Communications in Algebra®, 36: 546–563, 2008

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870701718971

APPROXIMATION PRESENTATIONS OF MODULES AND HOMOLOGICAL CONJECTURES

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In this article we give a sufficient condition of the existence of $\mathbb{W}^t$-approximation presentations. We also introduce property (W^k). As an application of the existence of $\mathbb{W}^t$-approximation presentations we give a connection between the finitistic dimension conjecture, the Auslander–Reiten conjecture, and property (W^k).

Key Words: Homological conjectures; Property (W^k); Right quasi k-Gorenstein rings; $\mathbb{W}^t$-approximation presentations.

2000 Mathematics Subject Classification: 16E10; 16E30; 16G10.

1. INTRODUCTION

In homological algebra and representation theory of algebras, the following is an important and interesting question, which is connected with the finitistic dimension conjecture.

Question. For a ring Λ, are the left and right self-injective dimensions of Λ identical?

Zaks (1969) proved that the answer is affirmative for a left and right noetherian ring if both dimensions are finite. Such rings are called *Gorenstein*.

For a positive integer k, Auslander and Reiten (1994) initiated the study of k-Gorenstein algebras, which has stimulated several investigations. They showed that the answer to the question above is positive in case Λ is an artin ∞-Gorenstein algebra (i.e., Λ is artin k-Gorenstein for all k). In Auslander and Reiten (1991), they also gave the relationship between the question above and the finitistic dimension conjecture (resp., the contravariant finiteness of the full subcategory of $\operatorname{mod}\Lambda$ consisting of the modules M with $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0$ for any $i \geq 1$).

It follows from Auslander and Reiten (1996, Theorem 0.1) and Hoshino and Nishida (2005, Theorem 4.1) that the following conditions are equivalent for a left and right noetherian ring Λ:

Received July 7, 2006; Revised March 15, 2007. Communicated by R. Wisbauer.
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(1) For a minimal injective resolution $0 \to \Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$ of Λ as a right Λ-module, the right flat dimension of I_i is at most $i+1$ for any $0 \leq i \leq k-1$;
(2) The strong grade of $\mathrm{Ext}_\Lambda^{i+1}(M, \Lambda)$ is at least i for any $M \in \mathrm{mod}\,\Lambda$ and $1 \leq i \leq k$;
(3) The grade of $\mathrm{Ext}_\Lambda^i(N, \Lambda)$ is at least i for any $N \in \mathrm{mod}\,\Lambda^{\mathrm{op}}$ and $1 \leq i \leq k$.

We call a ring *right quasi k-Gorenstein* provided it satisfies one of these equivalent conditions. A ring is called *right quasi* ∞*-Gorenstein* if it is right quasi k-Gorenstein for all k. From Huang (1999) we know that there are right quasi k-Gorenstein rings which are not k-Gorenstein; and contrary to the notion of k-Gorenstein, the notion of quasi k-Gorenstein is not left-right symmetric. We showed in Huang (2003) that the answer to the question above is also positive if Λ is an artin right quasi ∞-Gorenstein algebra.

For a ring Λ and a positive integer t, recall that a module $M \in \mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Lambda^{op}$) is called a W^t*-module* if $\mathrm{Ext}_\Lambda^i(M, \Lambda) = 0$ for any $1 \leq i \leq t$. We remark that $i = 0$ is not required in this definition. M is called a W^∞*-module* if it is a W^t-module for all t. Jans (1963) called a module $M \in \mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Lambda^{op}$) a *W-module* if $\mathrm{Ext}_\Lambda^1(M, \Lambda) = 0$. This is the motivation for us to give the above definition of W^t-modules. We use $\mathbb{W}^t$ (resp. $\mathbb{W}^\infty$) to denote the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of W^t-modules (resp. W^∞-modules). We call an exact sequence $0 \to K_t(M) \to E_t(M) \to M \to 0$ a $\mathbb{W}^t$*-approximation presentation* of M if $E_t(M) \to M$ is a right $\mathbb{W}^t$-approximation of M and the projective dimension of $K_t(M)$ is at most $t-1$ (see Huang, 1999).

One of the main results in Huang (1999) is that if Λ is a right quasi k-Gorenstein algebra, then every module in $\mathrm{mod}\,\Lambda$ has a $\mathbb{W}^t$-approximation presentation for any $1 \leq t \leq k$. In Section 3 we give a sufficient condition for the existence of $\mathbb{W}^t$-approximation presentations. Let Λ be a left and right noetherian ring and k a positive integer. For any $1 \leq t \leq k$, we show that a module M in $\mathrm{mod}\,\Lambda$ has a $\mathbb{W}^t$-approximation presentation if the strong grade of $\mathrm{Ext}_\Lambda^{i+1}(M, \Lambda)$ is at least i for any $1 \leq i \leq k-1$. This improves the main result in Huang (1999) and Auslander and Bridger (1969, Proposition 2.21). We then study the homological finiteness of the full subcategory of $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Lambda^{op}$) consisting of the modules with projective dimension at most k. In particular, we show that, over an artin right quasi k-Gorenstein algebra Λ, such subcategory is functorially finite in $\mathrm{mod}\,\Lambda^{op}$.

Auslander and Reiten (1994) posed the conjecture (**ARC**): An artin algebra is Gorenstein if it is ∞-Gorenstein. The famous Nakayama conjecture is a special case of this conjecture. We introduce in Section 2 *property* (W^k): Each W^k-module in $\mathrm{mod}\,\Lambda$ is torsionless. We then give some equivalent conditions for property (W^k), which we subsequently use to prove the following result: If the strong Nakayama conjecture holds true for Λ, and the left self-injective dimension of Λ is at most one, then the right self-injective dimension of Λ is at most one as well.

As an application of the results obtained in Section 3 we show in Section 4 the validity of **ARC** is equivalent to property (W^k), and we give the relationship with the finitistic dimension. Let Λ be an artin right quasi ∞-Gorenstein algebra and k a non-negative integer. We show that Λ is Gorenstein with self-injective dimension at most k if and only if Λ has property (W^k), and that the difference between the self-injective dimension and the finitistic dimension of Λ is at most one. We then conclude that in order to verify Nakayama conjecture (for any artin algebra) it suffices to verify the finitistic dimension conjecture for (right quasi) ∞-Gorenstein algebras.

Let $M \in \operatorname{mod}\Lambda$ and n a non-negative integer. If M admits a resolution (of finite length) $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ with all $X_i \in \mathbb{W}^\infty$, then set $\mathbb{W}^\infty\text{-dim}_\Lambda M = \inf\{n \mid$ there is an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ with all $X_i \in \mathbb{W}^\infty\}$. If no such resolution exists, set $\mathbb{W}^\infty\text{-dim}_\Lambda M = \infty$. We call $\mathbb{W}^\infty\text{-dim}_\Lambda M$ the *left orthogonal dimension* of M (see Huang, 2000).

In Section 5 we show that if Λ has property (W^∞), then for each $M \in \operatorname{mod}\Lambda$ the left orthogonal dimension of M and its Gorenstein dimension are identical, which yields that Λ has property (W^∞) if and only if each W^∞-module in $\operatorname{mod}\Lambda$ is reflexive. We also show that **ARC** is true if and only if $\mathbb{W}^\infty$ is contravariantly finite and the global dimension of Λ relative to $\mathbb{W}^\infty$ is finite.

According to the results obtained in the former sections, we pose in the final section two conjectures: (1) Any artin algebra has property (W^∞). (2) An artin algebra is Gorenstein if it is right quasi ∞-Gorenstein. The latter conjecture is clearly a generalized version of **ARC**.

Definitions and Notations

In the following, we give some definitions and notations which are often used in this article.

For a ring Λ, we use $\operatorname{mod}\Lambda$ to denote the category of finitely generated left Λ-modules and $\mathscr{P}^k(\Lambda)$ (resp. $\mathscr{P}^\infty(\Lambda)$) to denote the full subcategory of $\operatorname{mod}\Lambda$ consisting of the modules with projective dimension at most k (resp. the modules with finite projective dimension). For a left (resp. right) Λ-module M, $\text{l.pd}_\Lambda M$ and $\text{l.id}_\Lambda M$ (resp. $\text{r.pd}_\Lambda M$ and $\text{r.id}_\Lambda M$) are denoted the left projective dimension and the left injective dimension (resp. the right projective dimension and the right injective dimension) of M, respectively. In addition, we recall from Auslander and Smalϕ (1980) the following definition.

Assume that $\mathscr{C} \supset \mathscr{D}$ are full subcategories of $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Lambda^{op}$) and $C \in \mathscr{C}$, $D \in \operatorname{add}\mathscr{D}$, where $\operatorname{add}\mathscr{D}$ is the full subcategory of $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Lambda^{op}$) consisting of all Λ-modules (resp. Λ^{op}-modules) isomorphic to summands of finite direct sums of modules in $\mathscr{D}$. A morphism $D \to C$ is said to be a right $\mathscr{D}$-*approximation* of C if $\operatorname{Hom}_\Lambda(X, D) \to \operatorname{Hom}_\Lambda(X, C) \to 0$ is exact for all $X \in \operatorname{add}\mathscr{D}$. The subcategory $\mathscr{D}$ is said to be *contravariantly finite* in $\mathscr{C}$ if every module C in $\mathscr{C}$ has a right $\mathscr{D}$-approximation. Dually, we define the notions of left $\mathscr{D}$-*approximation* and *covariantly finite*. The subcategory $\mathscr{D}$ is said to be *functorially finite* in $\mathscr{C}$ if it is both contravariantly finite and covariantly finite in $\mathscr{C}$. The notions of contravariantly finite subcategories, covariantly finite subcategories and functorially finite subcategories are referred to as *homologically finite subcategories*.

