pure injective. Then a homomorphism $f\colon A \to C$ in* $\operatorname{Mod} R$ *with $C \in \mathcal{C}$ is a $\mathcal{C}$-preenvelope of A provided $f^+\colon C^+ \to A^+$ is a $\mathcal{D}$-precover of A^+ in* $\operatorname{Mod} R^{\mathrm{op}}$.

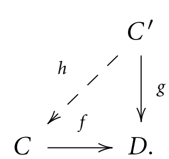
This paper is organized as follows. In Section 2, we give some terminology and some preliminary results.

Let R and S be rings and let ${}_SU_R$ be a given (S, R)-bimodule. For a subcategory $\mathcal{X}$ of $\operatorname{Mod} S$ (or $\operatorname{Mod} R^{\mathrm{op}}$), we write $\mathcal{X}^* = \{X^* \mid X \in \mathcal{X}\}$, where $(-)^* = \operatorname{Hom}(-, {}_SU_R)$. In Section 3, we first prove that if $\mathcal{C}$ is a subcategory of $\operatorname{Mod} S$ and $\mathcal{D}$ is a subcategory of $\operatorname{Mod} R^{\mathrm{op}}$ such that $\mathcal{C}^* \subseteq \mathcal{D}$ and the canonical evaluation homomorphism $X \to X^{**}$ is a split monomorphism for any $X \in \mathcal{C}$, then a homomorphism $f\colon A \to C$ in $\operatorname{Mod} S$ being a $\mathcal{C}$-preenvelope of A implies that $f^*\colon C^* \to A^*$ is a $\mathcal{D}$-precover of A^* in $\operatorname{Mod} R^{\mathrm{op}}$. As a consequence of this result we easily get Theorem 1.2. Then as applications of Theorem 1.2 we get the following results. For a ring R, a monomorphism $f\colon A \rightarrowtail C$ in $\operatorname{Mod} R$ with C pure injective is a pure injective (pre)envelope of A provided $f^+\colon C^+ \twoheadrightarrow A^+$ is a pure injective (or cotorsion) (pre)cover of A^+ in $\operatorname{Mod} R^{\mathrm{op}}$. For a left and right artinian ring R, a homomorphism $f\colon A \to P$ in $\operatorname{Mod} R$ is a projective preenvelope of A if and only if $f^+\colon P^+ \to A^+$ is an injective precover of A^+ in $\operatorname{Mod} R^{\mathrm{op}}$. In particular, we prove that for a left and right coherent ring R, an absolutely pure left R-module does not have a decomposition as a direct sum of indecomposable absolutely pure submodules in general. It means that a left (and right) coherent ring has no absolutely pure analogue of [M, Theorem 2.5], which states that for a left noetherian ring R, every injective left R-module has a decomposition as a direct sum of indecomposable injective submodules.

2 Preliminaries

In this section, we give some terminology and some preliminary results for later use.

Definition 2.1 ([E]) Let R be a ring and $\mathcal{C}$ a subcategory of $\operatorname{Mod} R$. The homomorphism $f\colon C \to D$ in $\operatorname{Mod} R$ with $C \in \mathcal{C}$ is said to be a $\mathcal{C}$-*precover* of D if for any homomorphism $g\colon C' \to D$ in $\operatorname{Mod} R$ with $C' \in \mathcal{C}$, there exists a homomorphism $h\colon C' \to C$ such that the following diagram commutes:



The homomorphism $f\colon C \to D$ is said to be *right minimal* if an endomorphism $h\colon C \to C$ is an automorphism whenever $f = fh$. A $\mathcal{C}$-precover $f\colon C \to D$ is called a $\mathcal{C}$-*cover* if f is right minimal. The notions of a $\mathcal{C}$-*preenvelope*, a *left minimal homomorphism* and *a* $\mathcal{C}$-*envelope* are defined dually.

