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ON THE NUMBER OF τ-TILTING MODULES OVER THE AUSLANDER ALGEBRAS OF RADICAL SQUARE ZERO NAKAYAMA ALGEBRAS

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Abstract. Let Λ_n be a radical square zero Nakayama algebra with n simple modules and Γ_n the Auslander algebra of Λ_n. We calculate the number $|\tau\text{-tilt}\,\Gamma_n|$ of τ-tilting modules and the number $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ of support τ-tilting modules over Γ_n. In particular, we prove the recurrence relations

$$|\tau\text{-tilt}\,\Gamma_n| = 3|\tau\text{-tilt}\,\Gamma_{n-1}| + |\tau\text{-tilt}\,\Gamma_{n-2}|,$$
$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-2}|,$$

from which the exact values of $|\tau\text{-tilt}\,\Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ are derived.

1. Introduction. The starting point of tilting theory was the introduction of tilting modules over a hereditary algebra by Happel and Ringel [10]. Ever since, the study of tilting modules has been an important branch in the representation theory of finite-dimensional algebras.

In 2014, Adachi, Iyama and Reiten [1] introduced τ-tilting theory replacing the rigidity condition $\mathrm{Ext}^1_\Lambda(M,M) = 0$ for a tilting module by the weaker condition $\mathrm{Hom}_\Lambda(M,\tau M) = 0$ for a τ-tilting module, where Λ is a finite-dimensional algebra and τ is the Auslander–Reiten translation. The support τ-tilting modules are in bijection with some important objects in representation theory including functorially finite torsion classes introduced in [5], 2-term silting complexes introduced in [13], cluster-tilting objects in the cluster category and left finite semibricks introduced in [3]. Therefore, it is important to calculate the number of support τ-tilting modules over a given algebra.

For hereditary algebras, the (support) τ-tilting modules are exactly the (support) tilting modules. For algebras of Dynkin type, the numbers of these

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modules were first calculated via cluster algebras [7], and later via representation theory [14]. In particular, over a hereditary algebra of type $\mathbb{A}_n$, the number of tilting modules is C_n and the number of support tilting modules is C_{n+1}, where C_i is the ith Catalan number $\frac{1}{i+1}\binom{2i}{i}$.

Recall from [4, V.3.2] that a finite-dimensional algebra is *Nakayama* if its quiver is one of the following:

$$\begin{tikzcd} A_n: & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n, & \widetilde{A}_n: & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n. \arrow[lllll, bend right] \end{tikzcd}$$

Adachi [2] gave a recurrence relation for the number of τ-tilting modules over Nakayama algebras of type A_n. Asai [3] also gave a recurrence relation for the number of support τ-tilting modules over Nakayama algebras KA_n/rad^r and $K\widetilde{A}_n/\mathrm{rad}^r$. More recently, Gao and Schiffler [9] extended the recurrence relation of Adachi to τ-tilting modules over $K\widetilde{A}_n/\mathrm{rad}^r$.

It was showed in [6] that the number of tilting modules over the Auslander algebra of $K[x]/(x^n)$ is $n!$. Kajita [12] calculated the number of tilting modules over the Auslander algebra of a hereditary algebra of Dynkin type. Iyama and Zhang [11] classified the support τ-tilting modules over the Auslander algebra of $K[x]/(x^n)$, and they also showed that there is a bijection between the set of support τ-tilting modules over the Auslander algebra of $K[x]/(x^n)$ and the symmetric group of degree n. More recently, Zhang [16] calculated the number of tilting modules over the Auslander algebra Γ_n of a radical square zero Nakayama algebra Λ_n. In particular, Zhang proved that the number of tilting modules over Γ_n is 2^{n-1} if Λ_n is of type A_n; and it is 2^n if Λ_n is of type $\widetilde{A}_n$.

In this paper, we calculate the number $|\tau\text{-tilt}\,\Gamma_n|$ of τ-tilting modules and the number $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ of support τ-tilting modules over the Auslander algebra Γ_n of a radical square zero Nakayama algebra Λ_n. Our result is as follows.