We also list some famous homological conjectures.

Finitistic Dimension Conjecture (FDC): $\text{fin.dim}\,\Lambda = \sup\{\text{l.pd}_\Lambda M \mid M \in \operatorname{mod}\Lambda$ and $\text{l.pd}_\Lambda M$ is finite$\}$ is finite for any artin algebra Λ.

A brief history and some recent development of **FDC** were given in Zimmermann-Huisgen (1992).

Auslander-Reiten Conjecture (ARC): Every artin ∞-Gorenstein algebra is Gorenstein.

Nakayama Conjecture (NC): An artin algebra Λ is self-injective if each term in a minimal injective resolution of Λ as a right Λ-module is projective.

Let us recall the relationship between the conjectures mentioned above. It is showed in Yamagata (1996) that **FDC** implies **NC**, and in Auslander and Reiten (1994) that **ARC** implies **NC**.

Throughout this article, unless stated otherwise, Λ is a left and right noetherian ring.

2. PROPERTY (W^K)

In this section we introduce property (W^k) and then give some characterizations and applications.

Let $A \in \operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Lambda^{op}$) and $\sigma_A : A \to A^{**}$ defined via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$, where $(\ )^* = \operatorname{Hom}_\Lambda(\ ,\Lambda)$. A is called *torsionless* if σ_A is a monomorphism; and A is called *reflexive* if σ_A is an isomorphism. Let $P_1 \to P_0 \to A \to 0$ be a projective resolution of A in $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Lambda^{op}$). For a positive integer k, A is called *k-torsionfree* if $\operatorname{Ext}^i_\Lambda(\operatorname{Tr} A, \Lambda) = 0$ for any $1 \le i \le k$, where $\operatorname{Tr} A = \operatorname{Coker}(P_0^* \to P_1^*)$ is the *transpose* of A (see Auslander and Bridger, 1969 or Auslander and Reiten, 1996). We remark that it is known that a module in $\operatorname{mod}\Lambda$ is torsionless (resp. reflexive) if and only if it is 1-torsionfree (resp. 2-torsionfree) (see Auslander and Bridger, 1969).

The following lemma, which is of independent interest, is useful for the rest of this article.

Lemma 2.1. *For a positive integer k, the following statements are equivalent*:

(1) *Each* W^k*-module in* $\operatorname{mod}\Lambda$ *is torsionless*;
(2) *Each* W^k*-module in* $\operatorname{mod}\Lambda$ *is reflexive*;
(3) *Each* W^k*-module in* $\operatorname{mod}\Lambda^{op}$ *is a* W^∞*-module.*

Proof. (2) $\Rightarrow$ (1) It is trivial.

(1) $\Rightarrow$ (3) The proof is essentially the same as that of (1) $\Rightarrow$ (2) in Huang and Tang (2001, Lemma 3.3). Let Y be a W^k-module in $\operatorname{mod}\Lambda^{op}$ and $P_{k+1} \xrightarrow{d_{k+1}} P_k \xrightarrow{d_k} \cdots \xrightarrow{d_1} P_0 \to Y \to 0$ a projective resolution of Y in $\operatorname{mod}\Lambda^{op}$. It is not difficult to verify that $\operatorname{Coker} d_{k+1}^*$ is a W^k-module in $\operatorname{mod}\Lambda$. By (1) $\operatorname{Coker} d_{k+1}^*$ is torsionless. Then by Huang and Tang (2001, Lemma 2.1), $\operatorname{Ext}^1_\Lambda(\operatorname{Coker} d_{k+1}, \Lambda) = 0$ and $\operatorname{Ext}^{k+1}_\Lambda(Y, \Lambda)(\cong \operatorname{Ext}^1_\Lambda(\operatorname{Coker} d_{k+1}, \Lambda)) = 0$.

Since $0 \to \operatorname{Im} d_1 \to P_0 \to Y \to 0$ is exact, $\operatorname{Ext}^i_\Lambda(\operatorname{Im} d_1, \Lambda) = 0$ for any $1 \le i \le k$. Repeating the above argument we have $\operatorname{Ext}^{k+1}_\Lambda(\operatorname{Im} d_1, \Lambda) = 0$ and thus $\operatorname{Ext}^{k+2}_\Lambda(Y, \Lambda) = 0$. Continuing this procedure, we get our conclusion.

(3) $\Rightarrow$ (2) By Huang and Tang (2001, Lemma 3.3). □

Definition 2.2. We say that Λ has *property* (W^k) if the condition (1) of Lemma 2.1 is satisfied for Λ. If k is infinite, we then say that Λ has *property* (W^∞). We say that Λ has *property* $(W^k)^{op}$ (resp. $(W^\infty)^{op}$) if Λ^{op} has property (W^k) (resp. (W^∞)).

Remark. (1) It is trivial that if Λ has property (W^k) (resp. $(W^k)^{op}$), then Λ has property (W^n) (resp. $(W^n)^{op}$) for any $n > k$ and property (W^∞) (resp. $(W^\infty)^{op}$).

(2) In Section 5 we will show that (1) and (2) in Lemma 2.1 are also equivalent when k is infinite. That is, Λ has property (W^∞) if and only if each W^∞-module in $\operatorname{mod}\Lambda$ is reflexive (see Corollary 5.2).

Let $N \in \operatorname{mod}\Lambda^{op}$ and

$$0 \to N \xrightarrow{\delta_0} E_0 \xrightarrow{\delta_1} E_1 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_i} E_i \xrightarrow{\delta_{i+1}} \cdots$$

be an injective resolution of N. Recall from Colby and Fuller (1990) that an injective resolution as above is called *ultimately closed* at n (where n is a positive integer) if $\operatorname{Im}\delta_n = \bigoplus_{j=0}^{m} W_j$, where each W_j is a direct summand of $\operatorname{Im}\delta_{i_j}$ with $i_j < n$.

Corollary 2.3. *Λ has property (W^k) if*:

(1) $\text{r.id}_\Lambda \Lambda \leq k$; *or*
(2) Λ_Λ *has a ultimately closed injective resolution at* k.

Proof. Our conclusions follow from Huang and Tang (2001, Theorems 2.2 and 2.4), respectively. □

Lemma 2.4 (Auslander and Bridger, 1969, Proposition 2.6)**.** *Let* $A \in \operatorname{mod}\Lambda$ (*resp.* $\operatorname{mod}\Lambda^{op}$). *Then we have the following exact sequences*:

$$0 \to \operatorname{Ext}^1_\Lambda(\operatorname{Tr} A, \Lambda) \to A \xrightarrow{\sigma_A} A^{**} \to \operatorname{Ext}^2_\Lambda(\operatorname{Tr} A, \Lambda) \to 0 \quad (1)$$

$$0 \to \operatorname{Ext}^1_\Lambda(A, \Lambda) \to \operatorname{Tr} A \xrightarrow{\sigma_{\operatorname{Tr} A}} (\operatorname{Tr} A)^{**} \to \operatorname{Ext}^2_\Lambda(A, \Lambda) \to 0 \quad (2)$$

We observe that in case $k = 1$, the converse of Corollary 2.3(1) holds true. That is, we have the following corollary.

Corollary 2.5. (Bass, 1962, Theorem 3.3 and Auslander and Reiten, 1991, Proposition 2.2). *The following statements are equivalent*:

(1) $\text{r.id}_\Lambda \Lambda \leq 1$;
(2) Λ *has property* (W^1);
(3) *Each torsionless module in* $\operatorname{mod}\Lambda$ *is reflexive*;
(4) *Each torsionless module in* $\operatorname{mod}\Lambda^{op}$ *is a* W^1*-module.*

Proof. The equivalence of (1), (3), and (4) follows from Bass (1962, Theorem 3.3). By Corollary 2.3 we have (1) implies (2).

(2) $\Rightarrow$ (1) Let A be torsionless in $\operatorname{mod}\Lambda^{op}$. Then there are exact sequences (1) and (2) as in Lemma 2.4. Since A is torsionless, by the the exactness of the sequence (1) we have $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} A, \Lambda) = 0$, and then by the assumption (2) $\operatorname{Tr} A$ is torsionless. So from the exact sequence (2), we have $\operatorname{Ext}^1_\Lambda(A, \Lambda) = 0$. Then it is easy to see that $\text{r.id}_\Lambda \Lambda \leq 1$. □

Following Colby and Fuller (1990), we say the strong Nakayama conjecture (**SNC**) is true for Λ if for any $M \in \operatorname{mod}\Lambda$ the condition $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0$ for all

$i \geq 0$ implies $M = 0$. By the proof of Yamagata (1996, Theorem 3.4.3), for any artin algebra we have **FDC** $\Rightarrow$ **SNC**.

