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Gorenstein algebras and recollements

Xin Ma[a], Tiwei Zhao[b], and Zhaoyong Huang[c]

[a]College of Science, Henan University of Engineering, Zhengzhou, Henan Province, P.R. China; [b]School of Mathematical Sciences, Qufu Normal University, Shandong Province, Qufu, P.R. China; [c]Department of Mathematics, Nanjing University, Nanjing, Jiangsu Province, P.R. China

ABSTRACT

Let A, A', and A'' be artin algebras. We prove that if there is a recollement of the bounded Gorenstein derived category $D^b_{\mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A))}(\mathrm{Mod}\ A)$ relative to the bounded Gorenstein derived categories $D^b_{\mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A'))}(\mathrm{Mod}\ A')$ and $D^b_{\mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A''))}(\mathrm{Mod}\ A'')$, then A is Gorenstein if and only if so are A' and A''. In addition, we prove that a virtually Gorenstein algebra A is Gorenstein if and only if the bounded homotopy category of (finitely generated) projective left A-modules and that of (finitely generated) injective left A-modules coincide.

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1. Introduction

Recollements play an important role in algebraic geometry and representation theory, see [1, 9, 25], which were introduced by Beĭlinson et al. in [9]. It is known that there are many interesting homological properties or invariants in the framework of recollements. For instance, for a recollement of bounded derived categories over finite dimensional algebras, Wiedemann proved in [32] that the finiteness of the global dimension is invariant in the recollement, and Happel proved in [21] that the finiteness of the finitistic dimension is also invariant in the recollement, and so on.

For an artin algebra A, we use Mod A and mod A to denote the category of left A-modules and the category of finitely generated left A-modules respectively. Let $\mathcal{X} = \mathrm{Mod}\ A$ or $\mathcal{X} = \mathrm{mod}\ A$. We use $\mathcal{G}(\mathcal{P}(\mathcal{X}))$ to denote the subcategory of $\mathcal{X}$ consisting of Gorenstein projective modules, and use $D^b_{\mathcal{G}(\mathcal{P}(\mathcal{X}))}(\mathcal{X})$ to denote the bounded Gorenstein derived categories of $\mathcal{X}$ [19].

Recently, the Gorensteinness of algebras in recollements was studied by several authors. Let A, A' and A'' be finite dimensional algebras and the bounded derived category $D^b(\mathrm{mod}\ A)$ admit a recollement

$$\begin{tikzcd} D^b(\mathrm{mod}\ A') \arrow[r, "i_*"] & D^b(\mathrm{mod}\ A) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D^b(\mathrm{mod}\ A''). \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Pan proved that if A is Gorenstein, then A' and A'' are also Gorenstein [28, Theorem 3.1]. Later, Chen and König proved that A is Gorenstein if and only if A' and A'' are also Gorenstein

CONTACT Zhaoyong Huang huangzy@nju.edu.cn Department of Mathematics, Nanjing University, Nanjing, Jiangsu Province 210093, P.R. China.
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plus an extra condition [15, Proposition 3.4]. Qin and Han proved that A is Gorenstein if and only if so are A' and A'' provided the recollement is what they call a 4-recollement, where they used the unbounded derived category of Mod [29, Theorem III]. On the other hand, Asadollahi, Bahiraei, Hafezi, and Vahed studied the Gorensteinness of algebras in recollements of relative derived categories and they showed that the Gorensteinness of certain artin algebras is an invariant of recollements; that is, for artin algebras A, A' and A'' with $\mathcal{G}(\mathcal{P}(\text{mod } A)), \mathcal{G}(\mathcal{P}(\text{mod } A'))$ and $\mathcal{G}(\mathcal{P}(\text{mod } A''))$ contravariantly finite in $\text{mod } A, \text{mod } A'$ and $\text{mod } A''$ respectively, if $D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A)$ admits a recollement

$$D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A'))}(\text{mod } A') \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A''))}(\text{mod } A''),$$

then A is Gorenstein if and only if so are A' and A'' [3, Theorem 4.7]. Independently, using a different proof from that in [3], Gao showed in [18, Theorem 3.5] that the Gorensteinness of virtually Gorenstein artin algebras is an invariant of recollements; that is, for virtually Gorenstein artin algebras A, A' and A'', if $D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A)$ admits a recollement as above, then A is Gorenstein if and only if so are A' and A''.

The following is one of the main results in this paper, which is a "big module" analog of the Asadollahi, Bahiraei, Hafezi, and Vahed's result mentioned above as well as a Gorenstein analog of a classical result of Wiedemann [32, Lemma 2.1].

Theorem 1.1. (Corollary 3.8) *For artin algebras A, A' and A'', if $D^b_{\mathcal{G}(\mathcal{P}(\text{Mod } A))}(\text{Mod } A)$ admits a recollement*

$$D^b_{\mathcal{G}(\mathcal{P}(\text{Mod } A'))}(\text{Mod } A') \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} D^b_{\mathcal{G}(\mathcal{P}(\text{Mod } A))}(\text{Mod } A) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} D^b_{\mathcal{G}(\mathcal{P}(\text{Mod } A''))}(\text{Mod } A''),$$

then A is Gorenstein if and only if so are A' and A''.

In fact, we prove this result in some more general setting (Theorem 3.7).

Let A be an artin algebra and $\mathcal{X} = \text{Mod } A$ or $\mathcal{X} = \text{mod } A$. For a subcategory $\mathcal{C}$ of $\mathcal{X}$, we use $K^b(\mathcal{C})$ to denote the bounded homotopy category of $\mathcal{C}$. Happel proved in [20, Lemma 1.5] that a finite-dimensional algebra A is Gorenstein if and only if $K^b(\mathcal{I}(\text{mod } A)) = K^b(\mathcal{P}(\text{mod } A))$, where $\mathcal{I}(\text{mod } A)$ and $\mathcal{P}(\text{mod } A)$ are the subcategories of $\text{mod } A$ consisting of injective modules and projective modules respectively. On the other hand, as an important generalization of Gorenstein algebras, Beligiannis and Reiten introduced in [11] virtually Gorenstein algebras. Note that any Gorenstein algebra is virtually Gorenstein and the converse is not true in general [11, 12]. We get some equivalent characterizations for virtually Gorenstein artin algebras being Gorenstein, which can be regarded as a Gorenstein analog of the above result of Happel.

