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HOMOLOGICAL ASPECTS OF THE ADJOINT COTRANSPOSE

BY

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Abstract. Let R and S be rings and ${}_R\omega_S$ a semidualizing bimodule. We introduce and study the adjoint cotransposes of modules and adjoint n-ω-cotorsionfree modules. We show that the Auslander class with respect to ${}_R\omega_S$ is the intersection of the class of adjoint ∞-ω-cotorsionfree modules and the right Tor-orthogonal class of ω_S. As a consequence, the classes of adjoint ∞-ω-cotorsionfree modules and of ∞-ω-cotorsionfree modules are equivalent under Foxby equivalence if and only if they coincide with the Auslander and Bass classes with respect to ω respectively. Moreover, we give some equivalent characterizations when the left and right projective dimensions of ${}_R\omega_S$ are finite in terms of the properties of (adjoint) ∞-ω-cotorsionfree modules.

1. Introduction. One of the most powerful tools of Auslander–Reiten theory in representation theory of artin algebras and in homological algebra is the Auslander transpose [ASS, AB, ARS]. In [TH1] we dualized it and introduced the notion of cotransposes of modules with respect to a semidualizing bimodule ${}_R\omega_S$ by applying the functor $\mathrm{Hom}_R(\omega, -)$ to minimal injective resolutions of left R-modules; and we showed that many results about the Auslander transpose have dual counterparts [TH1, TH2]. The motivation of this paper comes from the fact that $(\omega \otimes_S -, \mathrm{Hom}_R(\omega, -))$ naturally forms an adjoint pair. It is interesting to study what will happen if we apply the functor $\omega \otimes_S -$ to minimal flat resolutions of left S-modules. To this end, we introduce and study the so-called adjoint cotransposes of modules with respect to ${}_R\omega_S$. We show that many results about cotransposes of modules have adjoint counterparts. The paper is organized as follows.

In Section 2, we give some terminology and preliminary results.

Let R and S be rings and ${}_R\omega_S$ a semidualizing bimodule. In Section 3, as adjoint counterparts of cotransposes with respect to ${}_R\omega_S$ and n-ω-cotorsionfree modules of [TH1], we introduce the notions of adjoint cotransposes of modules with respect to ${}_R\omega_S$ and adjoint n-ω-cotorsionfree modules. We prove that the Auslander class with respect to ${}_R\omega_S$ is the intersection of

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the class of adjoint ∞-ω-cotorsionfree modules and the right Tor-orthogonal class of ω_S, which generalizes a result of Enochs and Holm [EH, Proposition 3.6]. As a consequence, the class of adjoint ∞-ω-cotorsionfree modules and that of ∞-ω-cotorsionfree modules are equivalent under Foxby equivalence if and only if they coincide with the Auslander and Bass classes with respect to ω respectively. Moreover, we prove that left S-modules with finite relative projective dimension with respect to adjoint ∞-ω-cotorsionfree modules are kernels and cokernels of homomorphisms from left S-modules with finite relative projective dimension with respect to ω-injective modules to adjoint ∞-ω-cotorsionfree modules.

In parallel to the contributions of the torsionfree dimensions of modules to the theory of Gorenstein rings [HH, Theorem 1.4], as applications of the results obtained in the previous section, we give in Section 4 some equivalent characterizations when the left and right projective dimensions of ${}_R\omega_S$ are finite in terms of the properties of the (adjoint) ∞-ω-cotorsionfree dimensions of modules. For any $n \geq 0$, we prove that the left and right projective dimensions of ${}_R\omega_S$ are at most n if and only if the ∞-ω-cotorsionfree injective dimensions of (finitely presented) left R-modules and (finitely presented) right S-modules are at most n; and these are equivalent to the adjoint ∞-ω-cotorsionfree projective dimensions of (finitely presented) right R-modules and (finitely presented) left S-modules being at most n when R and S are artin algebras.

2. Preliminaries. Throughout this paper, all rings are associative rings with unit. Let R be a ring. We use $\operatorname{Mod} R$ (resp. $\operatorname{Mod} R^{\mathrm{op}}$) to denote the category of left (resp. right) R-modules, and $\operatorname{mod} R$ (resp. $\operatorname{mod} R^{\mathrm{op}}$) to denote the category of finitely presented left (resp. right) R-modules. Let $M \in \operatorname{Mod} R$. We use $\operatorname{Add}_R M$ (resp. $\operatorname{add}_R M$) to denote the subcategory of $\operatorname{Mod} R$ consisting of all direct summands of direct sums (resp. finite direct sums) of copies of M.

Let $\mathcal{X}$ be a full subcategory of $\operatorname{Mod} R$. We write

$$\mathcal{X}^{\perp} := \{M \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geq 1}(X, M) = 0\},$$
$${}^{\perp}\mathcal{X} := \{M \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geq 1}(M, X) = 0\}.$$

A sequence

$$\mathbb{M} := \cdots \to M_1 \to M_2 \to M_3 \to \cdots$$

in $\operatorname{Mod} R$ is called $\operatorname{Hom}_R(\mathcal{X}, -)$-*exact* (resp. $\operatorname{Hom}_R(-, \mathcal{X})$-*exact*) if $\operatorname{Hom}_R(X, \mathbb{M})$ (resp. $\operatorname{Hom}_R(\mathbb{M}, X)$) is exact for any $X \in \mathcal{X}$. An exact sequence (of finite or infinite length)

$$\cdots \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

in $\operatorname{Mod} R$ is called an $\mathcal{X}$-*resolution* of M if all X_i are in $\mathcal{X}$. The $\mathcal{X}$-*projective* *dimension* $\mathcal{X}\text{-pd}_R M$ of M is defined as the infimum of n such that there exists an $\mathcal{X}$-resolution

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

of M in $\operatorname{Mod} R$. Dually, the notions of an $\mathcal{X}$*-coresolution* and the $\mathcal{X}$*-injective dimension* $\mathcal{X}\text{-id}_R M$ of M are defined. In particular, we use $\operatorname{pd}_R M$, $\operatorname{fd}_R M$ and $\operatorname{id}_R M$ to denote the projective, flat and injective dimensions of M respectively. We also write

$$\mathcal{X}\text{-pd}^{<\infty}(R) := \{M \in \operatorname{Mod} R \mid \mathcal{X}\text{-pd}_R M < \infty\},$$
$$\mathcal{X}\text{-id}^{<\infty}(R) := \{M \in \operatorname{Mod} R \mid \mathcal{X}\text{-id}_R M < \infty\}.$$

We first give the following

DEFINITION 2.1 ([HW]). Let R and S be rings. An (R-S)-bimodule ${}_R\omega_S$ is called *semidualizing* if

(a1) ${}_R\omega$ admits a degreewise finite R-projective resolution.
(a2) ω_S admits a degreewise finite S-projective resolution.
(b1) The homothety map ${}_RR_R \xrightarrow{{}_R\gamma} \operatorname{Hom}_{S^{\mathrm{op}}}(\omega,\omega)$ is an isomorphism.
(b2) The homothety map ${}_SS_S \xrightarrow{\gamma_S} \operatorname{Hom}_R(\omega,\omega)$ is an isomorphism.
(c1) $\operatorname{Ext}_R^{\geq 1}(\omega,\omega) = 0$, that is, ${}_R\omega$ is *self-orthogonal.*
(c2) $\operatorname{Ext}_{S^{\mathrm{op}}}^{\geq 1}(\omega,\omega) = 0$, that is, ω_S is *self-orthogonal.*

Typical examples of semidualizing bimodules include the free module of rank one, dualizing modules over a Cohen-Macaulay local ring.

From now on, R and S are arbitrary rings and we fix a semidualizing bimodule ${}_R\omega_S$. For convenience, we write $(-)_* := \operatorname{Hom}(\omega, -)$, and

$${}_R\omega^{\perp} := \{M \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{i\geq 1}(\omega, M) = 0\},$$
$${\omega_S}^{\top} := \{N \in \operatorname{Mod} S \mid \operatorname{Tor}^S_{i\geq 1}(\omega, N) = 0\}.$$

Following [HW], set

$$\mathcal{F}_\omega(R) := \{\omega \otimes_S F \mid F \text{ is flat in } \operatorname{Mod} S\},$$
$$\mathcal{P}_\omega(R) := \{\omega \otimes_S P \mid P \text{ is projective in } \operatorname{Mod} S\},$$
$$\mathcal{I}_\omega(S) := \{I_* \mid I \text{ is injective in } \operatorname{Mod} R\}.$$

The modules in $\mathcal{F}_\omega(R)$, $\mathcal{P}_\omega(R)$ and $\mathcal{I}_\omega(S)$ are called ω*-flat*, ω*-projective* and ω*-injective* respectively. Symmetrically, the classes of $\mathcal{F}_\omega(S^{\mathrm{op}})$, $\mathcal{P}_\omega(S^{\mathrm{op}})$ and $\mathcal{I}_\omega(R^{\mathrm{op}})$ are defined. Let $M \in \operatorname{Mod} R$ and $N \in \operatorname{Mod} S$. Then we have two canonical valuation homomorphisms:

$$\theta_M : \omega \otimes_S M_* \to M$$

defined by $\theta_M(x \otimes f) = f(x)$ for any $x \in \omega$ and $f \in M_*$, and

$$\mu_N : N \to (\omega \otimes_S N)_*$$

defined by $\mu_N(y)(x) = x \otimes y$ for any $y \in N$ and $x \in \omega$.