Let R be a ring. Recall that a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathrm{Mod}\, R$ is called *pure* if the functor $\mathrm{Hom}_R(M, -)$ preserves its exactness for any finitely presented left R-module M, and a module $E \in \mathrm{Mod}\, R$ is called *pure injective* if the functor $\mathrm{Hom}_R(-, E)$ preserves the exactness of a short pure exact sequence in $\mathrm{Mod}\, R$ (*cf.* [GT,K]). Recall from [K] that a subcategory $\mathcal{C}$ of $\mathrm{Mod}\, R$ is called *definable* if it is closed under direct limits, direct products, and pure submodules in $\mathrm{Mod}\, R$.

Lemma 2.2 ([K, Corollary 2.7]) *The following statements are equivalent for a definable subcategory $\mathcal{C}$ of* $\mathrm{Mod}\, R$.

(i) *Every module in $\mathcal{C}$ is pure injective.*
(ii) *Every module in $\mathcal{C}$ is a direct sum of indecomposable modules.*

Lemma 2.3

(i) [F, Theorem 2.1] *For a ring R, a module M in* $\mathrm{Mod}\, R$ *is flat if and only if M^+ is injective in* $\mathrm{Mod}\, R^{\mathrm{op}}$.
(ii) [DC, Theorem 4] *A ring R is left (resp. right) artinian if and only if a module A in* $\mathrm{Mod}\, R$ *(resp.* $\mathrm{Mod}\, R^{\mathrm{op}}$*) being injective is equivalent to A^+ being projective in* $\mathrm{Mod}\, R^{\mathrm{op}}$ *(resp.* $\mathrm{Mod}\, R$*).*

As a generalization of projective (resp. injective) modules, the notion of Gorenstein projective (resp. injective) modules was introduced by Enochs and Jenda in [EJ1] as follows.

Definition 2.4 ([EJ1]) Let R be a ring. A module M in $\mathrm{Mod}\, R$ is called *Gorenstein projective* if there exists an exact sequence:

$$\mathbb{P}\colon \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

in $\mathrm{Mod}\, R$ with all terms projective, such that $M = \mathrm{Im}(P_0 \to P^0)$ and the sequence $\mathrm{Hom}_R(\mathbb{P}, P)$ is exact for any projective left R-module P. Dually, the notion of Gorenstein injective modules is defined.

3 The Duality Between Preenvelopes and Precovers

Let R and S be rings and let ${}_SU_R$ be a given (S, R)-bimodule. We write $(-)^*$ for $\mathrm{Hom}(-, {}_SU_R)$. For a subcategory $\mathcal{X}$ of $\mathrm{Mod}\, S$ (or $\mathrm{Mod}\, R^{\mathrm{op}}$), we write $\mathcal{X}^* = \{X^* \mid X \in \mathcal{X}\}$. For any $X \in \mathrm{Mod}\, S$ (or $\mathrm{Mod}\, R^{\mathrm{op}}$), $\sigma_X\colon X \to X^{**}$ defined by $\sigma_X(x)(f) = f(x)$ for any $x \in X$ and $f \in X^*$ is the canonical evaluation homomorphism.

The following lemma plays a crucial role in proving the main result.

Lemma 3.1 *Let $\mathcal{C}$ be a subcategory of* $\mathrm{Mod}\, S$ *and $\mathcal{D}$ a subcategory of* $\mathrm{Mod}\, R^{\mathrm{op}}$ *such that $\mathcal{C}^* \subseteq \mathcal{D}$ and σ_X is a split monomorphism for any module $X \in \mathcal{C}$. For a homomorphism $f\colon A \to C$ in* $\mathrm{Mod}\, S$ *with $C \in \mathcal{C}$, if $f^*\colon C^* \to A^*$ is a $\mathcal{D}$-precover of A^* in* $\mathrm{Mod}\, R^{\mathrm{op}}$, *then $f\colon A \to C$ is a $\mathcal{C}$-preenvelope of A.*

Proof Let $f\colon A \to C$ be in $\mathrm{Mod}\, S$ with $C \in \mathcal{C}$ such that $f^*\colon C^* \to A^*$ is a $\mathcal{D}$-precover of A^* in $\mathrm{Mod}\, R^{\mathrm{op}}$. Assume that $A \in \mathrm{Mod}\, S$, $X \in \mathcal{C}$ and $g \in \mathrm{Hom}_S(A, X)$.