Theorem 1.2. (Theorem 4.6) *For a virtually Gorenstein artin algebra A, the following statements are equivalent.*

(1) A is Gorenstein.
(2) $K^b(\mathcal{G}(\mathcal{P}(\text{Mod } A))) = K^b(\mathcal{G}(\mathcal{I}(\text{Mod } A)))$ in $D^b_{\mathcal{G}(\mathcal{P}(\text{Mod } A))}(\text{Mod } A)$.
(3) $K^b(\mathcal{G}(\mathcal{P}(\text{mod } A))) = K^b(\mathcal{G}(\mathcal{I}(\text{mod } A)))$ in $D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A)$.

2. Preliminaries

Throughout this paper, $\mathcal{A}$ is an abelian category and all subcategories of $\mathcal{A}$ are additive, full and closed under isomorphisms. We use $\mathcal{P}(\mathcal{A})$ and $\mathcal{I}(\mathcal{A})$ to denote the subcategories of $\mathcal{A}$ consisting of projective and injective objects, respectively.

We fix a subcategory $\mathcal{C}$ of $\mathcal{A}$. A complex X in $\mathcal{A}$ is called $\mathcal{C}$-acyclic (resp. $\mathcal{C}$-coacyclic) if $H^i\mathrm{Hom}_{\mathcal{A}}(C, X) = 0$ (resp. $H^i\mathrm{Hom}_{\mathcal{A}}(X, C) = 0$) for any $C \in \mathcal{C}$ and $i \in \mathbb{Z}$ (the set of integers).

Definition 2.1. [6] Let $\mathcal{C} \subseteq \mathcal{D}$ be subcategories of $\mathcal{A}$. The morphism $f : C \to D$ in $\mathcal{A}$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$ is called a *right $\mathcal{C}$-approximation* of D if the complex $C \xrightarrow{f} D \to 0$ is $\mathcal{C}$-acyclic. If every object in $\mathcal{D}$ admits a right $\mathcal{C}$-approximation, then $\mathcal{C}$ is called *contravariantly finite* in $\mathcal{D}$. Dually, the notions of *left $\mathcal{C}$-approximations* and *covariantly finite subcategories* are defined.

Let $K^*_{\mathcal{C}-ac}(\mathcal{A})$ (resp. $K^*_{\mathcal{C}-coac}(\mathcal{A})$) denote the subcategory of the homotopy category $K^*(\mathcal{A})$ consisting of $\mathcal{C}$-acyclic (resp. $\mathcal{C}$-coacyclic) complexes, where $* \in \{blank, -, +, b\}$. A chain map $f : X \to Y$ is called a $\mathcal{C}$-quasi-isomorphism (resp. $\mathcal{C}$-coquasi-isomorphism) if $\mathrm{Hom}_{\mathcal{A}}(C, f)$ (resp. $\mathrm{Hom}_{\mathcal{A}}(f, C)$) is a quasi-isomorphism for any $C \in \mathcal{C}$. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then the *relative derived category*, denoted by $D^*_{\mathcal{C}}(\mathcal{A})$ (resp. $co\!-\!D^*_{\mathcal{C}}(\mathcal{A})$), is the Verdier quotient of the homotopy category $K^*(\mathcal{A})$ with respect to the thick subcategory $K^*_{\mathcal{C}-ac}(\mathcal{A})$ (resp. $K^*_{\mathcal{C}-coac}(\mathcal{A})$) [14]. Set

$$K^{-,\mathcal{C}b}(\mathcal{C}) := \left\{X \in K^-(\mathcal{C}) | \text{there exists } N \in \mathbb{Z} \text{ such that } H^i\mathrm{Hom}_{\mathcal{A}}(C, X) = 0 \text{ for any } C \in \mathcal{C} \text{ and } i \le N\right\}.$$
$$K^{+,\mathcal{C}b}(\mathcal{C}) := \left\{X \in K^+(\mathcal{C}) | \text{there exists } N \in \mathbb{Z} \text{ such that } H^i\mathrm{Hom}_{\mathcal{A}}(X, C) = 0 \text{ for any } C \in \mathcal{C} \text{ and } i \ge N\right\}.$$

Lemma 2.2. [19, Theorem 3.6] and [4, Theorem 3.3]

(1) *If $\mathcal{C}$ is contravariantly finite in $\mathcal{A}$, then $D^b_{\mathcal{C}}(\mathcal{A}) \cong K^{-,\mathcal{C}b}(\mathcal{C})$.*
(2) *If $\mathcal{C}$ is covariantly finite in $\mathcal{A}$, then $co\!-\!D^b_{\mathcal{C}}(\mathcal{A}) \cong K^{+,\mathcal{C}b}(\mathcal{C})$.*

Definition 2.3. [31] The *Gorenstein subcategory* $\mathcal{G}(\mathcal{C})$ of $\mathcal{A}$ is defined as $\mathcal{G}(\mathcal{C}) = \{M \in \mathcal{A}|$ there exists an exact sequence:

$$\cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots$$

in $\mathcal{A}$ with all C_i, C^i in $\mathcal{C}$, which is both $\mathcal{C}$-acyclic and $\mathcal{C}$-coacyclic, such that $M \cong Im(C_0 \to C^0)\}$.

If $\mathcal{C} = \mathcal{P}(\mathcal{A})$ (resp. $\mathcal{I}(\mathcal{A})$), then $\mathcal{G}(\mathcal{P}(\mathcal{A}))$ (resp. $\mathcal{G}(\mathcal{I}(\mathcal{A}))$) is exactly the subcategory of $\mathcal{A}$ consisting of Gorenstein projective (resp. Gorenstein injective) objects [16].