Definition 2.2 ([HW]).

(1) The *Auslander class* $\mathcal{A}_\omega(S)$ with respect to ω consists of all left S-modules N satisfying the following conditions:

(A1) $N \in {\omega_S}^\top$.
(A2) $\omega \otimes_S N \in {}_R\omega^\perp$.
(A3) μ_N is an isomorphism in $\operatorname{Mod} S$.

(2) The *Bass class* $\mathcal{B}_\omega(R)$ with respect to ω consists of all left R-modules M satisfying the following conditions:

(B1) $M \in {}_R\omega^\perp$.
(B2) $M_* \in {\omega_S}^\top$.
(B3) θ_M is an isomorphism in $\operatorname{Mod} R$.

Let $\mathcal{I}(R)$ be the subcategory of $\operatorname{Mod} R$ consisting of all injective modules.

Lemma 2.3 ([TH3, Lemma 2.5]).

(1) $\mathcal{I}(R) \cup \mathcal{F}_\omega(R)\text{-pd}^{<\infty}(R) \subseteq \mathcal{B}_\omega(R) \subseteq {}_R\omega^\perp = \mathcal{P}_\omega(R)^\perp$.
(2) $\mathcal{I}_\omega(R^{\mathrm{op}}) \subseteq {}^\perp\mathcal{I}_\omega(R^{\mathrm{op}})$ *and* $\mathcal{I}_\omega(S) \subseteq {}^\perp\mathcal{I}_\omega(S)$.

Let $M \in \operatorname{Mod} R$. We use

$$0 \to M \to I^0(M) \xrightarrow{f^0} I^1(M) \xrightarrow{f^1} \cdots \xrightarrow{f^{i-1}} I^i(M) \xrightarrow{f^i} \cdots$$

to denote a minimal injective resolution of M.

Definition 2.4 ([TH1]). Let $M \in \operatorname{Mod} R$ and $n \geq 1$.

(1) $\mathrm{cTr}_\omega M := \operatorname{Coker} {f^0}_*$ is called the *cotranspose* of M with respect to ${}_R\omega_S$.
(2) M is called *n-ω-cotorsionfree* if $\operatorname{Tor}^S_{1\leq i\leq n}(\omega, \mathrm{cTr}_\omega M) = 0$; and M is *∞-ω-cotorsionfree* if it is n-ω-cotorsionfree for all n. In particular, every module in $\operatorname{Mod} R$ is 0-ω-cotorsionfree.

We use $\mathrm{c}\mathcal{T}(R)$ to denote the subcategory of $\operatorname{Mod} R$ consisting of all ∞-ω-cotorsionfree modules.

3. Adjoint cotransposes of modules. Recall from [E] that a homomorphism $f : F \to N$ in $\operatorname{Mod} S$ with F flat is called a *flat cover* of N if $\operatorname{Hom}_S(F', f)$ is epic for any flat module F' in $\operatorname{Mod} S$, and an endomorphism $h : F \to F$ is an automorphism whenever $f = fh$. Let $N \in \operatorname{Mod} S$. Bican, El Bashir and Enochs [BBE] proved that N has a flat cover. We use

$$\cdots \xrightarrow{f_{n+1}} F_n(N) \xrightarrow{f_n} \cdots \xrightarrow{f_2} F_1(N) \xrightarrow{f_1} F_0(N) \xrightarrow{f_0} N \to 0$$

to denote a minimal flat resolution of N in $\operatorname{Mod} S$, where each $F_i(N) \to \operatorname{Im} f_i$ is a flat cover of $\operatorname{Im} f_i$. Note that $(\omega \otimes_S -, \operatorname{Hom}_R(\omega, -))$ is an adjoint pair. In view of Definition 2.4, we make the following

DEFINITION 3.1. Let $N \in \operatorname{Mod} S$ and $n \geq 1$.

(1) $\operatorname{acTr}_\omega N := \operatorname{Ker}(1_\omega \otimes f_1)$ is called the *adjoint cotranspose* of N with respect to ${}_R\omega_S$.
(2) N is called *adjoint n-ω-cotorsionfree* if $\operatorname{Ext}_R^{1\leq i\leq n}(\omega, \operatorname{acTr}_\omega N) = 0$; and N is *adjoint ∞-ω-cotorsionfree* if it is adjoint n-ω-cotorsionfree for all n. In particular, every left S-module is adjoint 0-ω-cotorsionfree.

Let $\operatorname{ac}\mathcal{T}(S)$ denote the subcategory of $\operatorname{Mod} S$ consisting of all adjoint ∞-ω-cotorsionfree modules. The following result is an adjoint counterpart of [TH1, Proposition 3.2].

PROPOSITION 3.2. *Let $N \in \operatorname{Mod} S$. Then there exists an exact sequence*

$$0 \to \operatorname{Ext}_R^1(\omega, \operatorname{acTr}_\omega N) \to N \xrightarrow{\mu_N} (\omega \otimes_S N)_* \to \operatorname{Ext}_R^2(\omega, \operatorname{acTr}_\omega N) \to 0.$$

Proof. Let $N \in \operatorname{Mod} S$. Then by [HW, Lemma 4.1], both $F_0(N)$ and $F_1(N)$ are in $\mathcal{A}_\omega(S)$, and so both $\mu_{F_0(N)}$ and $\mu_{F_1(N)}$ are isomorphisms. We also have an exact sequence

$$\omega \otimes_S F_1(N) \xrightarrow{1_\omega \otimes f_1} \omega \otimes_S F_0(N) \xrightarrow{1_\omega \otimes f_0} \omega \otimes_S N \to 0$$

in $\operatorname{Mod} R$ with both $\omega \otimes_S F_1(N)$ and $\omega \otimes_S F_0(N)$ in $\mathcal{F}_\omega(R)$. By Lemma 2.3(1), both $\omega \otimes_S F_0(N)$ and $\omega \otimes_S F_1(N)$ are in $\omega^\perp$. Now we get the desired exact sequence from [TH2, Proposition 6.7]. ■

By Proposition 3.2 and the definition of adjoint n-ω-cotorsionfree modules, we immediately have

COROLLARY 3.3. *Let $N \in \operatorname{Mod} S$.*

(1) *N is adjoint 1-ω-cotorsionfree if and only if μ_N is a monomorphism.*
(2) *N is adjoint 2-ω-cotorsionfree if and only if μ_N is an isomorphism.*
(3) *For $n \geq 3$, N is adjoint n-ω-cotorsionfree if and only if μ_N is an isomorphism and $\operatorname{Ext}_R^{1\leq i\leq n-2}(\omega, \omega \otimes_S N) = 0$.*

The following result gives an alternative description of the Auslander class, which is the adjoint counterpart of a characterization of the Bass class [TH1, Theorem 3.9].

PROPOSITION 3.4. $\mathcal{A}_\omega(S) = \operatorname{ac}\mathcal{T}(S) \cap {\omega_S}^\top$.

Proof. This follows from Corollary 3.3(3). ■

Let $\mathcal{F}(S)$ denote the subcategory of $\operatorname{Mod} S$ consisting of all flat modules. Compare the following result with Lemma 2.3(1).

COROLLARY 3.5.

(1) $\mathcal{F}(S)\cup\mathcal{I}_\omega(S)\text{-id}^{<\infty}(S)\subseteq \mathrm{ac}\mathcal{T}(S)$.
(2) $\mathcal{F}(S)\cup\mathcal{I}_\omega(S)\text{-id}^{<\infty}(S)\subseteq \mathcal{A}_\omega(S)\subseteq {\omega_S}^{\top}={}^{\perp}\mathcal{I}_\omega(S)$.

Proof. By [HW, Lemma 4.1 and Corollary 6.1] and Proposition 3.4, we have

$$\mathcal{F}(S)\cup\mathcal{I}_\omega(S)\text{-id}^{<\infty}(S)\subseteq \mathcal{A}_\omega(S)=\mathrm{ac}\mathcal{T}(S)\cap{\omega_S}^{\top},$$

and the first assertion follows.

By [GT, Lemma 2.16(b)], for any injective module $I\in\operatorname{Mod}R$ and $i\geq 1$, we have the following isomorphism of functors:

$$\operatorname{Hom}_R(\operatorname{Tor}_i^S(\omega,-),I)\cong\operatorname{Ext}_S^i(-,I_*).$$

Now by the definition of $\mathcal{I}_\omega(S)$, we have ${\omega_S}^{\top}={}^{\perp}\mathcal{I}_\omega(S)$, and the second assertion follows. ∎

Let

$$\mathbb{N}:=\cdots\to N_1\to N_2\to N_3\to\cdots$$

be a sequence in $\operatorname{Mod}S$. Then $\mathbb{N}$ is called $(\omega\otimes_S-)$-*exact* if $\omega\otimes_S\mathbb{N}$ is exact. We have the following easy observation.