Theorem 1.1 (Theorems 3.1, 3.5, 4.2 and 4.3). *Let Γ_n be the Auslander algebra of a radical square zero Nakayama algebra Λ_n.*

(1) *If Λ_n is of type A_n, then*

$$|\tau\text{-tilt}\,\Gamma_n| = \frac{(3+\sqrt{13})^n - (3-\sqrt{13})^n}{\sqrt{13}\cdot 2^n},$$

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = \frac{(3+2\sqrt{3})^n - (3-2\sqrt{3})^n}{2\sqrt{3}}.$$

(2) *If Λ_n is of type $\widetilde{A}_n$, then*

$$|\tau\text{-tilt}\,\Gamma_n| = \frac{(3+\sqrt{13})^n + (3-\sqrt{13})^n}{2^n},$$

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = (3+2\sqrt{3})^n + (3-2\sqrt{3})^n.$$

The paper is organized as follows. In Section 2, we fix some notations and recall several results about τ-tilting modules and Auslander algebras of radical square zero Nakayama algebras. In Section 3, we show that if Λ_n is of type A_n, then there are recurrence relations

$$|\tau\text{-tilt}\, \Gamma_n| = 3|\tau\text{-tilt}\, \Gamma_{n-1}| + |\tau\text{-tilt}\, \Gamma_{n-2}|,$$
$$|\mathrm{s}\tau\text{-tilt}\, \Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\, \Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\, \Gamma_{n-2}|.$$

In Section 4, we prove the same recurrence relations for Λ_n of type $\widetilde{A}_n$. From these recurrence relations the exact values of $|\tau\text{-tilt}\, \Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\, \Gamma_n|$ are derived. Finally, we list the values of $|\tau\text{-tilt}\, \Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\, \Gamma_n|$ for $1 \leq n \leq 8$ in a table in Section 5.

2. Preliminaries. Throughout this paper, all algebras are basic, connected, finite-dimensional K-algebras over an algebraically closed field K. For an algebra Λ, we denote by $\operatorname{mod} \Lambda$ the category of finitely generated right Λ-modules and by τ the Auslander–Reiten translation of Λ. We use P_i, I_i and S_i to denote the indecomposable projective, injective and simple modules of an algebra corresponding to the vertex i respectively. For any $i, j \in \{1, \ldots, n\}$, we write $[i, j] = \{i, i+1, \ldots, j\}$ if $i \leq j$; otherwise, $[i, j] = \emptyset$. Let e_i be the primitive idempotent element of an algebra corresponding to the vertex i. We write $e_{[i,j]} := e_i + e_{i+1} + \cdots + e_j$.

For a module $M \in \operatorname{mod} \Lambda$, we write $|M|$ for the number of pairwise non-isomorphic indecomposable summands of M, and use $l(M)$ and $\operatorname{pd}_\Lambda M$ to denote the Loewy length and projective dimension of M respectively. For a finite set X, we let $|X|$ denote the cardinality of X. For two sets X_1 and X_2, $X_1 \amalg X_2$ stands for their disjoint union.

Definition 2.1 ([1, Definition 0.1]). Let Λ be an algebra and $M \in \operatorname{mod} \Lambda$. Then M is called

- *τ-rigid* if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$;
- *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$;
- *support τ-tilting* if it is a τ-tilting $\Lambda/\Lambda e\Lambda$-module for some idempotent e of Λ;
- *proper support τ-tilting* if it is support τ-tilting but not τ-tilting.

Recall that $M \in \operatorname{mod} \Lambda$ is called *sincere* if every simple Λ-module appears as a composition factor in M. It is well-known that the τ-tilting modules are exactly the sincere support τ-tilting modules [1, Proposition 2.2(a)].

We denote by $\tau\text{-tilt}\, \Lambda$ (respectively, $\mathrm{s}\tau\text{-tilt}\, \Lambda$, $\mathrm{ps}\tau\text{-tilt}\, \Lambda$) the set of isomorphism classes of basic τ-tilting (respectively, support τ-tilting, proper support τ-tilting) Λ-modules.

Set

$$\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\,\Lambda := \{M \in \mathrm{ps}\tau\text{-tilt}\,\Lambda \mid M \text{ has no projective direct summands}\}.$$

THEOREM 2.2 ([2, Theorem 2.6]). *Let Λ be a Nakayama algebra. Then there is a bijection between τ-tilt Λ and* $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\,\Lambda$.

The following result is useful.

PROPOSITION 2.3 ([2, Proposition 2.32]). *Let Λ be a Nakayama algebra of type A_n. Then each τ-tilting Λ-module has P_1 as a direct summand.*

As a consequence, we get

LEMMA 2.4. *Let Λ be a Nakayama algebra of type A_n. Then each support τ-tilting Λ-module which has $S_1, \dots, S_{l(P_1)}$ as composition factors has P_1 as a direct summand.*

Proof. Let M be a support τ-tilting Λ-module which has $S_1, \dots, S_{l(P_1)}$ as composition factors. If M is τ-tilting, then it has P_1 as a direct summand by Proposition 2.3. Now, assume that M has $S_1, \dots, S_{l(P_1)}, \dots, S_j$ as composition factors but not S_{j+1}. Let N be the maximal direct summand of M which only has $S_1, \dots, S_{l(P_1)}, \dots, S_j$ as composition factors. Then N is a τ-tilting $\Lambda/\langle e_{[j+1,n]}\rangle$-module. By Proposition 2.3, N has P_1 as a direct summand. ∎