Let A be an artin algebra. Recall from [10, 11] that A is called *virtually Gorenstein* if

$$\mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A))^{\perp} = {}^{\perp}\mathcal{G}(\mathcal{I}(\mathrm{Mod}\ A)),$$

where

$$\mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A))^{\perp} = \left\{M \in \mathrm{Mod}\ A | \mathrm{Ext}^{\ge 1}_A(G, M) = 0 \text{ for any } G \in \mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A))\right\}$$

and

$${}^{\perp}\mathcal{G}(\mathcal{I}(\mathrm{Mod}\ A)) = \left\{M \in \mathrm{Mod}\ A | \mathrm{Ext}^{\ge 1}_A(M, H) = 0 \text{ for any } H \in \mathcal{G}(\mathcal{I}(\mathrm{Mod}\ A))\right\}.$$

Definition 2.4. [5, 7] Let $M \in \mathcal{A}$ and $n \ge 0$. The $\mathcal{C}$-dimension $\mathcal{C}\text{-}\dim M$ of M is said to be at most n if there exists an exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to M \to 0 \tag{2.1}$$

in $\mathcal{A}$ with all C_i objects in $\mathcal{C}$. Moreover, the sequence (2.1) is called a *proper $\mathcal{C}$-resolution* of M if it is $\mathcal{C}$-acyclic. Dually, The $\mathcal{C}$-codimension $\mathcal{C}$-codim M of M is said to be at most n if there exists an exact sequence

$$0 \to M \to C^0 \to \cdots \to C^{n-1} \to C^n \to 0 \tag{2.2}$$

in $\mathcal{A}$ with all C^i objects in $\mathcal{C}$. Moreover, the sequence (2.2) is called a *coproper $\mathcal{C}$-coresolution* of M if it is $\mathcal{C}$-coacyclic.

Definition 2.5. [9] Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be triangulated categories. A *recollement* of $\mathcal{T}$ relative to $\mathcal{T}'$ and $\mathcal{T}''$ is a diagram

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i_*"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{T}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

of triangle functors such that

(1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs.
(2) $j^* i_* = 0$.
(3) i_*, $j_!$ and j_* are fully faithful.
(4) For any object X in $\mathcal{T}$, there exist triangles

$$i_*i^!X \longrightarrow X \longrightarrow j_*j^*X \longrightarrow (i_*i^!X)[1]$$

and

$$j_!j^*X \longrightarrow X \longrightarrow i_*i^*X \longrightarrow (j_!j^*X)[1],$$

where the maps are given by adjunctions.

3. Relative singularity categories and recollements

Definition 3.1. [27] Let $\mathcal{D}$ be a triangulated category. An object $P \in \mathcal{D}$ is called *perfect* if for any object $Y \in \mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(P, Y[i]) = 0$ except for finitely many $i \in \mathbb{Z}$. Dually, the notion of *coperfect objects* is defined.

We use $\mathcal{D}_{\mathrm{perf}}$ (resp. $\mathcal{D}_{\mathrm{coperf}}$) to denote the triangulated subcategory consisting of perfect (resp. coperfect) objects, which is called the *perfect (resp. coperfect) subcategory* of $\mathcal{D}$ [27]. Obviously, $\mathcal{D}_{\mathrm{perf}}$ and $\mathcal{D}_{\mathrm{coperf}}$ are thick subcategories of $\mathcal{D}$, and both of them are invariants of triangle equivalence.

Let $\mathcal{C}$ be a contravariantly finite subcategory of $\mathcal{A}$. Li and Huang introduced in [26] the *relative singularity category* by the following Verdier quotient

$$D^b_{\mathcal{C}-sg}(\mathcal{A}) = D^b_{\mathcal{C}}(\mathcal{A})/K^b(\mathcal{C}).$$

On the other hand, Orlov [27] defined it in a different way.

Definition 3.2. [27, Definition 1.7] Let $\mathcal{D}$ be a triangulated category, the *singularity category* of $\mathcal{D}$ is defined by the Verdier quotient $\mathcal{D}/\mathcal{D}_{\mathrm{perf}}$. In particular, let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a subcategory of $\mathcal{A}$. The *relative singularity category* is defined by the following Verdier quotient

$$D^b_{\mathcal{C}-sg}(\mathcal{A}) = D^b_{\mathcal{C}}(\mathcal{A})/D^b_{\mathcal{C}}(\mathcal{A})_{\mathrm{perf}}.$$

In general, we have $K^b(\mathcal{C}) \subseteq D^b_{\mathcal{C}}(\mathcal{A})_{\mathrm{perf}}$. In the following, we will give some sufficient condition such that they are identical. In this case, the two definitions of the relative singularity categories mentioned above coincide.

Rickard proved in [30, Proposition 6.2] that $D^b(\mathrm{Mod}\ A)_{\mathrm{perf}} = K^b(\mathcal{P}(\mathrm{Mod}\ A))$ for any ring A. We generalize this result and [13, Lemma 1.2.1] as follows.

Proposition 3.3. *Assume that*

(a) $\mathcal{A}$ *has enough projective objects; and*
(b) $\mathcal{C}$ *is a contravariantly finite subcategory of* $\mathcal{A}$ *closed under direct summands, such that* $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}$.

Then for any $G \in D^b_{\mathcal{C}}(\mathcal{A})$, *the following statements are equivalent.*

(1) $G \in K^b(\mathcal{C})$.
(2) *There exists* $i(G) \in \mathbb{Z}$, *such that* $\mathrm{Hom}_{D^b_{\mathcal{C}}(\mathcal{A})}(G, M[j]) = 0$ *for any* $M \in \mathcal{A}$ *and* $j \geq i(G)$.
(3) *There exists a finite set* $I(G) \subseteq \mathbb{Z}$, *such that* $\mathrm{Hom}_{D^b_{\mathcal{C}}(\mathcal{A})}(G, M[j]) = 0$ *for any* $M \in \mathcal{A}$ *and* $j \notin I(G)$.

Furthermore, if $\mathcal{A}$ *is closed under direct sums, then* $D^b_{\mathcal{C}}(\mathcal{A})_{\mathrm{perf}} = K^b(\mathcal{C})$.

Proof. The implications $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$ are obvious,

$(2) \Rightarrow (1)$ Let $G \in D^b_{\mathcal{C}}(\mathcal{A})$. Then there exists a $\mathcal{C}$-quasi-isomorphism $Q \to G$ with $Q \in K^{-,\mathcal{C}b}(\mathcal{C})$ by Lemma 2.2, and hence there exists $N \in \mathbb{Z}$ such that $H^i\mathrm{Hom}_{\mathcal{A}}(C, Q) = 0$ for any $C \in \mathcal{C}$ and $i \leq N$. It follows that $H^iQ = 0$ for any $i \leq N$ since $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}$.