Then $X^* \in \mathcal{C}^* \subseteq \mathcal{D}$ and there exists $h \in \mathrm{Hom}_{R^{\mathrm{op}}}(X^*, C^*)$ such that the following diagram commutes:

$$\begin{array}{ccc} C^* & \xrightarrow{f^*} & A^* \\ & \nwarrow^{h} & \uparrow{\scriptstyle g^*} \\ & & X^*. \end{array}$$

Then $g^* = f^* h$ and $g^{**} = h^* f^{**}$.

We have the following diagram with each square commutative:

$$\begin{array}{ccc} X & \xrightarrow{\sigma_X} & X^{**} \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g^{**}} \\ A & \xrightarrow{\sigma_A} & A^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ C & \xrightarrow{\sigma_C} & C^{**}. \end{array}$$

Then $g^{**}\sigma_A = \sigma_X g$ and $f^{**}\sigma_A = \sigma_C g$. By assumption σ_X is a split monomorphism, so there exists $\alpha \in \mathrm{Hom}_S(X^{**}, X)$ such that $\alpha\sigma_X = 1_X$, and hence we have that $g = 1_X g = (\alpha\sigma_X)g = \alpha g^{**}\sigma_A = \alpha(h^* f^{**})\sigma_A = (\alpha h^* \sigma_C) f$, that is, we get a homomorphism $\alpha h^* \sigma_C \colon C \to X$ in Mod S such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & \swarrow_{\alpha h^* \sigma_C} & \\ X. & & \end{array}$$

Thus f is a $\mathcal{C}$-preenvelope of A. ∎

From now on, R is an arbitrary ring. For a subcategory $\mathcal{X}$ of Mod R (or Mod R^{op}), we write $\mathcal{X}^+ = \{X^+ \mid X \in \mathcal{X}\}$. The main result in this paper is the following theorem.

Theorem 3.2 *Let $\mathcal{C}$ be a subcategory of* Mod R *and $\mathcal{D}$ a subcategory of* Mod R^{op} *such that $\mathcal{C}^+ \subseteq \mathcal{D}$ and all modules in $\mathcal{C}$ are pure injective. Then a homomorphism $f \colon A \to C$ in* Mod R *with $C \in \mathcal{C}$ is a $\mathcal{C}$-(pre)envelope of A provided $f^+ \colon C^+ \to A^+$ is a $\mathcal{D}$-(pre)cover of A^+ in* Mod R^{op}.

Proof First note that $\mathrm{Hom}_{R^{\mathrm{op}}}(-, R^+) \cong (-)^+$ by the adjoint isomorphism theorem. Now let X be a module in $\mathcal{C}$. Then C is pure injective by assumption, and so $\sigma_X \colon X \to X^{++}$ is a split monomorphism by [GT, Theorem 1.2.19]. So the assertion follows from Lemma 3.1 and [EH, Corollary 3.2(2)]. ∎

In the rest of this section, we will give some applications of Theorem 3.2.

By [K, Example 3.16], any module in Mod R has a pure injective envelope. It is obvious that the pure injective (pre)envelope of any module is monic. Recall from [EJ2] that a module $N \in \operatorname{Mod} R^{\mathrm{op}}$ is called *cotorsion* if $\operatorname{Ext}^1_{R^{\mathrm{op}}}(F, N) = 0$ for any flat right R-module F. For any $M \in \operatorname{Mod} R$, M^+ is pure injective by [EJ2, Proposition 5.3.7], and hence cotorsion by [EJ2, Lemma 5.3.23]. Then by Theorem 3.2, we immediately have the following corollary.

Corollary 3.3 *A monomorphism $f: A \rightarrowtail C$ in* Mod R *with C pure injective is a pure injective (pre)envelope of A provided $f^+: C^+ \twoheadrightarrow A^+$ is a pure injective (or cotorsion) (pre)cover of A^+ in* Mod R^{op}.

Recall that R is called *left pure semisimple* if every left R-module is a direct sum of finitely generated modules, or equivalently, every left R-module is pure injective.