THEOREM 2.5 ([2, Theorem 2.33 and Corollary 2.34]). *Let Λ be a Nakayama algebra of type A_n. Then there are mutually inverse bijections*

$$\tau\text{-tilt}\,\Lambda \leftrightarrow \coprod_{i=1}^{l(P_1)} \tau\text{-tilt}(\Lambda/\langle e_i\rangle)$$

given by $\tau\text{-tilt}\,\Lambda \ni M \mapsto M/P_1$ *and* $N \mapsto N \oplus P_1 \in \tau\text{-tilt}\,\Lambda$. *In particular,*

$$|\tau\text{-tilt}\,\Lambda| = \sum_{i=1}^{l(P_1)} C_{i-1} \cdot |\tau\text{-tilt}(\Lambda/\langle e_{[1,i]}\rangle)|.$$

REMARK 2.6. Let Λ be a Nakayama algebra of type A_n. Then every τ-tilting Λ-module can be decomposed M as $M = P_1 \oplus N_1 \oplus N_2$ where N_1 is a maximal direct summand of M without S_1 as composition factors. Moreover, $N_1 \oplus N_2$ is a τ-tilting $\Lambda/\langle e_{j+1}\rangle$-module where $j := l(N_2)$ (see [2, proof of Theorem 2.33]).

An algebra Λ is of *finite representation type* if there are only finitely many indecomposable Λ-modules $X_1, \dots, X_m$ up to isomorphism. In this case, the endomorphism algebra $\mathrm{End}_\Lambda(\bigoplus_{i=1}^m X_i)$ is called the *Auslander algebra* of Λ.

By a straightforward calculation, we get the quiver of the Auslander algebra of radical square zero Nakayama algebras:

Proposition 2.7.

(1) *The Auslander algebra* Γ_n *of* $\Lambda_n := KA_n/\mathrm{rad}^2$ *is given by the quiver*

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1$$

with the relations $a_{2k-1}a_{2k} = 0$ *for* $1 \leq k \leq n-1$.

(2) *The Auslander algebra* Γ'_n *of* $\Lambda_n := K\widetilde{A}_n/\mathrm{rad}^2$ *is given by the quiver*

$$\begin{tikzcd}
1 \arrow[r, "a_1"'] & 2 \arrow[r, "a_2"'] & 3 \arrow[r, "a_3"'] & \cdots \arrow[r] & 2n-1 \arrow[r, "a_{2n-1}"'] & 2n \arrow[lllll, bend right, "a_{2n}"']
\end{tikzcd}$$

with the relations $a_{2k-1}a_{2k} = 0$ *for* $1 \leq k \leq n$.

3. The case for Γ_n. In this section, we will give formulas for $|\tau\text{-tilt}\,\Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$.

Let Δ_n be the algebra given by the quiver

$$0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1$$

with the relations $a_{2k-1}a_{2k} = 0$ for $1 \leq k \leq n-1$.

The following result gives a formula for $|\tau\text{-tilt}\,\Gamma_n|$.

Theorem 3.1. *We have*

$$|\tau\text{-tilt}\,\Gamma_n| = 3|\tau\text{-tilt}\,\Gamma_{n-1}| + |\tau\text{-tilt}\,\Gamma_{n-2}|$$

with $|\tau\text{-tilt}\,\Gamma_1| = 1$ *and* $|\tau\text{-tilt}\,\Gamma_2| = 3$. *Hence*

$$|\tau\text{-tilt}\,\Gamma_n| = \frac{(3+\sqrt{13})^n - (3-\sqrt{13})^n}{\sqrt{13}\cdot 2^n}.$$

Proof. Applying Theorem 2.5 to Γ_n and Δ_n, we have

$$\begin{aligned}
(1) \qquad |\tau\text{-tilt}\,\Gamma_n| &= C_0 \cdot |\tau\text{-tilt}(\Gamma_n/\langle e_1\rangle)| + C_1 \cdot |\tau\text{-tilt}(\Gamma_n/\langle e_1 + e_2\rangle)| \\
&= |\tau\text{-tilt}\,\Delta_{n-1}| + |\tau\text{-tilt}\,\Gamma_{n-1}|
\end{aligned}$$

and

$$\begin{aligned}
(2) \qquad |\tau\text{-tilt}\,\Delta_n| &= C_0 \cdot |\tau\text{-tilt}(\Delta_n/\langle e_0\rangle)| + C_1 \cdot |\tau\text{-tilt}(\Delta_n/\langle e_0 + e_1\rangle)| \\
&\quad + C_2 \cdot |\tau\text{-tilt}(\Delta_n/\langle e_0 + e_1 + e_2\rangle)| \\
&= |\tau\text{-tilt}\,\Gamma_n| + |\tau\text{-tilt}\,\Delta_{n-1}| + 2|\tau\text{-tilt}\,\Gamma_{n-1}|.
\end{aligned}$$