We claim that there exists $n \leq N$ such that $\mathrm{Im} d^n_Q \in \mathcal{C}$. Otherwise, there exists $-n \geq i(G)$ such that $\mathrm{Im} d^n_Q \notin \mathcal{C}$. Put $M := \mathrm{Im} d^n_Q$. Then there exists a non-zero epimorphism $\widetilde{d^n_Q} : Q^n \to M$ in $\mathcal{A}$, such that d^n_Q is identified with the composition of $Q^n \overset{\widetilde{d^n_Q}}{\twoheadrightarrow} M \overset{\lambda}{\hookrightarrow} Q^{n+1}$. It induces the following chain map

$$\begin{array}{ccc} Q & & \cdots \longrightarrow Q^{n-1} \xrightarrow{d_Q^{n-1}} Q^n \xrightarrow{d_Q^n} Q^{n+1} \longrightarrow \cdots \\ \downarrow f & & \qquad\quad \downarrow \qquad\quad \downarrow \widetilde{d_Q^n} \qquad \downarrow \\ M[-n] & & \cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots . \end{array}$$

Notice that $\mathrm{Im} d^n_Q \notin \mathcal{C}$, so f is not null homotopic. Otherwise, there exists a morphism $h : Q^{n+1} \to M$ such that $\widetilde{d^n_Q} = h d^n_Q = h\lambda\widetilde{d^n_Q}$. Since $\widetilde{d^n_Q}$ is epic, we have that $1_M = h\lambda$ and M is a isomorphic to a direct summand of Q^{n+1} in $\mathcal{C}$. Since $\mathcal{C}$ is closed under direct summands by assumption, it follows that $M \in \mathcal{C}$, a contradiction. So $\mathrm{Hom}_{K^-(\mathcal{A})}(Q, M[-n]) \neq 0$ and we have

$$\begin{aligned} \mathrm{Hom}_{D^b_{\mathcal{C}}(\mathcal{A})}(G, M[-n]) &\cong \mathrm{Hom}_{D^-_{\mathcal{C}}(\mathcal{A})}(Q, M[-n]) \\ &\cong \mathrm{Hom}_{K^-(\mathcal{A})}(Q, M[-n]) \neq 0, \end{aligned}$$

which contradicts the assumption. The claim is proved.

Now we have a $\mathcal{C}$-quasi-isomorphism

$$\begin{array}{ccc} Q & & \cdots \longrightarrow Q^{n-1} \xrightarrow{d_Q^{n-1}} Q^n \xrightarrow{d_Q^n} Q^{n+1} \longrightarrow \cdots \\ \downarrow & & \qquad\quad \downarrow \qquad\quad \downarrow \qquad\quad \downarrow \\ \tau_{\geq n+1}Q & & \cdots \longrightarrow 0 \longrightarrow \mathrm{Im}\, d_Q^n \longrightarrow Q^{n+1} \longrightarrow \cdots \end{array}$$

with $\tau_{\geq n+1}Q \in K^b(\mathcal{C})$. Then $G \cong Q \cong \tau_{\geq n+1}Q$ in $D^-_{\mathcal{C}}(\mathcal{A})$, and it follows that $G \cong \tau_{\geq n+1}Q \in K^b(\mathcal{C})$ in $D^b_{\mathcal{C}}(\mathcal{A})$.

Similarly, we get $(3) \Rightarrow (1)$.

Furthermore, the inclusion $K^b(\mathcal{C}) \subseteq D^b_{\mathcal{C}}(\mathcal{A})_{\text{perf}}$ is obvious. Conversely, let $P \in D^b_{\mathcal{C}}(\mathcal{A})_{\text{perf}}$. Then there exists a $\mathcal{C}$-quasi-isomorphism $Q \to P$ with $Q \in K^{-,\mathcal{C}b}(\mathcal{C})$ by Lemma 2.2, and hence there exists $N \in \mathbb{Z}$ such that $H^i\mathrm{Hom}_{\mathcal{A}}(C, Q) = 0$ for any $C \in \mathcal{C}$ and $i \leq N$. Since $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}$, we have $H^iQ = 0$ for any $i \leq N$.

We claim that there exists $n \leq N$ such that $\mathrm{Im} d^n_Q \in \mathcal{C}$. Otherwise, suppose that $\mathrm{Im} d^n_Q \notin \mathcal{C}$ for any $n \leq N$. Take $M := \bigoplus_{i \leq N} \mathrm{Im} d^i_Q \in \mathcal{C}$ and the non-zero morphism

$$\widetilde{d^n_Q} : Q^n \longrightarrow M = \mathrm{Im}\, d^n_Q \oplus (\bigoplus_{i \leq N, i \neq n} \mathrm{Im}\, d^i_Q),$$

such that d^n_Q is identified with the composition of $Q^n \overset{\widetilde{d^n_Q}}{\twoheadrightarrow} M \overset{\lambda}{\hookrightarrow} Q^{n+1}$. It induces the following chain map

$$\begin{tikzcd}
Q \arrow[d, "f"] & \cdots \arrow[r] & Q^{n-1} \arrow[r, "d^{n-1}_Q"] \arrow[d] & Q^n \arrow[r, "d^n_Q"] \arrow[d, "\widetilde{d^n_Q}"] & Q^{n+1} \arrow[r] \arrow[d] & \cdots \\
M[-n] & \cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

Note that $\mathrm{Im} d^n_Q \notin \mathcal{C}$, so f is not null homotopic. Otherwise, by an argument similar to the above, we have $\mathrm{Im} d^n_Q \in \mathcal{C}$, a contradiction. So $\mathrm{Hom}_{K^-(\mathcal{A})}(Q, M[-n]) \neq 0$ and we have

$$\begin{aligned}
\mathrm{Hom}_{D^b_{\mathcal{C}}(\mathcal{A})}(P, M[-n]) &\cong \mathrm{Hom}_{D^-_{\mathcal{C}}(\mathcal{A})}(Q, M[-n]) \\
&\cong \mathrm{Hom}_{K^-(\mathcal{A})}(Q, M[-n]) \neq 0.
\end{aligned}$$

So there are infinitely many n such that $\mathrm{Hom}_{D^b_{\mathcal{C}}(\mathcal{A})}(P, M[-n]) \neq 0$, which contradicts the assumption that $P \in D^b_{\mathcal{C}}(\mathcal{A})_{\text{perf}}$. The claim is proved.