OBSERVATION. *A sequence $\mathbb{S}$ in* $\operatorname{Mod}S$ *is $(\omega\otimes_S-)$-exact if and only if it is* $\operatorname{Hom}_S(-,\mathcal{I}_\omega(S))$*-exact.*

Proof. By the adjoint isomorphism theorem, for any injective module $I\in\operatorname{Mod}R$ we have the following isomorphism of functors:

$$\operatorname{Hom}_R(\omega\otimes_S-,I)\cong\operatorname{Hom}_S(-,I_*).$$

Now the assertion follows directly from the definition of $\mathcal{I}_\omega(S)$. ∎

The following result is an adjoint counterpart of [TH1, Proposition 3.5].

PROPOSITION 3.6. *Let $n\geq 1$, and let*

$$0\to L\to M\to N\to 0$$

be an $(\omega\otimes_S-)$-exact (equivalently $\operatorname{Hom}_S(-,\mathcal{I}_\omega(S))$*-exact) exact sequence in* $\operatorname{Mod}S$ *with N adjoint n-ω-cotorsionfree. Then L is adjoint n-ω-cotorsionfree if and only if so is M.*

Proof. By assumption we have an exact sequence

$$0\to\omega\otimes_S L\to\omega\otimes_S M\to\omega\otimes_S N\to 0$$

in $\operatorname{Mod}R$. Then we get the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\mu_L} & & \downarrow{\scriptstyle\mu_M} & & \downarrow{\scriptstyle\mu_N} & & \\
0 & \longrightarrow & (\omega\otimes_S L)_* & \longrightarrow & (\omega\otimes_S M)_* & \longrightarrow & (\omega\otimes_S N)_* & &
\end{array}$$

and the exact sequence

$$\mathrm{Ext}_R^{i-1}(\omega, \omega \otimes_S N) \to \mathrm{Ext}_R^{i}(\omega, \omega \otimes_S L) \to \mathrm{Ext}_R^{i}(\omega, \omega \otimes_S M) \\ \to \mathrm{Ext}_R^{i}(\omega, \omega \otimes_S N)$$

for any $i \geq 2$. Now the assertion follows easily from Corollary 3.3 and the snake lemma. ■

Next, we will give an equivalent characterization of adjoint n-ω-cotorsionfree modules in terms of special $\mathcal{I}_\omega(S)$-coresolutions of modules. First we prove

LEMMA 3.7. *Let* $N \in \mathrm{Mod}\, S$.

(1) N *is adjoint* 1-ω-*cotorsionfree if and only if there exists an* $(\omega \otimes_S -)$-*exact* (*equivalently* $\mathrm{Hom}_S(-, \mathcal{I}_\omega(S))$-*exact*) *exact sequence*

$$0 \to N \to U^0$$

in $\mathrm{Mod}\, S$ *with* $U^0 \in \mathcal{I}_\omega(S)$.

(2) M *is adjoint* 2-ω-*cotorsionfree if and only if there exists an* $(\omega \otimes_S -)$-*exact* (*equivalently* $\mathrm{Hom}_S(-, \mathcal{I}_\omega(S))$-*exact*) *exact sequence*

$$0 \to N \to U^0 \to U^1$$

in $\mathrm{Mod}\, S$ *with* $U^0, U^1 \in \mathcal{I}_\omega(S)$.

Proof. (1) Let $N \in \mathrm{Mod}\, S$ be adjoint 1-ω-cotorsionfree. Then μ_N is monic by Corollary 3.3(1). Since there exists a monomorphism $f : \omega \otimes_S N \rightarrowtail I^0$ in $\mathrm{Mod}\, R$ with I^0 injective, we get a monomorphism $f_* : (\omega \otimes_S N)_* \rightarrowtail {I^0}_*$ in $\mathrm{Mod}\, S$ with ${I^0}_* \in \mathcal{I}_\omega(S)$. So we have a monomorphism $f_* \cdot \mu_N : N \rightarrowtail {I^0}_*$ in $\mathrm{Mod}\, S$. Then by [HW, Proposition 5.3], N admits a monic $\mathcal{I}_\omega(S)$-preenvelope

$$g : N \rightarrowtail I_*$$

with I injective in $\mathrm{Mod}\, R$. Take E to be an injective cogenerator in $\mathrm{Mod}\, R$. Then $\mathrm{Ext}_S^1(\mathrm{Coker}\, g, E_*) = 0$ by Lemma 2.3(2). So, by [GT, Lemma 2.16(d)],

$$\mathrm{Hom}_R(\mathrm{Tor}_1^S(\omega, \mathrm{Coker}\, g), E) \cong \mathrm{Ext}_S^1(\mathrm{Coker}\, g, E_*) = 0.$$

Thus $\mathrm{Tor}_1^S(\omega, \mathrm{Coker}\, g) = 0$ and $1_\omega \otimes g$ is monic.

Conversely, assume that there exists an $(\omega \otimes_S -)$-exact exact sequence

$$0 \to N \to U^0$$

in $\mathrm{Mod}\, S$ with $U^0 \in \mathcal{I}_\omega(S)$. Because μ_{U_0} is an isomorphism by Corollary 3.5(2), from the commutative diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \longrightarrow & U^0 \\ & & \downarrow{\scriptstyle \mu_N} & & \downarrow{\scriptstyle \mu_{U^0}} \\ 0 & \longrightarrow & (\omega \otimes_S N)_* & \longrightarrow & (\omega \otimes_S U^0)_* \end{array}$$

we see that μ_N is monic and N is adjoint 1-ω-cotorsionfree.

(2) Let $N \in \operatorname{Mod} S$ be adjoint 2-ω-cotorsionfree. By (1), there exists an $(\omega \otimes_S -)$-exact exact sequence

$$0 \to N \to U^0 \to N^1 \to 0$$

in $\operatorname{Mod} S$ with $U^0 \in \mathcal{I}_\omega(S)$. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & U^0 & \longrightarrow & N^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_N} & & \downarrow{\scriptstyle \mu_{U^0}} & & \downarrow{\scriptstyle \mu_{N^1}} & & \\ 0 & \longrightarrow & (\omega \otimes_S N)_* & \longrightarrow & (\omega \otimes_S U^0)_* & \longrightarrow & (\omega \otimes_S N^1)_* & & \end{array} \tag{3.1}$$

Because both μ_N and μ_{U^0} are isomorphisms by assumption and Corollary 3.5(2), the snake lemma shows that μ_{N^1} is monic, and hence N^1 is adjoint 1-ω-cotorsionfree by Corollary 3.3(1). It follows from (1) that there exists an $(\omega \otimes_S -)$-exact exact sequence

$$0 \to N^1 \to U^1$$

in $\operatorname{Mod} S$ with $U^1 \in \mathcal{I}_\omega(S)$. Then the spliced sequence

$$0 \to N \to U^0 \to U^1$$

is as desired.

Conversely, let $0 \to N \to U^0 \to U^1$ be an $(\omega \otimes_S -)$-exact exact sequence in $\operatorname{Mod} S$ with $U^0, U^1 \in \mathcal{I}_\omega(S)$. Then $N^1 := \operatorname{Im}(U^0 \to U^1)$ is adjoint 1-ω-cotorsionfree by (1), and so μ_{N^1} is monic by Corollary 3.3(1). Now the diagram (3.1) above implies that μ_N is an isomorphism. Thus N is adjoint 2-ω-cotorsionfree by Corollary 3.3(2). ■

By induction we get the following result, which is an adjoint counterpart of [TH1, Proposition 3.7].

PROPOSITION 3.8. *Let $N \in \operatorname{Mod} S$ and $n \geq 1$. Then N is adjoint n-ω-cotorsionfree if and only if there exists an $(\omega \otimes_S -)$-exact (equivalently $\operatorname{Hom}_S(-, \mathcal{I}_\omega(S))$-exact) exact sequence*

$$0 \to N \to U^0 \to \cdots \to U^{n-1} \to U^n$$

in $\operatorname{Mod} S$ with all $U^i \in \mathcal{I}_\omega(S)$.

Proof. We proceed by induction on n. The case of $n \leq 2$ follows from Lemma 3.7.

Now suppose that $n \geq 3$ and $N \in \operatorname{Mod} S$ is adjoint n-ω-cotorsionfree. Then μ_N is an isomorphism and $\operatorname{Ext}_R^{1 \leq i \leq n-2}(\omega, \omega \otimes_S N) = 0$ by Corollary 3.3(3). In addition, by Lemma 3.7, there exists an exact sequence

$$0 \to N \to U^0 \to N^1 \to 0$$

in $\operatorname{Mod} S$ with $U^0 \in \mathcal{I}_\omega(S)$ such that

$$0 \to \omega \otimes_S N \to \omega \otimes_S U^0 \to \omega \otimes_S N^1 \to 0$$

in $\operatorname{Mod} R$ is also exact with $\omega \otimes_S U^0$ injective. Then

$$\operatorname{Ext}^i_R(\omega, \omega \otimes_S N^1) \cong \operatorname{Ext}^{i+1}_R(\omega, \omega \otimes_S N) = 0$$

for $1 \le i \le n-3$, and we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & U^0 & \longrightarrow & N^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_N} & & \downarrow{\scriptstyle \mu_{U^0}} & & \downarrow{\scriptstyle \mu_{N^1}} & & \\ 0 & \longrightarrow & (\omega \otimes_S N)_* & \longrightarrow & (\omega \otimes_S U^0)_* & \longrightarrow & (\omega \otimes_S N^1)_* & \longrightarrow & 0 \end{array}$$

Because both μ_N and μ_{U^0} are isomorphisms, so is μ_{N^1}. Thus N^1 is adjoint $(n-1)$-ω-cotorsionfree by Corollary 3.3. Now the assertion follows from the induction hypothesis.