The formula (1) implies

$$|\tau\text{-tilt}\,\Delta_{n-1}| = |\tau\text{-tilt}\,\Gamma_n| - |\tau\text{-tilt}\,\Gamma_{n-1}|.$$

Applying it to (2), we have

$$(3) \qquad |\tau\text{-tilt}\,\Gamma_n| = 3|\tau\text{-tilt}\,\Gamma_{n-1}| + |\tau\text{-tilt}\,\Gamma_{n-2}|$$

This is a linear homogeneous recurrence relation of degree 2 and its characteristic equation is $x^2 - 3x - 1 = 0$. The proof is finished. ■

Let Λ be an algebra. Recall that a module $M \in \operatorname{mod} \Lambda$ is called *tilting* if

- $\operatorname{pd}_{\Lambda} M \leq 1$;
- $\operatorname{Ext}^1_{\Lambda}(M, M) = 0$;
- $|M| = |\Lambda|$.

Thus a module $M \in \operatorname{mod} \Lambda$ is tilting if and only if it is τ-tilting and $\operatorname{pd}_{\Lambda} M \leq 1$, by the Auslander–Reiten formula.

The set of all tilting Λ-modules is denoted by $\operatorname{tilt} \Lambda$. The following result is part of [16, Theorem 2.8]. Here we give another proof.

Proposition 3.2. $|\operatorname{tilt} \Gamma_n| = 2^{n-1}$.

Proof. Note that P_1 is the unique Γ_n-module which has S_1 as a composition factor and its projective dimension is at most 1. By Remark 2.6 and the above argument, $P_1 \oplus N_1$ is a tilting Γ_n-module if and only if N_1 is a tilting $\Gamma_n/\langle e_1 \rangle$-module, since $\operatorname{pd}_{\Gamma_n} N_1 = \operatorname{pd}_{\Gamma_n/\langle e_1 \rangle} N_1$. Thus

$$|\operatorname{tilt} \Gamma_n| = |\operatorname{tilt}(\Gamma_n/\langle e_1 \rangle)| = |\operatorname{tilt} \Delta_{n-1}|.$$

Note that P_0 and S_0 are the only two Δ_n-modules which have S_0 as a composition factor and their projective dimension is at most 1. Similarly, we get

$$|\operatorname{tilt} \Delta_n| = |\operatorname{tilt}(\Delta_n/\langle e_0 \rangle)| + |\operatorname{tilt}(\Delta/\langle e_0 + e_1 \rangle)| = |\operatorname{tilt} \Gamma_n| + |\operatorname{tilt} \Delta_{n-1}|.$$

Thus $|\operatorname{tilt} \Gamma_n| = 2|\operatorname{tilt} \Gamma_{n-1}|$ with $|\operatorname{tilt} \Gamma_1| = 1$, and so $|\operatorname{tilt} \Gamma_n| = 2^{n-1}$. ■

As generalizations of simple modules and semisimple modules, bricks and semibricks were introduced and studied in [8, 15]. Let Λ be an algebra. A Λ-module M is called a *brick* if $\operatorname{Hom}_{\Lambda}(M, M)$ is a K-division algebra, and a *semibrick* is a set consisting of isoclasses of pairwise Hom-orthogonal bricks. Recall from [3] that a semibrick $\mathcal{S}$ is called *left finite* if the smallest torsion class $T(\mathcal{S})$ containing $\mathcal{S}$ is functorially finite. There exists a bijection between $\mathrm{s}\tau\text{-tilt}\, \Lambda$ and the set of left finite semibricks of Λ [3, Theorem 2.3]. Note that every torsion class is functorially finite for a representation-finite algebra. So, for a Nakayama algebra Λ, there exists a bijection between $\mathrm{s}\tau\text{-tilt}\, \Lambda$ and the set $\operatorname{sbrick} \Lambda$ of semibricks of Λ, and hence $|\mathrm{s}\tau\text{-tilt}\, \Lambda| = |\operatorname{sbrick} \Lambda|$. Asai gave a method to calculate the number of semibricks over $KA_n/\operatorname{rad}^r$. In fact, we have the following more general result.

