COLLOQUIUM MATHEMATICUM

VOL. 158 2019 NO. 2

TWO FILTRATION RESULTS FOR MODULES WITH APPLICATIONS TO THE AUSLANDER CONDITION

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Abstract. Under some strong cograde conditions, we obtain two different filtrations of modules in terms of the properties of cotransposes of modules with respect to a semidualizing bimodule. Then we apply these filtrations to investigate the relationship between artin algebras satisfying the Auslander condition and Gorenstein algebras, which is related to a conjecture of Auslander and Reiten.

1. Introduction. Let R be a left and right noetherian ring and M a finitely generated left R-module. In [4], Auslander devised the so-called *Auslander sequence*

$$0 \to \operatorname{Ext}^1_{R^{\mathrm{op}}}(\operatorname{Tr} M, R) \to M \to \operatorname{Hom}_{R^{\mathrm{op}}}(\operatorname{Hom}_R(M, R), R) \to \operatorname{Ext}^2_{R^{\mathrm{op}}}(\operatorname{Tr} M, R) \to 0,$$

where $\operatorname{Tr} M$ denotes the transpose of M. This sequence has proved very useful for the homological study of noetherian rings. Huang [19, Theorem 2.3] established a semidualizing version of this sequence. Under the Auslander condition, Hoshino and Nishida [18, Theorem 2.2] generalized the Auslander sequence by using a certain filtration of modules. Iyama and Jasso [23, Proposition 2.7] extended this sequence to a dualizing R-variety. Recently, in [28], for arbitrary associative rings R and S, we introduced the cotranspose $\operatorname{cTr}_\omega M$ of M with respect to a semidualizing bimodule ${}_R\omega_S$, and used it to provide the *dual Auslander sequence*

$$0 \to \operatorname{Tor}^S_2(\omega, \operatorname{cTr}_\omega M) \to \omega \otimes_S \operatorname{Hom}_R(\omega, M) \to M \to \operatorname{Tor}^S_1(\omega, \operatorname{cTr}_\omega M) \to 0.$$

In analogy with the philosophy of Hoshino and Nishida, one of our aims in this paper is to look for a special filtration of modules to generalize the dual Auslander sequence.

2010 *Mathematics Subject Classification*: 18G25, 16E10, 16E30.

Key words and phrases: n-ω-cotorsionfree modules, (strong) Ext-cograde, (strong) Tor-cograde, filtrations, Auslander condition, Auslander–Reiten conjecture, dual Evans–Griffith presentations.
Received 31 July 2018.
Published online 29 July 2019.

DOI: 10.4064/cm7678-10-2018

On the other hand, the grade condition for modules is linked with some interesting homological properties; see for example, [5, 6, 11, 21, 22]. In particular, Auslander and Bridger [5, Theorem 2.37] showed that if R satisfies some grade condition, then for any finitely generated left R-module M there exists a spherical filtration

$$M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M \oplus P$$

with P a finitely generated projective left R-module. Furthermore, Huang [20] gave a different filtration result for modules over right quasi k-Gorenstein rings. Along this direction, another aim of this paper is to see how the cograde condition induces some filtrations of modules.

The paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

Let R and S be rings and ${}_R\omega_S$ a semidualizing bimodule. In Section 3, we in particular describe a certain filtration of submodules of a left noetherian R-module M with finite Ext-cograde with respect to ω in case ω satisfies the ∞-cograde condition (Theorem 3.12). It is a dual version of [18, Theorem 2.2]. In Section 4, we prove that if ω satisfies the n-cograde condition with $n \geq 1$, then for any left R-module M, there exists an injective left R-module I and a chain of monomorphisms

$$M_n \rightarrowtail M_{n-1} \rightarrowtail \cdots \rightarrowtail M_1 \rightarrowtail M_0 = M \oplus I$$

with some interesting homological properties (Theorem 4.5).

Recall that an artin algebra R is said to satisfy the *Auslander condition* if the projective dimension of the ith term in a minimal injective resolution of ${}_RR$ is at most $i-1$ for any $i \geq 1$. Auslander and Reiten [6] conjectured that an artin algebra R satisfying the Auslander condition is Gorenstein (that is, the left and right self-injective dimensions of R are finite). In Section 5, we apply the two filtrations of modules obtained in Sections 3 and 4 to give some necessary (and sufficient) conditions for an artin algebra satisfying the Auslander condition to be Gorenstein (Theorems 5.2 and 5.4). Finally, we introduce the notion of dual Evans–Griffith presentation of modules. We prove that if ω satisfies the n-cograde condition with $n \geq 1$, then for any $0 \leq i \leq n-1$, each i-Bass-cosyzygy module admits a dual Evans–Griffith presentation (Proposition 5.6).

2. Preliminaries. Throughout, R and S are fixed associative rings with unit. We use $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) to denote the class of left R-modules (resp. finitely generated left R-modules). Let $M \in \operatorname{Mod} R$. We use $\operatorname{pd}_R M$ and $\operatorname{id}_R M$ to denote the projective and injective dimensions of M respectively, and $\operatorname{Add}_R M$ for the subclass of $\operatorname{Mod} R$ consisting of all direct summands of direct sums of copies of M.

Definition 2.1 ([2, 17]). An $(R\text{-}S)$-bimodule ${}_R\omega_S$ is called *semidualizing* if the following conditions are satisfied:

(1) ${}_R\omega$ admits a degreewise finite R-projective resolution.
(2) ω_S admits a degreewise finite S-projective resolution.
(3) The homothety map ${}_RR_R \xrightarrow{{}_R\gamma} \mathrm{Hom}_{S^{\mathrm{op}}}(\omega,\omega)$ is an isomorphism.
(4) The homothety map ${}_SS_S \xrightarrow{\gamma_S} \mathrm{Hom}_R(\omega,\omega)$ is an isomorphism.
(5) $\mathrm{Ext}_R^{\geq 1}(\omega,\omega) = 0$.
(6) $\mathrm{Ext}_{S^{\mathrm{op}}}^{\geq 1}(\omega,\omega) = 0$.

From now on, ${}_R\omega_S$ denotes a semidualizing bimodule. We write $(-)_* := \mathrm{Hom}(\omega,-)$. Following [17], set

$$\mathcal{P}_\omega(R) := \{\omega \otimes_S P \mid P \text{ is projective in } \mathrm{Mod}\, S\},$$
$$\mathcal{I}_\omega(S) := \{I_* \mid I \text{ is injective in } \mathrm{Mod}\, R\}.$$

The modules in $\mathcal{P}_\omega(R)$ and $\mathcal{I}_\omega(S)$ are called *ω-projective* and *ω-injective* respectively. We use $\mathcal{I}(R)$ to denote the subclass of $\mathrm{Mod}\, R$ consisting of injective modules, and $\mathcal{P}(S)$ for the subclass of $\mathrm{Mod}\, S$ consisting of projective modules. Let $M \in \mathrm{Mod}\, R$ and $N \in \mathrm{Mod}\, S$. Then we have two canonical valuation homomorphisms:

$$\theta_M : \omega \otimes_S M_* \to M \quad \text{defined by} \quad \theta_M(x \otimes f) = f(x)$$

for any $x \in \omega$ and $f \in M_*$; and

$$\mu_N : N \to (\omega \otimes_S N)_* \quad \text{defined by} \quad \mu_N(y)(x) = x \otimes y$$

for any $y \in N$ and $x \in \omega$.

Definition 2.2 ([17]). The *Bass class* $\mathcal{B}_\omega(R)$ with respect to ω consists of all left R-modules M satisfying the following conditions:

(1) $\mathrm{Ext}_R^{\geq 1}(\omega, M) = 0$.
(2) $\mathrm{Tor}^S_{\geq 1}(\omega, M_*) = 0$.
(3) θ_M is an isomorphism in $\mathrm{Mod}\, R$.

The *Auslander class* $\mathcal{A}_\omega(S)$ with respect to ω consists of all left S-modules N satisfying the following conditions:

(1) $\mathrm{Tor}^S_{i\geq 1}(\omega, N) = 0$.
(2) $\mathrm{Ext}_R^{\geq 1}(\omega, \omega \otimes_S N) = 0$.
(3) μ_N is an isomorphism in $\mathrm{Mod}\, S$.

Note that $\mathcal{I}_\omega(S) \cup \mathcal{P}(S) \subseteq \mathcal{A}_\omega(S)$ and $\mathcal{P}_\omega(R) \cup \mathcal{I}(R) \subseteq \mathcal{B}_\omega(R)$ [17, Lemma 4.1 and Corollary 6.1]. Let $M \in \mathrm{Mod}\, R$. We use

$$0 \to M \to I^0(M) \xrightarrow{f^0} I^1(M) \xrightarrow{f^1} \cdots \xrightarrow{f^{i-1}} I^i(M) \xrightarrow{f^i} \cdots \tag{2.1}$$

to denote a minimal injective resolution of M. For any $n \geq 1$, $\mathrm{co}\Omega^n(M) := \mathrm{Im}\, f^{n-1}$ is called the nth *cosyzygy* of M, and in particular $\mathrm{co}\Omega^0(M) = M$.

Definition 2.3 ([28]). Let $M \in \operatorname{Mod} R$ and $n \geq 1$.

(1) $\mathrm{cTr}_\omega M := \operatorname{Coker}(f^0{}_*)$ is called the *cotranspose* of M with respect to ${}_R\omega_S$, where f^0 is as in (2.1).

(2) M is called *n-ω-cotorsionfree* if $\operatorname{Tor}^S_{1\leq i\leq n}(\omega, \mathrm{cTr}_\omega M) = 0$. In particular, every module in $\operatorname{Mod} R$ is 0-ω-cotorsionfree.

We use $\mathrm{c}\mathcal{T}^n_\omega(R)$ to denote the subcategory of $\operatorname{Mod} S$ consisting of n-ω-cotorsionfree modules.

Dually, let $N \in \operatorname{Mod} S$. We use

$$\cdots \xrightarrow{f_1} F_1(N) \xrightarrow{f_0} F_0(N) \xrightarrow{f_{-1}} N \to 0 \tag{2.2}$$

to denote a minimal flat resolution of N in $\operatorname{Mod} S$, where each $F_i(N) \twoheadrightarrow \operatorname{Coker} f_i$ is the flat cover of $\operatorname{Coker} f_i$. The existence of such a resolution is guaranteed by the fact that any module has a flat cover (see [9]). Note that $(\omega \otimes_S -, \operatorname{Hom}_R(\omega, -))$ is an adjoint pair. So, it is reasonable to introduce the adjoint counterparts of the notions in Definition 2.3:

Definition 2.4 ([30]). Let $N \in \operatorname{Mod} S$ and $n \geq 1$.