Lemma 2.6. *Suppose that* **SNC** *holds true for* Λ. *If each torsionless module in* $\operatorname{mod}\Lambda^{op}$ *is reflexive, then each torsionless module in* $\operatorname{mod}\Lambda^{op}$ *is a* W^1*-module.*

Proof. Assume that A is a torsionless module in $\operatorname{mod}\Lambda^{op}$. Then A is reflexive by assumption. By Jans (1961, Theorem 1.1), there are a torsionless module B in $\operatorname{mod}\Lambda$ and an exact sequence:

$$0 \to B \xrightarrow{\sigma_B} B^{**} \to \operatorname{Ext}^1_\Lambda(A, \Lambda) \to 0. \tag{3}$$

Since B^* is torsionless in $\operatorname{mod}\Lambda^{op}$, B^* is reflexive (by assumption) and σ_{B^*} is an isomorphism. On the other hand, by Anderson and Fuller (1992, Proposition 20.14) we have $\sigma_B^*\sigma_{B^*} = 1_{B^*}$, it follows that σ_B^* is also an isomorphism.

By assumption and the dual version of Corollary 2.5, $\text{l.id}_\Lambda\Lambda \leq 1$. Because $B^{**} \in \operatorname{mod}\Lambda$ is torsionless, there is a module $C \in \operatorname{mod}\Lambda$ such that B^{**} is a 1-syzygy of C and $\operatorname{Ext}^1_\Lambda(B^{**}, \Lambda) \cong \operatorname{Ext}^2_\Lambda(C, \Lambda) = 0$.

From the exact sequence (3) we get a long exact sequence

$$0 \to [\operatorname{Ext}^1_\Lambda(A, \Lambda)]^* \to B^{***} \xrightarrow{\sigma_B^*} B^* \to \operatorname{Ext}^1_\Lambda(\operatorname{Ext}^1_\Lambda(A, \Lambda), \Lambda) \to \operatorname{Ext}^1_\Lambda(B^{**}, \Lambda) = 0.$$

So $[\operatorname{Ext}^1_\Lambda(A, \Lambda)]^* = 0$ and $\operatorname{Ext}^1_\Lambda(A, \Lambda)$ is a W^1-module in mod Λ. Because $\text{l.id}_\Lambda\Lambda \leq 1$, $\operatorname{Ext}^i_\Lambda(\operatorname{Ext}^1_\Lambda(A, \Lambda), \Lambda) = 0$ for any $i \geq 0$. Since **SNC** holds true for Λ, $\operatorname{Ext}^1_\Lambda(A, \Lambda) = 0$. □

We now give a partial answer to the question mentioned in the Introduction as follows.

Theorem 2.7. *If* **SNC** *holds true for* Λ, *then* $\text{l.id}_\Lambda\Lambda \leq 1$ *implies* $\text{r.id}_\Lambda\Lambda \leq 1$. *In particular, if* **SNC** *is always true, then* $\text{l.id}_\Lambda\Lambda \leq 1$ *if and only if* $\text{r.id}_\Lambda\Lambda \leq 1$.

Proof. Assume that **SNC** holds true for Λ and $\text{l.id}_\Lambda\Lambda \leq 1$. Then by the dual version of Corollary 2.5, each torsionless module in $\operatorname{mod}\Lambda^{op}$ is reflexive. It follows from Lemma 2.6 and Corollary 2.5 that each torsionless module in $\operatorname{mod}\Lambda^{op}$ is a W^1-module and $\text{r.id}_\Lambda\Lambda \leq 1$. □

It would be interesting to know whether the general case of Theorem 2.7 holds true. That is, if **SNC** holds true for Λ, then for any positive integer k, does $\text{l.id}_\Lambda\Lambda \leq k$ imply $\text{r.id}_\Lambda\Lambda \leq k$? If the answer is affirmative, then we get that **SNC** $\Rightarrow$ the Gorenstein Symmetric Conjecture (**GSC**). The latter conjecture states that the left and right self-injective dimensions are identical for any artin algebra.

For a positive integer t, we use $(\mathbb{W}^t)^{op}$ (resp. $(\mathbb{W}^\infty)^{op}$) to denote the subcategory of $\operatorname{mod}\Lambda^{op}$ consisting of W^t (resp. W^∞)-modules.

Proposition 2.8.

(1) **SNC** *is true for* Λ *if* Λ *has property* (W^∞).
(2) *If* $\text{l.id}_\Lambda\Lambda \leq k$ (*where* $k \leq 2$), *then* Λ *has property* (W^∞) *if and only if* $\text{r.id}_\Lambda\Lambda \leq k$.

Proof. (1) directly follows from the definition of property (W^∞).

(2) By Corollary 2.3, we get the sufficiency. In the following, we prove the necessity.

The case for $k = 0$ is trivial, and the case for $k = 1$ follows from (1) and Theorem 2.7.

Now suppose that $k = 2$ and $A \in \operatorname{mod} \Lambda^{op}$ and $P_1 \xrightarrow{f} P_0 \to A \to 0$ is a projective resolution of A in $\operatorname{mod} \Lambda^{op}$. Then we get the exact sequence in $\operatorname{mod} \Lambda$

$$0 \to A^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{Tr} A \to 0.$$

It is not difficult to see that $\operatorname{Ker} f \cong (\operatorname{Tr} A)^*$. Since $\operatorname{l.id}_\Lambda \Lambda \leq 2$, $\operatorname{Ext}_\Lambda^i(A^*, \Lambda) \cong \operatorname{Ext}_\Lambda^{i+2}(\operatorname{Tr} A, \Lambda) = 0$ for any $i \geq 1$ and $A^* \in \mathbb{W}^\infty$. By assumption, Λ has property (W^∞), so A^* is reflexive. Because $(\operatorname{Tr} A)^* \in \operatorname{mod} \Lambda^{op}$, we have that $(\operatorname{Tr} A)^{**}$ is reflexive by the above argument. Then $(\operatorname{Tr} A)^{***}$ is also reflexive. On the other hand, by Anderson and Fuller (1992, Proposition 20.14) we have $\sigma_{\operatorname{Tr} A}^* \sigma_{(\operatorname{Tr} A)^*} = 1_{(\operatorname{Tr} A)^*}$, it follows that $\sigma_{\operatorname{Tr} A}^*$ is a split epimorphism. Then, by applying the functor $\operatorname{Hom}_\Lambda(\ , \Lambda)$ to the exact sequence (2) in Lemma 2.4, we have that $(\operatorname{Tr} A)^*$ is a direct summand of $(\operatorname{Tr} A)^{***}$ and hence $(\operatorname{Tr} A)^*$ is reflexive. So, by assumption and Lemma 2.4, we have that $\operatorname{Tr}(\operatorname{Tr} A)^* \in \mathbb{W}^2 (= \mathbb{W}^\infty)$ and $\operatorname{Tr}(\operatorname{Tr} A)^*$ is reflexive. Then $(\operatorname{Tr} A)^* \in (\mathbb{W}^2)^{op}$ again by Lemma 2.4. Since $\operatorname{Ext}_\Lambda^{i+2}(A, \Lambda) \cong \operatorname{Ext}_\Lambda^i(\operatorname{Ker} f, \Lambda) \cong \operatorname{Ext}_\Lambda^i((\operatorname{Tr} A)^*, \Lambda)$ for any $i \geq 1$, $\operatorname{Ext}_\Lambda^3(A, \Lambda) = 0 = \operatorname{Ext}_\Lambda^4(A, \Lambda)$. Thus we conclude that $\operatorname{r.id}_\Lambda \Lambda \leq 2$. □

We wonder whether the assertion in Proposition 2.8(2) holds true when k is any positive integer. The answer is affirmative when Λ is an artin algebra (see Angeleri Hügel et al., 2006, Theorem 3.10).

3. THE EXISTENCE OF $\mathbb{W}^t$-APPROXIMATION PRESENTATIONS

Let M be in $\operatorname{mod} \Lambda$ (resp. $\operatorname{mod} \Lambda^{op}$) and i a non-negative integer. We say that the *grade* of M, written as grade$M$, is at least i if $\operatorname{Ext}_\Lambda^j(M, \Lambda) = 0$ for any $0 \leq j < i$. We say that the *strong grade* of M, written as s.grade$M$, is at least i if grade$X \geq i$ for each submodule X of M (see Auslander and Reiten, 1996). We showed in Huang (1999) that every module in $\operatorname{mod} \Lambda$ has a $\mathbb{W}^t$-approximation presentation for any $1 \leq t \leq k$ if Λ is a right quasi k-Gorenstein algebra. Actually, the argument we use in proving (Huang, 1999, Theorem 1) proves the following more general result (or c.f. Auslander and Bridger, 1969, Proposition 2.21).

Theorem 3.1. *Let M be in* $\operatorname{mod} \Lambda$ (*resp.* $\operatorname{mod} \Lambda^{op}$). *If* $\operatorname{grade} \operatorname{Ext}_\Lambda^t(M, \Lambda) \geq t$ *for any* $1 \leq t \leq k$, *then M has a $\mathbb{W}^t$-approximation presentation for any* $1 \leq t \leq k$.

In this section we develop this result and show that a module $M \in \operatorname{mod} \Lambda$ has a $\mathbb{W}^t$-approximation presentation for any $1 \leq t \leq k$ if $\operatorname{s.grade}\operatorname{Ext}_\Lambda^{t+1}(M, \Lambda) \geq t$ for any $1 \leq t \leq k - 1$.

