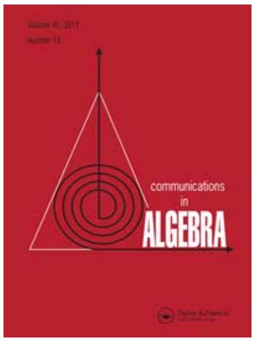

Taylor & Francis
Taylor & Francis Group

Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

A note on the stability of pure-injective modules

Chunhua Yang & Zhaoyong Huang

To cite this article: Chunhua Yang & Zhaoyong Huang (2017) A note on the stability of pure-injective modules, Communications in Algebra, 45:12, 5068-5072, DOI: 10.1080/00927872.2017.1291813

To link to this article: http://dx.doi.org/10.1080/00927872.2017.1291813

Accepted author version posted online: 16 Feb 2017.
Published online: 16 Feb 2017.

COMMUNICATIONS IN ALGEBRA®
2017, VOL. 45, NO. 12, 5068–5072
http://dx.doi.org/10.1080/00927872.2017.1291813

Taylor & Francis
Taylor & Francis Group

A note on the stability of pure-injective modules

Chunhua Yang[a] and Zhaoyong Huang[b]

[a]School of Mathematics and Information Science, Weifang University, Weifang, Shandong, P. R. China; [b]Department of Mathematics, Nanjing University, Nanjing, Jiangsu, P. R. China

ABSTRACT
Let R be a ring. Then a left R-module N is pure-injective if and only if $\mathrm{Hom}_R(M,N)$ is a pure-injective left S-module for any ring S and any (R,S)-bimodule $_RM_S$. If R is a commutative ring and M,N are R-modules with N pure-injective, then $\mathrm{Ext}^n_R(M,N)$ is a pure-injective R-module for any $n \geq 0$. Let R and S be rings and let $_SN_R$ be an (S,R)-bimodule and M a finitely presented left R-module. If N is pure-injective as a left S-module, then the left S-module $N \otimes_R M$ is pure-injective; and if R is left coherent, then the left S-module $\mathrm{Tor}^R_n(N,M)$ is pure-injective for any $n \geq 1$.

ARTICLE HISTORY
Received 22 June 2016
Revised 13 December 2016
Communicated by P. Gomez and L. Jose

KEYWORDS
Ext functors; Hom functors; Tor functors; pure-exact sequences; pure-injective modules; tensor functors

2010 MATHEMATICS SUBJECT CLASSIFICATION
16E30; 13D07

1. Introduction

Pure-injective modules were introduced in [3], and they were also called algebraically compact modules in [10]. Many authors have studied the properties of pure-injective modules, see [2–8, 10, 11] and references therein. Enochs proved in [4] that if R is a commutative Noetherian ring, then $\mathrm{Hom}_R(M,E)$ is a pure-injective R-module for any R-modules M and E with E injective. The aim of this note is to generalize this result.

In this note, all rings are associative rings with identity. Let R be a ring. We show that a left R-module N is pure-injective if and only if $\mathrm{Hom}_R(M,N)$ is a pure-injective left S-module for any associative ring S and any (R,S)-bimodule $_RM_S$. In particular, if R is a commutative ring, then an R-module N is pure-injective if and only if $\mathrm{Hom}_R(M,N)$ is a pure-injective R-module for any R-module M. Moreover, we get that if R is a commutative ring and M,N are R-modules with N pure-injective, then $\mathrm{Ext}^n_R(M,N)$ is a pure-injective R-module for any $n \geq 0$. Let R and S be rings and let $_SN_R$ be an (S,R)-bimodule and M a finitely presented left R-module. We show that if N is pure-injective as a left S-module, then the left S-module $N \otimes_R M$ is pure-injective; and if R is left coherent, then the left S-module $\mathrm{Tor}^R_n(N,M)$ is pure-injective for any $n \geq 1$.

2. Preliminaries

Throughout this note, R is an arbitrary ring, $\mathrm{Mod}\, R$ is the category of left R-modules and $\mathrm{mod}\, R$ is the category of finitely presented left R-modules.

Lemma 2.1 ([6, Lemma 1.2.13]). *Let*

$$0 \to A \to B \to C \to 0 \tag{2.1}$$

be an exact sequence in Mod R. *Then the following statements are equivalent.*

CONTACT Chunhua Yang chunhuayangwf@sina.com School of Mathematics and Information Science, Weifang University, Weifang, Shandong 261061, P. R. China.