Consider the following $\mathcal{C}$-quasi-isomorphism

$$\begin{tikzcd}
Q \arrow[d] & \cdots \arrow[r] & Q^{n-1} \arrow[r, "d^{n-1}_Q"] \arrow[d] & Q^n \arrow[r, "d^n_Q"] \arrow[d] & Q^{n+1} \arrow[r] \arrow[d] & \cdots \\
\tau_{\geq n+1}Q & \cdots \arrow[r] & 0 \arrow[r] & \mathrm{Im}\, d^n_Q \arrow[r] & Q^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

with $\tau_{\geq n+1}Q \in K^b(\mathcal{C})$. Then $P \cong Q \cong \tau_{\geq n+1}Q$ in $D^-_{\mathcal{C}}(\mathcal{A})$, and so $P \cong \tau_{\geq n+1}Q \in K^b(\mathcal{C})$ in $D^b_{\mathcal{C}}(\mathcal{A})$. It follows that $P \in K^b(\mathcal{C})$. The proof is finished.

Because $\mathcal{G}(\mathcal{C})$ is closed under direct summands by [23, Theorem 4.6(2)], the following is an immediate consequence of Proposition 3.3.

Corollary 3.4. *Assume that*

(a) *$\mathcal{A}$ has enough projective objects; and*
(b) *$\mathcal{G}(\mathcal{C})$ is a contravariantly finite subcategory of $\mathcal{A}$ and $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}$.*

Then for any $G \in D^b_{\mathcal{G}(\mathcal{C})}(\mathcal{A})$, the following statements are equivalent.

(1) *$G \in K^b(\mathcal{G}(\mathcal{C}))$.*
(2) *There exists $i(G) \in \mathbb{Z}$, such that $\mathrm{Hom}_{D^b_{\mathcal{G}(\mathcal{C})}(\mathcal{A})}(G, M[j]) = 0$ for any $M \in \mathcal{A}$ and $j \geq i(G)$.*
(3) *There exists a finite set $I(G) \subseteq \mathbb{Z}$, such that $\mathrm{Hom}_{D^b_{\mathcal{G}(\mathcal{C})}(\mathcal{A})}(G, M[j]) = 0$ for any $M \in \mathcal{A}$ and $j \notin I(G)$.*

Furthermore, if $\mathcal{A}$ is closed under direct sums, then $D^b_{\mathcal{G}(\mathcal{C})}(\mathcal{A})_{\text{perf}} = K^b(\mathcal{G}(\mathcal{C}))$.

Let $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}'}$ be triangulated subcategories of triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ respectively. Recall from [2] that a triangle functor $F : \mathcal{D} \to \mathcal{D}'$ is said to *restrict to* $\widetilde{\mathcal{D}}$ if F restricts to a triangle functor $\widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}'}$.

Lemma 3.5. *Let $F : \mathcal{D} \to \mathcal{D}'$ be a triangle functor of triangulated categories. If F admits a right adjoint G, then F restricts to $\mathcal{D}_{\mathrm{perf}}$.*

Proof. Let $X \in \mathcal{D}_{\mathrm{perf}}$. For any $Y \in \mathcal{D}'$, we have

$$\mathrm{Hom}_{\mathcal{D}'}\big(FX, Y[i]\big) \cong \mathrm{Hom}_{\mathcal{D}}\big(X, GY[i]\big) \cong \mathrm{Hom}_{\mathcal{D}}\big(X, (GY)[i]\big).$$

By the definition of perfect objects, we have $FX \in \mathcal{D}'_{\mathrm{perf}}$, and hence F restricts to $\mathcal{D}_{\mathrm{perf}}$.

Proposition 3.6. *Assume that*

(1) *both $\mathcal{A}$ and $\mathcal{A}'$ are abelian categories having enough projective objects and closed under direct sums.*
(2) *$\mathcal{C}$ and $\mathcal{C}'$ are contravariantly finite subcategories of $\mathcal{A}$ and $\mathcal{A}'$ closed under direct summands respectively, such that $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}$ and $\mathcal{P}(\mathcal{A}') \subseteq \mathcal{C}'$.*

If $F : D^b_{\mathcal{C}}(\mathcal{A}) \to D^b_{\mathcal{C}'}(\mathcal{A}')$ is a triangle functor admitting a right adjoint G, then F restricts to $K^b(\mathcal{C})$.

Proof. It follows from Proposition 3.3 and Lemma 3.5.

Our main result is the following

Theorem 3.7. *Assume that*

(1) *$\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ are abelian categories having enough projective objects and closed under direct sums; and*
(2) *$\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ are contravariantly finite subcategories of $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ closed under direct summands respectively, such that $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}$, $\mathcal{P}(\mathcal{A}') \subseteq \mathcal{C}'$ and $\mathcal{P}(\mathcal{A}'') \subseteq \mathcal{C}''$.*

If $D^b_{\mathcal{C}}(\mathcal{A})$ admits a recollement

$$\begin{tikzcd} D^b_{\mathcal{C}'}(\mathcal{A}') \arrow[r, "i_*"] & D^b_{\mathcal{C}}(\mathcal{A}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D^b_{\mathcal{C}''}(\mathcal{A}''), \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

then $D^b_{\mathcal{C}-sg}(\mathcal{A}) = 0$ if and only if $D^b_{\mathcal{C}'-sg}(\mathcal{A}') = 0 = D^b_{\mathcal{C}''-sg}(\mathcal{A}'')$.