As an immediate consequence of Theorem 3.2, we have the following corollary.

Corollary 3.4 *Let R be a left pure semisimple ring, let $\mathcal{C}$ be a subcategory of* Mod R*, and let $\mathcal{D}$ be a subcategory of* Mod R^{op} *such that $\mathcal{C}^+ \subseteq \mathcal{D}$. Then a homomorphism $f: A \to C$ in* Mod R *with $C \in \mathcal{C}$ is a $\mathcal{C}$-(pre)envelope of A provided $f^+: C^+ \to A^+$ is a $\mathcal{D}$-(pre)cover of A^+ in* Mod R^{op}.

The following are known facts:

(a) R is right coherent and left perfect if and only if every left R-module has a projective preenvelope ([DC, Proposition 3.14] and [AsM, Proposition 3.5]). A commutative ring R is artinian if and only if every R-module has a projective preenvelope ([AsM, Corollary 3.6]).

(b) R is right noetherian if and only if every right R-module has an injective (pre)cover ([E, Theorem 2.1]).

So for a right artinian ring R, every left R-module has a projective preenvelope and every right R-module has an injective (pre)cover.

We use $\operatorname{Proj}(R)$ (resp. $\operatorname{Inj}(R)$) to denote the subcategory of Mod R consisting of projective (resp. injective) left R-modules. As a consequence of Theorem 3.2, we have the following corollary.

Corollary 3.5 *Let R be a left artinian ring and let $\mathcal{D}$ be a subcategory of* Mod R^{op} *containing all injective modules. Then a homomorphism $f: A \to P$ in* Mod R *with P projective is a projective (pre)envelope of A provided $f^+: P^+ \to A^+$ is a $\mathcal{D}$-(pre)cover of A^+ in* Mod R^{op}.

Proof Let R be a left artinian ring. Then every projective left R-module has a decomposition as a direct sum of indecomposable projective submodules by [AF, Theorem 27.11]. By [SE, Theorem 5] $\operatorname{Proj}(R)$ is definable, so all projective modules in Mod R are pure injective by Lemma 2.2. Note that $[\operatorname{Proj}(R)]^+ \subseteq \operatorname{Inj}(R^{\mathrm{op}})$ by Lemma 2.3(i). So the assertion follows from Theorem 3.2. ■

Corollary 3.6 *The following statements are equivalent:*

(i) *R is a left artinian ring.*

(ii) *A monomorphism $f\colon A \rightarrowtail E$ in $\operatorname{Mod} R$ is an injective preenvelope of A if and only if $f^+\colon E^+ \to A^+$ is a projective precover of A^+ in $\operatorname{Mod} R^{\mathrm{op}}$.*

In addition, if R is a left and right artinian ring, then a homomorphism $f\colon A \to P$ in $\operatorname{Mod} R$ is a projective preenvelope of A if and only if $f^+\colon P^+ \to A^+$ is an injective precover of A^+ in $\operatorname{Mod} R^{\mathrm{op}}$.

Proof (ii) $\Rightarrow$ (i) follows from Lemma 2.3(ii).

(i) $\Rightarrow$ (ii) It is well known that a right R-module is flat if and only if it is projective over a left artinian ring R. So the assertion follows from [EH, Theorem 3.7].

Let R be a left and right artinian ring. Then

$$\big[\operatorname{Proj}(R)\big]^+ \subseteq \operatorname{Inj}(R^{\mathrm{op}}) \quad \text{and} \quad \big[\operatorname{Inj}(R^{\mathrm{op}})\big]^+ = \operatorname{Proj}(R)$$

by Lemma 2.3. Thus the last assertion follows from Corollary 3.5 and [EH, Corollary 3.2(1)]. ■

For a subcategory $\mathcal{C}$ of $\operatorname{Mod} R$, we write

$$\mathcal{C}^{\perp} = \big\{X \in \operatorname{Mod} R \mid \operatorname{Ext}^i_R(C,X) = 0 \text{ for any } C \in \mathcal{C} \text{ and } i \geq 1\big\},$$
$${}^{\perp}\mathcal{C} = \{X \in \operatorname{Mod} R \mid \operatorname{Ext}^i_R(X,C) = 0 \text{ for any } C \in \mathcal{C} \text{ and } i \geq 1\big\}.$$