(1) $\mathrm{acTr}_\omega N := \operatorname{Ker}(1_\omega \otimes f_0)$ is called the *adjoint cotranspose* of N with respect to ${}_R\omega_S$, where f_0 is as in (2.2).

(2) N is called *adjoint n-ω-cotorsionfree* if $\operatorname{Ext}^{1\leq i\leq n}_R(\omega, \mathrm{acTr}_\omega N) = 0$; and N is *adjoint ∞-ω-cotorsionfree* if it is adjoint n-ω-cotorsionfree for all n. In particular, every module in $\operatorname{Mod} S$ is adjoint 0-ω-cotorsionfree.

We use $\mathrm{ac}\mathcal{T}(S)$ to denote the subcategory of $\operatorname{Mod} S$ consisting of adjoint ∞-ω-cotorsionfree modules.

Definition 2.5 ([29, Definition 6.2]). Let $M \in \operatorname{Mod} R$, $N \in \operatorname{Mod} S$ and $n \geq 0$.

(1) The *Ext-cograde* of M with respect to ω is defined as $\text{E-cograde}_\omega M := \inf\{i \geq 0 \mid \operatorname{Ext}^i_R(\omega, M) \neq 0\}$; and the *strong Ext-cograde* of M with respect to ω, denoted by $\text{s.E-cograde}_\omega M$, is said to be at least n if $\text{E-cograde}_\omega X \geq n$ for any quotient module X of M.

(2) The *Tor-cograde* of N with respect to ω is defined as $\text{T-cograde}_\omega N := \inf\{i \geq 0 \mid \operatorname{Tor}^S_i(\omega, N) \neq 0\}$; and the *strong Tor-cograde* of N with respect to ω, denoted by $\text{s.T-cograde}_\omega N$, is said to be at least n if $\text{T-cograde}_\omega Y \geq n$ for any submodule Y of N.

Let $M \in \operatorname{Mod} R$. An exact sequence (finite or infinite)

$$\cdots \to X_1 \to X_0 \to M \to 0$$

in $\operatorname{Mod} R$ is called an *$\mathcal{X}$-resolution* of M if all X_i are in $\mathcal{X}$. The *$\mathcal{X}$-projective dimension* $\mathcal{X}\text{-pd}_R M$ of M is defined as $\inf\{n \mid$ there exists an $\mathcal{X}$-resolution $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ of M in $\operatorname{Mod} R\}$. We always take $\mathcal{X}$-$\mathrm{pd}_R\, 0 = -1$. Dually, the notions of *$\mathcal{X}$-coresolution* and *$\mathcal{X}$-injective dimension* $\mathcal{X}$-$\mathrm{id}_R\, M$ are defined.

3. A filtration of modules with finite Ext-cograde. Let $M \in \mathrm{Mod}\, R$ and let i, j be integers such that $i, j \geq 1$ or $i = j = 0$. Set

$$\begin{aligned} M_i^j &:= \mathrm{Tor}_i^S(\omega, \mathrm{cTr}_\omega\, \mathrm{co}\Omega^j(M)), \\ M_0^{-1} &:= M, \\ M_1^{-1} &:= 0, \\ M_i^0 &:= \mathrm{Tor}_i^S(\omega, \mathrm{cTr}_\omega\, M). \end{aligned}$$

The following result is a generalization of the dual Auslander formula demonstrated in Section 1.

PROPOSITION 3.1. *Let i, j be integers such that $i, j \geq 1$ or $i = j = 0$ and let $M \in \mathrm{Mod}\, R$ with $\mathrm{Tor}_{i-1}^S(\omega, \mathrm{Ext}_R^j(\omega, M)) = 0$. Then there exists an exact sequence*

$$M_{i+1}^{j-1} \to M_{i+2}^j \to \mathrm{Tor}_i^S(\omega, \mathrm{Ext}_R^j(\omega, M)) \to M_i^{j-1} \to M_{i+1}^j \to 0.$$

Proof. The case of $i = j = 0$ has been proved in [28, Proposition 3.2].

Now suppose $i, j \geq 1$. Applying the functor $(-)_*$ to the minimal injective resolution (2.1), we get a complex

$$0 \to M_* \to I^0(M)_* \to I^1(M)_* \to \cdots \xrightarrow{{f^{j-1}}_*} I^j(M)_* \xrightarrow{{f^j}_*} I^{j+1}(M)_* \to \cdots .$$

Consider the following commutative diagram with exact columns and rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}({f^j}_*) \arrow[r] & I^j(M)_* \arrow[r, "{f^j}_*"] \arrow[d, "\pi^j"] & I^{j+1}(M)_* \arrow[r, "\pi^{j+1}"] & \mathrm{Coker}({f^j}_*) \arrow[r] & 0 \\ & & \mathrm{Coker}({f^{j-1}}_*) \arrow[d] & & & \\ & & 0 & & & \end{tikzcd}$$

with π^j and π^{j+1} natural epimorphisms. As

$$\mathrm{Ker}\, \pi^j = \mathrm{Im}({f^{j-1}}_*) \subseteq \mathrm{Ker}({f^j}_*),$$

there exists $\alpha^j : \mathrm{Coker}({f^{j-1}}_*) \to I^{j+1}(M)_*$ such that ${f^j}_* = \alpha^j \cdot \pi^j$ by [1, Theorem 3.6(1)]. So we get the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & & & \\
 & \operatorname{Ker} g \arrow[r, dashed, "h"] \arrow[d, dashed] & \operatorname{Ker} \pi^j \arrow[d, dashed] & & & \\
0 \arrow[r] & \operatorname{Ker}({f^j}_*) \arrow[r] \arrow[d, dashed, "g"] & I^j(M)_* \arrow[r, "{f^j}_*"] \arrow[d, "\pi^j"] & I^{j+1}(M)_* \arrow[r, "\pi^{j+1}"] \arrow[d, equal] & \operatorname{Coker}({f^j}_*) \arrow[r] \arrow[d, dashed, "f"] & 0 \\
0 \arrow[r] & \operatorname{Ker} \alpha^j \arrow[r] & \operatorname{Coker}({f^{j-1}}_*) \arrow[r, "\alpha^j"] \arrow[d] & I^{j+1}(M)_* \arrow[r] & \operatorname{Coker} \alpha^j \arrow[r] & 0 \\
 & & 0 & & &
\end{tikzcd}$$

with f, g, h induced homomorphisms. By diagram chase, it is easy to see that f is an isomorphism and

$$\operatorname{Coker} \alpha^j \cong \operatorname{Coker}({f^j}_*) = \mathrm{cTr}_\omega \, \mathrm{co}\Omega^j(M).$$

Then g is an epimorphism and h is an isomorphism by the snake lemma. So

$$\begin{aligned} \operatorname{Ker} \alpha^j &\cong \operatorname{Ker}({f^j}_*)/\operatorname{Ker} g \cong \operatorname{Ker}({f^j}_*)/\operatorname{Ker} \pi^j \\ &\cong \operatorname{Ker}({f^j}_*)/\operatorname{Im}({f^{j-1}}_*) \cong \operatorname{Ext}^j_R(\omega, M), \end{aligned}$$

and hence we get the exact sequence

$$0 \to \operatorname{Ext}^j_R(\omega, M) \to \mathrm{cTr}_\omega \, \mathrm{co}\Omega^{j-1}(M) \xrightarrow{\alpha^j} I^{j+1}(M)_* \to \mathrm{cTr}_\omega \, \mathrm{co}\Omega^j(M) \to 0,$$

which induces exact sequences

$$0 \to \operatorname{Ext}^j_R(\omega, M) \to \mathrm{cTr}_\omega \, \mathrm{co}\Omega^{j-1}(M) \to \operatorname{Im} \alpha^j \to 0 \tag{3.1}$$

and

$$0 \to \operatorname{Im} \alpha^j \to I^{j+1}(M)_* \to \mathrm{cTr}_\omega \, \mathrm{co}\Omega^j(M) \to 0. \tag{3.2}$$

Note that $\operatorname{Tor}^S_{\geq 1}(\omega, I^{j+1}(M)_*) = 0$ by [17, Lemma 4.1]. So, applying the functor $\omega \otimes_S -$ to (3.2) yields

$$\operatorname{Tor}^S_i(\omega, \operatorname{Im} \alpha^j) \cong \operatorname{Tor}^S_{i+1}(\omega, \mathrm{cTr}_\omega \, \mathrm{co}\Omega^j(M)) = M^j_{i+1}$$

for any $i \geq 1$. Now applying the functor $\omega \otimes_S -$ to (3.1) yields the exact sequence

$$\begin{aligned} M^{j-1}_{i+1} \to \operatorname{Tor}^S_{i+1}(\omega, \operatorname{Im} \alpha^j) \ (\cong M^j_{i+2}) \to \operatorname{Tor}^S_i(\omega, \operatorname{Ext}^j_R(\omega, M)) \to M^{j-1}_i \\ \to \operatorname{Tor}^S_i(\omega, \operatorname{Im} \alpha^j) \ (\cong M^j_{i+1}) \to \operatorname{Tor}^S_{i-1}(\omega, \operatorname{Ext}^j_R(\omega, M)). \end{aligned} \tag{3.3}$$

As $\operatorname{Tor}^S_{i-1}(\omega, \operatorname{Ext}^j_R(\omega, M)) = 0$ by assumption, the assertion follows. ∎

The following proposition will be useful.

Proposition 3.2. *Let* $M \in \operatorname{Mod} R$ *with* $\text{E-cograde}_\omega M = n \geq 1$. *Then* $\mathcal{I}_\omega(S)\text{-pd}_S \operatorname{cTr}_\omega \operatorname{co}\Omega^{j-1}(M) \leq j$ *for any* $1 \leq j \leq n$ *and* $M \cong M_n^{n-1}$.