Proposition 3.3. *Let Λ be a Nakayama algebra of type A_n. Then*

$$(1)\quad |\mathrm{s}\tau\text{-tilt}\, \Lambda| = 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_n \rangle)| + \sum_{i=2}^{l(I_n)} C_{i-1} |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]} \rangle)|,$$

$$(2)\quad |\mathrm{s}\tau\text{-tilt}\, \Lambda| = 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_1 \rangle)| + \sum_{i=2}^{l(P_1)} C_{i-1} |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_{[1,i]} \rangle)|.$$

Proof. (1) For a given brick X of Λ with $\operatorname{top} X = S_i$ and $\operatorname{soc} X = S_j$, we will denote $S_{i,j} := X$.

We define W_0 as the subset of $\operatorname{sbrick}\Lambda$ consisting of the semibricks without S_n as a composition factor. It is clear that $|W_0| = |\operatorname{sbrick}(\Lambda/\langle e_n\rangle)|$.

Let W_i ($i = 1, \dots, l(I_n)$) be the subset of $\operatorname{sbrick}\Lambda$ consisting of the semibricks which contain the brick $S_{n-i+1,n}$.

First, there is a bijection

$$W_1 \mapsto \operatorname{sbrick}(\Lambda/\langle e_n\rangle)$$

defined by $\mathcal{S} \mapsto \mathcal{S} \setminus \{S_{n,n}\}$. So $|W_0| = |\operatorname{sbrick}(\Lambda/\langle e_n\rangle)|$.

Secondly, for $i = 2, 3, \dots, l(I_n)$, there exists a bijection

$$W_1 \mapsto \operatorname{sbrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle) \times \operatorname{sbrick}(\Lambda/\langle 1 - e_{[n-i+2,n-1]}\rangle)$$

defined by

$$\begin{aligned}\mathcal{S} \mapsto (\{S \in \mathcal{S} \mid \operatorname{Supp} S \cap [n-i+1, n] = \emptyset\},\\ \{S \in \mathcal{S} \mid \operatorname{Supp} S \subset [n-i+2, n-1]\}),\end{aligned}$$

where $\operatorname{Supp} S$ stands for the support of S. Note that $\operatorname{sbrick}\Lambda = \bigcup_{i=0}^{l(I_n)} W_i$. Thus we obtain

$$\begin{aligned}|\mathrm{s}\tau\text{-tilt}\,\Lambda| = |\operatorname{sbrick}\Lambda| = \sum_{i=0}^{l(I_n)} |W_i|\\ &= 2|\operatorname{sbrick}(\Lambda/\langle e_n\rangle)|\\ &\quad + \sum_{i=2}^{l(I_n)} |\operatorname{sbrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle)| \cdot |\operatorname{sbrick}(\Lambda/\langle 1 - e_{[n-i+2,n-1]}\rangle)|\\ &= 2|\operatorname{sbrick}(\Lambda/\langle e_n\rangle)| + \sum_{i=2}^{l(I_n)} |\operatorname{sbrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle)| \cdot |\operatorname{sbrick}(KA_{i-2})|\\ &= 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_n\rangle)| + \sum_{i=2}^{l(I_n)} |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]}\rangle)| \cdot |\mathrm{s}\tau\text{-tilt}(KA_{i-2})|\\ &= 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_n\rangle)| + \sum_{i=2}^{l(I_n)} C_{i-1} \cdot |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]}\rangle)|.\end{aligned}$$

(2) Note that there is a bijection between $\mathrm{s}\tau\text{-tilt}\,\Lambda$ and $\mathrm{s}\tau\text{-tilt}\,\Lambda^{\mathrm{op}}$ [1, Theorem 2.14]). Now the assertion follows from (1). ■

We give the following example to illustrate Proposition 3.3.

EXAMPLE 3.4. Let Λ be the algebra given by the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \rightarrow 4$$

with the relation $\alpha\beta = 0$. By Proposition 3.3(1), we have

$$\begin{aligned}|\mathrm{s}\tau\text{-tilt}\,\Lambda| &= 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_4\rangle)| + |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_3 + e_4\rangle)| \\ &\quad + 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_2 + e_3 + e_4\rangle)| \\ &= 2 \times 12 + 5 + 2 \times 2 = 33.\end{aligned}$$

On the other hand, by Proposition 3.2(2),

$$|\mathrm{s}\tau\text{-tilt}\,\Lambda| = 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_1\rangle)| + |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_1 + e_2\rangle)| = 2 \times 14 + 5 = 33.$$

The following result gives a formula for $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$.

