On the C-Flatness and Injectivity of Character Modules [1]

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Abstract

Let R, S be arbitrary associative rings and ${}_RC_S$ a semidualizing bimodule. We give some equivalent characterizations for R being left coherent (and right perfect) rings, left Noetherian rings and left Artinian rings in terms of the C-(FP-)injectivity, flatness and projectivity of character modules of certain left S-modules.

Key Words: Character modules, C-flatness, C-projectivity, C-(FP-)injectivity, left coherent (and right perfect) rings, left Noetherian rings, left Artinian rings.

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1 Introduction

Let R be an arbitrary associative ring and M a left R-module. The right R-module $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is called the *character module* of M, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers. Character modules are a kind of dual modules having nice properties, which played an important role in studying the classification and structure of rings in terms of their modules; see [7]–[9], [11], [15] and references therein. In particular, Cheatham and Stone [7] gave some equivalent characterizations for a ring R being left coherent (and right perfect), left Noetherian and left Artinian in terms of the (FP-)injectivity, flatness and projectivity of character modules of certain left R-modules.

On the other hand, the study of semidualizing modules in commutative rings was initiated by Foxby [10] and Golod [12]. Then Holm and White [14] extended it to arbitrary associative rings. Many authors have studied the properties of semidualizing modules and related modules; see [10], [12]–[14], [17] and [22]–[28], and so on. Among various research areas on semidualizing modules, one basic theme is to extend the "absolute" classical results in homological algebra to the "relative" setting with respect to semidualizing modules. The aim of this paper is to study whether those results of Cheatham and Stone [7] mentioned above have relative counterparts with respect to semidualizing modules. The paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

Let R and S be arbitrary associative rings and ${}_RC_S$ a semidualizing bimodule. Assume that R is a left coherent ring. In Section 3, we show that any FP-injective left R-module is in the Bass class $\mathcal{B}_C(R)$, and any left S-module with finite C-FP-injective dimension is in the Auslander class $\mathcal{A}_C(S)$ (Proposition 3.3). Then we get that for any module N in $\mathrm{Mod}\, S$, the FP-injective dimension of $C \otimes_S N$ is at most the C-FP-injective dimension of N, and with equality when N is in $\mathcal{A}_C(S)$ (Theorem 3.4).

In Section 4, we show that R is a left coherent (and right perfect) ring if and only if for any left S-module N, the C-FP-injective dimension of N and the C-flat (respectively, C-projective) dimension of N^+ are identical, and if and only if $(C_S^{(I)})^{++}$ is C-flat (respectively, C-projective)

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for any index set I (Theorems 4.1 and 4.5). Moreover, we get that R is a left Noetherian (respectively, Artinian) ring if and only if for any left S-module N, the C-injectivity of N coincides with the C-flatness (respectively, C-projectivity) of N^+ (Theorems 4.3 and 4.6).

2 Preliminaries

Throughout this paper, all rings are associative rings with unit and all modules are unital. For a ring R, we use $\operatorname{Mod} R$ to denote the category of left R-modules.

Recall from [18, 21] that a module $Q \in \operatorname{Mod} R$ is called FP*-injective* (or *absolutely pure*) if $\operatorname{Ext}^1_R(X, Q) = 0$ for any finitely presented left R-module X. The FP*-injective dimension* FP-$\operatorname{id}_R M$ of a module $M \in \operatorname{Mod} R$ is defined as $\inf\{n \geq 0 \mid \operatorname{Ext}^{\geq n+1}_R(X, M) = 0$ for any finitely presented left R-module $X\}$, and set FP-$\operatorname{id}_R M = \infty$ if no such an integer exists. For a module $B \in \operatorname{Mod} R^{op}$, we use $\operatorname{fd}_{R^{op}} B$ to denote the flat dimension of B.

Definition 2.1. ([2, 14]). Let R and S be arbitrary rings. An (R-S)-bimodule ${}_RC_S$ is called *semidualizing* if the following conditions are satisfied.

(1) ${}_RC$ admits a degreewise finite R-projective resolution and C_S admits a degreewise finite S^{op}-projective resolution.
(2) $R = \operatorname{End}(C_S)$ and $S = \operatorname{End}({}_RC)$.
(3) $\operatorname{Ext}^{\geq 1}_R(C, C) = 0 = \operatorname{Ext}^{\geq 1}_{S^{op}}(C, C)$.

Wakamatsu [26] introduced and studied the so-called *generalized tilting modules*, which are usually called *Wakamatsu tilting modules*, see [5, 19]. Note that a bimodule ${}_RC_S$ is semidualizing if and only if it is Wakamatsu tilting ([28, Corollary 3.2]). Typical examples of semidualizing bimodules include the free module of rank one and the dualizing module over a Cohen-Macaulay local ring. More examples of semidualizing bimodules can be found in [14, 24, 27].