Let M be in $\operatorname{mod} \Lambda$ and $P \xrightarrow{f} M^* \to 0$ an exact sequence in $\operatorname{mod} \Lambda^{op}$ with P projective, and let h be the composition: $M \xrightarrow{\sigma_M} M^{**} \overset{f^*}{\hookrightarrow} P^*$. Set $\mathcal{P}^0(\Lambda) = \{$projective modules in $\operatorname{mod} \Lambda\}$. From the proof of Huang (2000, Lemma 1) we have the following lemma.

Lemma 3.2. *h is a left $\mathscr{P}^0(\Lambda)$-approximation of M.*

Theorem 3.3. *Let M be in* $\operatorname{mod}\Lambda$ *and k a positive integer. If* $\text{s.gradeExt}_\Lambda^{t+1}(M, \Lambda) \geq t$ *for any* $1 \leq t \leq k-1$, *then M has a $\mathbb{W}^t$-approximation presentation for any* $1 \leq t \leq k$.

Proof. We proceed by induction on k. The case $k = 1$ follows from Trlifaj (1996, Lemma 6.9). Now suppose that $k \geq 2$ and a module $M \in \operatorname{mod}\Lambda$ satisfies $\text{s.gradeExt}_\Lambda^{t+1}(M, \Lambda) \geq t$ for any $1 \leq t \leq k-1$.

Let $0 \to L \overset{f}{\to} P \overset{g}{\to} M \to 0$ be an exact sequence in $\operatorname{mod}\Lambda$ with P projective. Then $\operatorname{Ext}_\Lambda^t(L, \Lambda) \cong \operatorname{Ext}_\Lambda^{t+1}(M, \Lambda)$ for any $t \geq 1$ and $\text{s.gradeExt}_\Lambda^t(L, \Lambda) \geq t$ for any $1 \leq t \leq k-1$. By inductive hypothesis, L has a $\mathbb{W}^{k-1}$-approximation presentation:

$$0 \to K_{k-1}(L) \overset{\alpha}{\to} E_{k-1}(L) \overset{\beta}{\to} L \to 0,$$

with $\text{l.pd}_\Lambda K_{k-1}(L) \leq k-2$ and $E_{k-1}(L) \in \mathbb{W}^{k-1}$. So $\operatorname{Ext}_\Lambda^t(K_{k-1}(L), \Lambda) \cong \operatorname{Ext}_\Lambda^{t+1}(L, \Lambda)$ for any $1 \leq t \leq k-2$ and hence $\text{s.gradeExt}_\Lambda^t(K_{k-1}(L), \Lambda) \geq t+1$ for any $1 \leq t \leq k-2$.

Notice that $K_{k-1}(L)$ is torsionless by Auslander and Bridger (1969, Proposition 3.17) (we remark that $K_{k-1}(L)$ is trivially torsionless even for $k = 1$). On the other hand, L is torsionless since it is a submodule of the projective module P. Then by Auslander and Reiten (1996, Theorem 1.1), $E_{k-1}(L)$ is torsionless.

Let $Q \overset{h_1}{\to} [E_{k-1}(L)]^* \to 0$ be an exact sequence in $\operatorname{mod}\Lambda^{op}$ with Q projective. Then $0 \to [E_{k-1}(L)]^{**} \overset{h_1^*}{\to} Q^*$ is exact in $\operatorname{mod}\ \Lambda$. Let h be the composition: $E_{k-1}(L) \overset{\sigma_{E_{k-1}(L)}}{\longrightarrow} [E_{k-1}(L)]^{**} \overset{h_1^*}{\to} Q^*$. Since $E_{k-1}(L)$ is torsionless, $\sigma_{E_{k-1}(L)}$ and h are monomorphisms. By Lemma 3.2, h is a left $\mathscr{P}^0(\Lambda)$-approximation of $E_{k-1}(L)$. So there is a homomorphism $\delta : Q^* \to P$ such that $\delta h = f\beta$ and hence we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{k-1}(L) & \xrightarrow{\alpha} & E_{k-1}(L) & \xrightarrow{\beta} & L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \binom{h}{0}} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & Q^* & \longrightarrow & Q^* \oplus P & \xrightarrow{(\delta,\ 0)} & P & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle g} & & \\
 & & & & & & M & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

where γ is an induced homomorphism. Put $K_k(M) = \text{Coker}\,\gamma$ and $E_k(M) = \text{Coker}\binom{h}{0}$. By the Snake Lemma we have the exact sequence

$$0 \to K_k(M) \to E_k(M) \to M \to 0. \tag{4}$$

By the exactness of $0 \to K_{k-1}(L) \xrightarrow{\gamma} Q^* \to K_k(M) \to 0$ we have $\text{l.pd}_\Lambda K_k(M) \leq k - 1$. On the other hand, by the exactness of $0 \to E_{k-1}(L) \xrightarrow{\binom{h}{0}} Q^* \oplus P \to E_k(M) \to 0$ we have $\text{Ext}^t_\Lambda(E_{k-1}(L), \Lambda) \cong \text{Ext}^{t+1}_\Lambda(E_k(M), \Lambda)$ for any $t \geq 1$, which implies that $\text{Ext}^t_\Lambda(E_k(M), \Lambda) = 0$ for any $2 \leq t \leq k$. In addition, It is easy to see that $\binom{h}{0}$ is also a left $\mathscr{P}^0(\Lambda)$-approximation of $E_{k-1}(L)$, thus $\text{Ext}^1_\Lambda(E_k(M), \Lambda) = 0$ and we conclude that $\text{Ext}^t_\Lambda(E_k(M), \Lambda) = 0$ for any $1 \leq t \leq k$. So $E_k(M) \in \mathbb{W}^t$ and the exact sequence (4) is as required. □

Let

$$0 \to \Lambda_\Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$$

be a minimal injective resolution of Λ as a right Λ-module.

Recall that Λ is called *k-Gorenstein* if the right flat dimension of I_i is at most i for any $0 \leq i \leq k-1$, and Λ is called *∞-Gorenstein* if it is k-Gorenstein for all k. It is well known that the notion of k-Gorenstein rings (resp. ∞-Gorenstein rings) is left-right symmetric (see Fossum et al., 1975, Auslander's Theorem 3.7).

Definition 3.4. Λ is called *right quasi* k-Gorenstein if the right flat dimension of I_i is at most $i+1$ for any $0 \leq i \leq k-1$, and Λ is called *right quasi* ∞-Gorenstein if it is right quasi k-Gorenstein for all k.

By Auslander and Reiten (1996, Theorem 0.1) and Hoshino and Nishida (2005, Theorem 4.1), we have that Λ is right quasi k-Gorenstein if and only if $\text{s.grade}\,\text{Ext}^{i+1}_\Lambda(M, \Lambda) \geq i$ for any $M \in \text{mod}\,\Lambda$ and $1 \leq i \leq k$, if and only if $\text{grade}\,\text{Ext}^i_\Lambda(N, \Lambda) \geq i$ for any $N \in \text{mod}\,\Lambda^{\text{op}}$ and $1 \leq i \leq k$.

Remark. A k-Gorenstein ring (resp. an ∞-Gorenstein ring) is clearly right quasi k-Gorenstein (resp. right quasi ∞-Gorenstein). However, we gave examples in Huang (1999) to explain that there are right quasi k-Gorenstein rings (resp. right quasi ∞-Gorenstein rings) which are not k-Gorenstein (resp. ∞-Gorenstein); and contrary to the notion of k-Gorenstein (resp. ∞-Gorenstein), the notion of quasi k-Gorenstein (resp. quasi ∞-Gorenstein) is not left-right symmetric.

The following corollary develops (Huang, 1999, Theorem 1).

Corollary 3.5. *Let Λ be a right quasi k-Gorenstein ring. Then we have*:

(1) *Each module in* $\text{mod}\,\Lambda$ *has a* $\mathbb{W}^t$*-approximation presentation for any* $1 \leq t \leq k+1$;
(2) *Each module in* $\text{mod}\,\Lambda^{op}$ *has a* $(\mathbb{W}^t)^{op}$*-approximation presentation for any* $1 \leq t \leq k$.

Proof. (1) and (2) follow from Theorems 3.3 and 3.1, respectively. □

It is interesting to know when $\mathscr{P}^k(\Lambda)$ is homologically finite in mod Λ. For an artin algebra Λ the following results are known:

(1) $\mathscr{P}^0(\Lambda)$ is functorially finite;
(2) $\mathscr{P}^1(\Lambda)$ is covariantly finite (see Auslander and Reiten, 1991); $\mathscr{P}^1(\Lambda)$ is contravariantly finite if the projective dimension of the injective envelope of Λ as a left Λ-module is at most one (see Igusa et al., 1990) (Dually, we have that $\mathscr{P}^1(\Lambda^{op})$ is contravariantly finite in mod Λ^{op} if Λ is a right quasi 1-Gorenstein algebra).
(3) $\mathscr{P}^k(\Lambda)$ is functorially finite if Λ is of finite representation type, where $k \in \mathbb{N}\bigcup\{\infty\}$ ($\mathbb{N}$ denotes positive integers) (see Auslander and Reiten, 1991).
(4) $\mathscr{P}^k(\Lambda)$ is contravariantly finite if Λ is stably equivalent to a hereditary algebra, where $k \in \mathbb{N}\bigcup\{\infty\}$ (see Auslander and Reiten, 1991; Deng, 1996).