(1) $0 \to N \otimes_R A \to N \otimes_R B \to N \otimes_R C \to 0$ *is exact for any* $N \in \operatorname{Mod} R^{op}$.
(2) $0 \to N \otimes_R A \to N \otimes_R B \to N \otimes_R C \to 0$ *is exact for any* $N \in \operatorname{mod} R^{op}$.
(3) $0 \to \operatorname{Hom}_R(M, A) \to \operatorname{Hom}_R(M, B) \to \operatorname{Hom}_R(M, C) \to 0$ *is exact for any* $M \in \operatorname{mod} R$.
(4) *The exact sequence* (2.1) *is a direct limit of split short exact sequences in* $\operatorname{Mod} R$.

The exact sequence (2.1) is called **pure-exact** if one of the equivalent conditions in Lemma 2.1 is satisfied.

Definition 2.2 (cf. [6]). A module $M \in \operatorname{Mod} R$ is called **pure-injective** if it is an injective object with respect to pure-exact sequences; that is, any pure-exact sequence

$$0 \to M \to B \to C \to 0$$

starting from M splits; equivalently, for any pure-exact sequence

$$0 \to A \to B \to C \to 0$$

in $\operatorname{Mod} R$, the induced sequence

$$0 \to \operatorname{Hom}_R(C, M) \to \operatorname{Hom}_R(B, M) \to \operatorname{Hom}_R(A, M) \to 0$$

is exact.

It is easy to see that the class of pure-injective modules is closed under direct summands and products. We write $(-)^+ := \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive and $\mathbb{Q}$ is the additive group of rational numbers.

Lemma 2.3 ([5, Propositions 5.3.7 and 5.3.9]). *For any* $M \in \operatorname{Mod} R$, *we have*
(1) $M^+ \in \operatorname{Mod} R^{op}$ *is pure-injective.*
(2) $\sigma_M : M \to M^{++}$ *via* $\sigma_M(m)(f) = f(m)$ *for any* $m \in M$ *and* $f \in M^+$ *induces a pure-exact sequence*

$$0 \to M \stackrel{\sigma_M}{\to} M^{++} \to \operatorname{Coker} \sigma_M \to 0.$$

Thus M *is pure-injective if and only if* σ_M *is a section.*

Lemma 2.4 ([1, Proposition I.10.1]). *Let* $M \in \operatorname{Mod} R$ *be pure-injective and* $\{N_i\}_{i \in I}$ *a direct system in* $\operatorname{Mod} R$. *Then for any* $n \geq 0$, *the canonical morphism*

$$\operatorname{Ext}_R^n(\varinjlim N_i, M) \to \varprojlim \operatorname{Ext}_R^n(N_i, M)$$

is an isomorphism of abelian groups.

3. Main results

We first prove the following

Theorem 3.1.
(1) *A module* $N \in \operatorname{Mod} R$ *is pure-injective if and only if* $\operatorname{Hom}_R(M, N) \in \operatorname{Mod} S$ *is pure-injective for any ring* S *and any* (R, S)*-bimodule* ${}_RM_S$.
(2) *If* R *is a commutative ring, then* $N \in \operatorname{Mod} R$ *is pure-injective if and only if* $\operatorname{Hom}_R(M, N)$ *is a pure-injective* R*-module for any* $M \in \operatorname{Mod} R$.

Proof.
(1) The sufficiency is obvious by taking $M = {}_RR_R$. Conversely, let

$$\xi : 0 \to A \to B \to C \to 0$$

be a pure-exact sequence in Mod S. Then by Lemma 2.1, ξ is a direct limit of split short exact sequences

$$\xi_\alpha : 0 \to A_\alpha \to B_\alpha \to C_\alpha \to 0$$

in Mod S. Let ${}_RM_S$ be an (R, S)-bimodule. By Lemma 2.1, we get the following split exact sequence

$$0 \to M \otimes_S A_\alpha \to M \otimes_S B_\alpha \to M \otimes_S C_\alpha \to 0$$

in Mod R and the following exact sequence

$$0 \to \mathrm{Ext}^1_R(M \otimes_S C_\alpha, N) \to \mathrm{Ext}^1_R(M \otimes_S B_\alpha, N) \to \mathrm{Ext}^1_R(M \otimes_S A_\alpha, N)$$

of abelian groups for any α. Then by the left exactness of the functor $\varprojlim$, we have that

$$0 \to \varprojlim \mathrm{Ext}^1_R(M \otimes_S C_\alpha, N) \to \varprojlim \mathrm{Ext}^1_R(M \otimes_S B_\alpha, N) \to \varprojlim \mathrm{Ext}^1_R(M \otimes_S A_\alpha, N)$$

is an exact sequence of abelian groups. By Lemma 2.4, we have that

$$0 \to \mathrm{Ext}^1_R(M \otimes_S C, N) \to \mathrm{Ext}^1_R(M \otimes_S B, N) \to \mathrm{Ext}^1_R(M \otimes_S A, N) \tag{3.1}$$