Conversely, assume that there exists an $(\omega \otimes_S -)$-exact exact sequence

$$0 \to N \to U^0 \to \cdots \to U^{n-1} \to U^n$$

in $\operatorname{Mod} S$ with $U^i \in \mathcal{I}_\omega(S)$. Set $N^1 = \operatorname{Im}(U^0 \to U^1)$. Then

$$0 \to \omega \otimes_S N \to \omega \otimes_S U^0 \to \omega \otimes_S N^1 \to 0$$

in $\operatorname{Mod} R$ is exact with $\omega \otimes_S U^0$ injective. Because N^1 is adjoint $(n-1)$-ω-cotorsionfree by the induction hypothesis, μ_{N^1} is an isomorphism and $\operatorname{Ext}^{1 \le i \le n-3}_R(\omega, \omega \otimes_S N^1) = 0$ by Corollary 3.3.

Consider the following commutative diagram with the top row exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & U^0 & \longrightarrow & N^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_N} & & \downarrow{\scriptstyle \mu_{U^0}} & & \downarrow{\scriptstyle \mu_{N_1}} & & \\ 0 & \longrightarrow & (\omega \otimes_S N)_* & \longrightarrow & (\omega \otimes_S U^0)_* & \longrightarrow & (\omega \otimes_S N^1)_* & \longrightarrow & 0 \end{array}$$

Because μ_{U^0} is an isomorphism, so is μ_N, and the bottom row in the above diagram is exact. So $\operatorname{Ext}^1_R(\omega, \omega \otimes_S N) = 0$ and $\operatorname{Ext}^{i+1}_R(\omega, \omega \otimes_S N) \cong \operatorname{Ext}^i_R(\omega, \omega \otimes_S N^1) = 0$ for $1 \le i \le n-3$, that is, $\operatorname{Ext}^{1 \le i \le n-2}_R(\omega, \omega \otimes_S N) = 0$. Thus N is adjoint n-ω-cotorsionfree by Corollary 3.3(3). ∎

The following result is an immediate consequence of Proposition 3.8.

COROLLARY 3.9. *For $N \in \operatorname{Mod} S$, $N \in \operatorname{ac}\mathcal{T}(S)$ if and only if there exists an $(\omega \otimes_S -)$-exact (equivalently $\operatorname{Hom}_S(-, \mathcal{I}_\omega(S))$-exact) exact sequence*

$$0 \to N \to U^0 \to U^1 \to U^2 \to \cdots$$

in $\operatorname{Mod} S$ with all $U^i \in \mathcal{I}_\omega(S)$. In this case, $\operatorname{Tor}^S_1(\omega, N^i) = 0$, where $N^i = \operatorname{Im}(U^{i-1} \to U^i)$ for any $i \ge 1$.

As an adjoint counterpart of [TH2, Proposition 3.1], we have

Proposition 3.10.

(1) *If* $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega < \infty$*, then* $\mathrm{ac}\mathcal{T}(S) \subseteq {\omega_S}^{\top}$*.*
(2) *If* $\mathrm{pd}_R\,\omega < \infty$*, then* ${\omega_S}^{\top} \subseteq \mathrm{ac}\mathcal{T}(S)$*.*

Proof. (1) Let $N \in \mathrm{ac}\mathcal{T}(S)$. By Corollary 3.9, there exists an exact sequence

$$0 \to N \to U^0 \to U^1 \to U^2 \to \cdots$$

in $\mathrm{Mod}\,S$ with all $U^i \in \mathcal{I}_\omega(S)$. Set $N^i = \mathrm{Im}(U^{i-1} \to U^i)$ for any $i \geq 1$. Note that $U^i \in {\omega_S}^{\top}$ for any $i \geq 0$ by Corollary 3.5(2). So, if $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega = n$ $(< \infty)$, then $\mathrm{Tor}_i^S(\omega, N) \cong \mathrm{Tor}_{i+n}^S(\omega, N^n) = 0$ for any $i \geq 1$. Thus $N \in {\omega_S}^{\top}$.

(2) Let $\mathrm{pd}_R\,\omega = n$ $(< \infty)$ and $N \in {\omega_S}^{\top}$. Set $\Omega_0(N) = N$ and $\Omega_i(N) = \mathrm{Im}(F_i(N) \to F_{i-1}(N))$ for any $i \geq 1$. Then we get an exact sequence

$$0 \to \omega \otimes_S \Omega_{i+1}(N) \to \omega \otimes_S F_i(N) \to \omega \otimes_S \Omega_i(N) \to 0$$

in $\mathrm{Mod}\,R$ for $i \geq 0$. It follows from Lemma 2.3(1) that

$$\mathrm{Ext}_R^j(\omega, \omega \otimes_S \Omega_i(N)) \cong \mathrm{Ext}_R^{j+n}(\omega, \omega \otimes_S \Omega_{i+n}(N)) = 0$$

for any $i \geq 0$ and $j \geq 1$; in particular, $\mathrm{Ext}_R^1(\omega, \omega \otimes_S \Omega_2(N)) = 0$. Thus we get the following diagram with exact rows:

$$\begin{array}{ccccc}
F_1(N) & \longrightarrow & \Omega_1(N) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mu_{F_1(N)}} & & \downarrow{\scriptstyle \mu_{\Omega_1(N)}} & & \\
(\omega \otimes_S F_1(N))_* & \longrightarrow & (\omega \otimes_S \Omega_1(N))_* & \longrightarrow & 0
\end{array}$$

Because $\mu_{F_1(N)}$ is an isomorphism by Corollary 3.5(2), $\mu_{\Omega_1(N)}$ is an epimorphism. But $\Omega_1(N)$ is a submodule of $F_0(N)$, so $\mu_{\Omega_1(N)}$ is a monomorphism and hence an isomorphism. Then $\Omega_1(N)$ is adjoint 2-ω-cotorsionfree by Corollary 3.3(2). On the other hand, because $\mathrm{Ext}_R^1(\omega, \omega \otimes_S \Omega_1(N)) = 0$ by the above argument, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_1(N) & \longrightarrow & F_0(N) & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_{\Omega_1(N)}} & & \downarrow{\scriptstyle \mu_{F_0(N)}} & & \downarrow{\scriptstyle \mu_N} & & \\
0 & \longrightarrow & (\omega \otimes_S \Omega_1(N))_* & \longrightarrow & (\omega \otimes_S F_0(N))_* & \longrightarrow & (\omega \otimes_S N)_* & \longrightarrow & 0
\end{array}$$

Since $\mu_{F_0(N)}$ is an isomorphism, the snake lemma shows that so is μ_N and N is 2-ω-cotorsionfree. So by Lemma 3.7, there exists an $(\omega \otimes_S -)$-exact exact sequence

$$0 \to N \to U^0 \to N^1 \to 0$$

in $\mathrm{Mod}\,S$ with $U^0 \in \mathcal{I}_\omega(S)$. Then $N^1 \in {\omega_S}^{\top}$. Now by an argument similar to the above, we get an $(\omega \otimes_S -)$-exact exact sequence

$$0 \to N^1 \to U^1 \to N^2 \to 0$$

in $\operatorname{Mod} S$ with $U^1 \in \mathcal{I}_\omega(S)$ and $N^2 \in {\omega_S}^{\top}$. Continuing this procedure, we get an $(\omega \otimes_S -)$-exact exact sequence

$$0 \to N \to U^0 \to \cdots \to U^{n-1} \to U^n \to \cdots$$

in $\operatorname{Mod} S$ with all U^i in $\mathcal{I}_\omega(S)$. Then Corollary 3.9 shows that $N \in \mathrm{ac}\mathcal{T}(S)$. ■

Summarizing Proposition 3.4, Corollary 3.9 and Proposition 3.10, we have the following result, in which the first assertion means that [EH, Proposition 3.6] still holds true without assuming the given ring is commutative Noetherian.

THEOREM 3.11.

(1) $\mathcal{A}_\omega(S) = \mathrm{ac}\mathcal{T}(S) \cap {\omega_S}^{\top} = \{N \in \operatorname{Mod} S \mid$ *there exists an* $(\omega \otimes_S -)$*-exact* (*equivalently* $\operatorname{Hom}_S(-, \mathcal{I}_\omega(S))$*-exact*) *exact sequence*

$$\cdots \to F_2 \to F_1 \to F_0 \to U^0 \to U^1 \to U^2 \to \cdots$$

in $\operatorname{Mod} S$ *with all* F_i *in* $\mathcal{F}(S)$, *all* U^i *in* $\mathcal{I}_\omega(S)$ *and* $N \cong \operatorname{Im}(F_0 \to U^0)\}$.

(2) *If* $\operatorname{pd}_{S^{\mathrm{op}}} \omega < \infty$, *then* $\mathcal{A}_\omega(S) = \mathrm{ac}\mathcal{T}(S)$.

(3) *If* $\operatorname{pd}_R \omega < \infty$, *then* $\mathcal{A}_\omega(S) = {\omega_S}^{\top}$.