THEOREM 3.5. *We have*

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-2}|$$

with $|\mathrm{s}\tau\text{-tilt}\,\Gamma_1| = 2$ *and* $|\mathrm{s}\tau\text{-tilt}\,\Gamma_2| = 12$. *Hence*

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = \frac{(3 + 2\sqrt{3})^n - (3 - 2\sqrt{3})^n}{2\sqrt{3}}.$$

Proof. Applying Proposition 3.3(2) to Γ_n and Δ_n respectively, we have

$$\begin{aligned}|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| &= 2|\mathrm{s}\tau\text{-tilt}(\Gamma_n/\langle e_1\rangle)| + C_1 \cdot |\mathrm{s}\tau\text{-tilt}(\Gamma_n/\langle e_1 + e_2\rangle)| \\ &= 2|\mathrm{s}\tau\text{-tilt}\,\Delta_{n-1}| + |\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}|\end{aligned} \tag{4}$$

and

$$\begin{aligned}|\mathrm{s}\tau\text{-tilt}\,\Delta_n| &= 2|\mathrm{s}\tau\text{-tilt}(\Delta_n/\langle e_0\rangle)| + C_1 \cdot |\mathrm{s}\tau\text{-tilt}(\Delta_n/\langle e_0 + e_1\rangle)| \\ &\quad + C_2 \cdot |\mathrm{s}\tau\text{-tilt}(\Delta_n/\langle e_0 + e_1 + e_2\rangle)| \\ &= 2|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| + |\mathrm{s}\tau\text{-tilt}\,\Delta_{n-1}| + 2|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}|.\end{aligned}$$

This implies

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-2}|. \tag{5}$$

This is a linear homogeneous recurrence relation of degree 2 and its characteristic equation is $x^2 - 6x - 3 = 0$. The proof is finished. ■

Let $\overline{\Gamma}_n$ be the algebra given by the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n$$

with the relations $a_{2k-1}a_{2k} = 0$ for $1 \le k \le n-1$, and let $\overline{\Delta}_n$ be the algebra given by the quiver

$$0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n$$

with the relations $a_{2k-1}a_{2k} = 0$ for $1 \le k \le n-1$. By using the same argument as in Theorem 3.5, we can obtain

$$|\mathrm{s}\tau\text{-tilt}\,\overline{\Delta}_n| = 6|\mathrm{s}\tau\text{-tilt}\,\overline{\Delta}_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\overline{\Delta}_{n-2}|.$$

4. The case for Γ'_n. In this section, we will give formulas for $|\tau\text{-tilt}\,\Gamma'_n|$ and $|\text{s}\tau\text{-tilt}\,\Gamma'_n|$.

Let X_n be the set of all support τ-tilting Γ_n-modules which do not have $P_1, \ldots, P_{2n-3}$ as direct summands, and let Y_n be the set of all support τ-tilting Δ_n-modules which do not have $P_0, P_1, \ldots, P_{2n-3}$ as direct summands. Let X'_n be the set of all support τ-tilting $\overline{\Gamma}_n$-modules which do not have $P_1, \ldots, P_{2n-2}$ as direct summands, and let Y'_n be the set of all support τ-tilting $\overline{\Delta}_n$-modules which do not have $P_0, P_1, \ldots, P_{2n-2}$ as direct summands.

We need the following lemma.

Lemma 4.1.

(1) $|X_n| = 3|X_{n-1}| + |X_{n-2}|$ *and* $|Y_n| = 3|Y_{n-1}| + |Y_{n-2}|$.
(2) $|X'_n| = 3|X'_{n-1}| + |X'_{n-2}|$ *and* $|Y'_n| = 3|Y'_{n-1}| + |Y'_{n-2}|$.

Proof. (1) By Lemma 2.4, all support τ-tilting Γ_n-modules which have S_1, S_2 as composition factors must have P_1 as a direct summand. Hence X_n consists of two parts: the first part comes from all support τ-tilting Γ_n-modules which do not have $P_1, \ldots, P_{2n-3}$ as direct summands and do not have S_1 as a composition factor (their number is exactly $|Y_{n-1}|$); the second part comes from all support τ-tilting Γ_n-modules which do not have $P_1, \ldots, P_{2n-3}$ as direct summands and have S_1 as a composition factor but not S_2 (their number is exactly $|X_{n-1}|$). Hence, $|X_n| = |Y_{n-1}| + |X_{n-1}|$. Similarly, we have $|Y_n| = |X_n| + |Y_{n-1}| + 2|X_{n-1}|$. These two equalities imply $|X_n| = 3|X_{n-1}| + |X_{n-2}|$ and $|Y_n| = 3|Y_{n-1}| + |Y_{n-2}|$.

(2) The proof is similar. ■

The following result gives a formula for $|\tau\text{-tilt}\,\Gamma'_n|$.