is an exact sequence of abelian groups. On the other hand, the purity of ξ gives an exact sequence

$$0 \to M \otimes_S A \to M \otimes_S B \to M \otimes_S C \to 0$$

in Mod R, which induces a long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M \otimes_S C, N) &\to \mathrm{Hom}_R(M \otimes_S B, N) \to \mathrm{Hom}_R(M \otimes_S A, N) \\ \to \mathrm{Ext}^1_R(M \otimes_S C, N) &\to \mathrm{Ext}^1_R(M \otimes_S B, N) \to \mathrm{Ext}^1_R(M \otimes_S A, N) \to \cdots \end{aligned} \tag{3.2}$$

of abelian groups. By (3.1) and (3.2), we get an exact sequence

$$0 \to \mathrm{Hom}_R(M \otimes_S C, N) \to \mathrm{Hom}_R(M \otimes_S B, N) \to \mathrm{Hom}_R(M \otimes_S A, N) \to 0$$

of abelian groups. Then by the adjoint isomorphism theorem, we have that

$$0 \to \mathrm{Hom}_S(C, \mathrm{Hom}_R(M, N)) \to \mathrm{Hom}_S(B, \mathrm{Hom}_R(M, N)) \to \mathrm{Hom}_S(A, \mathrm{Hom}_R(M, N)) \to 0$$

is an exact sequence of abelian groups. Thus $\mathrm{Hom}_R(M, N) \in \mathrm{Mod}\, S$ is pure-injective.

(2) It is a direct consequence of (1). □

Lemma 3.2.

(1) *Let R and S be rings. Then for any (S, R)-bimodule ${}_SM_R$ and $L \in \mathrm{Mod}\, S^{op}$, the adjoint isomorphism*

$$\theta : {}_R\mathrm{Hom}_{\mathbb{Z}}(L \otimes_S M, \mathbb{Q}/\mathbb{Z}) \to {}_R\mathrm{Hom}_S(M, \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z}))$$

is an isomorphism of left R-modules.

(2) *Let R be a commutative ring. Then for any $M, L \in \mathrm{Mod}\, R$ and $n \geq 0$,*

$$\mathrm{Ext}^n_R(M, \mathrm{Hom}_Z(L, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_n(L, M), \mathbb{Q}/\mathbb{Z})$$

is an isomorphism of R-modules.

Proof.

(1) By [9, Theorem 2.76], θ defined as $\theta(f)(m)(l) = f(l \otimes m)$ for any $f \in \mathrm{Hom}_{\mathbb{Z}}(L \otimes_S M, \mathbb{Q}/\mathbb{Z})$, $m \in M$ and $l \in L$ is an isomorphism of abelian groups. For any $r \in R$, we have

$$\begin{aligned} (r\theta(f))(m)(l) &= \theta(f)(mr)(l) = f_{mr}(l) = f(l \otimes mr) \\ &= (rf)(l \otimes m) = (rf)_m(l) = \theta(rf)(m)(l). \end{aligned}$$

So $(r\theta(f))(m) = \theta(rf)(m)$ and $r\theta(f) = \theta(rf)$. The assertion follows.

(2) The case for $n = 0$ follows from (1). Now let $n \geq 1$ and let $\mathbb{P}_\bullet$ be a deleted projective resolution of ${}_RM$. Then we have

$$\begin{aligned} H^n(\mathrm{Hom}_R(\mathbb{P}_\bullet, \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z}))) &\cong H^n(\mathrm{Hom}_{\mathbb{Z}}(L \otimes_R \mathbb{P}_\bullet, \mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(H_n(L \otimes_R \mathbb{P}_\bullet), \mathbb{Q}/\mathbb{Z}). \end{aligned}$$

Now the assertion follows from (1). □

For commutative rings, we have the following

Theorem 3.3. *Let R be a commutative ring and M, N in* Mod R *with N pure-injective. Then* $\mathrm{Ext}^n_R(M, N)$ *is a pure-injective R-module for any $n \geq 0$.*

Proof. The case for $n = 0$ follows from Theorem 3.1(2). Now let $n \geq 1$ and let M, N be in Mod R with N pure-injective. By Lemma 3.2(2), we have an isomorphism

$$\mathrm{Ext}^n_R(M, N^{++}) \cong (\mathrm{Tor}^R_n(N^+, M))^+$$

in Mod R. Then by Lemma 2.3(1), $\mathrm{Ext}^n_R(M, N^{++})$ is a pure-injective R-module. Note that N is a direct summand of N^{++} by Lemma 2.3(2). Now the assertion follows from the fact that the class of pure-injective modules is closed under direct summands. □

To obtain the adjoint counterparts of Theorems 3.1 and 3.3, we need the following standard observation.