Proof. Since $\text{E-cograde}_\omega M = n$ by assumption, we get the exact sequence

$$(3.4(j)) \quad 0 \to I^0(M)_* \to I^1(M)_* \to \cdots \to I^j(M)_* \to \operatorname{cTr}_\omega \operatorname{co}\Omega^{j-1}(M) \to 0$$

for any $1 \leq j \leq n$. It implies that

$$\mathcal{I}_\omega(S)\text{- pd}_S \operatorname{cTr}_\omega \operatorname{co}\Omega^{j-1}(M) \leq j.$$

Note that $\operatorname{Tor}^S_{\geq 1}(\omega, I^i(M)_*) = 0$ for any $i \geq 0$ by [17, Lemma 4.1]. So, applying the functor $\omega \otimes_S -$ to $(3.4(n))$ gives the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M_n^{n-1} \arrow[r] \arrow[d, dashed, "f"] & \omega \otimes_S I^0(M)_* \arrow[r] \arrow[d, "\theta_{I^0(M)}"] & \omega \otimes_S I^1(M)_* \arrow[d, "\theta_{I^1(M)}"] \\
0 \arrow[r] & M \arrow[r] & I^0(M) \arrow[r] & I^1(M)
\end{tikzcd}$$

Since $\theta_{I^0(M)}$ and $\theta_{I^1(M)}$ are isomorphisms, so is f and $M \cong M_n^{n-1}$. ■

The following result establishes a relation between the strong Ext-cograde and the strong Tor-cograde of modules.

Lemma 3.3 ([29, Theorem 6.9]). *For any* $n \geq 1$, *the following statements are equivalent:*

(1) $\text{s.E-cograde}_\omega \operatorname{Tor}^S_i(\omega, N) \geq i$ *for any* $N \in \operatorname{Mod} S$ *and* $1 \leq i \leq n$.
(2) $\text{s.T-cograde}_\omega \operatorname{Ext}^i_R(\omega, M) \geq i$ *for any* $M \in \operatorname{Mod} R$ *and* $1 \leq i \leq n$.

Relying on this lemma, we introduce the following

Definition 3.4. For $n \geq 1$, we say that ω satisfies the *n-cograde condition* if one of the equivalent conditions in Lemma 3.3 is satisfied; and ω satisfies the *∞-cograde condition* if it satisfies the n-cograde condition for any $n \geq 1$.

Let R be an artin algebra and D the usual duality between $\operatorname{mod} R$ and $\operatorname{mod} R^{\mathrm{op}}$. Then $D(R)$ is a typical semidualizing (R, R)-bimodule. Recall from [15] that R is said to satisfy the *Auslander condition* if $\operatorname{pd}_R I^i(R) \leq i$ for any $i \geq 0$; equivalently, $\operatorname{id}_{R^{\mathrm{op}}} \operatorname{Hom}_R(P_i(R), D(R)) \leq i$ for any $i \geq 0$, where $P_i(R)$ is the $(i+1)$st term in the minimal projective resolution of ${}_RR$. Note that an artin algebra R satisfies the Auslander condition if and only if DR satisfies the ∞-cograde condition by [31, Proposition 7.7].

Let Q be the quiver

$$\begin{tikzcd} & 3 \arrow[dl, "\beta"'] & \\ 1 & & 5 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & 4 \arrow[ul, "\delta"] \arrow[dl, "\varepsilon"'] & \\ 2 & & \end{tikzcd}$$

and $R = KQ/\langle \beta\alpha - \delta\gamma, \varepsilon\gamma\rangle$ with K a field. Take

$$\omega := \begin{array}{c} 0 \\ 1\ 0 \\ 0 \\ 0 \end{array} \oplus \begin{array}{c} 0 \\ 1\ 0 \\ 1 \\ 1 \end{array} \oplus \begin{array}{c} 0 \\ 1\ 0 \\ 1 \\ 0 \end{array} \oplus \begin{array}{c} 1 \\ 1\ 1 \\ 1 \\ 0 \end{array} \oplus \begin{array}{c} 0 \\ 0\ 1 \\ 1 \\ 0 \end{array}.$$

By [3, Example VI.2.8(a)], ${}_R\omega$ is a non-injective tilting module with $\mathrm{pd}_R\,\omega = 1$. Thus ω is a semidualizing $(R, \mathrm{End}_R(\omega))$-bimodule. It is straightforward to verify that the projective cover $P_0(\omega)$ of ω is $P(1) \oplus P(4)^2 \oplus P(5)^2$. So $P_0(\omega) \in \mathcal{P}_\omega(R)$, and hence ω satisfies the 1-cograde condition by [31, Proposition 7.7]. Since $\mathrm{pd}_R\,\omega = 1$, we have $\mathrm{Ext}_R^{\geq 2}(\omega, M) = 0$ for any M in $\mathrm{Mod}\,R$. Thus ω satisfies the ∞-cograde condition.

From Propositions 3.1 and 3.2, we get the following two corollaries.

COROLLARY 3.5. *Assume ω satisfies the $(n+1)$-cograde condition with $n \geq 0$. If $M \in \mathrm{Mod}\,R$ with $\text{E-cograde}_\omega\, M = n$, then $\text{T-cograde}_\omega\, \mathrm{Ext}_R^n(\omega, M) = n$.*

Proof. If $n = 0$, then $M_* \neq 0$. It follows from [29, Lemma 6.1(1)] that $\omega \otimes_S M_* \neq 0$ and $\text{T-cograde}_\omega\, M_* = 0$.

Now suppose $n \geq 1$. Then by Proposition 3.1, we have an exact sequence

$$\mathrm{Tor}_n^S(\omega, \mathrm{Ext}_R^n(\omega, M)) \to M_n^{n-1} \to M_{n+1}^n \to 0.$$

By Lemma 3.3, $\text{T-cograde}_\omega\, \mathrm{Ext}_R^n(\omega, M) \geq n$. If $\text{T-cograde}_\omega\, \mathrm{Ext}_R^n(\omega, M) > n$, then the above exact sequence implies $M_n^{n-1} \cong M_{n+1}^n$. So by Proposition 3.2,

$$M \cong M_{n+1}^n = \mathrm{Tor}_{n+1}^S(\omega, \mathrm{cTr}_\omega\, \mathrm{co}\Omega^n(M)).$$

Then Lemma 3.3 yields $\text{E-cograde}_\omega\, M \geq n+1$, contrary to assumption. ∎

COROLLARY 3.6. *Let $M \in \mathrm{Mod}\,R$ with $\text{E-cograde}_\omega\, M = n \geq 1$. Then $\mathrm{Tor}_i^S(\omega, \mathrm{Ext}_R^n(\omega, M)) \cong M_{i+2}^n$ for any $i \geq n+1$.*

Proof. By the proof of Proposition 3.1, we have an exact sequence

$$M_{i+1}^{n-1} \to M_{i+2}^n \to \mathrm{Tor}_i^S(\omega, \mathrm{Ext}_R^n(\omega, M)) \to M_i^{n-1}.$$

Because $i \geq n+1$, Proposition 3.2 yields $M_{i+1}^{n-1} = M_i^{n-1} = 0$ and the assertion follows. ∎

Applying Corollary 3.5, we get the following lemma which shows how the Ext-cograde and the Tor-cograde of modules behave in short exact sequences. Because the argument is standard, we omit it.

LEMMA 3.7. *Assume that ω satisfies the $(n+1)$-cograde condition with $n \geq 0$.*

(1) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in $\mathrm{Mod}\, R$ with $n_i = \text{E-cograde}_\omega M_i$ for $i = 1,2,3$ and $\max\{n_1, n_2, n_3\} \leq n$. Then $n_2 = \min\{n_1, n_3\}$.*
(2) *Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be an exact sequence in $\mathrm{Mod}\, S$ with $n_i = \text{T-cograde}_\omega N_i$ for $i = 1,2,3$ and $\max\{n_1, n_2, n_3\} \leq n$. Then $n_2 = \min\{n_1, n_3\}$.*

We say that a module $M \in \mathrm{Mod}\, R$ is *pure of Ext-cograde k* if $\text{E-cograde}_\omega M = \text{E-cograde}_\omega M/M' = k$ for any proper R-submodule M' of M; dually, a module $N \in \mathrm{Mod}\, S$ is *pure of Tor-cograde l* if $\text{T-cograde}_\omega N = \text{T-cograde}_\omega N' = l$ for any non-zero S-submodule N' of N.

EXAMPLE 3.8. Let R be a finite-dimensional algebra over an algebraically closed field given by the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3.$$

Then $\omega := I(1) \oplus I(2) \oplus I(3)$ is a semidualizing $(R\text{-}R)$-bimodule. Set $M := S(2)$. It is easy to see that $M_* = 0$. By [3, IV.2 Theorem 2.13] and [8, VII.1 Example], we have $\mathrm{Ext}^1_R(I(1), M) \cong D\, \overline{\mathrm{Hom}}_R(M, M) \neq 0$ and $\text{E-cograde}_\omega M = 1$. Because M is simple, it follows that M is pure of Ext-cograde 1 and $D(M)$ is pure of Tor-cograde 1.

On the other hand, because $N := I(3)$ is a direct summand of ω, it follows that $\text{E-cograde}_\omega N = 0$. Thus $\text{E-cograde}_\omega M \oplus N = 0$ and $M \oplus N$ is not pure of Ext-cograde 0. Because $\mathrm{Tor}^R_i(D(M), \omega) \cong D(\mathrm{Ext}^i_R(\omega, M))$ and $\mathrm{Tor}^R_i(D(M \oplus N), \omega) \cong D(\mathrm{Ext}^i_R(\omega, M \oplus N))$ for any $i \geq 0$, we find that $\text{T-cograde}_\omega D(M \oplus N) = 0$ and $M \oplus N$ is not pure of Tor-cograde 0.

PROPOSITION 3.9. *Assume that ω satisfies the ∞-cograde condition.*

(1) *If $M \in \mathrm{Mod}\, R$ with $\text{E-cograde}_\omega M = k$ and $\mathrm{Tor}^S_i(\omega, \mathrm{Ext}^i_R(\omega, M)) = 0$ for any $i \geq k+1$, then M is pure of Ext-cograde k.*
(2) *If $N \in \mathrm{Mod}\, S$ with $\text{T-cograde}_\omega N = l$ and $\mathrm{Ext}^i_R(\omega, \mathrm{Tor}^S_i(\omega, N)) = 0$ for any $i \geq l+1$, then N is pure of Tor-cograde l.*

Proof. (1) Let M' be a proper R-submodule of M and $\text{E-cograde}_\omega M/M' = t$. Then $\text{T-cograde}_\omega \mathrm{Ext}^t_R(\omega, M/M') = t$ by Corollary 3.5.