Auslander and Reiten (1991) showed that if $\mathscr{P}^\infty(\Lambda)$ is contravariantly finite in mod Λ then **FDC** holds true for Λ. However, $\mathscr{P}^k(\Lambda)$ and $\mathscr{P}^\infty(\Lambda)$ need not to be contravariantly finite in mod Λ when **FDC** holds true (see Igusa et al., 1990).

In the following, we will study the homological finiteness of $\mathscr{P}^k(\Lambda^{op})$ and $\mathscr{P}^k(\Lambda)$ over a right quasi k-Gorenstein algebra (or ring) Λ.

For a non-negative integer n, we use Ω^n (mod Λ) to denote the full subcategory of mod Λ consisting of n-syzygy modules, and use $\mathscr{I}^n(\Lambda)$ to denote the full subcategory of mod Λ consisting of the modules with injective dimension at most n.

The following result generalizes and develops Igusa et al. (1990, Theorem 2.1) and Auslander and Reiten (1994, Proposition 5.8).

Theorem 3.6. *Let Λ be an artin right quasi k-Gorenstein algebra. Then $\mathscr{P}^k(\Lambda^{op})$ is functorially finite in mod Λ^{op} and has almost split sequences.*

Proof. Ω^k (mod Λ) is functorially finite in mod Λ by Auslander and Solberg (1993, Section 3) and closed under extensions by Auslander and Reiten (1996, Theorem 4.7). So, by the dual version of Auslander and Reiten (1991, Remark after Proposition 1.8), $\mathscr{I}^k(\Lambda) = \{C \in \text{mod } \Lambda \mid \text{Ext}^1_\Lambda(\Omega^k(\text{mod } \Lambda), C) = 0\}$ is covariantly finite in mod Λ. Hence $\mathscr{P}^k(\Lambda^{op})$ is contravariantly finite in mod Λ^{op}. Then $\mathscr{P}^k(\Lambda^{op})$ is also covariantly finite in mod Λ^{op} by Krause and Solberg (2003, Corollary 2.6). By Auslander and Smalϕ (1981, Theorem 2.4), $\mathscr{P}^k(\Lambda^{op})$ has almost split sequences. □

It is still open when $\mathscr{P}^k(\Lambda)$ is covariantly (or contravariantly) finite in mod Λ for an artin (right quasi k-Gorenstein) algebra Λ (see Auslander and Reiten, 1991). However, as an application of Theorem 3.3 we have the following result.

Proposition 3.7. *Let $M \in \text{mod } \Lambda$ with $\text{s.gradeExt}^{t+1}_\Lambda(M, \Lambda) \geq t$ for any $1 \leq t \leq k-1$. If Λ has property (W^k), then M has a left $\mathscr{P}^k(\Lambda)$-approximation.*

Proof. Following Theorem 3.3, we assume that $0 \to K_k(M) \to E_k(M) \to M \to 0$ is a $\mathbb{W}^k$-approximation presentation of M with $\text{l.pd}_\Lambda K_k(M) \leq k-1$ and $E_k(M) \in \mathbb{W}^k$. Because Λ has property (W^k), $E_k(M)$ is torsionless. From the proof of Theorem 3.3 we know that there is an exact sequence $0 \to E_k(M) \to P \to E \to 0$ such that

$E_k(M) \to P$ is a left $\mathscr{P}^0(\Lambda)$-approximation and $E \in \mathbb{W}^{k+1}$. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_k(M) & \longrightarrow & E_k(M) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_k(M) & \longrightarrow & P & \longrightarrow & F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E & = & E & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the middle row we know that $\text{l.pd}_\Lambda F \leq k$. Since $E \in \mathbb{W}^{k+1}$, it is easy to see that the exact sequence $0 \to M \to F \to E \to 0$ is a left $\mathscr{P}^k(\Lambda)$-approximation. □

Corollary 3.8. *Let Λ be a right quasi k-Gorenstein ring.*

(1) *If Λ has property (W^k), then $\mathscr{P}^k(\Lambda)$ is covariantly finite in* $\text{mod}\,\Lambda$.
(2) *If Λ has property (W^{k+1}), then $\mathscr{P}^{k+1}(\Lambda)$ is covariantly finite in* $\text{mod}\,\Lambda$.

Proof. Notice that for a right quasi k-Gorenstein ring Λ we have $\text{s.gradeExt}_\Lambda^{t+1}(M, \Lambda) \geq t$ for any $M \in \text{mod}\,\Lambda$ and $1 \leq t \leq k$, so (1) and (2) follow from Proposition 3.7. □

In the rest of this section, Λ is an artin algebra. We will give some further properties for right quasi k-Gorenstein algebras.

Proposition 3.9. *Let Λ be a right quasi k-Gorenstein algebra. Then $\mathbb{W}^k \subseteq \{M \in \text{mod}\Lambda \mid \text{Ext}_\Lambda^1(M, \mathscr{P}^{k-1}(\Lambda)) = 0\}$ and $(\mathbb{W}^k)^{op} = \{N \in \text{mod}\,\Lambda^{op} \mid \text{Ext}_\Lambda^1(N, \mathscr{P}^{k-1}(\Lambda^{op})) = 0\}$.*

Proof. It is trivial that $\mathbb{W}^k \subseteq \{M \in \text{mod}\,\Lambda \mid \text{Ext}_\Lambda^1(M, \mathscr{P}^{k-1}(\Lambda)) = 0\}$ and $(\mathbb{W}^k)^{op} \subseteq \{N \in \text{mod}\,\Lambda^{op} \mid \text{Ext}_\Lambda^1(N, \mathscr{P}^{k-1}(\Lambda^{op})) = 0\}$. Now, to show the inclusion $(\mathbb{W}^k)^{op} \supseteq \{N \in \text{mod}\,\Lambda^{op} \mid \text{Ext}_\Lambda^1(N, \mathscr{P}^{k-1}(\Lambda^{op})) = 0\}$, consider the minimal injective resolution of Λ as a right Λ-module: $0 \to \Lambda_\Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$. Because Λ is an artin right quasi k-Gorenstein algebra, $\text{l.pd}_\Lambda I_i \leq i + 1$ for any $0 \leq i \leq k - 1$. Then inductively we have $\text{l.pd}_\Lambda \text{Im}(I_i \to I_{i+1}) \leq i + 1$ for any $0 \leq i \leq k - 1$. Now it is not difficult to verify that if a module $N \in \text{mod}\,\Lambda^{op}$ satisfies $\text{Ext}_\Lambda^1(N, \mathscr{P}^{k-1}(\Lambda^{op})) = 0$, then $N \in (\mathbb{W}^k)^{op}$. □

Since the notion of k-Gorenstein algebras is left-right symmetric, by Proposition 3.9 we have the following corollary.

Corollary 3.10. *Let* Λ *be a k-Gorenstein algebra. Then* $\mathbb{W}^k = \{M \in \operatorname{mod}\Lambda \,|\, \operatorname{Ext}^1_\Lambda(M, \mathscr{P}^{k-1}(\Lambda)) = 0\}$ *and* $(\mathbb{W}^k)^{op} = \{N \in \operatorname{mod}\Lambda^{op} \,|\, \operatorname{Ext}^1_\Lambda(N, \mathscr{P}^{k-1}(\Lambda^{op})) = 0\}$.

Theorem 3.11. *Let* Λ *be a right quasi k-Gorenstein algebra. Then for any* $N \in \operatorname{mod}\Lambda^{op}$ *there is an exact sequence*

$$N \to K^{(k)}(N) \to E^{(k)}(N) \to 0 \tag{5}$$

where $N \to K^{(k)}(N)$ *is a minimal left* $\mathscr{P}^{k-1}(\Lambda^{op})$*-approximation of* N *and* $E^{(k)}(N) \in (\mathbb{W}^k)^{op}$.

Proof. By Theorem 3.6, $\mathscr{P}^{k-1}(\Lambda^{op})$ is covariantly finite in $\operatorname{mod}\Lambda^{op}$ and any $N \in \operatorname{mod}\Lambda^{op}$ has a minimal left $\mathscr{P}^{k-1}(\Lambda^{op})$-approximation $N \to K^{(k)}(N)$. Put $E^{(k)}(N) = \operatorname{Coker}(N \to K^{(k)}(N))$. Then by Wakamatsu's Lemma (see Auslander and Reiten, 1991, Lemma 1.3), we have $\operatorname{Ext}^1_\Lambda(E^{(k)}(N), \mathscr{P}^{k-1}(\Lambda^{op})) = 0$. So $E^{(k)}(N) \in (\mathbb{W}^k)^{op}$ by Proposition 3.9 and we get the exact sequence (5). □

A natural question is: when is the minimal left $\mathscr{P}^{k-1}(\Lambda^{op})$-approximation of N: $N \to K^{(k)}(N)$ in the exact sequence (5) monomorphic? If N is torsionless in $\operatorname{mod}\Lambda^{op}$, then the answer to this question is clearly affirmative.

4. PROPERTY (W^K) AND HOMOLOGICAL CONJECTURES

Auslander and Reiten (1994) conjectured that an artin ∞-Gorenstein algebra is Gorenstein (**ARC**). The famous **NC** is a special case of **FDC** (resp. **ARC**). In this section we give a connection between **FDC**, **ARC**, and property (W^k) as follows.