The following result characterizes when $\mathrm{ac}\mathcal{T}(S)$ and $\mathrm{c}\mathcal{T}(R)$ are equivalent under Foxby equivalence.

THEOREM 3.12. *The following statements are equivalent:*

(1) *There exists an equivalence of categories*

$$\mathrm{ac}\mathcal{T}(S) \underset{\operatorname{Hom}_R(\omega,-)}{\overset{\omega\otimes_S -}{\rightleftarrows}} \mathrm{c}\mathcal{T}(R). \quad (\sim)$$

(2) $\mathrm{ac}\mathcal{T}(S) = \mathcal{A}_\omega(S)$ *and* $\mathrm{c}\mathcal{T}(R) = \mathcal{B}_\omega(R)$.

Proof. (2)⇒(1) follows from [HW, Theorem 5.1].

(1)⇒(2). By Theorem 3.11(1) and [TH1, Theorem 3.9], it suffices to prove that $\mathrm{ac}\mathcal{T}(S) \subseteq \mathcal{A}_\omega(S)$ and $\mathrm{c}\mathcal{T}(R) \subseteq \mathcal{B}_\omega(R)$.

Let $N \in \mathrm{ac}\mathcal{T}(S)$. Then $N \cong (\omega \otimes_S N)_*$. By Corollary 3.9, there exists an $(\omega \otimes_S -)$-exact exact sequence

$$0 \to N \to U^0 \to U^1 \to U^2 \to \cdots \tag{3.2}$$

in $\operatorname{Mod} S$ with all $U^i \in \mathcal{I}_\omega(S)$ and $\operatorname{Tor}_1^S(\omega, N^i) = 0$, where $N^i = \operatorname{Im}(U^{i-1} \to U^i)$ for any $i \geq 1$. Applying the functor $\omega \otimes_S -$ to (3.2) yields an exact sequence

$$0 \to \omega \otimes_S N \to \omega \otimes_S U^0 \to \omega \otimes_S U^1 \to \omega \otimes_S U^2 \to \cdots \tag{3.3}$$

in $\operatorname{Mod} R$. All $\omega \otimes_S U^i$ are injective in $\operatorname{Mod} R$ by [HW, Lemma 5.1(c)]. Because the functor $\operatorname{Hom}_R(\omega, -)$ sends (3.3) to (3.2), it is easy to see that $N^2 \cong \mathrm{cTr}_\omega(\omega \otimes_S N) \oplus U$ for some $U \in \mathcal{I}_\omega(S)$. Note that $\omega \otimes_S N \in \mathrm{c}\mathcal{T}(R)$ and $U \in {\omega_S}^{\top}$ by assumption and Corollary 3.5(2). So $\mathrm{cTr}_\omega(\omega \otimes_S N) \in {\omega_S}^{\top}$

and $N^2 \in {\omega_S}^{\top}$, and hence $N \in {\omega_S}^{\top}$. Now it follows from Theorem 3.11(1) that $N \in \mathcal{A}_\omega(S)$ and $\mathrm{ac}\mathcal{T}(S) \subseteq \mathcal{A}_\omega(S)$. Similarly, $\mathrm{c}\mathcal{T}(R) \subseteq \mathcal{B}_\omega(R)$. ∎

The following result shows that any module in $\mathrm{Mod}\, S$ with finite $\mathrm{ac}\mathcal{T}(S)$-projective dimension is isomorphic to a kernel (resp. a cokernel) of a homomorphism from a module with finite $\mathcal{I}_\omega(S)$-projective dimension to an adjoint ∞-ω-cotorsionfree module.

Theorem 3.13. *Let* $N \in \mathrm{Mod}\, S$ *with* $\mathrm{ac}\mathcal{T}(S)\text{-}\mathrm{pd}_S\, N \leq n$ $(< \infty)$*. Then there exists an exact sequence*

$$0 \to U_N \to V_N \to U^N \to V^N \to 0 \tag{3.4}$$

in $\mathrm{Mod}\, S$ *such that* $N \cong \mathrm{Im}(V_N \to U^N)$ *and the following conditions are satisfied:*

(1) $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S\, U^N \leq n$, $V^N \in \mathrm{ac}\mathcal{T}(S)$ *and*

$$0 \to N \to U^N \to V^N \to 0$$

is exact and $\mathrm{Hom}_S(-, \mathcal{I}_\omega(S))$*-exact* (*equivalently* $(\omega \otimes_S -)$*-exact*).

(2) $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S\, U_N \leq n-1$ *and* $V_N \in \mathrm{ac}\mathcal{T}(S)$.

Proof. By Proposition 3.6 and Corollary 3.9, $\mathrm{ac}\mathcal{T}(S)$ is an $\mathcal{I}_\omega(S)$-coresolving subcategory of $\mathrm{Mod}\, S$ admitting an $\mathcal{I}_\omega(S)$-coproper cogenerator $\mathcal{I}_\omega(S)$ in the sense of [H]. Then by [H, Corollary 4.5], we have a $\mathrm{Hom}_S(-, \mathcal{I}_\omega(S))$-exact (equivalently $(\omega \otimes_S -)$-exact) exact sequence

$$0 \to N \to U^N \to V^N \to 0 \tag{3.5}$$

in $\mathrm{Mod}\, S$ such that $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S\, U^N \leq n$ and $V^N \in \mathrm{ac}\mathcal{T}(S)$. On the other hand, by [H, Theorem 4.7] we have an exact sequence

$$0 \to U_N \to V_N \to N \to 0 \tag{3.6}$$

in $\mathrm{Mod}\, S$ such that $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S\, U_N \leq n-1$ and $V_N \in \mathrm{ac}\mathcal{T}(S)$. Now splicing (3.5) and (3.6) we get the desired exact sequence (3.4). ∎

The following result, as an adjoint counterpart of Theorem 3.13, shows that any module in $\mathrm{Mod}\, R$ with finite $\mathrm{c}\mathcal{T}(R)$-injective dimension is isomorphic to a kernel (resp. a cokernel) of a homomorphism from an ∞-ω-cotorsionfree module to a module with finite $\mathcal{P}_\omega(R)$-injective dimension.

Theorem 3.14. *Let* $M \in \mathrm{Mod}\, R$ *with* $\mathrm{c}\mathcal{T}(R)\text{-}\mathrm{id}_R\, M \leq n$ $(< \infty)$*. Then there exists an exact sequence*

$$0 \to Y_M \to X_M \to Y^M \to X^M \to 0 \tag{3.7}$$

in $\mathrm{Mod}\, R$ *such that* $M \cong \mathrm{Im}(X_M \to Y^M)$ *and the following conditions are satisfied:*

(1) $\mathcal{P}_\omega(R)$-$\mathrm{id}_R\, X_M \le n$, $Y_M \in \mathrm{c}\mathcal{T}(R)$ *and*

$$0 \to Y_M \to X_M \to M \to 0$$

is exact and $\mathrm{Hom}_R(\mathcal{P}_\omega(R), -)$*-exact.*

(2) $\mathcal{P}_\omega(R)$-$\mathrm{id}_R\, X^M \le n-1$ *and* $Y^M \in \mathrm{c}\mathcal{T}(R)$.

Proof. By [TH1, Propositions 3.5 and 3.7], $\mathrm{c}\mathcal{T}(R)$ is a $\mathcal{P}_\omega(R)$-resolving subcategory of $\mathrm{Mod}\, R$ admitting a $\mathcal{P}_\omega(R)$-proper generator $\mathcal{P}_\omega(R)$ in the sense of [H]. Then by [H, Corollary 3.5], we have a $\mathrm{Hom}_R(\mathcal{P}_\omega(R), -)$-exact exact sequence

$$0 \to Y_M \to X_M \to M \to 0 \tag{3.8}$$

in $\mathrm{Mod}\, R$ such that $\mathcal{P}_\omega(R)$-$\mathrm{id}_R\, X_M \le n$ and $Y_M \in \mathrm{c}\mathcal{T}(R)$. On the other hand, by [H, Theorem 3.7] we have an exact sequence

$$0 \to M \to Y^M \to X^M \to 0 \tag{3.9}$$

in $\mathrm{Mod}\, R$ such that $\mathcal{P}_\omega(R)$-$\mathrm{id}_R\, X^M \le n-1$ and $Y^M \in \mathrm{c}\mathcal{T}(R)$. Now splicing (3.8) and (3.9) we get the desired exact sequence (3.7). ∎

We end this section with a non-trivial example of adjoint ∞-ω-cotorsionfree modules. The following example is due to Jorgensen and Şega [JS].

Example 3.15. Let k be a field which is not algebraic over a finite field and let $\alpha \in k$ be an element of infinite multiplicative order. Suppose that $R_\alpha = k[V, X, Y, Z]/I_\alpha$, where $I_\alpha = \langle V^2, Z^2, XY, VX + \alpha XZ, VY + YZ, VX + Y^2, VY - X^2 \rangle$. Let m denote the unique maximal ideal of the local artinian ring R_α, $\omega = I^0(R_\alpha/m)$ and ${R_\alpha}^2 = R_\alpha \oplus R_\alpha$. For $i \le 0$, let $d_i : {R_\alpha}^2 \to {R_\alpha}^2$ denote the map given by the matrix

$$\begin{pmatrix} v & \alpha^{-i} x \\ y & z \end{pmatrix},$$

where v, x, y, z denote the residue classes of the variables modulo I_α respectively. Set $M = \mathrm{Coker}\, d_{-1}$. Then M is an adjoint ∞-ω-cotorsionfree module, but $M \notin \mathcal{A}_\omega(R_\alpha)$.