We claim $t \leq k$. If $t > k$, then by assumption, $\text{T-cograde}_\omega \mathrm{Ext}^t_R(\omega, M) \geq t$ and $\mathrm{Tor}^S_t(\omega, \mathrm{Ext}^t_R(\omega, M)) = 0$. So we have $\text{T-cograde}_\omega \mathrm{Ext}^t_R(\omega, M) \geq t+1$.

Consider the exact sequence

$$\mathrm{Ext}_R^t(\omega, M) \xrightarrow{f} \mathrm{Ext}_R^t(\omega, M/M') \xrightarrow{g} \mathrm{Ext}_R^{t+1}(\omega, M').$$

By Lemma 3.7(2), we have

$$\text{T-cograde}_\omega \operatorname{Im} f \geq \text{T-cograde}_\omega \mathrm{Ext}_R^t(\omega, M) \geq t+1,$$
$$\text{T-cograde}_\omega \operatorname{Im} g \geq \text{T-cograde}_\omega \mathrm{Ext}_R^{t+1}(\omega, M') \geq t+1.$$

Thus $\text{T-cograde}_\omega \mathrm{Ext}_R^t(\omega, M/M') \geq t+1$, a contradiction. The claim follows. Then by Lemma 3.7(1), $\text{E-cograde}_\omega M = \text{E-cograde}_\omega M/M'$.

(2) is dual to (1). ∎

As a consequence, we get

Corollary 3.10. *Assume that ω satisfies the ∞-cograde condition. Then:*

(1) $\mathrm{Ext}_R^k(\omega, M)$ *is pure of Tor-cograde k for any $M \in \operatorname{Mod} R$ satisfying* $\text{E-cograde}_\omega M = k$.

(2) $\mathrm{Tor}_l^S(\omega, N)$ *is pure of Ext-cograde l for any $N \in \operatorname{Mod} S$ satisfying* $\text{T-cograde}_\omega N = l$.

Proof. (1) Let $M \in \operatorname{Mod} R$ with $\text{E-cograde}_\omega M = k$. It follows from Corollary 3.5 that $\text{T-cograde}_\omega \mathrm{Ext}_R^k(\omega, M) = k$.

We claim that $\mathrm{Ext}_R^i(\omega, \mathrm{Tor}_i^S(\omega, \mathrm{Ext}_R^k(\omega, M))) = 0$ for any $i \geq k+1$. If $k = 0$, then $\text{E-cograde}_\omega \mathrm{Tor}_i^S(\omega, M_*) = \text{E-cograde}_\omega \mathrm{Tor}_{i+2}^S(\omega, \mathrm{cTr}_\omega M) \geq i+2$ for any $i \geq 1$. If $k \geq 1$, then $\text{E-cograde}_\omega \mathrm{Tor}_i^S(\omega, \mathrm{Ext}_R^k(\omega, M)) = \text{E-cograde}_\omega M_{i+2}^k \geq i+2$ for any $i \geq k+1$ by Corollary 3.6. The claim follows. Thus $\mathrm{Ext}_R^k(\omega, M)$ is pure of Tor-cograde k by Proposition 3.9(2).

(2) is dual to (1). ∎

Recall that a sequence $\cdots \to M_1 \to M_2 \to M_3 \to \cdots$ in $\operatorname{Mod} R$ is called $\mathrm{Hom}_R(\omega, -)$-*exact* if it is exact after applying the functor $\mathrm{Hom}_R(\omega, -)$.

Lemma 3.11. *If $M \in \mathcal{B}_\omega(R)$, then $\mathrm{cTr}_\omega M \in \mathcal{A}_\omega(S)$.*

Proof. Let $M \in \mathcal{B}_\omega(R)$. Then by [28, Proposition 3.7 and Theorem 3.9], there exists a $\mathrm{Hom}_R(\omega, -)$-exact exact sequence

$$\cdots \to W_1 \to W_0 \to I^0(M) \to I^1(M) \to \cdots \tag{3.5}$$

in $\operatorname{Mod} R$ with all W_i in $\operatorname{Add}_R \omega$ such that $M \cong \operatorname{Im}(W_0 \to I^0(M))$. Applying the functor $(-)_*$ to (3.5) yields an exact sequence

$$\cdots \to W_{1*} \to W_{0*} \to I^0(M)_* \to I^1(M)_* \to \cdots . \tag{3.6}$$

Applying $\omega_S \otimes -$ to (3.6), it is easy to verify that it remains exact. This implies that $\mathrm{Tor}_{\geq 1}^S(\omega, \mathrm{cTr}_\omega M) = 0$ and $\mathrm{cTr}_\omega M \in \mathrm{ac}\mathcal{T}(S)$ by [30, Corollary 3.9]. Then [30, Theorem 3.11(1)] yields $\mathrm{cTr}_\omega M \in \mathcal{A}_\omega(S)$. ∎

Now we can present the main theorem in this section, which is useful in providing information about noetherian modules with finite Ext-cograde.

THEOREM 3.12. *Assume that ω satisfies the ∞-cograde condition. If M is a noetherian left R-module with* $\operatorname{E-cograde}_\omega M = k < \infty$, *then there exists a filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \tag{3.7}$$

of R-submodules of M such that:

(1) $M_1 = \cdots = M_k = 0$ *and we have an exact sequence*
$$0 \to \operatorname{Tor}^S_{k+2}(\omega, \operatorname{cTr}_\omega \operatorname{co}\Omega^k(M)) \to \operatorname{Tor}^S_k(\omega, \operatorname{Ext}^k_R(\omega, M)) \\ \to M/M_k \to M/M_{k+1} \to 0.$$
(2) *If* $\operatorname{Tor}^S_i(\omega, \operatorname{Ext}^i_R(\omega, M)) \neq 0$, *then* $\operatorname{E-cograde}_\omega M/M_i = i$, $M_i \neq M_{i+1}$ *and M_{i+1}/M_i is pure of Ext-cograde i.*
(3) *If* $\operatorname{Tor}^S_i(\omega, \operatorname{Ext}^i_R(\omega, M)) = 0$, *then* $M_i = M_{i+1}$.
(4) *If* $\mathcal{B}_\omega(R)\text{-}\mathrm{id}_R M = d < \infty$, *then*
$$M = M_{d+1} \quad \text{and} \quad M/M_d \cong \operatorname{Tor}^S_d(\omega, \operatorname{Ext}^d_R(\omega, M)).$$
(5) *If* $\mathcal{B}_\omega(R)\text{-}\mathrm{id}_R M = d < \infty$, *then* $\operatorname{fil}(M) \leq d - k + 1$, *and equality holds whenever* $\operatorname{T-cograde}_\omega \operatorname{Ext}^i_R(\omega, M) = i$ *for any $k \leq i \leq d$, where* $\operatorname{fil}(M)$ *is the number of strict inclusions in* (3.7).

Proof. By Proposition 3.1, there exists a chain of epimorphisms
$$M_0^{-1}\ (= M) \to M_1^0 \to \cdots \to M_i^{i-1} \to \cdots.$$
Then we get a filtration $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of R-submodules of M with $M/M_i = M_i^{i-1}$.

(1) The case $k = 0$ is trivial. If $k \geq 1$, then Proposition 3.2 yields $M \cong M_k^{k-1}$. Since there exists an exact sequence
$$0 \to M_k \to M \to M_k^{k-1}\ (\cong M) \to 0$$
and M is noetherian, we deduce from [25, Proposition 1.14] that $M_k = 0$, and hence $M_1 = \cdots = M_k = 0$. Since $\mathcal{I}_\omega(S)\text{-}\mathrm{pd}_R \operatorname{cTr}_\omega \operatorname{co}\Omega^{k-1}(M) \leq k$ by Proposition 3.2 again, we have $M_{k+1}^{k-1} = 0$ by dimension shifting. Now we get the desired exact sequence from Proposition 3.1.

(2) If $\operatorname{Tor}^S_i(\omega, \operatorname{Ext}^i_R(\omega, M)) \neq 0$, then $\operatorname{T-cograde}_\omega \operatorname{Ext}^i_R(\omega, M) = i$ by assumption. It follows from Corollary 3.10(2) that $\operatorname{Tor}^S_i(\omega, \operatorname{Ext}^i_R(\omega, M))$ is pure of Ext-cograde i. By Proposition 3.1 we have an exact sequence

$$M_{i+2}^i \xrightarrow{f} \operatorname{Tor}^S_i(\omega, \operatorname{Ext}^i_R(\omega, M)) \to M_i^{i-1} \to M_{i+1}^i \to 0. \tag{3.8}$$

If $M_i = M_{i+1}$, then $M_i^{i-1} = M_{i+1}^i$ and f is an epimorphism. Consqeuently, $\operatorname{E-cograde}_\omega M_{i+2}^i \leq i$ by Lemma 3.7(1), a contradiction. Thus $M_i \neq M_{i+1}$. Since $M_{i+1}/M_i \cong \operatorname{Coker} f$ is a quotient module of $\operatorname{Tor}^S_i(\omega, \operatorname{Ext}^i_R(\omega, M))$, we see that M_{i+1}/M_i is pure of Ext-cograde i. Notice that $\operatorname{E-cograde}_\omega M_{i+1}^i \geq i + 1$ by assumption, so $\operatorname{E-cograde}_\omega M/M_i = i$ by Lemma 3.7(1).

(3) follows directly from the exact sequence (3.8).

(4) If $\mathcal{B}_\omega(R)$-$\mathrm{id}_R M = d$, then $\mathrm{co}\Omega^d(M) \in \mathcal{B}_\omega(R)$ by [29, Theorem 4.2]. Lemma 3.11 yields $\mathrm{cTr}_\omega \,\mathrm{co}\Omega^d(M) \in \mathcal{A}_\omega(S)$ and $M^d_{d+1} = M^d_{d+2} = 0$. Thus $M = M_{d+1}$ and $M/M_d \cong \mathrm{Tor}^S_d(\omega, \mathrm{Ext}^d_R(\omega, M))$ by the exact sequence (3.8).

(5) is a consequence of the former assertions. ∎

4. Another filtration of modules

DEFINITION 4.1. Let $n \geq 1$. A module M in $\mathrm{Mod}\, R$ is called *n-Bass-cosyzygy* if there exists an exact sequence

$$B^{-(n-1)} \to \cdots \to B^{-1} \to B^0 \to M \to 0$$

in $\mathrm{Mod}\, R$ with all B^i in $\mathcal{B}_\omega(R)$.

We use $\mathrm{co}\Omega^n_{\mathcal{B}}(R)$ to denote the subclass of $\mathrm{Mod}\, R$ consisting of n-Bass-cosyzygy modules.