Theorem 4.1. *Let k be a non-negative integer. The following statements are equivalent for an artin right quasi* ∞*-Gorenstein algebra* Λ:

(1) Λ *is Gorenstein with self-injective dimension at most* k;
(2) $\operatorname{l.id}_\Lambda\Lambda \le k$;
$(2)^{op}$ $\operatorname{r.id}_\Lambda\Lambda \le k$;
(3) Λ *has property* (W^k);
$(3)^{op}$ Λ *has property* $(W^k)^{op}$.

In particular, we have $\operatorname{fin.dim}\Lambda \le \operatorname{l.id}_\Lambda\Lambda \le \operatorname{fin.dim}\Lambda + 1$.

Auslander and Reiten (1994, Corollary 5.5) showed that the conditions (1) and (2) in Theorem 4.1 are equivalent for an artin ∞-Gorenstein algebra (also see Beligiannis, 2000, Corollary 6.21). Notice that **NC** is a special case of **ARC**, so we know from Theorem 4.1 that in order to verify **NC** (for any artin algebra) it suffices to verify **FDC** for (right quasi) ∞-Gorenstein algebras.

To prove this theorem we need some lemmas.

Lemma 4.2. *Let* Λ *be a right quasi k-Gorenstein ring* (*especially, a right quasi* ∞*-Gorenstein ring*) *with property* $(W^k)^{op}$. *Then* $\operatorname{l.id}_\Lambda\Lambda \le k$.

Proof. Let Λ be a right quasi k-Gorenstein ring and M any module in $\operatorname{mod}\Lambda$. By Corollary 3.5(1) there is a $\mathbb{W}^k$-approximation presentation of M:

$$0 \to K_k(M) \to E_k(M) \to M \to 0$$

with $\mathrm{l.pd}_\Lambda K_k(M) \leq k-1$ and $E_k \in \mathbb{W}^k$. Since Λ has property $(W^k)^{op}$, it follows from the dual version of Lemma 2.1 that E_k is a W^∞-module and we then have $\mathrm{Ext}_\Lambda^{i+1}(M, \Lambda) \cong \mathrm{Ext}_\Lambda^i(K_k(M), \Lambda) = 0$ for any $i \geq k$, which implies that $\mathrm{l.id}_\Lambda \Lambda \leq k$. □

By using Corollary 3.5(2) and an argument similar to that in proving Lemma 4.2, we have the following result, which says that the converse of Corollary 2.3(1) holds true for a right quasi k-Gorenstein ring Λ.

Lemma 4.3. *Let Λ be a right quasi k-Gorenstein ring (especially, a right quasi ∞-Gorenstein ring) with property (W^k). Then* $\mathrm{r.id}_\Lambda \Lambda \leq k$.

Lemma 4.4 (Huang, 2003, Corollary 4). *If Λ is an artin right quasi ∞-Gorenstein algebra, then* $\mathrm{l.id}_\Lambda \Lambda = \mathrm{r.id}_\Lambda \Lambda$.

The following result is well known.

Lemma 4.5. $\mathrm{l.id}_\Lambda \Lambda \geq \mathrm{fin.dim}\, \Lambda$.

Lemma 4.6. *Let Λ be a right quasi ∞-Gorenstein ring. Then* $\mathrm{l.id}_\Lambda \Lambda \leq \mathrm{fin.dim}\, \Lambda + 1$.

Proof. Without loss of generality, we assume that $\mathrm{fin.dim}\, \Lambda = k < \infty$ and M is any module in mod Λ. By Corollary 3.5(1), for any $i \geq 1$ there is a $\mathbb{W}^{k+i+1}$-approximation presentation of M

$$0 \to K_{k+i+1}(M) \to E_{k+i+1}(M) \to M \to 0$$

with $\mathrm{l.pd}_\Lambda K_{k+i+1}(M) \leq k+i$ and $E_{k+i+1} \in \mathbb{W}^{k+i+1}$. Then $\mathrm{l.pd}_\Lambda K_{k+i+1}(M) \leq k$ since $\mathrm{fin.dim}\, \Lambda = k$. So $\mathrm{Ext}_\Lambda^{k+i+1}(M, \Lambda) \cong \mathrm{Ext}_\Lambda^{k+i}(K_{k+i+1}(M), \Lambda) = 0$ and $\mathrm{l.id}_\Lambda \Lambda \leq k+1$. We are done. □

Proof of Theorem 4.1. By Lemma 4.4 we get the equivalence of (1), (2), and $(2)^{op}$. That (2) (resp. $(2)^{op}$) implies $(3)^{op}$ (resp. (3)) follows from the dual version of Huang and Tang (2001, Theorem 2.2) (resp. Corollary 2.3), and $(3)^{op}$ (resp. (3)) implies (2) (resp. $(2)^{op}$) follows from Lemma 4.2 (resp. Lemma 4.3). The last assertion follows from Lemmas 4.5 and 4.6. □

Corollary 4.7. *Let Λ be an artin right quasi ∞-Gorenstein algebra. If Λ_Λ has a ultimately closed injective resolution, then Λ is Gorenstein.*

Proof. By Theorem 4.1 and Corollary 2.3. □

5. FINITE HOMOLOGICAL DIMENSIONS

Let $M \in \operatorname{mod}\Lambda$ and n a non-negative integer. Recall that M is said to have *Gorenstein dimension zero* if it satisfies the conditions: (1) M is reflexive; (2) $M \in \mathbb{W}^\infty$ and $M^* \in (\mathbb{W}^\infty)^{op}$; and M is said to have *Gorenstein dimension* n, written as $\text{G-dim}_\Lambda M = n$, if n is the least non-negative integer such that there is an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ with all X_i having Gorenstein dimension zero. If no such n exists set $\text{G-dim}_\Lambda M = \infty$ (see Auslander and Bridger, 1969). Gorenstein dimension of modules is an important invariant in homological algebra (see Christensen, 2000). It is clear that in general $\mathbb{W}^\infty\text{-dim}_\Lambda M \leq \text{G-dim}_\Lambda M$. Huang (2000) showed that for an artin algebra Λ, $\text{l.id}_\Lambda \Lambda < \infty$ if and only if $\mathbb{W}^\infty$-$\dim_\Lambda M < \infty$ for each $M \in \operatorname{mod}\Lambda$, if and only if $\text{G-dim}_\Lambda M < \infty$ for each $M \in \operatorname{mod}\Lambda$. Here we give the following proposition.

Proposition 5.1. *If* Λ *has property* (W^∞), *then for any* $M \in \operatorname{mod}\Lambda$ *we have* $\mathbb{W}^\infty\text{-}\dim_\Lambda M = \text{G-}\dim_\Lambda M$.

Proof. For any $M \in \operatorname{mod}\Lambda$, it is enough to show that $\text{G-dim}_\Lambda M \leq \mathbb{W}^\infty\text{-dim}_\Lambda M$ in case $\mathbb{W}^\infty\text{-dim}_\Lambda M = k < \infty$, because $\mathbb{W}^\infty\text{-dim}_\Lambda M \leq \text{G-dim}_\Lambda M$ by definition. We prove it by induction on k.

If $k = 0$, then $M \in \mathbb{W}^\infty$ and by assumption M is torsionless. Let $P \xrightarrow{f} M^* \to 0$ be exact in $\operatorname{mod}\Lambda^{op}$ with P projective. Then by Lemma 3.2, the composition: $M \overset{\sigma_M}{\hookrightarrow} M^{**} \overset{f^*}{\hookrightarrow} P^*$ is a left $\mathscr{P}^0(\Lambda)$-approximation of M. Put $g = f^*\sigma_M$ and $N = \operatorname{Coker} g$. Then $\operatorname{Ext}^1_\Lambda(N, \Lambda) = 0$. Thus from the exactness of $0 \to M \to P^* \to N \to 0$ we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P^* & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_M} & & \downarrow{\scriptstyle \sigma_P^*} & & \downarrow{\scriptstyle \sigma_N} & & \\ 0 & \longrightarrow & M^{**} & \longrightarrow & P^{***} & \longrightarrow & N^{**} & & \end{array}$$

where σ_M is a monomorphism and σ_{P^*} is an isomorphism. Since $M \in \mathbb{W}^\infty$, it is easy to see that $N \in \mathbb{W}^\infty$ and by assumption N is also torsionless. Thus σ_N is a monomorphism and therefore σ_M is an isomorphism and M is reflexive. Similarly we know that N is also reflexive. So, also from the exact sequence $0 \to M \to P^* \to N \to 0$ we have an exact sequence $0 \to M^{**} \to P^{***} \to N^{**} \to 0$ in $\operatorname{mod}\Lambda$. Notice that $0 \to N^* \to P^{**} \to M^* \to 0$ is exact in $\operatorname{mod}\Lambda^{op}$, thus $\operatorname{Ext}^1_\Lambda(M^*, \Lambda) = 0$. Similar to the above argument, we have $\operatorname{Ext}^1_\Lambda(N^*, \Lambda) = 0$ and so $\operatorname{Ext}^2_\Lambda(M^*, \Lambda) = 0$. Continuing this process, we finally get that $M^* \in (\mathbb{W}^\infty)^{op}$ and $\text{G-dim}_\Lambda M = 0$.