Lemma 3.4. *Let ${}_RL_S$ be an (R, S)-bimodule and $M \in$* mod R*. Then we have*
(1) *There exists an isomorphism of left S-modules*

$$\tau : {}_S\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z}) \otimes_R M \to {}_S\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(M, L), \mathbb{Q}/\mathbb{Z}).$$

(2) *If R is left coherent, then for any $n \geq 1$, there exists an isomorphism of left S-modules*

$$\mathrm{Tor}^R_n(\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z}), M) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ext}^n_R(M, L), \mathbb{Q}/\mathbb{Z}).$$

Proof.
(1) By [5, Theorem 3.2.11], τ defined as $\tau(f \otimes a)(g) = f(g(a))$ for any $f \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z})$, $a \in M$ and $g \in \mathrm{Hom}_R(M, L)$ is an isomorphism of abelian groups. For any $s \in S$, we have

$$\begin{aligned} \tau(s(f \otimes a))(g) &= \tau((sf) \otimes a)(g) = (sf)(g(a)) \\ &= f(g(a)s) = f((gs)(a)) = \tau(f \otimes a)(gs) = s\tau(f \otimes a)(g). \end{aligned}$$

Then $\tau(s(f \otimes a)) = s\tau(f \otimes a)$. The assertion follows.

(2) Since R is left coherent and $M \in$ mod R, M admits a degreewise finite R-projective resolution $\mathbb{P}_\bullet$. Then we have

$$\begin{aligned} H_n(\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z}) \otimes_R \mathbb{P}_\bullet) &\cong H_n\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(\mathbb{P}_\bullet, L), \mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(H^n\mathrm{Hom}_R(\mathbb{P}_\bullet, L), \mathbb{Q}/\mathbb{Z}). \end{aligned}$$

Now the assertion follows from (1). □

Theorem 3.5. *Let ${}_SN_R$ be an (S, R)-bimodule and $M \in$* mod R*. If N is pure-injective as a left S-module, then we have*
(1) *The left S-module $N \otimes_R M$ is pure-injective.*
(2) *If R is left coherent, then the left S-module $\mathrm{Tor}^R_n(N, M)$ is pure-injective for any $n \geq 1$.*

Proof.

(1) By Lemma 3.4(1), $N^{++} \otimes_R M \cong (\mathrm{Hom}_R(M, N^+))^+$ is a pure-injective left S-module. Note that N is a direct summand of N^{++} by Lemma 2.3(2). Now the assertion follows from the fact that the class of pure-injective modules is closed under direct summands.

(2) It is similar to the proof of (1) in view of Lemma 3.4(2). □

Funding

This research was partially supported by National Natural Science Foundation of China (Grant nos. 11226062 and 11571164) and a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions. The authors thank the referee for the useful suggestions.

References

[1] Auslander, M. (1978). Functors and morphisms determined by objects. In: *Representation Theory of Algebras* (Proceeding Conference, Temple University, Philadelphia, PA, 1976). Lecture Notes in Pure Application Mathematics, Vol. 37, New York: Dekker, pp. 1–244.
[2] Cartan, H., Eilenberg, S. (1999). *Homological Algebra*. Reprint of the 1956 original, Princeton Landmarks in Mathematics. Princeton: Princeton University Press.
[3] Cohn, P. M. (1959). On the free product of associative rings. *Math. Z.* 71:380–398.
[4] Enochs, E. E. (1984). Flat cover and flat cotorsion modules. *Proc. Am. Math. Soc.* 92:179–184.
[5] Enochs, E. E., Jenda, O. M. G. (2000). *Relative Homological Algebra*. De Gruyter Expositions in Mathematics, Vol. 30. Berlin: Walter de Gruyter.
[6] Göbel, R., Trlifaj, J. (2006). *Approximations and Endomorphism Algebras of Modules*. De Gruyter Expositions in Mathematics, Vol. 41. Berlin: Walter de Gruyter GmbH & Co. KG.
[7] Herzog, I., Rothmaler, P. (2009). When cotorsion modules are pure injective. *J. Math. Log.* 9:63–102.
[8] Prest, M. (2009). *Purity, Spectra and Localisation*. Encyclopedia of Mathematics and its Applicaition, Vol. 121. Cambridge: Cambridge University Press.
[9] Rotman, J. J. (2009). *An Introduction to Homological Algebra*. 2nd ed. New York: Universitext, Springer.
[10] Warfield, R. B. Jr. (1969). Purity and algebraic compactness for modules. *Pacific J. Math.* 28:699–719.
[11] Xu, J. Z. (1996). *Flat Covers of Modules*. Lecture Notes in Mathematics, Vol. 1634. Berlin: Springer-Verlag.