From now on, R and S are arbitrary rings and we fix a semidualizing bimodule ${}_RC_S$. We write $(-)_* := \operatorname{Hom}(C, -)$, and write

$$\mathcal{P}_C(S^{op}) := \{P \otimes_R C \mid P \text{ is projective in } \operatorname{Mod} R^{op}\},$$

$$\mathcal{F}_C(S^{op}) := \{F \otimes_R C \mid F \text{ is flat in } \operatorname{Mod} R^{op}\},$$

$$\mathcal{I}_C(S) := \{I_* \mid I \text{ is injective in } \operatorname{Mod} R\},$$

$$\mathcal{FI}_C(S) := \{Q_* \mid Q \text{ is FP-injective in } \operatorname{Mod} R\}.$$

The modules in $\mathcal{P}_C(S^{op})$, $\mathcal{F}_C(S^{op})$, $\mathcal{I}_C(S)$ and $\mathcal{FI}_C(S)$ are called *C-projective*, *C-flat*, *C-injective* and *C-*FP*-injective* respectively. By [17, Proposition 2.4(1)], we have $\mathcal{P}_C(S^{op}) = \operatorname{Add} C_S$, where $\operatorname{Add} C_S$ is the subcategory of $\operatorname{Mod} S^{op}$ consisting of direct summands of direct sums of copies of C_S. When ${}_RC_S = {}_RR_R$, C-projective, C-flat, C-injective and C-FP-injective modules are exactly projective, flat, injective and and FP-injective modules respectively.

Lemma 2.2. ([16, Theorem 4.17(1) and Corollary 4.18(1)])

(1) *A right S-module $N \in \mathcal{F}_C(S^{op})$ if and only if $N^+ \in \mathcal{I}_C(S)$.*
(2) *The class $\mathcal{F}_C(S^{op})$ is closed under pure submodules and pure quotients.*

The following definition is cited from [14].

Definition 2.3.

(1) The *Auslander class* $\mathcal{A}_C(R^{op})$ with respect to C consists of all modules N in $\operatorname{Mod} R^{op}$ satisfying the following conditions.

(a1) $\mathrm{Tor}^R_{\geq 1}(N, C) = 0$.
(a2) $\mathrm{Ext}^{\geq 1}_{S^{op}}(C, N \otimes_R C) = 0$.
(a3) The canonical evaluation homomorphism

$$\mu_N : N \to (N \otimes_R C)_*$$

defined by $\mu_N(x)(c) = x \otimes c$ for any $x \in N$ and $c \in C$ is an isomorphism in $\mathrm{Mod}\, R^{op}$.

(2) The *Bass class* $\mathcal{B}_C(R)$ with respect to C consists of all modules M in $\mathrm{Mod}\, R$ satisfying the following conditions.

(b1) $\mathrm{Ext}^{\geq 1}_R(C, M) = 0$.
(b2) $\mathrm{Tor}^S_{\geq 1}(C, M_*) = 0$.
(b3) The canonical evaluation homomorphism

$$\theta_M : C \otimes_S M_* \to M$$

defined by $\theta_M(c \otimes f) = f(c)$ for any $c \in C$ and $f \in M_*$ is an isomorphism in $\mathrm{Mod}\, R$.

(3) The *Auslander class* $\mathcal{A}_C(S)$ in $\mathrm{Mod}\, S$ and the *Bass class* $\mathcal{B}_C(S^{op})$ in $\mathrm{Mod}\, S^{op}$ are defined symmetrically.

The following lemma will be used frequently in the sequel.

Lemma 2.4. ([15, Proposition 3.2])

(1) For a module $N \in \mathrm{Mod}\, R^{op}$ (resp. $\mathrm{Mod}\, S$), $N \in \mathcal{A}_C(R^{op})$ (resp. $\mathcal{A}_C(S)$) if and only if $N^+ \in \mathcal{B}_C(R)$ (resp. $\mathcal{B}_C(S^{op})$).

(2) For a module $M \in \mathrm{Mod}\, R$ (resp. $\mathrm{Mod}\, S^{op}$), $M \in \mathcal{B}_C(R)$ (resp. $\mathcal{B}_C(S^{op})$) if and only if $M^+ \in \mathcal{A}_C(R^{op})$ (resp. $\mathcal{A}_C(S)$).

Let $\mathscr{X}$ be a subcategory of $\mathrm{Mod}\, S$. For a module $A \in \mathrm{Mod}\, S$. The *$\mathscr{X}$-injective dimension* $\mathscr{X}\text{-id}\, A$ of A is defined as $\inf\{n \geq 0 \mid$ there exists an exact sequence

$$0 \to A \to X^0 \to X^1 \to \cdots \to X^n \to 0$$

in $\mathrm{Mod}\, S$ with all $X^i \in \mathscr{X}\}$, and set $\mathscr{X}\text{-id}\, A = \infty$ if no such integer exists. Dually, for a subcategory $\mathscr{Y}$ of $\mathrm{Mod}\, S^{op}$ and a module $B \in \mathrm{Mod}\, S^{op}$, the *$\mathscr{Y}$-projective dimension* $\mathscr{Y}\text{-pd}\, B$ of B is defined.

3 The (C-)FP-injectivity of modules

Recall that a ring R is called *left coherent* if any finitely generated left ideal of R is finitely presented. We begin with the following lemma.