Theorem 4.2. *We have*

$$|\tau\text{-tilt}\,\Gamma'_n| = 3|\tau\text{-tilt}\,\Gamma'_{n-1}| + |\tau\text{-tilt}\,\Gamma'_{n-2}|$$

with $|\tau\text{-tilt}\,\Gamma'_1| = 3$ *and* $|\tau\text{-tilt}\,\Gamma'_2| = 11$. *Hence*

$$|\tau\text{-tilt}\,\Gamma'_n| = \frac{(3+\sqrt{13})^n + (3-\sqrt{13})^n}{2^n}.$$

Proof. We claim that every proper support τ-tilting Γ'_n-module M which has S_1, S_2 as composition factors must have a projective Γ'_n-module as a direct summand. Indeed, if M does not have S_{2n} as a composition factor, then it has P_1 as a direct summand by Lemma 2.4. Now, assume that M has $S_i, S_{i+1}, \ldots, S_{2n}, S_1, S_2$ as composition factors, but not S_{i-1}. Then M has P_i as a direct summand by Lemma 2.4.

Now, $\text{ps}\tau\text{-tilt}_{\text{np}}\,\Gamma'_n$ consists of the following two parts:

(i) U_1: the subset of modules which do not have S_2 as a composition factor.

(ii) U_2: the subset of modules which have S_2 as a composition factor, but not S_1.

Since $\overline{\Lambda} := \Gamma'_n/\langle e_2\rangle$ is the quiver

$$3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n \xrightarrow{a_{2n}} 1$$

with the relations $a_{2k-1}a_{2k} = 0$ for $2 \le k \le n$, U_1 is exactly the set of support τ-tilting $\overline{\Lambda}$-modules which do not have $P_3, P_4, \dots, P_{2n-1}$ as direct summands, and so $|U_1| = |X_n|$. Note that $\overline{\Gamma} := \Gamma'_n/\langle e_1\rangle$ is the quiver

$$2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n$$

with the relations $a_{2k-1}a_{2k} = 0$ for $2 \le k \le n-1$. Thus, the number of support τ-tilting $\overline{\Gamma}$-modules which do not have $P_2, P_4, \dots, P_{2n-2}$ as direct summands is exactly $|Y'_{n-1}|$. Moreover, the number of support τ-tilting $\overline{\Gamma}$-modules which do not have $P_2, P_4, \dots, P_{2n-2}$ as direct summands and do not have S_2 as a composition factor is exactly $|X'_{n-1}|$. Therefore, $|U_2| = |Y'_{n-1}| - |X'_{n-1}|$. By Theorem 2.2, we obtain

$$|\tau\text{-tilt}\, \Gamma'_n| = |\text{ps}\tau\text{-tilt}_{\text{np}}\, \Gamma'_n| = |U_1| + |U_2| = |X_n| + |Y'_{n-1}| - |X'_{n-1}|.$$

Now, the recurrence relation for $|\tau\text{-tilt}\, \Gamma'_n|$ follows from Lemma 4.1. ∎

The following result gives a formula for $|\text{s}\tau\text{-tilt}\, \Gamma'_n|$.

Theorem 4.3. *We have*

$$|\text{s}\tau\text{-tilt}\, \Gamma'_n| = 6|\text{s}\tau\text{-tilt}\, \Gamma'_{n-1}| + 3|\text{s}\tau\text{-tilt}\, \Gamma'_{n-2}|$$

with $|\text{s}\tau\text{-tilt}\, \Gamma'_1| = 6$ *and* $|\text{s}\tau\text{-tilt}\, \Gamma'_2| = 42$. *Hence*

$$|\text{s}\tau\text{-tilt}\, \Gamma'_n| = (3+2\sqrt{3})^n + (3-2\sqrt{3})^n.$$

Proof. The set $\text{sbrick}\, \Gamma'_n$ of semibricks of Γ'_n consists of five parts:

(i) V_0: the semibricks without S_1 as a composition factor.
(ii) V_1: the semibricks which contain S_1 but not the brick I_2.
(iii) V_2: the semibricks which contain I_1.
(iv) V_3: the semibricks which contain P_1.
(v) V_4: the semibricks which contain I_2.

Obviously, $|V_0| = |\text{sbrick}(\Gamma'_n/\langle e_1\rangle)| = |\text{sbrick}\, \overline{\Delta}_{n-1}|$.