Proof. By assumption, we have $D^b_{\mathcal{C}}(\mathcal{A}) \cong K^b(\mathcal{C})$. Let $X', Y \in D^b_{\mathcal{C}'}(\mathcal{A}')$. Since i_* is fully faithful and $i_*X' \in D^b_{\mathcal{C}}(\mathcal{A})(\cong K^b(\mathcal{C}))$, it follows from Proposition 3.3 that

$$\mathrm{Hom}_{D^b_{\mathcal{C}'}(\mathcal{A}')}\big(X', Y[i]\big) \cong \mathrm{Hom}_{D^b_{\mathcal{C}}(\mathcal{A})}\big(i_*X', i_*Y[i]\big) \cong \mathrm{Hom}_{D^b_{\mathcal{C}}(\mathcal{A})}\big(i_*X', (i_*Y)[i]\big) = 0$$

except for finitely many $i \in \mathbb{Z}$ and so $X' \in K^b(\mathcal{C}')$. Thus $K^b(\mathcal{C}') \cong D^b_{\mathcal{C}'}(\mathcal{A}')$ and $D^b_{\mathcal{C}'-sg}(\mathcal{A}') = 0$. Similarly, we have $D^b_{\mathcal{C}''-sg}(\mathcal{A}'') = 0$.

Conversely, let $X \in D^b_{\mathcal{C}}(\mathcal{A})$. Then there exists a triangle

$$j_!j^*X \longrightarrow X \longrightarrow i_*i^*X \longrightarrow (j_!j^*X)[1]\ .$$

By assumption, $j^*X \in K^b(\mathcal{C}'')$ and $i^*X \in K^b(\mathcal{C}')$. Since i_* restricts to $K^b(\mathcal{C}')$ and $j_!$ restricts to $K^b(\mathcal{C}'')$ by Proposition 3.6, we have $j_!j^*X \in K^b(\mathcal{C})$ and $i_*i^*X \in K^b(\mathcal{C})$, and hence $X \in K^b(\mathcal{C})$, which implies that $K^b(\mathcal{C}) \cong D^b_{\mathcal{C}}(\mathcal{A})$ and $D^b_{\mathcal{C}-sg}(\mathcal{A}) = 0$.

Recall that an artin algebra A is called *Gorenstein* if its left self-injective dimension $\mathrm{id}_A A$ and right self-injective dimension $\mathrm{id}_{A^{op}} A$ are finite. We have the following facts: (1) An artin algebra A is Gorenstein if and only if $D^b_{\mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A))-sg}(\mathrm{Mod}\ A) = 0$ [8]; (2) $\mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A))$ is contravariantly finite in $\mathrm{Mod}\ A$ [10, Theorem 3.5]. So, by taking $\mathcal{A} = \mathrm{Mod}\ A$ and $\mathcal{C} = \mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A))$ in Theorem 3.7, we have the following result, which is a Gorenstein analog of [32, Lemma 2.1] as well as a "big module" analog of [3, Theorem 4.7] and [18, Theorem 3.5].

Corollary 3.8. *Let A, A' and A'' be artin algebras. If $D^b_{\mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A))}(\mathrm{Mod}\ A)$ admits a recollement*

$$\begin{tikzcd} D^b_{\mathcal{G}(\mathcal{P}(\mathrm{Mod}\,A'))}(\mathrm{Mod}\,A') \arrow[r, "i_*"] & D^b_{\mathcal{G}(\mathcal{P}(\mathrm{Mod}\,A))}(\mathrm{Mod}\,A) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D^b_{\mathcal{G}(\mathcal{P}(\mathrm{Mod}\,A''))}(\mathrm{Mod}\,A''), \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

then A is Gorenstein if and only if so are A' and A''.

It is well known that a ring A has finite left global dimension if and only if $D^b(\mathrm{Mod}\ A) = K^b(\mathcal{P}(\mathrm{Mod}\ A))$ (that is, $D^b_{\mathcal{P}(\mathrm{Mod}\ A)-sg}(\mathrm{Mod}\ A) = 0$). So, by taking $\mathcal{A} = \mathrm{Mod}\ A$ and $\mathcal{C} = \mathcal{P}(\mathrm{Mod}\ A)$ in Theorem 3.7, we have the following "big module" analog of [32, Lemma 2.1].

Corollary 3.9. (cf. [25, Corollaries 5 and 6]) *Let A, A' and A'' be rings. If $D^b(\mathrm{Mod}\ A)$ admits a recollement*

$$\begin{tikzcd} D^b(\mathrm{Mod}\,A') \arrow[r, "i_*"] & D^b(\mathrm{Mod}\,A) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D^b(\mathrm{Mod}\,A''), \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

then A has finite left global dimension if and only if so do A' and A''.

We remark that the dual counterparts of all results in this section also hold true.

4. Characterizing Gorenstein algebras

In this section, A is an artin algebra.

Lemma 4.1. *We have*

(1) *For any $X \in K^{-,\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A))b}(\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A)))$, the following statements are equivalent.*
(1.1) $X \in K^b(\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A)))$.
(1.2) For any $Y \in K^{-,\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A))b}(\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A)))$,

$$\mathrm{Hom}_{K^{-,\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A))b}(\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A)))}(X, Y[i]) = 0$$

except for finitely many $i \in \mathbb{Z}$.

(2) *For any $G \in K^{+,\mathcal{G}(\mathcal{I}(\mathrm{mod}\ A))b}(\mathcal{G}(\mathcal{I}(\mathrm{mod}\ A)))$, the following statements are equivalent.*
(2.1) $G \in K^b(\mathcal{G}(\mathcal{I}(\mathrm{mod}\ A)))$.
(2.2) for any $Y \in K^{+,\mathcal{G}(\mathcal{I}(\mathrm{mod}\ A))b}(\mathcal{G}(\mathcal{I}(\mathrm{mod}\ A)))$,

$$\mathrm{Hom}_{K^{+,\mathcal{G}(\mathcal{I}(\mathrm{mod}\ A))b}(\mathcal{G}(\mathcal{I}(\mathrm{mod}\ A)))}(Y[i], G) = 0$$

except for finitely many $i \in \mathbb{Z}$.

Proof. The first assertion is [3, Lemma 4.5], and the second one is its dual.

Compare the following result with the last assertion in Corollary 3.4 and its dual counterpart.

Proposition 4.2. *Let A be virtually Gorenstein. Then we have*

(1) $D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A)_{\text{perf}} = K^b(\mathcal{G}(\mathcal{P}(\text{mod } A)))$.
(2) $co{-}D^b_{\mathcal{G}(\mathcal{I}(\text{mod } A))}(\text{mod } A)_{\text{coperf}} = K^b(\mathcal{G}(\mathcal{I}(\text{mod } A)))$.