Lemma 3.1. *Let R, S, T be arbitrary rings and consider the situation $({}_TA_S, {}_RB_S)$ with A_S and and ${}_RB$ finitely presented.*

(1) If $\mathrm{Hom}_{S^{op}}(A, B)$ is a finitely generated left R-module, then for any FP-injective left R-module E, there exists a natural isomorphism

$$\tau_{A,B,E} : A \otimes_S \mathrm{Hom}_R(B, E) \to \mathrm{Hom}_R(\mathrm{Hom}_{S^{op}}(A, B), E)$$

in $\mathrm{Mod}\, T$ defined by $\tau_{A,B,E}(x \otimes f)(g) = fg(x)$ for any $x \in A$, $f \in \mathrm{Hom}_R(B, E)$ and $g \in \mathrm{Hom}_{S^{op}}(A, B)$.

(2) *If R is a left coherent ring, then $\operatorname{Hom}_{S^{op}}(A,B)$ is a finitely presented left R-module. Moreover, if there exists an exact sequence*

$$S^{t_{n+1}} \to \cdots \to S^{t_1} \to S^{t_0} \to A \to 0 \tag{2.1}$$

in $\operatorname{Mod} S^{op}$ with $n \geq 0$ and all t_i positive integers, then $\operatorname{Ext}^i_{S^{op}}(A,B)$ is a finitely presented left R-module for any $0 \leq i \leq n$.

Proof. Since A_S is finitely presented, there exists an exact sequence

$$S^{t_1} \xrightarrow{f_0} S^{t_0} \to A \to 0$$

in $\operatorname{Mod} S^{op}$ with s_0, s_1 positive integers. Then we get two exact sequences of abelian groups:

$$S^{t_1} \otimes_S \operatorname{Hom}_R(B,E) \to S^{t_0} \otimes_S \operatorname{Hom}_R(B,E) \to A \otimes_S \operatorname{Hom}_R(B,E) \to 0, \text{ and}$$

$$0 \to \operatorname{Hom}_{S^{op}}(A,B) \to \operatorname{Hom}_{S^{op}}(S^{t_0},B) \xrightarrow{\operatorname{Hom}_{S^{op}}(f_0,B)} \operatorname{Hom}_{S^{op}}(S^{t_1},B) \tag{2.2}$$

with $\operatorname{Hom}_{S^{op}}(S^{t_i},B) \cong B^{t_i}$ finitely presented left R-modules for $i = 0,1$.

(1) If $\operatorname{Hom}_{S^{op}}(A,B)$ is a finitely generated left R-module, then $\operatorname{Im}(\operatorname{Hom}_{S^{op}}(f_0,B))$ and $\operatorname{Coker}(\operatorname{Hom}_{S^{op}}(f_0,B))$ are finitely presented left R-modules by [3, Proposition 1.6(ii)]. Thus for any FP-injective left R-module E, applying the functor $\operatorname{Hom}_R(-,E)$ to (2.2) yields the following exact sequence of abelian groups:

$$\operatorname{Hom}_R(\operatorname{Hom}_{S^{op}}(S^{t_1},B),E) \to \operatorname{Hom}_R(\operatorname{Hom}_{S^{op}}(S^{t_0},B),E) \to \operatorname{Hom}_R(\operatorname{Hom}_{S^{op}}(A,B),E) \to 0.$$

By [20, Lemma 3.55(i)], there exists the following commutative diagram

$$\begin{array}{ccccccc}
S^{t_1} \otimes_S \operatorname{Hom}_R(B,E) & \longrightarrow & S^{t_0} \otimes_S \operatorname{Hom}_R(B,E) & \longrightarrow & A \otimes_S \operatorname{Hom}_R(B,E) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \tau_{S^{t_1},B,E}} & & \downarrow{\scriptstyle \tau_{S^{t_0},B,E}} & & \downarrow{\scriptstyle \tau_{A,B,E}} & & \\
\operatorname{Hom}_R(\operatorname{Hom}_{S^{op}}(S^{t_1},B),E) & \longrightarrow & \operatorname{Hom}_R(\operatorname{Hom}_{S^{op}}(S^{t_0},B),E) & \longrightarrow & \operatorname{Hom}_R(\operatorname{Hom}_{S^{op}}(A,B),E) & \longrightarrow & 0
\end{array}$$

with both $\tau_{S^{t_0},B,E}$ and $\tau_{S^{t_1},B,E}$ being isomorphisms of abelian groups. So $\tau_{A,B,E}$ is also an isomorphisms of abelian groups. Notice that A is a (T,S)-bimodule, so $\tau_{A,B,E}$ is an isomorphism of left T-modules.

(2) If R is a left coherent ring, then $\operatorname{Im}(\operatorname{Hom}_{S^{op}}(f_0,B))$ is a finitely generated left R-submodule of the finitely presented left R-module $\operatorname{Hom}_{S^{op}}(S^{t_1},B)$, and so $\operatorname{Im}(\operatorname{Hom}_{S^{op}}(f_0,B))$ is a finitely presented left R-module. It follows from [3, Proposition 1.6(i)] that $\operatorname{Hom}_{S^{op}}(A,B)$ is a finitely generated left R-submodule of the finitely presented left R-module $\operatorname{Hom}_{S^{op}}(S^{t_0},B)$, and hence $\operatorname{Hom}_{S^{op}}(A,B)$ is a finitely presented left R-module.