LEMMA 4.2. *Let $n \geq 1$. If* $\text{T-cograde}_\omega \mathrm{Ext}^i_R(\omega, M) \geq i-1$ *for any $M \in \mathrm{Mod}\, R$ and $1 \leq i \leq n$, then* $\mathrm{co}\Omega^i_{\mathcal{B}}(R) = \mathrm{c}\mathcal{T}^i_\omega(R)$ *for any $1 \leq i \leq n$.*

Proof. Because $\mathcal{P}_\omega(R) \subseteq \mathcal{B}_\omega(R)$ by [17, Corollary 6.1], we have $\mathrm{c}\mathcal{T}^i_\omega(R) \subseteq \mathrm{co}\Omega^i_{\mathcal{B}}(R)$ by [28, Proposition 3.7].

Assume that $\text{T-cograde}_\omega \mathrm{Ext}^i_R(\omega, M) \geq i-1$ for any $M \in \mathrm{Mod}\, R$ and $1 \leq i \leq n$. We now prove by induction on n that $\mathrm{co}\Omega^i_{\mathcal{B}}(R) \subseteq \mathrm{c}\mathcal{T}^i_\omega(R)$ for any $1 \leq i \leq n$. Let $M \in \mathrm{co}\Omega^1_{\mathcal{B}}(R)$. Then there exists an exact sequence $B^0 \xrightarrow{f^0} M \to 0$ in $\mathrm{Mod}\, R$ with $B^0 \in \mathcal{B}_\omega(R)$, and we get the following commutative diagram with the bottom row exact:

$$\begin{tikzcd}
\omega \otimes_S B^0{}_* \arrow[r, "1_\omega \otimes f^0{}_*"] \arrow[d, "\theta_{B^0}"] & \omega \otimes_S M_* \arrow[d, "\theta_M"] & \\
B^0 \arrow[r, "f^0"] & M \arrow[r] & 0
\end{tikzcd}$$

Since θ_{B^0} is an isomorphism, we see that θ_M is an epimorphism and M is in $\mathrm{c}\mathcal{T}^1_\omega(R)$. The case $n = 1$ is proved.

Now let $M \in \mathrm{co}\Omega^n_{\mathcal{B}}(R)$ with $n \geq 2$. Then there exists an exact sequence

$$B^{-(n-1)} \xrightarrow{f^{n-1}} \cdots \to B^{-1} \xrightarrow{f^1} B^0 \xrightarrow{f^0} M \to 0 \tag{4.1}$$

in $\mathrm{Mod}\, R$ with all B^i in $\mathcal{B}_\omega(R)$. By the induction hypothesis, we know that $\mathrm{Im}\, f^1 \in \mathrm{c}\mathcal{T}^{n-1}_\omega(R)$. Applying the functor $(-)_*$ to (4.1) gives an exact sequence

$$0 \to (\mathrm{Im}\, f^1)_* \to B^0{}_* \xrightarrow{f^0{}_*} M_* \to \mathrm{Ext}^n_R(\omega, \mathrm{Ker}\, f^{n-1}) \to 0. \tag{4.2}$$

Set $N := \mathrm{Im}\, f^0{}_*$ and let $f^0{}_* := \alpha \cdot \pi$ be the natural epic-monic decompositions of $f^0{}_*$ with $\pi : B^0{}_* \twoheadrightarrow N$ and $\alpha : N \hookrightarrow M_*$. Then we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^S(\omega, N) \arrow[r] & \omega \otimes_S (\mathrm{Im}\, f^1)_* \arrow[r] \arrow[d, "\theta_{\mathrm{Im}\, f^1}"] & \omega \otimes_S {B^0}_* \arrow[r, "1_\omega \otimes \pi"] \arrow[d, "\theta_{B^0}"] & \omega \otimes_S N \arrow[r] \arrow[d, dashed, "g"] & 0 \\
 & 0 \arrow[r] & \mathrm{Im}\, f^1 \arrow[r] & B^0 \arrow[r, "f^0"] & M \arrow[r] & 0
\end{tikzcd} \tag{4.3}$$

So $\theta_M \cdot (1_\omega \otimes \alpha) \cdot (1_\omega \otimes \pi) = \theta_M \cdot (1_\omega \otimes {f^0}_*) = f^0 \cdot \theta_{B^0} = g \cdot (1_\omega \otimes \pi)$. Because $1_\omega \otimes \pi$ is epic, we have the relation $\theta_M \cdot (1_\omega \otimes \alpha) = g$ and the following commutative diagram with the top row exact:

$$\begin{tikzcd}
\omega \otimes_S N \arrow[r, "1_\omega \otimes \alpha"] \arrow[d, "g"] & \omega \otimes_S M_* \arrow[r] \arrow[d, "\theta_M"] & \omega \otimes_S \mathrm{Ext}_R^n(\omega, \mathrm{Ker}\, f^{n-1}) \arrow[r] & 0 \\
M \arrow[r, equal] & M
\end{tikzcd} \tag{4.4}$$

Since $\theta_{\mathrm{Im}\, f^1}$ is an epimorphism by the above argument, the snake lemma shows that g is a monomorphism. As $\omega \otimes_S \mathrm{Ext}_R^n(\omega, \mathrm{Ker}\, f^{n-1}) = 0$ by assumption, we find that θ_M is an isomorphism and $M \in \mathrm{c}\mathcal{T}_\omega^2(R)$ by the diagram (4.4). This means that the assertion holds for $n = 2$. If $n \geq 3$, then $\mathrm{Im}\, f^1 \in \mathrm{c}\mathcal{T}_\omega^{n-1}(R)$ implies $\theta_{\mathrm{Im}\, f^1}$ is an isomorphism. So $\mathrm{Tor}_1^S(\omega, N) = 0$ by the diagram (4.3). In addition, we have $\mathrm{Tor}_{1 \leq i \leq n-3}^S(\omega, (\mathrm{Im}\, f^1)_*) = 0$ by [28, Corollary 3.4(3)]. Since $\text{T-cograde}_\omega\, \mathrm{Ext}_R^n(\omega, \mathrm{Ker}\, f^{n-1}) \geq n-1$ by assumption, applying dimension shifting to (4.2) we obtain $\mathrm{Tor}_{1 \leq i \leq n-2}^S(\omega, M_*) = 0$. Therefore $M \in \mathrm{c}\mathcal{T}_\omega^n(R)$ by [28, Corollary 3.4(3)] again. ∎

The following result shows how the strong Tor-cograde conditions on modules affect the extension closure of $\mathrm{c}\mathcal{T}_\omega^n(R)$. It is a dual version of [7, Theorem 1.1].

Lemma 4.3. *Let $n \geq 1$ and*

$$0 \to A \to B \to C \to 0 \tag{4.5}$$

be an exact sequence in $\mathrm{Mod}\, R$ *with $A, C \in \mathrm{c}\mathcal{T}_\omega^n(R)$. If*

$$\text{s.T-cograde}_\omega\, \mathrm{Ext}_R^1(\omega, A) \geq n,$$

then $B \in \mathrm{c}\mathcal{T}_\omega^n(R)$.

Proof. Applying $(-)_*$ to (4.5) gives the exact sequence

$$0 \to A_* \to B_* \to C_* \to \mathrm{Ext}_R^1(\omega, A).$$

Set $L = \mathrm{Coker}(B_* \to C_*)$ and $K := \mathrm{Im}(B_* \to C_*)$.

Let $n = 1$. Since $\text{s.T-cograde}_\omega\, \mathrm{Ext}_R^1(\omega, A) \geq 1$ and $L \subseteq \mathrm{Ext}_R^1(\omega, A)$, we have $\omega \otimes_S L = 0$. This yields an epimorphism $\omega \otimes_S B_* \to \omega \otimes_S C_*$ and the following commutative diagram with the bottom row exact:

$$\begin{tikzcd} & \omega \otimes_S A_* \arrow[r] \arrow[d, "\theta_A"] & \omega \otimes_S B_* \arrow[r] \arrow[d, "\theta_B"] & \omega \otimes_S C_* \arrow[r] \arrow[d, "\theta_C"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Because $A, C \in \mathrm{c}\mathcal{T}^1_\omega(R)$ by assumption, θ_A and θ_C are epimorphisms. Then by diagram chase, θ_B is also an epimorphism and $B \in \mathrm{c}\mathcal{T}^1_\omega(R)$.

Let $n = 2$. Since s.T-cograde$_\omega \operatorname{Ext}^1_R(\omega, A) \geq 2$ and $L \subseteq \operatorname{Ext}^1_R(\omega, A)$, we obtain an isomorphism $\omega \otimes_S K \to \omega \otimes_S C_*$. It yields the exact sequence

$$\omega \otimes_S A_* \to \omega \otimes_S B_* \to \omega \otimes_S C_* \to 0$$

and the commutative diagram with exact rows

$$\begin{tikzcd} & \omega \otimes_S A_* \arrow[r] \arrow[d, "\theta_A"] & \omega \otimes_S B_* \arrow[r] \arrow[d, "\theta_B"] & \omega \otimes_S C_* \arrow[r] \arrow[d, "\theta_C"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Because $A, C \in \mathrm{c}\mathcal{T}^2_\omega(R)$ by assumption, θ_A and θ_C are isomorphisms. So θ_B is also an isomorphism and $B \in \mathrm{c}\mathcal{T}^2_\omega(R)$.

Let $n \geq 3$. Since s.T-cograde$_\omega \operatorname{Ext}^1_R(\omega, A) \geq n \geq 3$, we have $B \in \mathrm{c}\mathcal{T}^2_\omega(R)$ by the above argument. Consider an exact sequence

$$0 \to K \to C_* \to L \to 0.$$

Since $L \subseteq \operatorname{Ext}^1_R(\omega, A)$, we have $\operatorname{Tor}^S_{0 \leq i \leq n-1}(\omega, L) = 0$. Then it follows that $\operatorname{Tor}^S_i(\omega, K) \cong \operatorname{Tor}^S_i(\omega, C_*)$ for any $0 \leq i \leq n-2$. Because $A, C \in \mathrm{c}\mathcal{T}^n_\omega(R)$ by assumption, we have $\operatorname{Tor}^S_{1 \leq i \leq n-2}(\omega, A_*) = 0 = \operatorname{Tor}^S_{1 \leq i \leq n-2}(\omega, C_*) = 0$ by [28, Corollary 3.4]. Now applying the functor $\omega \otimes_S -$ to the exact sequence

$$0 \to A_* \to B_* \to K \to 0$$

yields $\operatorname{Tor}^S_{1 \leq i \leq n-2}(\omega, B_*) = 0$. Thus $B \in \mathrm{c}\mathcal{T}^n_\omega(R)$ by [28, Corollary 3.4] again. ■

The following proposition is crucial in proving the main result of this section.