Proof. (1) Let A be virtually Gorenstein. It follows from [12, Theorem 5] that $\mathcal{G}(\mathcal{P}(\text{mod } A))$ is contravariantly finite in mod A. Then by Lemma 2.2, we have

$$D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A)_{\text{perf}} = K^{-,\mathcal{G}(\mathcal{P}(\text{mod } A))b}(\mathcal{G}(\mathcal{P}(\text{mod } A))).$$

Now the assertion follows from Lemma 4.1(1).
(2) It is dual to (1).

We have the following easy observation.

Lemma 4.3. *The following statements are equivalent.*

(1) *A is Gorenstein.*
(2) *Any object in $\mathcal{G}(\mathcal{P}(\text{Mod } A))$ has finite $\mathcal{G}(\mathcal{I}(\text{Mod } A))$-codimension and any object in $\mathcal{G}(\mathcal{I}(\text{Mod } A))$ has finite $\mathcal{G}(\mathcal{P}(\text{Mod } A))$-dimension.*
(3) *Any object in $\mathcal{G}(\mathcal{P}(\text{mod } A))$ has finite $\mathcal{G}(\mathcal{I}(\text{mod } A))$-codimension and any object in $\mathcal{G}(\mathcal{I}(\text{mod } A))$ has finite $\mathcal{G}(\mathcal{P}(\text{mod } A))$-dimension.*

Proof. $(1) \Rightarrow (2)+(3)$ Let A be Gorenstein. Then any object in Mod A (resp. mod A) has finite $\mathcal{G}(\mathcal{P}(\text{Mod } A))$-dimension (resp. $\mathcal{G}(\mathcal{P}(\text{mod } A))$-dimension) by [17, Theorem 11.5.1] (resp. [22, Theorem]). Dually, any object in Mod A (resp. mod A) has finite $\mathcal{G}(\mathcal{I}(\text{Mod } A))$-dimension (resp. $\mathcal{G}(\mathcal{I}(\text{mod } A))$-dimension).

$(2) \Rightarrow (1)$ (resp. $(3) \Rightarrow (1)$) By assumption, we have that $D(A_A)$ has finite $\mathcal{G}(\mathcal{P}(\text{Mod } A))$-dimension (resp. $\mathcal{G}(\mathcal{P}(\text{mod } A))$-dimension). So $D(A_A)$ has finite projective dimension by [24, Corollary 3.12], and hence $\text{id}_{A^{op}}A<\infty$. Dually, because ${}_AA$ has finite $\mathcal{G}(\mathcal{I}(\text{Mod } A))$-dimension (resp. $\mathcal{G}(\mathcal{I}(\text{mod } A))$-dimension) by assumption, we have $\text{id}_A A<\infty$ by [24, Corollary 4.12]. Thus A is Gorenstein.

Lemma 4.4. *Let A be virtually Gorenstein. Then there exist triangle equivalences*

$$D^b_{\mathcal{G}(\mathcal{P}(\text{mod } A))}(\text{mod } A) \cong co{-}D^b_{\mathcal{G}(\mathcal{I}(\text{mod } A))}(\text{mod } A),$$
$$D^b_{\mathcal{G}(\mathcal{P}(\text{Mod } A))}(\text{Mod } A) \cong co{-}D^b_{\mathcal{G}(\mathcal{I}(\text{Mod } A))}(\text{Mod } A).$$

Proof. The first equivalence follows from [3, Theorem 6.3].

By [33, Proposition 3.2], we have that $(\mathcal{G}(\mathcal{P}(\text{Mod } A)), \mathcal{G}(\mathcal{I}(\text{Mod } A)))$ is a balanced pair. Then by [14, Proposition 2.2], we have that an exact sequence is $\mathcal{G}(\mathcal{P}(\text{Mod } A))$-acyclic if and only if it is $\mathcal{G}(\mathcal{I}(\text{Mod } A))$-coacyclic, which yields the second equivalence.

As a consequence, we have the following

Proposition 4.5. *Let A be virtually Gorenstein and either $\mathcal{X} = \text{Mod } A$ or $\mathcal{X} = \text{mod } A$. Then we have*

(1) *The following statements are equivalent.*
(1.1) Any object in $\mathcal{G}(\mathcal{P}(\mathcal{X}))$ has finite $\mathcal{G}(\mathcal{I}(\mathcal{X}))$-codimension.
(1.2) $K^b(\mathcal{G}(\mathcal{P}(\mathcal{X}))) \subseteq K^b(\mathcal{G}(\mathcal{I}(\mathcal{X})))$ in $D^b_{\mathcal{G}(\mathcal{P}(\mathcal{X}))}(\mathcal{X})$.

(2) *The following statements are equivalent.*
(2.1) Any object in $\mathcal{G}(\mathcal{I}(\mathcal{X}))$ has finite $\mathcal{G}(\mathcal{P}(\mathcal{X}))$-dimension.
(2.2) $K^b(\mathcal{G}(\mathcal{I}(\mathcal{X}))) \subseteq K^b(\mathcal{G}(\mathcal{P}(\mathcal{X})))$ in $D^b_{\mathcal{G}(\mathcal{P}(\mathcal{X}))}(\mathcal{X})$.

Proof. (1.1) $\Rightarrow$ (1.2) Let $X \in K^b(\mathcal{G}(\mathcal{P}(\mathcal{X})))(\subseteq D^b_{\mathcal{G}(\mathcal{P}(\mathcal{X}))}(\mathcal{X}))$. Then $X \in co{-}D^b_{\mathcal{G}(\mathcal{I}(\mathcal{X}))}(\mathcal{X})$ by Lemma 4.4. We will prove $X \in K^b(\mathcal{G}(\mathcal{I}(\mathcal{X})))$ by induction on the width $w(X)$ of X. Let $w(X) = 1$. By the dual version of [23, Corollary 5.12], there exists a finite coproper $\mathcal{G}(\mathcal{I}(\mathcal{X}))$-coresolution

$$0 \longrightarrow X \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots \longrightarrow G^{n-1} \longrightarrow G^n \longrightarrow 0$$

of X, which can be viewed as a $\mathcal{G}(\mathcal{I}(\mathcal{X}))$-coquasi-isomorphism