Assume that there exists an exact sequence as in (2.1). We prove the latter assertion by induction on n. The case for $n = 0$ follows from the former assertion. Suppose $n \geq 1$ and set $K_1 := \operatorname{Im} f_0$. Then we get an exact sequence

$$\operatorname{Hom}_{S^{op}}(S^{t_0},B) \to \operatorname{Hom}_{S^{op}}(K_1,B) \to \operatorname{Ext}^1_{S^{op}}(A,B) \to 0$$

and an isomorphism

$$\operatorname{Ext}^{i+1}_{S^{op}}(A,B) \cong \operatorname{Ext}^i_{S^{op}}(K_1,B)$$

in $\operatorname{Mod} R$ for any $i \geq 1$. Now the assertion follows easily from the induction hypothesis. □

The following result is a generalization of [20, Theorem 10.66].

Lemma 3.2. *Let R, S, T be arbitrary rings and consider the situation $({}_TA_S, {}_RB_S)$ such that ${}_RB$ is finitely presented and there exists an exact sequence as in (2.1). If R is a left coherent ring, then for any FP-injective left R-module E, there exists a natural isomorphism*

$$\mathrm{Tor}_i^S(A, \mathrm{Hom}_R(B, E)) \cong \mathrm{Hom}_R(\mathrm{Ext}^i_{S^{op}}(A, B), E)$$

in $\mathrm{Mod}\, T$ *for any* $0 \leq i \leq n$.

Proof. Applying Lemma 3.1, we get the assertion by using an argument similar to that in the proof of [20, Theorem 10.66]. □

It was shown in [14, Lemma 4.1] that any injective left R-module is in $\mathcal{B}_C(R)$. The assertion (1) in the following proposition extends this result.

Proposition 3.3. *Let R be a left coherent ring. Then we have*
(1) *Any* FP*-injective left R-module is in $\mathcal{B}_C(R)$.*
(2) *If $N \in \mathrm{Mod}\, S$ with $\mathcal{FI}_C(S)$-$\mathrm{id}\, N < \infty$, then $N \in \mathcal{A}_C(S)$.*

Proof. (1) Let E be an FP-injective left R-module. Then $\mathrm{Ext}_R^{\geq 1}(C, E) = 0$. By Lemma 3.2, we have

$$\mathrm{Tor}_i^S(C, E_*) \cong \mathrm{Hom}_R(\mathrm{Ext}^i_{S^{op}}(C, C), E) = 0$$

for any $i \geq 1$. Finally, consider the following sequence of left R-module homomorphisms:

$$C \otimes_S E_* \overset{\tau_{C,C,E}}{\longrightarrow} \mathrm{Hom}_R(C_*, E) = \mathrm{Hom}_R(R, E) \overset{\alpha}{\longrightarrow} E,$$

where α is the canonical evaluation homomorphism defined by $\alpha(h) = h(1_R)$ for any $h \in C_*$. It is well known that α is an isomorphism with the inverse $\beta : E \to \mathrm{Hom}_R(R, E)$ defined by $\beta(e)(r) = re$ for any $e \in E$ and $r \in R$. Note that the unit 1_R of R coincides with the identity homomorphism id_C of C_S. So, for any $x \in C$ and $f \in E_*$, we have

$$\alpha\tau_{C,C,E}(x \otimes f) = \tau_{C,C,E}(x \otimes f)(1_R) = \tau_{C,C,E}(x \otimes f)(\mathrm{id}_C) = f\,\mathrm{id}_C(x) = f(x),$$

which implies $\theta_E = \alpha\tau_{C,C,E}$. Since $\tau_{C,C,E}$ is an isomorphism by Lemma 3.1(1), it follows that θ_E is also an isomorphism. Thus we conclude that $E \in \mathcal{B}_C(R)$.

(2) Let $Q \in \mathcal{FI}_C(S)$. Then $Q = E_*$ for some FP-injective left R-module E. By (1) and [14, Proposition 4.1], we have $Q \in \mathcal{A}_C(S)$. Now the assertion follows from [14, Theorem 6.2]. □

Now we are in a position to prove the following result.

Theorem 3.4. *Let R be a left coherent ring and $N \in \mathrm{Mod}\, S$. Then* FP-$\mathrm{id}_R\, C \otimes_S N \leq \mathcal{FI}_C(S)$-$\mathrm{id}\, N$ *with equality when $N \in \mathcal{A}_C(S)$.*

Proof. Let $N \in \mathrm{Mod}\, S$ with $\mathcal{FI}_C(S)$-$\mathrm{id}\, N = n < \infty$. Then there exists an exact sequence

$$0 \to N \to {E^0}_* \to {E^1}_* \to \cdots \to {E^n}_* \to 0 \tag{2.3}$$

in $\mathrm{Mod}\, S$ with all E^i FP-injective left R-modules. By Proposition 3.3(2), we have ${E^i}_* \in \mathcal{A}_C(S)$ and $\mathrm{Tor}^S_{\geq 1}(C, {E^i}_*) = 0$ for any $0 \leq i \leq n$. Then applying the functor $C \otimes_S -$ to the exact sequence (2.3) yields the following exact sequence

$$0 \to C \otimes_S N \to C \otimes_S {E^0}_* \to C \otimes_S {E^1}_* \to \cdots \to C \otimes_S {E^n}_* \to 0$$

in $\operatorname{Mod} R$. By Proposition 3.3(1), we have that $E^i \in \mathcal{B}_C(R)$ and $C \otimes_S {E^i}_* \cong E^i$ is FP-injective for any $0 \leq i \leq n$. Thus FP-$\mathrm{id}_R\, C \otimes_S N \leq n$.