Now suppose $k \geq 1$ and $0 \to L \to P_{k-1} \to \cdots \to P_0 \to M \to 0$ is exact in $\operatorname{mod}\Lambda$ with all P_i projective. Then $\mathbb{W}^\infty\text{-dim}_\Lambda L = 0$ and by the argument above we have $\text{G-dim}_\Lambda L = 0$. It follows from Auslander and Bridger (1969, Theorem 3.13) that $\text{G-dim}_\Lambda M \leq k$. This completes the proof. □

Assume that Λ has property (W^∞). If M is a W^∞-module in $\operatorname{mod}\Lambda$, then $\mathbb{W}^\infty\text{-dim}_\Lambda M = 0$ and by Proposition 5.1 we have $\text{G-dim}_\Lambda M = 0$. Thus M is reflexive and consequently we get the following interesting result (compare it with Lemma 2.1).

Corollary 5.2. *The following statements are equivalent*:

(1) Λ *has property* (W^∞), *that is, each* W^∞*-module in* $\mathrm{mod}\,\Lambda$ *is torsionless*;
(2) *Each* W^∞*-module in* $\mathrm{mod}\,\Lambda$ *is reflexive.*

Let $\mathscr{X}$ be a contravariantly finite subcategory of $\mathrm{mod}\,\Lambda$. Then for each module M in $\mathrm{mod}\,\Lambda$ we have a complex:

$$\cdots \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0 \tag{6}$$

with $X_0 \to M$ a right $\mathscr{X}$-approximation of M and $X_i \to \mathrm{Ker} f_{i-1}$ a right $\mathscr{X}$-approximation of $\mathrm{Ker} f_{i-1}$ for any $i \geq 1$. Since $\mathscr{X}$ is contravariantly finite, we have an exact sequence

$$\cdots \to \mathrm{Hom}_\Lambda(\mathscr{X}, X_n) \to \cdots \to \mathrm{Hom}_\Lambda(\mathscr{X}, X_1) \to \mathrm{Hom}_\Lambda(\mathscr{X}, X_0) \to \mathrm{Hom}_\Lambda(\mathscr{X}, M) \to 0.$$

We define $\mathrm{pd}_{(\mathscr{X},\mathrm{mod}\,\Lambda)}M$ to be $\inf\{n \mid 0 \to \mathrm{Hom}_\Lambda(\mathscr{X}, X_n) \to \cdots \to \mathrm{Hom}_\Lambda(\mathscr{X}, X_1) \to \mathrm{Hom}_\Lambda(\mathscr{X}, X_0) \to \mathrm{Hom}_\Lambda(\mathscr{X}, M) \to 0$ is exact$\}$. If no such n exists set $\mathrm{pd}_{(\mathscr{X},\mathrm{mod}\,\Lambda)}M = \infty$. We also define $\mathrm{gl.dim}\,(\mathscr{X}, \mathrm{mod}\,\Lambda)$ to be $\sup\{\,\mathrm{pd}_{(\mathscr{X},\mathrm{mod}\,\Lambda)}M \mid M \in \mathrm{mod}\,\Lambda\}$.

On the other hand, if there is some n such that the complex (6) stops after n steps, that is, we have a complex $0 \to X_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0$ with $X_0 \to M$ a right $\mathscr{X}$-approximation of M and $X_i \to \mathrm{Ker} f_{i-1}$ a right $\mathscr{X}$-approximation of $\mathrm{Ker} f_{i-1}$ for any $1 \leq i \leq n$, then $\mathrm{pd}_{\mathscr{X}}M$ is defined to be the smallest non-negative n such that the complex (6) stops after n steps. If no such n exists set $\mathrm{pd}_{\mathscr{X}}M = \infty$. In addition, $\mathrm{gl.dim}_{\mathscr{X}}\Lambda$ is defined to be $\sup\{\mathrm{pd}_{\mathscr{X}}M \mid M \in \mathrm{mod}\,\Lambda\}$.

Proposition 5.3. *Let* $\mathscr{X}$ *be a contravariantly finite subcategory of* $\mathrm{mod}\,\Lambda$ *containing* $\mathscr{P}^0(\Lambda)$. *Then* $pd_{(\mathscr{X},\mathrm{mod}\,\Lambda)}M = \mathrm{pd}_{\mathscr{X}}M$ *for any* $M \in \mathrm{mod}\,\Lambda$.

Proof. Clearly, we have $\mathrm{pd}_{(\mathscr{X},\mathrm{mod}\,\Lambda)}M \leq \mathrm{pd}_{\mathscr{X}}M$. Now suppose $\mathrm{pd}_{(\mathscr{X},\mathrm{mod}\,\Lambda)}M = n < \infty$. Because $\mathscr{P}^0(\Lambda)$ is contained in $\mathscr{X}$, the complex (6) is exact. By definition of $\mathrm{pd}_{(\mathscr{X},\mathrm{mod}\,\Lambda)}M$,

$$0 \to \mathrm{Hom}_\Lambda(\mathscr{X}, X_n) \to \mathrm{Hom}_\Lambda(\mathscr{X}, X_{n-1}) \to \cdots \to \mathrm{Hom}_\Lambda(\mathscr{X}, X_1)$$
$$\to \mathrm{Hom}_\Lambda(\mathscr{X}, X_0) \to \mathrm{Hom}_\Lambda(\mathscr{X}, M) \to 0$$

is also exact. On the other hand, if we put $K_i = \mathrm{Ker} f_{i-1}$ for any $i \geq 1$, we also have an exact sequence:

$$0 \to \mathrm{Hom}_\Lambda(\mathscr{X}, K_n) \to \mathrm{Hom}_\Lambda(\mathscr{X}, X_{n-1}) \to \cdots \to \mathrm{Hom}_\Lambda(\mathscr{X}, X_1)$$
$$\to \mathrm{Hom}_\Lambda(\mathscr{X}, X_0) \to \mathrm{Hom}_\Lambda(\mathscr{X}, M) \to 0.$$

So $\mathrm{Hom}_\Lambda(\mathscr{X}, X_n) \cong \mathrm{Hom}_\Lambda(\mathscr{X}, K_n)$ and hence $\mathrm{Hom}_\Lambda(\mathscr{X}, K_{n+1}) = 0$. Now consider the exact sequence $0 \to K_{n+2} \xrightarrow{\alpha} X_{n+1} \to K_{n+1} \to 0$. We then have that $\mathrm{Hom}_\Lambda(\mathscr{X}, K_{n+2}) \xrightarrow{\mathrm{Hom}_\Lambda(\mathscr{X},\alpha)} \mathrm{Hom}_\Lambda(\mathscr{X}, X_{n+1})$ is an isomorphism, which yields in particular that $\mathrm{Hom}_\Lambda(X_{n+1}, K_{n+2}) \xrightarrow{\mathrm{Hom}_\Lambda(X_{n+1},\alpha)} \mathrm{Hom}_\Lambda(X_{n+1}, X_{n+1})$ is an isomorphism and α is an epimorphism and hence an isomorphism. So $K_{n+1} = 0$ and $\mathrm{pd}_{\mathscr{X}} M \leq n$. This finishes the proof. □

Corollary 5.4. *Under the assumptions of Proposition* 5.3, *we have* $\mathrm{gl.dim}\,(\mathscr{X}, \mathrm{mod}\,\Lambda) = \mathrm{gl.dim}_{\mathscr{X}} \Lambda$.

The following corollary is an immediate consequence of Corollary 3.5.

Proposition 5.5. *Let Λ be a right quasi k-Gorenstein ring. Then for any $M \in \mathrm{mod}\,\Lambda$ and $1 \leq t \leq k+1$, we have $\mathrm{pd}_{(\mathbb{W}^t, \mathrm{mod}\,\Lambda)} M = \mathrm{pd}_{\mathbb{W}^t} M \leq t$ and so $\mathrm{gl.dim}\,(\mathbb{W}^t, \mathrm{mod}\,\Lambda) = \mathrm{gl.dim}_{\mathbb{W}^t} \Lambda \leq t$.*

Proof. By Corollary 3.5(1), for any $1 \leq t \leq k+1$, $\mathbb{W}^t$ is a contravariantly finite subcategory of $\mathrm{mod}\,\Lambda$ containing $\mathscr{P}^0(\Lambda)$. Then by Proposition 5.3 and Corollary 5.4, we have that $\mathrm{pd}_{(\mathbb{W}^t, \mathrm{mod}\,\Lambda)} M = \mathrm{pd}_{\mathbb{W}^t} M$ for any $M \in \mathrm{mod}\,\Lambda$ and $\mathrm{gl.dim}\,(\mathbb{W}^t, \mathrm{mod}\,\Lambda) = \mathrm{gl.dim}_{\mathbb{W}^t} \Lambda$.

By Corollary 3.5(1), for any $M \in \mathrm{mod}\,\Lambda$ and $1 \leq t \leq k+1$ there is a $\mathbb{W}^t$-approximation presentation of M:

$$0 \to K_t(M) \to E_t(M) \to M \to 0$$

with $\mathrm{l.pd}_\Lambda K_t(M) \leq t-1$ and $E_t(M) \in \mathbb{W}^t$.

For a W^t-module A, it is easy to see that A is projective if $\mathrm{l.pd}_\Lambda A \leq t-1$. Now, considering the $\mathbb{W}^t$-approximation presentation of $K_t(M)$ we get easily our conclusion. □

For a subcategory $\mathscr{X}$ of $\mathrm{mod}\,\Lambda$ we use $\widehat{\mathscr{X}}$ to denote the subcategory of $\mathrm{mod}\,\Lambda$ consisting of the modules C for which there is an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to C \to 0$ with all X_i in $\mathscr{X}$ (see Auslander and Reiten, 1991).