PROPOSITION 4.4. *Assume that ω satisfies the n-cograde condition with $n \geq 1$ and $M \in \mathrm{co}\Omega^i_{\mathcal{B}}(R)$ with $0 \leq i \leq n-1$. Then there exists a $\operatorname{Hom}_R(\omega, -)$-exact exact sequence*

$$0 \to A \to M \oplus I \to B \to 0$$

in $\operatorname{Mod} R$ satisfying the following conditions:

(1) *$A \in \mathrm{co}\Omega^{i+1}_{\mathcal{B}}(R)$, $I \in \mathcal{I}(R)$ and $B \cong \mathrm{co}\Omega^i(\operatorname{Tor}^S_{i+1}(\omega, \mathrm{cTr}_\omega M))$.*
(2) *$\mathcal{I}_\omega(S)$-$\mathrm{pd}_S B_* \leq i-1$.*

Proof. Let $i = 0$. Set $A := \operatorname{Im}\theta_M$, $I := 0$ and $B := \operatorname{Tor}_1^S(\omega, \mathrm{cTr}_\omega M)$. Then by [28, Proposition 3.2], we have an exact sequence

$$0 \to A \to M \to B \to 0. \tag{4.6}$$

Since $\theta_{\omega\otimes_S M_*}$ is an epimorphism by [29, Lemma 6.1], we have $\omega \otimes_S M_* \in \mathrm{c}\mathcal{T}_\omega^1(R)$. Note that A is a quotient module of $\omega \otimes_S M_*$. So $A \in \mathrm{c}\mathcal{T}_\omega^1(R)$ by [28, Lemma 3.6], and hence $A \in \mathrm{co}\Omega_{\mathcal{B}}^1(R)$ by Lemma 4.2. On the other hand, since $\text{E-cograde}_\omega B \geq 1$ by assumption, we have $B_* = 0$. So (4.6) is the desired exact sequence.

Let $i = 1$. Consider the exact sequence of R-modules

$$0 \to \operatorname{Tor}_2^S(\omega, \mathrm{cTr}_\omega M) \to I \to B \to 0$$

with $I = I^0(\operatorname{Tor}_2^S(\omega, \mathrm{cTr}_\omega M)) \in \mathcal{I}(R)$ and $B = \mathrm{co}\Omega^1(\operatorname{Tor}_2^S(\omega, \mathrm{cTr}_\omega M))$. Then by [28, Proposition 3.2], we have the following push-out diagram with the middle column splitting:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \operatorname{Tor}_2^S(\omega, \mathrm{cTr}_\omega M) \arrow[r] \arrow[d] & I \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \omega \otimes_S M_* \arrow[r] \arrow[d] & M \oplus I \arrow[r, "g"] \arrow[d] & B \arrow[r] & 0 \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Because $\text{E-cograde}_\omega(\operatorname{Tor}_2^S(\omega, \mathrm{cTr}_\omega M)) \geq 2$ by assumption, we see that f_* is an isomorphism. So B_* $(\cong I_*) \in \mathcal{I}_\omega(S)$ and g_* is an epimorphism. Now let

$$Q_1 \to Q_0 \to M_* \to 0$$

be an exact sequence in $\operatorname{Mod} S$ with $Q_0, Q_1 \in \mathcal{P}(S)$. Then

$$\omega \otimes_S Q_1 \to \omega \otimes_S Q_0 \to \omega \otimes_S M_* \to 0$$

is exact in $\operatorname{Mod} R$ and $\omega \otimes_S M_* \in \mathrm{co}\Omega_{\mathcal{B}}^2(R)$. Thus the middle row in the above diagram is the desired exact sequence.

Now suppose $i \geq 2$. By Lemma 4.2, we have $M \in \mathrm{co}\Omega_{\mathcal{B}}^i(R) = \mathrm{c}\mathcal{T}_\omega^i(R)$. Then by [28, Proposition 3.7], there exists a $\operatorname{Hom}_R(\omega, -)$-exact exact sequence

$$0 \to N \to W_{i-1} \xrightarrow{f} W_{i-2} \to \cdots \to W_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all W_j in $\operatorname{Add}_R \omega$ and $N = \operatorname{Ker} f$. As $\operatorname{Tor}_{i+1}^S(\omega, \mathrm{cTr}_\omega M) \cong \operatorname{Tor}_1^S(\omega, \operatorname{Coker} f_*)$ and $\operatorname{Tor}_{i+2}^S(\omega, \mathrm{cTr}_\omega M) \cong \operatorname{Tor}_2^S(\omega, \operatorname{Coker} f_*)$, by [31, Proposition 5.1] we have the exact sequence

$$0 \to \operatorname{Tor}^S_{i+2}(\omega, \operatorname{cTr}_\omega M) \to \omega \otimes_S N_* \xrightarrow{\theta_N} N \to \operatorname{Tor}^S_{i+1}(\omega, \operatorname{cTr}_\omega M) \to 0.$$

Set $X := \operatorname{Im}\theta_N$ and $L := \operatorname{Tor}^S_{i+1}(\omega, \operatorname{cTr}_\omega M)$. Consider the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0(X) \arrow[r] \arrow[d] & I^0(X) \oplus I^0(L) \arrow[r] \arrow[d] & I^0(L) \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & I^{i-1}(X) \arrow[r] \arrow[d] & I^{i-1}(X) \oplus I^{i-1}(L) \arrow[r] \arrow[d] & I^{i-1}(L) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D \arrow[r, "\alpha"] \arrow[d] & H \arrow[r, "\beta"] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since X is a quotient module of $\omega \otimes_S N_*$, we have $X \in \operatorname{co}\Omega^1_{\mathcal{B}}(R)$ and $D \in \operatorname{co}\Omega^{i+1}_{\mathcal{B}}(R)$. Because $\text{E-cograde}_\omega L \geq i+1$ by assumption, we get the exact sequence

$$0 \to I^0(L)_* \to I^1(L)_* \to \cdots \to I^{i-1}(L)_* \to B_* \to 0.$$

Thus $\mathcal{I}_\omega(S)\text{-pd}_S B_* \leq i-1$ and β_* is an epimorphism. Next we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & W_{i-1} \arrow[r] \arrow[d, dashed, "g_{i-1}"] & W_{i-2} \arrow[r] \arrow[d, dashed, "g_{i-2}"] & \cdots \arrow[r] & W_0 \arrow[r] \arrow[d, dashed, "g_0"] & M \arrow[r] \arrow[d, dashed, "g_{-1}"] & 0 \\
0 \arrow[r] & N \arrow[r] & E^0 \arrow[r] & E^1 \arrow[r] & \cdots \arrow[r] & E^{i-1} \arrow[r] & H \arrow[r] & 0
\end{tikzcd}$$

where $E^j = I^j(X) \oplus I^j(L)$ for any $0 \leq j \leq i-1$. The injectivity of E^j guarantees the existence of all g_j. Now we view the sequence $(g_{i-1}, g_{i-2}, \ldots, g_{-1})$ as a quasi-isomorphism between the complexes

$$0 \to W_{i-1} \to W_{i-2} \to \cdots \to W_0 \to M \to 0$$

and

$$0 \to E^0 \to E^1 \to \cdots \to E^{i-1} \to H \to 0.$$

We then obtain an exact sequence

$$0 \to W_{i-1} \to W_{i-2} \oplus E^0 \to W_{i-3} \oplus E^1 \to \cdots \to W_0 \oplus E^{i-2} \to M \oplus E^{i-1} \to H \to 0.$$

Set

$$K := \mathrm{Im}(W_0 \oplus E^{i-2} \to M \oplus E^{i-1}).$$

It is not hard to see that $\mathrm{Ext}_R^{\geq 1}(\omega, K) = 0$ and $K \in \mathrm{co}\Omega^j_{\mathcal{B}}(R)$ for $j \geq 1$. Consider the pull-back diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & K \arrow[r] & M \oplus E^{i-1} \arrow[r, "\gamma"] \arrow[d] & H \arrow[r] \arrow[d, "\beta"] & 0 \\
 & & B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $K, D \in \mathrm{c}\mathcal{T}^{i+1}_{\omega}(R)$, we have $A \in \mathrm{c}\mathcal{T}^{i+1}_{\omega}(R)$ by Lemma 4.3. Thus $A \in \mathrm{co}\Omega^{i+1}_{\mathcal{B}}(R)$ by Lemma 4.2. It follows from $\mathrm{Ext}_R^{\geq 1}(\omega, K) = 0$ that γ_* is an epimorphism. Notice that β_* is also an epimorphism, so

$$0 \to A_* \to (M \oplus E^{i-1})_* \to B_* \to 0$$

is exact. The proof is finished. ■

We are now in a position to give the main result in this section.