Now suppose $N \in \mathcal{A}_C(S)$. Then $N \cong (C \otimes_S N)_*$ and $\operatorname{Ext}_R^{\geq 1}(C, C \otimes_S N) = 0$. If FP-$\mathrm{id}_R\, C \otimes_S N = n < \infty$, then there exists an exact sequence

$$0 \to C \otimes_S N \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

in $\operatorname{Mod} R$ with all E^i FP-injective. Applying the functor $\operatorname{Hom}_R(C, -)$ to it yields the following exact sequence

$$0 \to (C \otimes_S N)_* (\cong N) \to {E^0}_* \to {E^1}_* \to \cdots \to {E^n}_* \to 0$$

in $\operatorname{Mod} S$ with all ${E^i}_* \in \mathcal{FI}_C(S)$, and so $\mathcal{FI}_C(S)$-$\mathrm{id}\, N \leq n$. □

We also need the following lemma.

Lemma 3.5.
(1) *For any $M \in \operatorname{Mod} R^{op}$, we have $(M \otimes_R C)^{++} \cong M^{++} \otimes_R C$.*
(2) *For any $N \in \operatorname{Mod} S$, we have $(C \otimes_S N)^{++} \cong C \otimes_S N^{++}$.*

Proof. (1) By [11, Lemma 2.16(a)(c)], we have $(M \otimes_R C)^{++} \cong [(M^+)_*]^+ \cong M^{++} \otimes_R C$.
Symmetrically, we get (2). □

The following observation is useful in the next section.

Proposition 3.6.
(1) *For any $N \in \operatorname{Mod} S$, if $N^+ \in \mathcal{F}_C(S^{op})$, then $N \in \mathcal{FI}_C(S)$.*
(2) *For any $N \in \operatorname{Mod} S$, if $N^{++} \in \mathcal{I}_C(S)$, then $N \in \mathcal{FI}_C(S)$.*
(3) *For any $Q \in \operatorname{Mod} S^{op}$, if $Q^{++} \in \mathcal{F}_C(S^{op})$, then $Q \in \mathcal{F}_C(S^{op})$.*

Proof. (1) Let $N^+ \in \mathcal{F}_C(S^{op})$. Then $N^+ \in \mathcal{B}_C(S^{op})$ and $N \in \mathcal{A}_C(S)$ by [14, Corollary 6.1] and Lemma 2.4(1). On the other hand, $N^{++} \in \mathcal{I}_C(S)$ by Lemma 2.2(1). Then $C \otimes_S N^{++}$ is an injective left R-module by [14, Lemma 5.1(c)]. Since $(C \otimes_S N)^{++} \cong C \otimes_S N^{++}$ by Lemma 3.5(2), it follows that $(C \otimes_S N)^{++}$ is also an injective left R-module. Notice that $C \otimes_S N$ is a pure submodule of $(C \otimes_S N)^{++}$ by [11, Corollary 2.21(b)], so $C \otimes_S N$ is an FP-injective left R-module by [18, Lemma 4]. Since $N \in \mathcal{A}_C(S)$, we have $N \cong (C \otimes_S N)_* \in \mathcal{FI}_C(S)$.

(2) It follows from Lemma 2.2(1) and (1).

(3) Let $Q \in \operatorname{Mod} S^{op}$. Note that Q is a pure submodule of Q^{++} by [8, Proposition 5.3.9]. Thus, if $Q^{++} \in \mathcal{F}_C(S^{op})$, then $Q \in \mathcal{F}_C(S^{op})$ by Lemma 2.2(2). □

4 Equivalent characterizations of several kinds of rings

In the following result, we give some equivalent characterizations for R being left coherent in terms of the C-FP-injectivity and flatness of character modules of certain left S-modules, in which the equivalence between (1) and (3) has been obtained in [23, Lemma 4.1] when ${}_R C_S$ is faithful.