Proof. It follows from [JS, Lemma 1.4] that there exists an exact sequence

$$\mathbf{A} : 0 \to M \to {R_\alpha}^2 \xrightarrow{d_{-3}} {R_\alpha}^2 \xrightarrow{d_{-4}} \cdots .$$

Since R_α is a commutative artinian local ring, ω is a semidualizing module and $\mathcal{I}_\omega(R_\alpha) = \mathrm{Add}_{R_\alpha} R_\alpha$. This implies that ${R_\alpha}^2 \in \mathcal{I}_\omega(R_\alpha)$. By [JS, Lemma 1.5], the sequence $\mathrm{Hom}_{R_\alpha}(\mathbf{A}, R_\alpha)$ remains exact and $M \notin {}^{\perp_{R_\alpha}} R_\alpha$. By Corollary 3.9, M is an adjoint ∞-ω-cotorsionfree module. Note that ω is an injective cogenerator for $\mathrm{Mod}\, R_\alpha$. So by [CE, Proposition VI.5.3], we have

$$\mathrm{Tor}_i^{R_\alpha}(\omega, M) \cong \mathrm{Tor}_i^{R_\alpha}(\mathrm{Hom}_{R_\alpha}(R_\alpha, \omega), M) \cong \mathrm{Hom}_{R_\alpha}(\mathrm{Ext}^i_{R_\alpha}(M, R_\alpha), \omega)$$

for any $i \geq 0$. Thus $M \notin {\omega_{R_\alpha}}^{\top}$, and therefore $M \notin \mathcal{A}_\omega(R_\alpha)$ by Theorem 3.11(1). ■

Remark 3.16. By Theorem 3.12, the categories $\mathrm{ac}\mathcal{T}(R)$ and $\mathrm{c}\mathcal{T}(R)$ are not equivalent under Foxby equivalence when $R = R_\alpha$ is the ring of Example 3.15 and ω is the semidualizing R-module given in Example 3.15.

4. Finiteness of $\mathrm{pd}_R\,\omega$ and $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega$. As applications of Theorems 3.13 and 3.14, in this section we will characterize when $\mathrm{pd}_R\,\omega = \mathrm{pd}_{S^{\mathrm{op}}}\,\omega < \infty$ in terms of the properties of the (adjoint) ∞-ω-cotorsionfree dimensions of modules. We begin with the following result, which was proved by Wakamatsu [W, Proposition 7] when R and S are artin algebras.

Proposition 4.1. *If $\mathrm{pd}_R\,\omega < \infty$ and $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega < \infty$, then $\mathrm{pd}_R\,\omega = \mathrm{pd}_{S^{\mathrm{op}}}\,\omega$.*

Proof. Let $\mathrm{pd}_R\,\omega = m < \infty$ and $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega = n < \infty$. It is easy to see that $\mathrm{pd}_R\,\omega = \mathrm{add}\,\omega_S\text{-}\mathrm{id}_{S^{\mathrm{op}}}\,S$ and $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega = \mathrm{add}_R\,\omega\text{-}\mathrm{id}_R\,R$. So we have an exact sequence

$$0 \to R \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

in $\mathrm{Mod}\,R$ with all C^i in $\mathrm{add}_R\,\omega$. Set $K^i = \mathrm{Ker}(C^i \to C^{i+1})$ for any $0 \leq i \leq n-1$. If $m < n$, then $\mathrm{Ext}^1_R(\omega, K^{n-1}) \cong \mathrm{Ext}^2_R(\omega, K^{n-2}) \cong \cdots \cong \mathrm{Ext}^{m+1}_R(\omega, K^{n-m-1}) = 0$. So the exact sequence

$$0 \to K^{n-1} \to C^{n-1} \to C^n \to 0$$

splits and K^{n-1} is isomorphic to a direct summand of C^{n-1}. This implies that $K^{n-1} \in \mathrm{add}_R\,\omega$ and $\mathrm{add}_R\,\omega\text{-}\mathrm{id}_R\,R \leq n-1$, which is a contradiction. So $m \geq n$. Similarly, $n \geq m$. ■

The aim of this section is to prove the following result.

Theorem 4.2. *The following statements are equivalent for any $n \geq 0$:*

(1) $\mathrm{pd}_R\,\omega = \mathrm{pd}_{S^{\mathrm{op}}}\,\omega \leq n$.
(2) $\mathcal{P}_\omega(R)\text{-}\mathrm{id}_R\,R = \mathcal{P}_\omega(S^{\mathrm{op}})\text{-}\mathrm{id}_{S^{\mathrm{op}}}\,S \leq n$.
(3) $\mathcal{B}_\omega(R)\text{-}\mathrm{id}_R\,R = \mathcal{B}_\omega(S^{\mathrm{op}})\text{-}\mathrm{id}_{S^{\mathrm{op}}}\,S \leq n$.
(4) $\mathrm{c}\mathcal{T}(R)\text{-}\mathrm{id}_R\,R = \mathrm{c}\mathcal{T}(S^{\mathrm{op}})\text{-}\mathrm{id}_{S^{\mathrm{op}}}\,S \leq n$.
(5) $\mathcal{B}_\omega(R)\text{-}\mathrm{id}_R\,M \leq n$ *for any* $M \in \mathrm{Mod}\,R$, *and* $\mathcal{B}_\omega(S^{\mathrm{op}})\text{-}\mathrm{id}_{S^{\mathrm{op}}}\,N \leq n$ *for any* $N \in \mathrm{Mod}\,S^{\mathrm{op}}$.
(6) $\mathcal{B}_\omega(R)\text{-}\mathrm{id}_R\,M \leq n$ *for any* $M \in \mathrm{mod}\,R$, *and* $\mathcal{B}_\omega(S^{\mathrm{op}})\text{-}\mathrm{id}_{S^{\mathrm{op}}}\,N \leq n$ *for any* $N \in \mathrm{mod}\,S^{\mathrm{op}}$.
(7) $\mathrm{c}\mathcal{T}(R)\text{-}\mathrm{id}_R\,M \leq n$ *for any* $M \in \mathrm{Mod}\,R$, *and* $\mathrm{c}\mathcal{T}(S^{\mathrm{op}})\text{-}\mathrm{id}_{S^{\mathrm{op}}}\,N \leq n$ *for any* $N \in \mathrm{Mod}\,S^{\mathrm{op}}$.
(8) $\mathrm{c}\mathcal{T}(R)\text{-}\mathrm{id}_R\,M \leq n$ *for any* $M \in \mathrm{mod}\,R$, *and* $\mathrm{c}\mathcal{T}(S^{\mathrm{op}})\text{-}\mathrm{id}_{S^{\mathrm{op}}}\,N \leq n$ *for any* $N \in \mathrm{mod}\,S^{\mathrm{op}}$.

(9) ${}_R\omega^{\perp}$-$\mathrm{id}_R\, M \le n$ *for any* $M \in \operatorname{Mod} R$, *and* $\omega_S{}^{\perp}$-$\mathrm{id}_{S^{\mathrm{op}}}\, N \le n$ *for any* $N \in \operatorname{Mod} S^{\mathrm{op}}$.

(10) ${}_R\omega^{\perp}$-$\mathrm{id}_R\, M \le n$ *for any* $M \in \operatorname{mod} R$, *and* $\omega_S{}^{\perp}$-$\mathrm{id}_{S^{\mathrm{op}}}\, M \le n$ *for any* $M \in \operatorname{mod} S^{\mathrm{op}}$.

To prove this theorem, we need the following three lemmas.

LEMMA 4.3. *We have*

$$\begin{aligned}\mathrm{pd}_R\, \omega = \mathcal{P}_\omega(S^{\mathrm{op}})\text{-}\mathrm{id}_{S^{\mathrm{op}}}\, S &= \sup\{{}_R\omega^{\perp}\text{-}\mathrm{id}_R\, M \mid M \in \operatorname{Mod} R\} \\ &= \sup\{{}_R\omega^{\perp}\text{-}\mathrm{id}_R\, M \mid M \in \operatorname{mod} R\}.\end{aligned}$$

Proof. Since $\mathrm{pd}_R\, \omega = \operatorname{add} \omega_S\text{-}\mathrm{id}_{S^{\mathrm{op}}}\, S$, it is straightforward to verify $\mathrm{pd}_R\, \omega = \mathcal{P}_\omega(S^{\mathrm{op}})\text{-}\mathrm{id}_{S^{\mathrm{op}}}\, S$ by [TH2, Lemma 4.7]. It remains to prove that $\sup\{{}_R\omega^{\perp}\text{-}\mathrm{id}_R\, M \mid M \in \operatorname{Mod} R\} \le \mathrm{pd}_R\, \omega \le \sup\{{}_R\omega^{\perp}\text{-}\mathrm{id}_R\, M \mid M \in \operatorname{mod} R\}$.