Theorem 5.6. *Let Λ be an artin right quasi ∞-Gorenstein algebra. Then the following statements are equivalent.*

(1) Λ *is Gorenstein*;
(2) $\mathbb{W}^\infty$ *is contravariantly finite and* $\mathrm{gl.dim}(\mathbb{W}^\infty, \mathrm{mod}\,\Lambda)$ *is finite*;
(3) $\mathbb{W}^\infty$ *is contravariantly finite and* $\mathrm{pd}_{(\mathbb{W}^\infty, \mathrm{mod}\,\Lambda)} M$ *is finite for any* $M \in \mathrm{mod}\,\Lambda$;
(4) $\mathbb{W}^\infty$ *is contravariantly finite and* $\mathrm{gl.dim}_{\mathbb{W}^\infty} \Lambda$ *is finite*;
(5) $\mathbb{W}^\infty$ *is contravariantly finite and* $\mathrm{pd}_{\mathbb{W}^\infty} M$ *is finite for any* $M \in \mathrm{mod}\,\Lambda$;
(6) $\mathrm{fin.dim}\,\Lambda$ *is finite.*

Proof. The equivalence of (1) and (6) follows from Theorem 4.1, and that (2) implies (3) and (4) implies (5) are trivial. Because $\mathscr{P}^0(\Lambda)$ is contained in $\mathbb{W}^\infty$, from Corollary 5.4 (resp. Proposition 5.3) we know that (2) and (4) (resp. (3) and (5)) are equivalent. Assume that Λ is Gorenstein. Then $\mathbb{W}^\infty = \mathbb{W}^t$ for some positive integer t and from Proposition 5.5 we know that (1) $\Rightarrow$ (2) holds. Now suppose (5) holds. Then we have that $\widehat{\mathbb{W}^\infty} = \mathrm{mod}\,\Lambda$. Moreover, $\mathbb{W}^\infty$ is clearly resolving (i.e., $\mathbb{W}^\infty$ is closed under extensions, kernels of epimorphisms, and contains $\mathscr{P}^0(\Lambda)$). It follows from Auslander and Reiten (1991, Theorem 5.5) that ${}_\Lambda\Lambda$ is cotilting and so Λ is Gorenstein. This proves (5) implies (1). We are done. □

In view of Theorem 5.6 it would be interesting to know when $\mathbb{W}^\infty$ is contravariantly finite.

6. CONJECTURES

From the result obtained above, as far as $\mathbb{W}^\infty$ is concerned, two problems are worth being studied: When is a module in $\mathbb{W}^\infty$ torsionless (i.e., when has a ring Λ property (W^∞)) and what does property (W^∞) imply about Λ? In particular, we pose the following conjecture.

Conjecture I. Over an artin algebra, each module in $\mathbb{W}^\infty$ is torsionless (equivalently, each module in $\mathbb{W}^\infty$ is reflexive), that is, any artin algebra has property (W^∞).

Remark. (1) We know from Proposition 5.1 that if a left and right noetherian ring Λ has property (W^∞) then for any $M \in \operatorname{mod}\Lambda$ we have $\mathbb{W}^\infty\text{-}\dim_\Lambda M = \text{G-}\dim_\Lambda M$. So, we also conjecture that over an artin algebra the notion of the left orthogonal dimension and that of the Gorenstein dimension of any module in $\operatorname{mod}\Lambda$ coincide. This is an immediate corollary of Conjecture I.

(2) Assume that Conjecture I holds true for Λ. If $\text{l.id}_\Lambda \Lambda \leq k$, then every W^k-module in $\operatorname{mod}\Lambda$ is a W^∞-module and hence it is reflexive (compare this result with Theorem 4.1).

(3) Assume that Conjecture I holds true for an artin right quasi k-Gorenstein algebra Λ. If Λ has property $(W^k)^{op}$, then $\text{l.id}_\Lambda \Lambda \leq k$ by Lemma 4.2 and each W^k-module in $\operatorname{mod}\Lambda$ is a W^∞-module. By assumption (Conjecture I holds true for Λ), we know that each W^k-module in $\operatorname{mod}\Lambda$ is torsionless. Similar to the argument of Proposition 3.7, we then have that every module in $\operatorname{mod}\Lambda$ has a left $\mathcal{P}^k(\Lambda)$-approximation and $\mathcal{P}^k(\Lambda)$ is covariantly finite in $\operatorname{mod}\Lambda$ (compare this result with Corollary 3.8).

(4) The validity of Conjecture I would imply the validity of **GSC** (see Angeleri Hügel et al., 2006, Proposition 3.10).

In view of the results obtained in Section 4 and Theorem 5.6 we pose the following conjecture.

Conjecture II (Generalized **ARC**). An artin algebra is Gorenstein if it is right quasi ∞-Gorenstein.

ACKNOWLEDGMENTS

The research was partially supported by Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20060284002), NSFC (Grant No. 10771095), and NSF of Jiangsu Province of China (Grant No. BK2007517). The author thanks the referee for the useful and detailed suggestions on this article.

REFERENCES

Anderson, F. W., Fuller, K. R. (1992). *Rings and Categories of Modules*. 2nd ed. Graduate Texts in Mathematics 13. Berlin-Heidelberg-New York: Springer-Verlag.

Angeleri Hügel, L., Herbera, D., Trlifaj, J. (2006). Tilting modules and Gorenstein rings. *Forum Math.* 18:211–229.

Auslander, M., Bridger, M. (1969). *Stable Module Theory*. Memoirs Amer. Math. Soc. Vol. 94. Providence, Rhode Island: American Mathematical Society.

Auslander, M., Reiten, I. (1991). Applications of contravariantly finite subcategories. *Adv. Math.* 86:111–152.

Auslander, M., Reiten, I. (1994). k-Gorenstein algebras and syzygy modules. *J. Pure Appl. Algebra* 92:1–27.

Auslander, M., Reiten, I. (1996). Syzygy modules for noetherian rings. *J. Algebra* 183:167–185.

Auslander, M., Smalϕ, S. O. (1980). Preprojective modules over artin algebras. *J. Algebra* 66:61–122.

Auslander, M., Smalϕ, S. O. (1981). Almost split sequences in subcategories. *J. Algebra* 69:426–454.

Auslander, M., Solberg, ϕ. (1993). Relative homology and representation theory I, Relative homology and homologically finite subcategories. *Comm. Algebra* 21:2995–3031.

Bass, H. (1962). Injective dimension in noetherian rings. *Trans. Amer. Math. Soc.* 102:18–29.

Beligiannis, A. (2000). The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co-)stabilization. *Comm. Algebra* 28:4547–4596.

Christensen, L. W. (2000). *Gorenstein Dimensions*. Lecture Notes in Mathematics 1747. Berlin-Heidelberg-New York: Springer-Verlag.

Colby, R. R., Fuller, K. R. (1990). A note on the Nakayama Conjecture. *Tsukuba J. Math.* 14:343–352.

Deng, B. M. (1996). On contravariant finiteness of subcategories of modules of projective dimension $\leq i$. *Proc. Amer. Math. Soc.* 124:1673–1677.

Fossum, R. M., Griffith, P. A., Reiten, I. (1975). *Trivial Extensions of Abelian Categories*. Lecture Notes in Mathematics 456. Berlin-Heidelberg-New York: Springer-Verlag.

Hoshino, M., Nishida, K. (2005). A generalization of the Auslander formula. In: Representations of Algebras and Related Topics, Fields Institute Communications 45. Amer. Math. Soc., Providence, Rhode Island, pp.175–186.

Huang, Z. Y. (1999). $\mathbb{W}^t$-approximation presentations over quasi k-Gorenstein algebras. *Science in China (Series A)* 42:945–956.

Huang, Z. Y. (2000). Selforthogonal modules with finite injective dimension. *Science in China (Series A)* 43:1174–1181.

Huang, Z. Y. (2003). Selforthogonal modules with finite injective dimension II. *J. Algebra* 264:262–268.

Huang, Z. Y., Tang, G. H. (2001). Self-orthogonal modules over coherent rings. *J. Pure Appl. Algebra* 161:167–176.

Igusa, K., Smalϕ, S. O., Todorov, G. (1990). Finite projectivity and contravariant finiteness. *Proc. Amer. Math. Soc.* 109:937–941.

Jans, J. P. (1961). Duality in noetherian rings. *Proc. Amer. Math. Soc.* 12:829–835.

Jans, J. P. (1963). On finitely generated modules over noetherian rings. *Trans. Amer. Math. Soc.* 106:330–340.

Krause, H., Solberg, O. (2003). Applications of cotorsion pairs. *J. London Math. Soc.* 68:631–650.

Trlifaj, J. (1996). Whitehead test modules. *Trans. Amer. Math. Soc.* 348:1521–1554.

Yamagata, K. (1996). *Frobenius algebras, Handbook of Algebra 1*. Hazewinkel, M., ed. Amsterdam: North-Holland Publishing Co, pp. 841–887.

Zaks, A. (1969). Injective dimension of semi-primary rings. *J. Algebra* 13:73–86.

Zimmermann-Huisgen, B. (1992). Homological domino effects and the first finitistic dimension conjecture. *Invent. Math.* 108:369–383.