Theorem 4.1. *The following statements are equivalent.*
(1) *R is a left coherent ring.*
(2) *$\mathcal{FI}_C(S)$-$\mathrm{id}\, N = \mathcal{F}_C(S^{op})$-$\mathrm{pd}\, N^+$ for any $N \in \operatorname{Mod} S$.*
(3) *A left S-module N is C-FP-injective (if and) only if N^+ is a C-flat right S-module.*
(4) *A left S-module N is C-FP-injective (if and) only if N^{++} is a C-injective left S-module.*

(5) *A right S-module Q is C-flat (if and) only if Q^{++} is a C-flat right S-module.*
(6) *If Q is a C-projective right S-module, then Q^{++} is a C-flat right S-module.*
(7) *$(C_S^{(I)})^{++}$ is a C-flat right S-module for any index set I.*

Proof. (1) $\Rightarrow$ (2) Let $N \in \operatorname{Mod} S$. Then for any finitely presented left R-module A and $i \geq 1$, we have

$$\operatorname{Tor}_i^R((N^+)_*, A) \cong \operatorname{Tor}_i^R((C \otimes_S N)^+, A) \cong [\operatorname{Ext}_R^i(A, C \otimes_S N)]^+$$

by [11, Lemma 2.16(a)(d)], and so $\operatorname{Tor}_i^R((N^+)_*, A) = 0$ if and only if $\operatorname{Ext}_R^i(A, C \otimes_S N) = 0$. It implies that

$$\operatorname{fd}_{R^{op}}(N^+)_* = \operatorname{FP-id}_R C \otimes_S N. \quad (3.1)$$

By Proposition 3.3(2) and Lemma 2.4(1), we have that if $\mathcal{FI}_C(S)$-$\operatorname{id} N < \infty$, then $N \in \mathcal{A}_C(S)$ and $N^+ \in \mathcal{B}_C(S^{op})$. On the other hand, by [14, Corollary 6.1] and Lemma 2.4(1), we have that if $\mathcal{F}_C(S^{op})$-$\operatorname{pd} N^+ < \infty$, then $N^+ \in \mathcal{B}_C(S^{op})$ and $N \in \mathcal{A}_C(S)$. Then for any $n \geq 0$, we have

$$\begin{aligned}
&\mathcal{FI}_C(S)\text{-}\operatorname{id} N = n \\
\Leftrightarrow\ &\operatorname{FP-id}_R C \otimes N = n \quad \text{(by Theorem 3.4)} \\
\Leftrightarrow\ &\operatorname{fd}_{R^{op}}(N^+)_* = n \quad \text{(by (3.1))} \\
\Leftrightarrow\ &\mathcal{F}_C(S^{op})\text{-}\operatorname{pd} N^+ = n. \quad \text{(by [25, Lemma 2.6(1)])}
\end{aligned}$$

The implications (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are trivial.

(3) $\Rightarrow$ (4) If $N^{++} \in \mathcal{I}_C(S)$, then $N^+ \in \mathcal{F}_C(S^{op})$ by Lemma 2.2(1), and hence $N \in \mathcal{FI}_C(S)$ by Proposition 3.6(1). Conversely, if $N \in \mathcal{FI}_C(S)$, then $N^+ \in \mathcal{F}_C(S^{op})$ by (3), and hence $N^{++} \in \mathcal{I}_C(S)$ by Lemma 2.2(1) again.

(4) $\Rightarrow$ (5) If $Q^{++} \in \mathcal{F}_C(S^{op})$, then $Q \in \mathcal{F}_C(S^{op})$ by Proposition 3.6(3). Conversely, if $Q \in \mathcal{F}_C(S^{op})$, then $Q^+ \in \mathcal{I}_C(S)$ by Lemma 2.2(1). Thus $Q^{+++} \in \mathcal{I}_C(S)$ by (4), and therefore $Q^{++} \in \mathcal{F}_C(S^{op})$ by Lemma 2.2(1) again.

(7) $\Rightarrow$ (1) By [6, Theorem 2.1], it suffices to prove that $(R_R)^I$ is a flat right R-module for any index set I. By (7), we have $[((C_S)^+)^I]^+ \cong (C_S^{(I)})^{++} \in \mathcal{F}_C(S^{op})$. Since there exists a pure monomorphism $\lambda : [(C_S)^+]^{(I)} \to [(C_S)^+]^I$ by [7, Lemma 1(1)], it follows from [8, Proposition 5.3.8] that λ^+ is a split epimorphism and $[(C_S)^{++}]^I (\cong [((C_S)^+)^{(I)}]^+)$ is a direct summand of $[((C_S)^+)^I]^+$. Then $[(C_S)^{++}]^I \in \mathcal{F}_C(S^{op})$ by [14, Proposition 5.1(a)]. By [8, Theorem 3.2.22] and Lemma 3.5(1), we have

$$[(R_R)^{++}]^I \otimes_R C \cong [(R_R)^{++} \otimes_R C]^I \cong [(R \otimes_R C)^{++}]^I \cong [(C_S)^{++}]^I \in \mathcal{F}_C(S^{op}).$$

Since $R_R \in \mathcal{A}_C(R^{op})$, both $(R_R)^{++}$ and $[(R_R)^{++}]^I$ are in $\mathcal{A}_C(R^{op})$ by Lemma 2.4 and [14, Proposition 4.2(a)]. So $[(R_R)^{++}]^I \cong ([(R_R)^{++}]^I \otimes_R C)_*$ is a flat right R-module by [25, Lemma 2.6(1)]. Since R_R is a pure submodule of $(R_R)^{++}$ by [8, Proposition 5.3.9], it follows from [7, Lemma 1(2)] that $(R_R)^I$ is a pure submodule of $[(R_R)^{++}]^I$, and hence $(R_R)^I$ is also a flat right R-module. □

We need the following lemma.