Let $\mathrm{pd}_R\, \omega = n$ $(< \infty)$ and pick $M \in \operatorname{Mod} R$. Define $K^n = \operatorname{Im}(I^{n-1}(M) \to I^n(M))$. Then $\operatorname{Ext}^i_R(\omega, K^n) \cong \operatorname{Ext}^{n+i}_R(\omega, M) = 0$ for any $i \ge 1$. So $K^n \in {}_R\omega^{\perp}$ and ${}_R\omega^{\perp}$-$\mathrm{id}_R\, M \le n$.

Now let $\sup\{{}_R\omega^{\perp}\text{-}\mathrm{id}_R\, M \mid M \in \operatorname{mod} R\} = n$ $(< \infty)$. Then by dimension shifting, it is easy to see that $\operatorname{Ext}^{\ge n+1}_R(\omega, M) = 0$ for any $M \in \operatorname{mod} R$. Let $X \in \operatorname{Mod} R$. Then $X = \varinjlim M_i$ with all M_i in $\operatorname{mod} R$ by [GT, Lemma 2.5]. It follows from [GT, Lemma 6.6] that $\operatorname{Ext}^{\ge n+1}_R(\omega, X) = 0$, which implies $\mathrm{pd}_R\, \omega \le n$. ■

LEMMA 4.4. *If* $\mathrm{pd}_R\, \omega = \mathrm{pd}_{S^{\mathrm{op}}}\, \omega \le n$ $(< \infty)$, *then* $\mathcal{B}_\omega(R)\text{-}\mathrm{id}_R\, M = \mathrm{c}\mathcal{T}(R)\text{-}\mathrm{id}_R\, M \le n$ *for any* $M \in \operatorname{Mod} R$.

Proof. If $\mathrm{pd}_R\, \omega = \mathrm{pd}_{S^{\mathrm{op}}}\, \omega \le n$, then $\mathcal{B}_\omega(R) = {}_R\omega^{\perp} = \mathrm{c}\mathcal{T}(R)$ by [TH2, Corollary 3.2]. Now the assertion follows from Lemma 4.3. ■

LEMMA 4.5. $\mathrm{c}\mathcal{T}(R)\text{-}\mathrm{id}_R\, R = \mathcal{B}_\omega(R)\text{-}\mathrm{id}_R\, R = \mathcal{P}_\omega(R)\text{-}\mathrm{id}_R\, R.$

Proof. By Lemma 2.3(1) and [TH1, Theorem 3.9], we have $\mathcal{P}_\omega(R) \subseteq \mathcal{B}_\omega(R) \subseteq \mathrm{c}\mathcal{T}(R)$. So $\mathrm{c}\mathcal{T}(R)\text{-}\mathrm{id}_R\, R \le \mathcal{B}_\omega(R)\text{-}\mathrm{id}_R\, R \le \mathcal{P}_\omega(R)\text{-}\mathrm{id}_R\, R$. Now let $\mathrm{c}\mathcal{T}(R)\text{-}\mathrm{id}_R\, R = n$ $(< \infty)$. It follows from Theorem 3.14 that there exists a module $X \in \operatorname{Mod} R$ with $\mathcal{P}_\omega(R)\text{-}\mathrm{id}_R\, X \le n$ such that ${}_RR$ is isomorphic to a direct summand of X. Thus $\mathcal{P}_\omega(R)\text{-}\mathrm{id}_R\, R \le n$ by [TH2, Lemma 4.6], and therefore $\mathcal{P}_\omega(R)\text{-}\mathrm{id}_R\, R \le \mathrm{c}\mathcal{T}(R)\text{-}\mathrm{id}_R\, R$. ■

Proof of Theorem 4.2. By Proposition 4.1, Lemma 4.3 and its symmetric version, we have (1)⇔(2)⇔(9)⇔(10). By Lemma 4.4 and its symmetric version, we deduce (1)⇒(5) and (1)⇒(7). By Lemma 4.5 and its symmetric version, we obtain (2)⇔(3)⇔(4). The implications (7)⇒(8)⇒(4) and (5)⇒(6)⇒(3) are clear. ■

It should be pointed out that a semidualizing bimodule ${}_R\omega_S$ satisfying condition (1) in Theorem 4.2 is actually a tilting bimodule in the sense of [M]. In the following, we will give an adjoint counterpart of Theorem 4.2. We need some lemmas.

LEMMA 4.6.
(1) $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega = \sup\{{\omega_S}^{\top}\text{-}\mathrm{pd}_S\,N \mid N \in \mathrm{Mod}\,S\} = \sup\{{\omega_S}^{\top}\text{-}\mathrm{pd}_S\,N \mid N \in \mathrm{mod}\,S\}$.
(2) *If* $\mathrm{pd}_R\,\omega = \mathrm{pd}_{S^{\mathrm{op}}}\,\omega \le n\ (<\infty)$, *then* $\mathcal{A}_\omega(S)\text{-}\mathrm{pd}_S\,N = \mathrm{ac}\mathcal{T}(S)\text{-}\mathrm{pd}_S\,N \le n$ *for any* $N \in \mathrm{Mod}\,S$.

Proof. (1) It suffices to prove

$$\sup\{{\omega_S}^{\top}\text{-}\mathrm{pd}_S\,N \mid N \in \mathrm{Mod}\,S\} \le \mathrm{pd}_{S^{\mathrm{op}}}\,\omega \le \sup\{{\omega_S}^{\top}\text{-}\mathrm{pd}_S\,N \mid N \in \mathrm{mod}\,S\}.$$

Let $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega \le n\ (<\infty)$ and $N \in \mathrm{Mod}\,S$. Set $K_n = \mathrm{Coker}(F_{n+1}(N) \to F_n(N))$. Then $\mathrm{Tor}_i^S(\omega, K_n) \cong \mathrm{Tor}_{n+i}^S(\omega, N) = 0$ for any $i \ge 1$. It follows that $K_n \in {\omega_S}^{\top}$ and ${\omega_S}^{\top}\text{-}\mathrm{pd}_S\,N \le n$. Conversely, note that ω_S admits a degreewise finite S-projective resolution. Then by dimension shifting, it is easy to get $\mathrm{pd}_{S^{\mathrm{op}}}\,\omega = \mathrm{fd}_{S^{\mathrm{op}}}\,\omega \le \sup\{{\omega_S}^{\top}\text{-}\mathrm{pd}_S\,N \mid N \in \mathrm{mod}\,S\}$.

(2) Let $\mathrm{pd}_R\,\omega = \mathrm{pd}_{\omega^{\mathrm{op}}}\,\omega \le n$. Then $\mathcal{A}_\omega(S) = {\omega_S}^{\top} = \mathrm{ac}\mathcal{T}(S)$ by Theorem 3.11. Now the assertion follows from (1). ∎

LEMMA 4.7. *Both* $\mathcal{I}_\omega(R^{\mathrm{op}})\text{-}\mathrm{pd}^{\le n}(R^{\mathrm{op}})$ *and* $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}^{\le n}(S)$ *are closed under direct summands.*

Proof. By Lemma 2.3(2), we have $\mathcal{I}_\omega(S) \subseteq \mathcal{I}_\omega(S)^{\perp}$. It is trivial that $\mathcal{I}_\omega(S)$ is an $\mathcal{I}_\omega(S)$-resolving subcategory of $\mathrm{Mod}\,S$ with an $\mathcal{I}_\omega(S)$-proper generator $\mathcal{I}_\omega(S)$ in the sense of [H]. Note that $\mathcal{I}_\omega(S)$ is closed under direct summands by [HW, Proposition 5.1(c)]. So $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}^{\le n}(S)$ is closed under direct summands by [H, Corollary 3.9]. Symmetrically, we deduce that $\mathcal{I}_\omega(R^{\mathrm{op}})\text{-}\mathrm{pd}^{\le n}(R^{\mathrm{op}})$ is closed under direct summands. ∎

LEMMA 4.8. *For any injective module* I *in* $\mathrm{Mod}\,S$, *we have*

$$\mathrm{ac}\mathcal{T}(S)\text{-}\mathrm{pd}_S\,I = \mathcal{A}_\omega(S)\text{-}\mathrm{pd}_S\,I = \mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S\,I.$$

Proof. By Theorem 3.11(1) and Corollary 3.5, we have $\mathcal{I}_\omega(S) \subseteq \mathcal{A}_\omega(S) \subseteq \mathrm{ac}\mathcal{T}(S)$. So $\mathrm{ac}\mathcal{T}(S)\text{-}\mathrm{pd}_S\,N \le \mathcal{A}_\omega(S)\text{-}\mathrm{pd}_S\,N \le \mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S\,N$ for any $N \in \mathrm{Mod}\,S$. Now let $I \in \mathrm{Mod}\,S$ be injective with $\mathrm{ac}\mathcal{T}(S)\text{-}\mathrm{pd}_S\,I = n\ (<\infty)$. It follows from Theorem 3.13 that there exists $U \in \mathrm{Mod}\,S$ with $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S\,U \le n$ such that I is isomorphic to a direct summand of U. Thus $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S\,I \le n$ by Lemma 4.7, and therefore $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S\,I \le \mathrm{ac}\mathcal{T}(S)\text{-}\mathrm{pd}_S\,I$. ∎

Let R be an artin k-algebra over a commutative artin ring k. We denote by D the ordinary Matlis duality, that is, $D(-) := \mathrm{Hom}_k(-, I^0(k/J(k)))$, where $J(k)$ is the Jacobson radical of k. It is well known that D induces an equivalence between $\mathrm{mod}\,R$ and $\mathrm{mod}\,R^{\mathrm{op}}$.