There is a bijection $V_1 \mapsto \text{sbrick}(\Gamma'_n/\langle e_1\rangle)$ defined by $\mathcal{S} \mapsto \mathcal{S} \setminus \{S_1\}$, so

$$|V_1| = |\text{sbrick}(\Gamma'_n/\langle e_1\rangle)| = |\text{sbrick}\, \overline{\Delta}_{n-1}|.$$

Similarly, there are bijections

$$V_2 \mapsto \text{sbrick}(\Gamma'_n/\langle e_1 + e_{2n}\rangle) \quad \text{and} \quad V_3 \mapsto \text{sbrick}(\Gamma'_n/\langle e_1 + e_2\rangle),$$

so

$$|V_2| = |\text{sbrick}(\Gamma'_n/\langle e_1 + e_{2n}\rangle)| = |\text{sbrick}\, \Delta_{n-1}|,$$
$$|V_3| = |\text{sbrick}(\Gamma'_n/\langle e_1 + e_2\rangle)| = |\text{sbrick}\, \Delta^{\text{op}}_{n-1}|.$$

Finally, we can define a bijection

$$V_4 \mapsto \operatorname{sbrick}(\mathit{\Gamma}'_n/\langle e_1 + e_2 + e_{2n}\rangle) \times \operatorname{sbrick}(\mathit{\Gamma}'_n/\langle 1 - e_1\rangle)$$

by $V_4 \ni \mathcal{S} \mapsto (\mathcal{S} \setminus \{S_1, I_2\}, S_1 \cap \mathcal{S})$. Thus

$$|V_4| = |\operatorname{sbrick}(\mathit{\Gamma}'_n/\langle e_1 + e_2 + e_{2n}\rangle)| \cdot |\operatorname{sbrick}(\mathit{\Gamma}'_n/\langle 1 - e_1\rangle)| = 2|\operatorname{sbrick} \mathit{\Gamma}_{n-1}|.$$

Therefore

$$\begin{aligned}
|\text{s}\tau\text{-tilt}\, \mathit{\Gamma}'_n| = |\operatorname{sbrick} \mathit{\Gamma}'_n| = \sum_{i=0}^{4} |V_i| \\
&= 2|\operatorname{sbrick} \overline{\mathit{\Delta}}_{n-1}| + |\operatorname{sbrick} \mathit{\Delta}_{n-1}| + |\operatorname{sbrick} \mathit{\Delta}^{\mathrm{op}}_{n-1}| + 2|\operatorname{sbrick} \mathit{\Gamma}_{n-1}| \\
&= 2|\text{s}\tau\text{-tilt}\, \overline{\mathit{\Delta}}_{n-1}| + |\text{s}\tau\text{-tilt}\, \mathit{\Delta}_{n-1}| + |\text{s}\tau\text{-tilt}\, \mathit{\Delta}^{\mathrm{op}}_{n-1}| + 2|\text{s}\tau\text{-tilt}\, \mathit{\Gamma}_{n-1}| \\
&= 2|\text{s}\tau\text{-tilt}\, \overline{\mathit{\Delta}}_{n-1}| + 2|\text{s}\tau\text{-tilt}\, \mathit{\Delta}_{n-1}| + 2|\text{s}\tau\text{-tilt}\, \mathit{\Gamma}_{n-1}|.
\end{aligned}$$

Note that $|\text{s}\tau\text{-tilt}\, \mathit{\Delta}_{n-1}|$ is a linear combination of $|\text{s}\tau\text{-tilt}\, \mathit{\Gamma}_n|$ and $|\text{s}\tau\text{-tilt}\, \mathit{\Gamma}_{n-1}|$ by (4), so $|\text{s}\tau\text{-tilt}\, \mathit{\Delta}_n|$ has the same recurrence relation as $|\text{s}\tau\text{-tilt}\, \mathit{\Gamma}_n|$. In particular, $|\text{s}\tau\text{-tilt}\, \overline{\mathit{\Delta}}_n|$, $|\text{s}\tau\text{-tilt}\, \mathit{\Delta}_n|$, $|\text{s}\tau\text{-tilt}\, \mathit{\Gamma}_n|$ have the same recurrence relations, and so $|\text{s}\tau\text{-tilt}\, \mathit{\Gamma}'_n|$ also has the same recurrence relation. ■

5. Examples. In this section, we list the numbers of (support) τ-tilting modules over $\mathit{\Gamma}_n$ and $\mathit{\Gamma}'_n$ in the following table. The sequence $|\tau\text{-tilt}\, \mathit{\Gamma}_n|$ is listed in the On-line Encyclopedia of Integer Sequences (OEIS) as the sequence A006190 and $|\tau\text{-tilt}\, \mathit{\Gamma}'_n|$ as A006497.

n	1	2	3	4	5	6	7	8
$\lvert\tau\text{-tilt}\, \mathit{\Gamma}_n\rvert$	1	3	10	33	109	360	1189	3927
$\lvert\text{s}\tau\text{-tilt}\, \mathit{\Gamma}_n\rvert$	2	12	78	504	3258	21060	136134	879984
$\lvert\tau\text{-tilt}\, \mathit{\Gamma}'_n\rvert$	3	11	36	119	393	1298	4287	114159
$\lvert\text{s}\tau\text{-tilt}\, \mathit{\Gamma}'_n\rvert$	6	42	270	17464	11286	72954	471582	3048354

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