Lemma 4.2. *For any $U \in \mathcal{FI}_C(S)$, there exists a module $N \in \mathcal{I}_C(S)$ such that U^+ is a direct summand of N^+.*

Proof. Let $U \in \mathcal{FI}_C(S)$ such that $U = E_*$ with E being FP-injective in $\operatorname{Mod} R$. Then there exists a pure exact sequence

$$0 \to E \to I \to L \to 0$$

in $\operatorname{Mod} R$ with I injective. By [8, Proposition 5.3.8], the induced exact sequence

$$0 \to L^+ \to I^+ \to E^+ \to 0$$

in $\operatorname{Mod} R^{op}$ splits and E^+ is a direct summand of I^+. Then $E^+ \otimes_R C$ is a direct summand of $I^+ \otimes_R C$. By [11, Lemma 2.16(c)], we have

$$U^+ = (E_*)^+ \cong E^+ \otimes_R C \text{ and } (I_*)^+ \cong I^+ \otimes_R C.$$

Thus $U^+ (\cong E^+ \otimes_R C)$ is a direct summand of $(I_*)^+ (\cong I^+ \otimes_R C)$. □

We give some equivalent characterizations for R being left Noetherian in terms of the C-injectivity and flatness of character modules of certain left S-modules as follows.

Theorem 4.3. *The following statements are equivalent.*
(1) *R is a left Noetherian ring.*
(2) $\mathcal{I}_C(S)$-id $N = \mathcal{F}_C(S^{op})$-pd N^+ *for any* $N \in \operatorname{Mod} S$.
(3) *A left S-module N is C-injective if and only if N^+ is a C-flat right S-module.*
(4) *A left S-module N is C-injective if and only if N^{++} is a C-injective left S-module.*

Proof. (1) ⇒ (2) Let R be a left Noetherian ring. Then a left R-module is FP-injective if and only if it is injective, and so a left S-module is C-FP-injective if and only if it is C-injective. Thus the assertion follows from Theorem 4.1.

(2) ⇒ (3) It is trivial.

By Lemma 2.2(1), we have that for a left S-module N, $N^+ \in \mathcal{F}_C(S^{op})$ if and only if $N^{++} \in \mathcal{I}_C(S)$. Thus the assertion (3) ⇔ (4) follows.

(3) ⇒ (1) Let $U \in \mathcal{FI}_C(S)$. By Lemma 4.2, there exists a module $N \in \mathcal{I}_C(S)$ such that U^+ is a direct summand of N^+. Then $U^+ \in \mathcal{F}_C(S^{op})$ by (3) and [14, Proposition 5.1(a)]. Thus R is a left coherent ring by Theorem 4.1.

To prove that R is a left Noetherian ring, it suffices to prove that the class of injective left R-modules is closed under direct sums by [4, Theorem 2.1]. Let $\{E_i \mid i \in I\}$ be a family of injective left R-modules with I any index set. By [11, Lemma 2.7], we have

$$[(\oplus_{i\in I} E_i)_*]^+ \cong [\oplus_{i\in I}(E_i)_*]^+ \cong \Pi_{i\in I}[(E_i)_*]^+.$$

Since R is a left coherent ring and since all $[(E_i)_*]^+$ are in $\mathcal{F}_C(S^{op})$ by (3), we have that $\Pi_{i\in I}[(E_i)_*]^+$, and hence $[(\oplus_{i\in I} E_i)_*]^+$, is also in $\mathcal{F}_C(S^{op})$ by [14, Proposition 5.1(a)]. Then $(\oplus_{i\in I} E_i)_* \in \mathcal{I}_C(S)$ by (3) again. Since all E_i are in $\mathcal{B}_C(R)$, we have $\oplus_{i\in I} E_i \in \mathcal{B}_C(R)$ by [14, Proposition 4.2(a)]. It follows from [14, Lemma 5.1(c)] that $\oplus_{i\in I} E_i \cong C \otimes_S (\oplus_{i\in I} E_i)_*$ is an injective left R-module. □

As a consequence of Theorems 4.1 and 4.3, we get the following corollary, which generalizes [14, Lemma 5.2(c)].

Corollary 4.4.
(1) *Let R be a left coherent ring and $n \geq 0$. Then the subcategory of* $\operatorname{Mod} S$ *consisting of modules N with* $\mathcal{FI}_C(S)$-id $N \leq n$ *is closed pure submodules and pure quotients.*

(2) *Let R be a left Noetherian ring and $n \geq 0$. Then the subcategory of* $\operatorname{Mod} S$ *consisting of modules N with $\mathcal{I}_C(S)$-id $N \leq n$ is closed pure submodules and pure quotients.*