Theorem 4.5. *Assume that ω satisfies the n-cograde condition with $n \geq 1$. Then for any $M \in \mathrm{Mod}\, R$, there exists an injective left R-module I and a chain of monomorphisms*

$$M_n \rightarrowtail M_{n-1} \rightarrowtail \cdots \rightarrowtail M_1 \to M_0 = M \oplus I$$

in $\mathrm{Mod}\, R$ such that for any $0 \leq i \leq n-1$, we have:

(1) $B_i = \mathrm{Coker}(M_{i+1} \to M_i) \cong \mathrm{co}\Omega^i(\mathrm{Tor}^S_{i+1}(\omega, \mathrm{cTr}_{\omega} M))$.
(2) $M_i \in \mathrm{co}\Omega^i_{\mathcal{B}}(R)$.
(3) $\mathcal{I}_{\omega}(S)$-$\mathrm{pd}_S\, B_{i*} \leq i-1$.
(4) *The exact sequence* $0 \to M_{i+1} \to M_i \to B_i \to 0$ *is* $\mathrm{Hom}_R(\omega, -)$*-exact.*

Proof. From the proof of Proposition 4.4, we get a $\mathrm{Hom}_R(\omega, -)$-exact exact sequence

$$0 \to A_{i+1} \to A_i \oplus I_i \to B_i \to 0$$

in $\operatorname{Mod} R$ such that $A_0 = M$, $A_i \in \operatorname{co}\Omega^i_{\mathcal{B}}(R)$, $I_i \in \mathcal{I}(R)$, and $B_i \cong \operatorname{co}\Omega^i(\operatorname{Tor}^S_{i+1}(\omega, \operatorname{cTr}_\omega M))$ with $\mathcal{I}_\omega(S)$-$\operatorname{pd}_S B_{i*} \le i-1$ for $0 \le i \le n-1$. Set $I := \bigoplus_{i=0}^{n-1} I_i$, $M_0 := M \oplus I$, $M_n := A_n$ and $M_i := A_i \oplus \bigoplus_{j=i}^{n-1} I_j$ for any $1 \le i \le n-1$. Now the assertion follows easily. ∎

As a consequence of Theorem 4.5, we have

Corollary 4.6. *Assume that ω satisfies the n-cograde condition with $n \ge 2$. Then for any $N \in \operatorname{Mod} S$, there exists an injective left R-module I and a chain of monomorphisms*

$$N_n \rightarrowtail N_{n-1} \rightarrowtail \cdots \rightarrowtail N_2 \rightarrowtail (\omega \otimes_S N)_* \oplus I_*$$

in $\operatorname{Mod} R$ such that:

(1) $\mathcal{I}_\omega(S)$-$\operatorname{pd}_S Y_i \le i$, *where $Y_i = ((\omega \otimes_S N)_* \oplus I_*)/N_{i+2}$ for any $0 \le i \le n-2$.*
(2) $0 \to \omega \otimes_S N_{i+2} \to \omega \otimes_S ((\omega \otimes_S N)_* \oplus I_*) \to \omega \otimes_S Y_i \to 0$ *in $\operatorname{Mod} R$ is exact for any $0 \le i \le n-2$.*
(3) *For $1 \le i \le n-2$, the natural epimorphism $(\omega \otimes_S N)_* \oplus I_* \twoheadrightarrow Y_i$ in $\operatorname{Mod} S$ induces an isomorphism*

$$\operatorname{Tor}^S_j(\omega, (\omega \otimes_S N)_*) \xrightarrow{\cong} \operatorname{Tor}^S_j(\omega, Y_i) \quad \textit{for any } 1 \le j \le n-2.$$

Proof. Let $M = \omega \otimes_S N$. By Theorem 4.5, there exists a $\operatorname{Hom}_R(\omega, -)$-exact exact sequence

$$0 \to M_1 \to M_0\ (\cong M \oplus I) \to B_0 \to 0$$

in $\operatorname{Mod} R$ such that $B_0 \cong \operatorname{Tor}^S_1(\omega, \operatorname{cTr}_\omega M)$, $M_1 \in \operatorname{co}\Omega^1_{\mathcal{B}}(R)$ and B_{0*} $(= 0) \in \mathcal{I}_\omega(S)$. By Theorem 4.5 again, we further have $\operatorname{Hom}_R(\omega, -)$-exact exact sequences

$$0 \to M_2 \to M_1 \to B_1 \to 0, \qquad 0 \to M_3 \to M_2 \to B_2 \to 0$$

in $\operatorname{Mod} R$ such that $M_2 \in \operatorname{co}\Omega^2_{\mathcal{B}}(R)$, $M_3 \in \operatorname{co}\Omega^3_{\mathcal{B}}(R)$, $B_{1*} \in \mathcal{I}_\omega(S)$ and $\mathcal{I}_\omega(S)$-$\operatorname{pd}_S B_{2*} \le 1$. Now consider the push-out diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M_{3*} \arrow[r, equal] \arrow[d, "\alpha"] & M_{3*} \arrow[d, "\beta"] & & \\
0 \arrow[r] & M_{2*} \arrow[r, "\delta"] \arrow[d] & M_{1*} \arrow[r] \arrow[d, "\gamma"] & B_{1*} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B_{2*} \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & B_{1*} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By [17, Theorem 6.4], we have $\mathrm{Ext}_R^{\geq 1}(V, B_{2*}) = 0$ for any $V \in \mathcal{I}_\omega(S)$. So $\mathcal{I}_\omega(S)$-$\mathrm{pd}_S Y_1 \leq 1$ by [12, Lemma 8.2.1]. Moreover, there exists a commutative diagram with exact rows

$$\begin{tikzcd}
 & \omega \otimes_S M_{3*} \arrow[r, "1_\omega \otimes \alpha"] \arrow[d, "\theta_{M_3}"] & \omega \otimes_S M_{2*} \arrow[r] \arrow[d, "\theta_{M_2}"] & \omega \otimes_S B_{2*} \arrow[r] \arrow[d, "\theta_{B_2}"] & 0 \\
0 \arrow[r] & M_3 \arrow[r] & M_2 \arrow[r] & B_2 \arrow[r] & 0
\end{tikzcd}$$

Because $M_3 \in \mathrm{co}\Omega^3_{\mathcal{B}}(R) = \mathrm{c}\mathcal{T}^3_\omega(R)$ by Lemma 4.2, we infer that θ_{M_3} is an isomorphism and $1_\omega \otimes \alpha$ is a monomorphism. Similarly $1_\omega \otimes \delta$ is a monomorphism, and hence so is $1_\omega \otimes \beta$. Since $\mathrm{Tor}_1^S(\omega, M_{3*}) = 0$ by [28, Corollary 3.4], the sequence

$$0 \to \omega \otimes_S M_{3*} \to \omega \otimes_S M_{1*} \ (\cong \omega \otimes_S ((\omega \otimes_S N)_* \oplus I_*)) \to \omega \otimes_S Y_1 \to 0$$

is exact and γ induces an isomorphism $\mathrm{Tor}_1^S(\omega, (\omega \otimes_S N)_*) \xrightarrow{\cong} \mathrm{Tor}_1^S(\omega, Y_1)$. Now put $Y_0 := B_{1*}$ and $N_i := M_{i*}$ for $i = 2, 3$. Continuing this process, we may construct a submodule chain of $(\omega \otimes_S N)_* \oplus I_*$ satisfying the desired properties. ∎

5. Applications. In this section, we apply the two filtrations of modules obtained in Sections 3 and 4 to study mainly the relationship between artin algebras satisfying the Auslander condition and Gorenstein algebras.

Following [12, Definition 10.1.1], a module M in $\mathrm{Mod}\, R$ is called *Gorenstein injective* if there exists an exact sequence

$$\mathbf{I} : \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

in $\mathrm{Mod}\, R$ with all $I_i, I^i \in \mathcal{I}(R)$ such that $\mathrm{Hom}_R(E, \mathbf{I})$ is exact for any E in $\mathcal{I}(R)$ and $M \cong \mathrm{Im}(I_0 \to I^0)$. We use $\mathcal{GI}$ to denote the class of Gorenstein injective modules, and $\mathrm{Gid}_R\, M$ for the $\mathcal{GI}$-injective dimension (that is, the Gorenstein injective dimension) of M.

Note that a module M in $\mathrm{mod} R$ belongs to $\mathcal{B}_{D(R)}(R)$ if and only if M is in $\mathcal{GI}$ by [28, Theorem 3.9 and Corollary 5.2]. So, putting $\omega := D(R)$ in Theorem 3.12, we get

Corollary 5.1. *Let R be an artin algebra satisfying the Auslander condition. If $M \in \mathrm{mod} R$ with $\text{E-cograde}_{D(R)}\, M = k < \infty$, then there exists a filtration $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of R-submodules of M such that:*

(1) $M_1 = \cdots = M_k = 0$ *and there exists an exact sequence*

$$0 \to \mathrm{Tor}_{k+2}^R(D(R), \mathrm{cTr}_{D(R)}\, \mathrm{co}\Omega^k(M)) \to \mathrm{Tor}_k^R(D(R), \mathrm{Ext}_R^k(D(R), M)) \to M/M_k \to M/M_{k+1} \to 0.$$

(2) *If* $\mathrm{Tor}_i^R(D(R), \mathrm{Ext}_R^i(D(R), M)) \neq 0$, *then* $\text{E-cograde}_{D(R)}\, M/M_i = i$, $M_i \neq M_{i+1}$ *and* M_{i+1}/M_i *is pure of Ext-cograde* i.

(3) *If* $\operatorname{Tor}_i^R(D(R), \operatorname{Ext}_R^i(D(R), M)) = 0$, *then* $M_i = M_{i+1}$.
(4) *If* $\operatorname{Gid}_R M = d < \infty$, *then*

$$M = M_{d+1} \quad \text{and} \quad M/M_d \cong \operatorname{Tor}_d^R(D(R), \operatorname{Ext}_R^d(D(R), M)).$$

(5) *If* $\operatorname{Gid}_R M = d < \infty$, *then* $\operatorname{fil}(M) \leq d - k + 1$, *and equality holds whenever* $\text{T-cograde}_{D(R)} \operatorname{Ext}_R^i(D(R), M) = i$ *for any* $k \leq i \leq d$.

Auslander and Reiten [6] conjectured that any artin algebra R satisfying the Auslander condition is Gorenstein.

THEOREM 5.2. *Let R be an artin algebra satisfying the Auslander condition. If R is Gorenstein with* $\operatorname{id}_R R = \operatorname{id}_{R^{\mathrm{op}}} R = n$, *then* $\operatorname{fil}(\mathrm{co}\Omega^2(R/J)) \leq n - 1$, *and equality holds if* $\operatorname{Tor}_i^R(D(R), \operatorname{Ext}_R^i(D(R), \mathrm{co}\Omega^2(R/J))) \neq 0$ *for any* $0 \leq i \leq n-2$ *or* $\mathrm{co}\Omega^2(R/J)$ *is Gorenstein injective.*

Proof. Since $\operatorname{id}_R R = n$, it follows from [12, Theorem 12.3.1] and [27, Theorem 2.1] that $\operatorname{Gid}_R R/J = n$.