Lemma 4.9. *Let R and S be artin algebras. Then*

$$\mathrm{pd}_R\,\omega = \mathcal{I}_\omega(S)\text{-}\mathrm{pd}_S\,D(S_S).$$

Proof. If $\mathcal{I}_\omega(S)$-$\mathrm{pd}_S\,D(S_S) = n$ $(< \infty)$, then there exists an exact sequence

$$0 \to U_n \to \cdots \to U_1 \to U_0 \to D(S_S) \to 0 \tag{4.1}$$

in $\mathrm{Mod}\,S$ with all U_i in $\mathcal{I}_\omega(S)$. Applying the duality $D(-)$ to (4.1) yields the following exact sequence:

$$0 \to S_S \to D(U_0) \to D(U_1) \to \cdots \to D(U_n) \to 0 \tag{4.2}$$

in $\mathrm{Mod}\,R$ with $D(U_i) \in \mathcal{P}_\omega(S^{\mathrm{op}})$. Now applying $\mathrm{Hom}_{S^{\mathrm{op}}}(-,\omega)$ to (4.2) we get $\mathrm{pd}_R\,\omega \le n$. Therefore $\mathrm{pd}_R\,\omega \le \mathcal{I}_\omega(S)$-$\mathrm{pd}_S\,D(S_S)$. By a dual argument, $\mathcal{I}_\omega(S)$-$\mathrm{pd}_S\,D(S_S) \le \mathrm{pd}_R\,\omega$. ∎

Now we are ready to prove

Theorem 4.10. *Let R and S be artin algebras and $n \ge 0$. Then the following statements are equivalent:*

(1) $\mathrm{pd}_R\,\omega = \mathrm{pd}_{S^{\mathrm{op}}}\,\omega \le n$.
(2) $\mathcal{I}_\omega(R^{\mathrm{op}})$-$\mathrm{pd}_{R^{\mathrm{op}}}\,D({}_RR) = \mathcal{I}_\omega(S)$-$\mathrm{pd}_S\,D(S_S) \le n$.
(3) $\mathcal{A}_\omega(R^{\mathrm{op}})$-$\mathrm{pd}_{R^{\mathrm{op}}}\,D({}_RR) = \mathcal{A}_\omega(S)$-$\mathrm{pd}_S\,D(S_S) \le n$.
(4) $\mathrm{ac}\mathcal{T}(R^{\mathrm{op}})$-$\mathrm{pd}_{R^{\mathrm{op}}}\,D({}_RR) = \mathrm{ac}\mathcal{T}(S)$-$\mathrm{pd}_S\,D(S_S) \le n$.
(5) $\mathcal{A}_\omega(R^{\mathrm{op}})$-$\mathrm{pd}_{R^{\mathrm{op}}}\,M \le n$ *for any* $M \in \mathrm{Mod}\,R^{\mathrm{op}}$, *and* $\mathcal{A}_\omega(S)$-$\mathrm{pd}_S\,N \le n$ *for any* $N \in \mathrm{Mod}\,S$.
(6) $\mathcal{A}_\omega(R^{\mathrm{op}})$-$\mathrm{pd}_{R^{\mathrm{op}}}\,M \le n$ *for any* $M \in \mathrm{mod}\,R^{\mathrm{op}}$, *and* $\mathcal{A}_\omega(S)$-$\mathrm{pd}_S\,N \le n$ *for any* $N \in \mathrm{mod}\,S$
(7) $\mathrm{ac}\mathcal{T}(R^{\mathrm{op}})$-$\mathrm{pd}_{R^{\mathrm{op}}}\,M \le n$ *for any* $M \in \mathrm{Mod}\,R^{\mathrm{op}}$, *and* $\mathrm{ac}\mathcal{T}(S)$-$\mathrm{pd}_S\,N \le n$ *for any* $N \in \mathrm{Mod}\,S$.
(8) $\mathrm{ac}\mathcal{T}(R^{\mathrm{op}})$-$\mathrm{pd}_{R^{\mathrm{op}}}\,M \le n$ *for any* $M \in \mathrm{mod}\,R^{\mathrm{op}}$, *and* $\mathrm{ac}\mathcal{T}(S)$-$\mathrm{pd}_S\,N \le n$ *for any* $N \in \mathrm{mod}\,S$.
(9) ${}_R\omega^{\top}$-$\mathrm{pd}_{R^{\mathrm{op}}}\,M \le n$ *for any* $M \in \mathrm{Mod}\,R^{\mathrm{op}}$, *and* ${\omega_S}^{\top}$-$\mathrm{pd}_S\,N \le n$ *for any* $N \in \mathrm{Mod}\,S$.
(10) ${}_R\omega^{\top}$-$\mathrm{pd}_{R^{\mathrm{op}}}\,M \le n$ *for any* $M \in \mathrm{mod}\,R^{\mathrm{op}}$, *and* ${\omega_S}^{\top}$-$\mathrm{pd}_S\,N \le n$ *for any* $N \in \mathrm{mod}\,S$.

Proof. By Proposition 4.1, Lemma 4.6 and its symmetric version, we have (9)⇔(10)⇔(1)⇒(5) and (1)⇒(7). By Lemma 4.8 and its symmetric version, we obtain (2)⇔(3)⇔(4). The implications (7)⇒(8)⇒(4) and (5)⇒(6)⇒(3) are clear. By Proposition 4.1, Lemma 4.9 and its symmetric version, we deduce (1)⇔(2). ∎

As a consequence of Theorems 4.2 and 4.10, we have the following

Corollary 4.11. *Let R and S be artin algebras and $n \geq 0$. Then the following statements are equivalent:*

(1) $\mathrm{pd}_R\, \omega = \mathrm{pd}_{S^{\mathrm{op}}}\, \omega \leq n$.
(2) $\mathrm{c}\mathcal{T}(R)$-$\mathrm{id}_R\, M \leq n$ *for any* $M \in \mathrm{Mod}\, R$, *and* $\mathrm{ac}\mathcal{T}(S)$-$\mathrm{pd}_S\, N \leq n$ *for any* $N \in \mathrm{Mod}\, S$.
(3) $\mathrm{c}\mathcal{T}(R)$-$\mathrm{id}_R\, M \leq n$ *for any* $M \in \mathrm{mod}\, R$, *and* $\mathrm{ac}\mathcal{T}(S)$-$\mathrm{pd}_S\, N \leq n$ *for any* $N \in \mathrm{mod}\, S$.
(4) $\mathrm{c}\mathcal{T}(R)$-$\mathrm{id}_R\, R \leq n$ *and* $\mathrm{ac}\mathcal{T}(S)$-$\mathrm{pd}_S\, D(S_S) \leq n$.

Proof. (1)⇒(2) follows from Theorems 4.2 and 4.10.

(2)⇒(3)⇒(4) are trivial.

(4)⇒(1). Since $\mathrm{c}\mathcal{T}(R)$-$\mathrm{id}_R\, R \leq n$ and $\mathrm{ac}\mathcal{T}(S)$-$\mathrm{pd}_S\, D(S_S) \leq n$ by (4), we have $\mathrm{pd}_{S^{\mathrm{op}}}\, \omega = \mathcal{P}_\omega(R)$-$\mathrm{id}_R\, R \leq n$ by Lemma 4.5 and the symmetric version of Lemma 4.3, and $\mathrm{pd}_R\, \omega \leq n$ by Lemmas 4.8 and 4.9. Now the assertion follows from Proposition 4.1. ∎

Recall that an artin algebra R is called *Gorenstein* if $\mathrm{id}_R\, R = \mathrm{id}_{R^{\mathrm{op}}}\, R < \infty$. It is easy to see that the (R, R)-bimodule $D(R)$ is semidualizing. Taking $R = S$ and $\omega = D(R)$ in Corollary 4.11, we immediately get

Corollary 4.12. *Let R be an artin algebra and $n \geq 0$. Then the following statements are equivalent:*

(1) *R is Gorenstein with* $\mathrm{id}_R\, R = \mathrm{id}_{R^{\mathrm{op}}}\, R \leq n$.
(2) $\mathrm{c}\mathcal{T}(R)$-$\mathrm{id}_R\, M \leq n$ *for any* $M \in \mathrm{Mod}\, R$, *and* $\mathrm{ac}\mathcal{T}(R)$-$\mathrm{pd}_R\, N \leq n$ *for any* $N \in \mathrm{Mod}\, S$.
(3) $\mathrm{c}\mathcal{T}(R)$-$\mathrm{id}_R\, M \leq n$ *for any* $M \in \mathrm{mod}\, R$, *and* $\mathrm{ac}\mathcal{T}(R)$-$\mathrm{pd}_R\, N \leq n$ *for any* $N \in \mathrm{mod}\, S$.
(4) $\mathrm{c}\mathcal{T}(R)$-$\mathrm{id}_R\, R \leq n$ *and* $\mathrm{ac}\mathcal{T}(R)$-$\mathrm{pd}_R\, D(R_R) \leq n$.

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