Proof. (1) Let

$$0 \to K \to N \to L \to 0$$

be a pure exact sequence in $\operatorname{Mod} S$ with $\mathcal{FI}_C(S)$-pd $N \leq n$. Then by [8, Proposition 5.3.8], the induced exact sequence

$$0 \to L^+ \to N^+ \to K^+ \to 0$$

in $\operatorname{Mod} S^{op}$ splits and both K^+ and L^+ are direct summands of N^+. By Theorem 4.1, we have $\mathcal{F}_C(S^{op})$-pd $N^+ \leq n$. Since $\mathcal{F}_C(S^{op})$ is closed under direct summands by [14, Proposition 5.1(a)], the class of right S-modules with $\mathcal{F}_C(S^{op})$-projective dimension at most n is closed under direct summands by [16, Corollary 3.9]. It follows that $\mathcal{F}_C(S^{op})$-pd $K^+ \leq n$ and $\mathcal{F}_C(S^{op})$-pd $L^+ \leq n$. Thus $\mathcal{FI}_C(S)$-pd $K \leq n$ and $\mathcal{FI}_C(S)$-pd $L \leq n$ by Theorem 4.1 again.

(2) From the proof of (1) $\Rightarrow$ (2) in Theorem 4.3, we know that if R is a left Noetherian ring, then $\mathcal{FI}_C(S) = \mathcal{I}_C(S)$. Now the assertion follows from (1). □

In the following result, we give some equivalent characterizations for R being left coherent and right perfect in terms of the C-FP-injectivity and projectivity of character modules of certain left S-modules.

Theorem 4.5. *The following statements are equivalent.*

(1) *R is a left coherent and right perfect ring.*
(2) *$\mathcal{FI}_C(S)$-id $N = \mathcal{P}_C(S^{op})$-pd N^+ for any $N \in \operatorname{Mod} S$.*
(3) *A left S-module N is C-FP-injective (if and) only if N^+ is a C-projective right S-module.*
(4) *A right S-module Q is C-flat (if and) only if Q^{++} is a C-projective right S-module.*
(5) *If Q is a C-projective right S-module, then Q^{++} is a C-projective right S-module.*
(6) *$(C_S^{(I)})^{++}$ is a C-projective right S-module for any index set I.*

Proof. (1) $\Rightarrow$ (2) Let R be a left coherent and right perfect ring. Then a right R-module is flat and only if it is projective by [1, Theorem 28.4], and hence $\mathcal{F}_C(S^{op}) = \mathcal{P}_C(S^{op})$. Thus the assertion follows from Theorem 4.1.

The implications (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are trivial.

(3) $\Rightarrow$ (4) If $Q^{++} \in \mathcal{P}_C(S^{op})$, then $Q \in \mathcal{F}_C(S^{op})$ by Proposition 3.6(3). Conversely, if $Q \in \mathcal{F}_C(S^{op})$, then $Q^+ \in \mathcal{I}_C(S)$ by Lemma 2.2(1), and hence $Q^{++} \in \mathcal{P}_C(S^{op})$ by (3).

(6) $\Rightarrow$ (1) It follows from (6) and Theorem 4.1 that R is a left coherent ring. Let I be an infinite set such that its cardinality is greater than the cardinality of R. By using an argument similar to that in the proof (7) $\Rightarrow$ (1) in Theorem 4.1, we get that $[(R_R)^{++}]^I$ is a projective right R-module and $(R_R)^I$ is a pure submodule of $[(R_R)^{++}]^I$, and hence $(R_R)^I$ is a pure submodule of a free right R-module. It follows from [6, Theorems 3.1 and 3.2] that R is a right perfect ring. □

Observe from [1, Corollary 15.23 and Theorem 28.4] that R is a left Artinian ring if and only if R is a left Noetherian and right (or left) perfect ring. Finally, we give some equivalent characterizations for R being left Artinian in terms of the C-injectivity and projectivity of character modules of certain left S-modules as follows.

Theorem 4.6. *The following statements are equivalent.*

(1) *R is a left Artinian ring.*
(2) *$\mathcal{I}_C(S)$-id $N = \mathcal{P}_C(S^{op})$-pd N^+ for any $N \in \operatorname{Mod} S$.*

(3) *A left S-module N is C-injective if and only if N^+ is a C-projective right S-module.*

Proof. The implication (2) $\Rightarrow$ (3) is trivial.

If R is a left Artinian ring, then $\mathcal{FI}_C(S) = \mathcal{I}_C(S)$ and $\mathcal{F}_C(S^{op}) = \mathcal{P}_C(S^{op})$. Thus the implication (1) $\Rightarrow$ (2) follows from Theorems 4.3 and 4.5.

(3) $\Rightarrow$ (1) Let E be an FP-injective left R-module. Then by Lemma 4.2, there exists an injective left R-module I such that $(E_*)^+$ is a direct summand of $(I_*)^+$. So $(E_*)^+ \in \mathcal{P}_C(S^{op})$ by (3) and [14, Proposition 5.1(b)], and hence R is a left coherent and right perfect ring by Theorem 4.5.

On the other hand, $E_* \in \mathcal{I}_C(S)$ by (3) again. It follows from Lemma 3.1 and [14, Lemma 5.1(c)] that $E \cong C \otimes_S E_*$ is an injective left R-module. Then R is left Noetherian ring by [18, Theorem 3]. Thus we conclude that R is a left Noetherian and right perfect ring, and hence a left Artinian ring. □

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