If $n \geq 2$, we deduce from [12, Theorem 12.3.1] that $\mathrm{co}\Omega^n(R/J)$ is Gorenstein injective. Thus $\operatorname{Gid}_R \mathrm{co}\Omega^2(R/J) \leq n-2$. Because $\operatorname{Gid}_R R/J = n$, we have $\operatorname{Gid}_R \mathrm{co}\Omega^2(R/J) = n-2$. So $\mathrm{co}\Omega^2(R/J) \neq 0$ and $D(\mathrm{co}\Omega^2(R/J)) \neq 0$. Because $D(\mathrm{co}\Omega^2(R/J))$ is 2-syzygy, it follows from [7, Theorems 1.7 and 4.7] that $\operatorname{Hom}_R(\operatorname{Hom}_{R^{\mathrm{op}}}(D(\mathrm{co}\Omega^2(R/J)), R), R) \cong D(\mathrm{co}\Omega^2(R/J)) \neq 0$. Take $\omega := D(R)$. Then

$$\begin{aligned}\operatorname{Hom}_R(D(R), \mathrm{co}\Omega^2(R/J)) &\cong \operatorname{Hom}_{R^{\mathrm{op}}}(D(\mathrm{co}\Omega^2(R/J)), DD(R)) \\ &\cong \operatorname{Hom}_{R^{\mathrm{op}}}(D(\mathrm{co}\Omega^2(R/J)), R) \neq 0\end{aligned}$$

and $\text{E-cograde}_{D(R)} \mathrm{co}\Omega^2(R/J) = 0$. Now the first assertion follows from Corollary 5.1.

If $n < 2$, then $\mathrm{co}\Omega^1(R/J)$ is Gorenstein injective. So $\mathrm{co}\Omega^2(R/J)$ is also Gorenstein injective by [12, Theorem 10.1.4]. ∎

Secondly, we turn to an application of the filtration of modules obtained in Section 4. Inspired by [24, Definition 2.15], we give its dual version.

DEFINITION 5.3.

(1) Two homomorphisms $f : A \to B$ and $f' : A' \to B'$ in $\operatorname{Mod} R$ are said to be *isomorphic up to a direct sum of injective modules* if there exist injective modules I, E, U, I', E' and U' such that

$$A \oplus I \oplus E \xrightarrow{g} B \oplus E \oplus U$$

and

$$A' \oplus I' \oplus E' \xrightarrow{h} B' \oplus E' \oplus U'$$

are isomorphic, where g and h are given by the following matrices:

$$g = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} f' & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(2) For an integer $k \geq 0$, a module $M \in \operatorname{Mod} R$ is called *injectively stationary of type k* if for any $i > k$, the inclusions $\lambda_i : M_i \to M_0$ and $\lambda_k : M_k \to M_0$ are isomorphic up to a direct sum of injective modules, where the M_i are as in Theorem 4.5.

We use $\underline{\operatorname{mod}}\, R$ to denote the stable category of $\operatorname{mod} R$ modulo projectives.

THEOREM 5.4. *Let R be an artin algebra satisfying the Auslander condition. Then the following statements are equivalent:*

(1) *R is Gorenstein.*
(2) *For some $k \geq 0$, any 2-$D(R)$-cotorsionfree left R-module is injectively stationary of type k.*
(3) *For some $k \geq 0$, any finitely generated 2-$D(R)$-cotorsionfree left R-module is injectively stationary of type k.*

Proof. (1)⇒(2). Let $M \in \operatorname{mod} R$. Since R is Gorenstein, we have $\operatorname{Gid}_R M < \infty$ by [12, Theorem 12.3.1]. Then $M \in \mathcal{B}_{D(R)}(R)$ by [10, Theorem 4.4]. So $\operatorname{cTr}_{D(R)} M \in \mathcal{A}_{D(R)}(R)$ by Lemma 3.11. This implies

$$B_i \cong \operatorname{co}\Omega^i(\operatorname{Tor}_{i+1}^S(D(R), \operatorname{cTr}_{D(R)} M)) = 0$$

for any $i \geq 0$. Thus all the M_i equal M_0 and the assertion follows.

(2)⇒(3) is trivial.

(3)⇒(1). By [16, Theorem 4.1], we only need to show that $\operatorname{pd}_R M \leq k+2$ for any $M \in \operatorname{mod} R$ with $\operatorname{pd}_R M < \infty$. Let $M \in \operatorname{mod} R$ with $\operatorname{pd}_R M = l < \infty$ and let

$$0 \to Q_l \to \cdots \to Q_1 \to Q_0 \to M \to 0$$

be a minimal projective resolution of M in $\operatorname{mod} R$. Let $M' := \operatorname{Ker}(Q_1 \to Q_0)$ and $\omega := D(R)$. Because M' is 2-syzygy, we have

$$\begin{aligned}(D(R) \otimes_R M')_* &\cong (D(\operatorname{Hom}_R(M', R)))_* \quad \text{(by [12, Theorem 3.2.13])} \\ &\cong \operatorname{Hom}_{R^{\mathrm{op}}}(\operatorname{Hom}_R(M', R), R) \\ &\cong M' \quad \text{(by [7, Theorems 1.7 and 4.7])}.\end{aligned}$$

So M' is adjoint 2-$D(R)$-cotorsionfree and $(D(R) \otimes_R M')_* \cong M'$. Note that I_* is a projective left R-module for any injective left R-module I by [12, Theorem 3.2.9]. So, putting $N = M'$ in Corollary 4.6, from the proof of that corollary we infer that there exists an exact sequence

$$0 \to N_{i+2} \to M' \oplus P \xrightarrow{f_i} Y_i \to 0$$

in $\operatorname{mod} R$ with $P \in \mathcal{P}(R)$ and $\operatorname{pd}_R Y_i \leq i$ for any $i \geq 0$, and f_i also induces an isomorphism

$$\operatorname{Tor}_j^R(D(R), M' \oplus P) \xrightarrow{\cong} \operatorname{Tor}_j^R(D(R), Y_i)$$

for any $j \geq 1$. By [12, Theorem 3.2.13], we have

$$\operatorname{Tor}_j^R(D(R), M' \oplus P) \cong D(\operatorname{Ext}_R^j(M' \oplus P, R)),$$
$$\operatorname{Tor}_j^R(D(R), Y_i) \cong D(\operatorname{Ext}_R^j(Y_i, R)).$$

Then by [5, Lemma 2.42], any homomorphism $\underline{M'} \to \underline{L}$ in $\underline{\operatorname{mod}}\, R$ with $\operatorname{pd}_R L \leq i$ factors through $\underline{f_i}$. As $D(R) \otimes_R (D(R) \otimes_R M')_* \cong D(R) \otimes_R M'$, we see that $D(R) \otimes_R M' \in \operatorname{mod} R$ is 2-$D(R)$-cotorsionfree. By the construction of N_i and [24, Lemma 2.16], we have $\underline{Y_k} \cong \underline{Y_i}$ for any $i > k$ by the assumption of (3). We immediately get a homomorphism $g : Y_{l-2} \to M'$ of left R-modules such that $\underline{1_{M'}} = \underline{g} \cdot \underline{f_{l-2}}$. Hence there exists a projective left R-module Q such that M' is isomorphic to a direct summand of $Y_{l-2} \oplus Q$. As $\underline{Y_k} \cong \underline{Y_{l-2}}$ in $\underline{\operatorname{mod}}\, R$, by [14, Proposition 3.1] there exist projective left R-modules P_1 and P_2 such that $Y_{l-2} \oplus P_1 \cong Y_k \oplus P_2$. Thus $\operatorname{pd}_R M' \leq k$ and $\operatorname{pd}_R M \leq k+2$. ∎

For a commutative noetherian ring R and an n-syzygy module M in $\operatorname{mod} R$ with $n \geq 0$, an *Evans–Griffith presentation* of M is defined to be an exact sequence

$$0 \to S \to B \to M \to 0$$

in $\operatorname{mod} R$ with B an nth syzygy of $\operatorname{Ext}_{R^{\mathrm{op}}}^{n+1}(\operatorname{Tr} M, R)$ and S an $(n+2)$-syzygy module [13, 26]. We introduce the dual version of this notion:

Definition 5.5. Let $n \geq 0$ and $M \in \operatorname{co}\Omega_{\mathcal{B}}^n(R)$. A *dual Evans–Griffith presentation* of M is an exact sequence

$$0 \to M \to B \to C \to 0$$

in $\operatorname{Mod} R$ with B an nth cosyzygy of $\operatorname{Tor}_{n+1}^S(\omega, \operatorname{cTr}_\omega M)$ and $C \in \operatorname{co}\Omega_{\mathcal{B}}^{n+2}(R)$.

As an application of Proposition 4.4, we have

Proposition 5.6. *Assume that ω satisfies the n-cograde condition with $n \geq 1$. Then for any $0 \leq i \leq n-1$, each module in $\operatorname{co}\Omega_{\mathcal{B}}^i(R)$ admits a dual Evans–Griffith presentation.*

Proof. Let $M \in \operatorname{co}\Omega_{\mathcal{B}}^i(R)$ with $0 \leq i \leq n-1$. Then by Proposition 4.4, there exists an exact sequence

$$0 \to A \xrightarrow{\alpha} M \oplus I \xrightarrow{\beta} B \to 0$$

in $\operatorname{Mod} R$ with $A \in \operatorname{co}\Omega^{i+1}_{\mathcal{B}}(R)$, $I \in \mathcal{I}(R)$ and $B \cong \operatorname{co}\Omega^{i}(\operatorname{Tor}^{S}_{i+1}(\omega, \operatorname{cTr}_{\omega} M))$. Let $\gamma := \beta\left(\begin{smallmatrix} 1_M \\ 0 \end{smallmatrix}\right)$ and $\lambda : M \hookrightarrow E$ be an embedding in $\operatorname{Mod} R$ with E injective. Then we have the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & M \arrow[r, equal] \arrow[d, "\left(\begin{smallmatrix} 1_M \\ 0 \\ \lambda \end{smallmatrix}\right)"] & M \arrow[d, "\left(\begin{smallmatrix} \gamma \\ \lambda \end{smallmatrix}\right)"] & \\
0 \arrow[r] & A \arrow[r, "\left(\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix}\right)"] \arrow[d, equal] & M \oplus I \oplus E \arrow[r, "\left(\begin{smallmatrix} \beta & 0 \\ 0 & 1_E \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & 1_I & 0 \\ -\lambda & 0 & 1_E \end{smallmatrix}\right)"] & B \oplus E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & I \oplus E \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where $C = \operatorname{Coker}\left(\begin{smallmatrix} \gamma \\ \lambda \end{smallmatrix}\right)$. It follows from the bottom row in the above diagram that $C \in \operatorname{co}\Omega^{i+2}_{\mathcal{B}}(R)$. Thus the rightmost column is a dual Evans–Griffith presentation of M. ∎

Acknowledgements. The authors thank the referee for the useful suggestions.

This research was partially supported by NSFC (Grant Nos. 11571164, 11501144), a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions and NSF of Guangxi Province of China (Grant No. 2016GXNSFAA380151) and China Scholarship Council (Grant No. 201708455034).

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