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & G^0 \arrow[r] & G^1 \arrow[r] & \cdots \arrow[r] & G^{n-1} \arrow[r] & G^n \arrow[r] & 0. \end{tikzcd}$$

It follows that $X \in K^b(\mathcal{G}(\mathcal{I}(\mathcal{X})))$. Now suppose $w(X) = n \geq 2$. From the following diagram

$$\begin{tikzcd} X_1 \arrow[d] & 0 \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{n-2} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ X \arrow[d] & 0 \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{n-2} \arrow[r] \arrow[d] & X^{n-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\ X_2 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & X^{n-1} \arrow[r] & 0, \end{tikzcd}$$

we get a triangle

$$X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow X_1[1]$$

with $w(X_1) = n{-}1$ and $w(X_2) = 1$. By the induction hypothesis, we have $X_1 \in K^b(\mathcal{G}(\mathcal{I}(\mathcal{X})))$ and $X_2 \in K^b(\mathcal{G}(\mathcal{I}(\mathcal{X})))$, and hence $X \in K^b(\mathcal{G}(\mathcal{I}(\mathcal{X})))$.

(1.2) $\Rightarrow$ (1.1) Let $G \in \mathcal{G}(\mathcal{P}(\mathcal{X}))$, as a stalk complex, be an object in $K^b(\mathcal{G}(\mathcal{P}(\mathcal{X})))$. Then $G \in K^b(\mathcal{G}(\mathcal{I}(\mathcal{X})))(\subseteq co{-}D^b_{\mathcal{G}(\mathcal{I}(\mathcal{X}))}(\mathcal{X}))$ by (1.2). Then there exists a $\mathcal{G}(\mathcal{I}(\mathcal{X}))$-coquasi-isomorphism

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & G^0 \arrow[r] & G^1 \arrow[r] & \cdots \arrow[r] & G^{n-1} \arrow[r] & G^n \arrow[r] & 0; \end{tikzcd}$$

in particular, $0 \longrightarrow G \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots \longrightarrow G^n \longrightarrow 0$ is exact, and hence G has finite $\mathcal{G}(\mathcal{I}(\mathcal{X}))$-codimension.

Dually, we get (2.1) $\Longleftrightarrow$ (2.2).

Now we are in a position to prove the following

Theorem 4.6. *For a virtually Gorenstein algebra A, the following statements are equivalent.*

(1) *A is Gorenstein.*
(2) $K^b(\mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A))) = K^b(\mathcal{G}(\mathcal{I}(\mathrm{Mod}\ A)))$ *in* $D^b_{\mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A))}(\mathrm{Mod}\ A)$.
(3) $D^b_{\mathcal{G}(\mathcal{P}(\mathrm{Mod}\ A))}(\mathrm{Mod}\ A)_{\mathrm{perf}} = co{-}D^b_{\mathcal{G}(\mathcal{I}(\mathrm{Mod}\ A))}(\mathrm{Mod}\ A)_{\mathrm{coperf}}$ *in* $D^b_{\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A))}(\mathrm{Mod}\ A)$.
(4) $K^b(\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A))) = K^b(\mathcal{G}(\mathcal{I}(\mathrm{mod}\ A)))$ *in* $D^b_{\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A))}(\mathrm{mod}\ A)$.
(5) $D^b_{\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A))}(\mathrm{mod}\ A)_{\mathrm{perf}} = co{-}D^b_{\mathcal{G}(\mathcal{I}(\mathrm{mod}\ A))}(\mathrm{mod}\ A)_{\mathrm{coperf}}$ *in* $D^b_{\mathcal{G}(\mathcal{P}(\mathrm{mod}\ A))}(\mathrm{mod}\ A)$.

Proof. By Lemma 4.3 and Proposition 4.5, we have (2) $\Longleftrightarrow$ (1) $\Longleftrightarrow$ (4).

By Corollary 3.4 and its dual counterpart, we have $D^b_{\mathcal{G}(\mathcal{P}(\text{Mod } A))}(\text{Mod } A)_{\text{perf}} = K^b(\mathcal{G}(\mathcal{P}(\text{Mod } A)))$ and $co{-}D^b_{\mathcal{G}(\mathcal{I}(\text{Mod } A))}(\text{Mod } A)_{\text{coperf}} = K^b(\mathcal{G}(\mathcal{I}(\text{Mod } A)))$. So the assertion (2) $\Longleftrightarrow$ (3) follows. The assertion (4) $\Longleftrightarrow$ (5) follows from Proposition 4.2.

In summary, if A is Gorenstein, then we have the following diagram of identifications

$$\begin{tikzcd}
D^b_{\mathcal{G}(\mathcal{P}(\mathrm{Mod}\,A))}(\mathrm{Mod}\,A)_{\mathrm{perf}} \arrow[r, equal, "\text{Theorem 4.6(3)}"] \arrow[d, equal, "\text{Corollary 3.4}"'] & co\text{-}D^b_{\mathcal{G}(\mathcal{I}(\mathrm{Mod}\,A))}(\mathrm{Mod}\,A)_{\mathrm{coperf}} \arrow[d, equal, "\text{The dual of Corollary 3.4}"] \\
K^b(\mathcal{G}(\mathcal{P}(\mathrm{Mod}\,A))) \arrow[r, equal, "\text{Theorem 4.6(2)}"] & K^b(\mathcal{G}(\mathcal{I}(\mathrm{Mod}\,A))) \\
K^b(\mathcal{G}(\mathcal{P}(\mathrm{mod}\,A))) \arrow[u, hook] \arrow[r, equal, "\text{Theorem 4.6(4)}"] \arrow[d, equal, "\text{Proposition 4.2(1)}"'] & K^b(\mathcal{G}(\mathcal{I}(\mathrm{mod}\,A))) \arrow[u, hook] \arrow[d, equal, "\text{Proposition 4.2(2)}"] \\
D^b_{\mathcal{G}(\mathcal{P}(\mathrm{mod}\,A))}(\mathrm{mod}\,A)_{\mathrm{perf}} \arrow[r, equal, "\text{Theorem 4.6(5)}"] & co\text{-}D^b_{\mathcal{G}(\mathcal{I}(\mathrm{mod}\,A))}(\mathrm{mod}\,A)_{\mathrm{coperf}}.
\end{tikzcd}$$

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