We use $\operatorname{GProj}(R)$ (resp. $\operatorname{GInj}(R)$) to denote the subcategory of $\operatorname{Mod} R$ consisting of Gorenstein projective (resp. injective) modules. For an artinian algebra R, recall from [B1] that R is called *virtually Gorenstein* if $[\operatorname{GProj}(R)]^{\perp} = {}^{\perp}[\operatorname{GInj}(R)]$, and R is said of *finite Cohen-Macaulay type* (*finite CM-type* for short) if there exist only finitely many non-isomorphic finitely generated indecomposable Gorenstein projective left R-modules. The notion of virtually Gorenstein algebras is a common generalization of that of Gorenstein algebras and algebras of finite representation type ([B2, Example 4.5]).

Note that for a Gorenstein ring (that is, a left and right noetherian ring with finite left and right self-injective dimensions) R, every finitely generated left R-module has a Gorenstein projective preenvelope ([EJ2, Corollary 11.8.3]).

Corollary 3.7 *Let R be a virtually Gorenstein artinian algebra of finite CM-type and let $\mathcal{D}$ be a subcategory of $\operatorname{Mod} R^{\mathrm{op}}$ containing all Gorenstein injective modules. Then a homomorphism $f\colon A \to G$ in $\operatorname{Mod} R$ with G Gorenstein projective is a Gorenstein projective (pre)envelope of A provided $f^+\colon G^+ \to A^+$ is a $\mathcal{D}$-(pre)cover of A^+ in $\operatorname{Mod} R^{\mathrm{op}}$.*

Proof Let R be a virtually Gorenstein artinian algebra of finite CM-type. Then $\operatorname{GProj}(R)$ is definable by [B1], and every Gorenstein projective module in $\operatorname{Mod} R$ is a direct sum of indecomposable submodules by [B2, Theorem 4.10]. So all Gorenstein projective modules in $\operatorname{Mod} R$ are pure injective by Lemma 2.2. Note that $[\operatorname{GProj}(R)]^+ \subseteq \operatorname{GInj}(R^{\mathrm{op}})$ by [HuX, Corollary 2.6] and [H, Theorem 3.6]. So the assertion follows from Theorem 3.2. ■

Recall from [Me] that a module M in Mod R is called *absolutely pure* if it is a pure submodule in every module in Mod R that contains it, or equivalently, if it is pure in every injective module in Mod R that contains it. Absolutely pure modules are also known as *FP-injective modules*. It is trivial that an injective module is absolutely pure. By [Me, Theorem 3], a ring R is left noetherian if and only if every absolutely pure module in Mod R is injective.

For a left noetherian ring R, every injective left R-module has a decomposition as a direct sum of indecomposable injective submodules ([M, Theorem 2.5]). It is well known that many results about finitely generated modules or injective modules over noetherian rings should have a counterpart about finitely presented modules or absolutely pure modules (see [G, EJ2, GT, Me, P] and so on). The following corollary shows that the result just mentioned above is one of the exceptions.

Corollary 3.8 *For a (left and right coherent) ring R, an absolutely pure left R-module does not in general have a decomposition as a direct sum of indecomposable absolutely pure submodules.*

Proof Let R be a left and right coherent ring. Then the subcategory of Mod R consisting of absolutely pure modules is definable by [K, Proposition 15.1]. If any absolutely pure left R-module has a decomposition as a direct sum of indecomposable absolutely pure submodules, then any absolutely pure left R-module is pure injective by Lemma 2.2. So by Lemma 2.3(ii) and Theorem 3.2, a homomorphism $f\colon A \to C$ in Mod R with C absolutely pure is an absolutely pure preenvelope of A provided $f^+\colon C^+ \to A^+$ is a flat precover of A^+ in Mod R^{op}, which contradicts [EH, Example 3.6]. ■

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