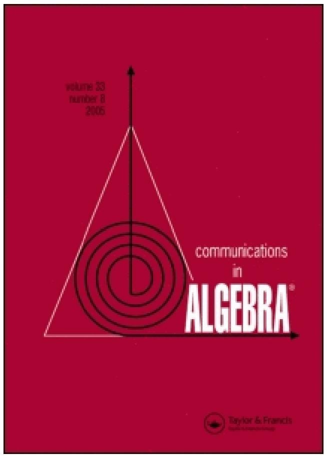

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

On a generalization of the auslander-bridger transpose

Huang Zhaoyong [a]
[a] Department of Mathematics. Nanjing. People's Republic of China

Online Publication Date: 01 January 1999
To cite this Article: Zhaoyong, Huang , (1999) 'On a generalization of the auslander-bridger transpose', Communications in Algebra, 27:12, 5791 - 5812
To link to this article: DOI: 10.1080/00927879908826791
URL: http://dx.doi.org/10.1080/00927879908826791

COMMUNICATIONS IN ALGEBRA, 27(12), 5791–5812 (1999)

ON A GENERALIZATION OF THE AUSLANDER-BRIDGER TRANSPOSE*

Huang Zhaoyong

Department of Mathematics,
Nanjing University,
Nanjing 210093,
People's Republic of China

ABSTRACT Let Λ be an artin algebra and ${}_\Lambda\omega_\Lambda$ a faithfully balanced self-orthogonal bimodule. We generalize the notion of the Auslander-Bridger transpose to that of the transpose with respect to ${}_\Lambda\omega_\Lambda$ and obtain some properties about dual modules with respect to ${}_\Lambda\omega_\Lambda$. Further, we characterize cotilting bimodules and give criteria for computing generalized Gorenstein dimension.

1. INTRODUCTION

Let Λ be an artin algebra and mod Λ (resp. mod Λ^{op}) the category of finitely generated left (resp. right) Λ-modules. For any $T \in$ mod Λ (resp. mod Λ^{op}), we use l. $\mathrm{id}_\Lambda(T)$ (resp. r. $\mathrm{id}_\Lambda(T)$) to denote the left (resp. right) injective dimension of T.

Definition 1.1. For an algebra Λ, a bimodule ${}_\Lambda\omega_\Lambda$ is called a cotilting bimodule if it satisfies the following conditions:

(C_1) The natural maps $\Lambda \to \mathrm{End}_\Lambda({}_\Lambda\omega)^{op}$ and $\Lambda \to \mathrm{End}_\Lambda(\omega_\Lambda)$ are isomorphisms.

---

* 1991 *Mathematics Subject Classification.* 16E10, 16G30, 16G50.
Keywords. faithfully balanced selforthogonal bimodules, ω-reflexive moudules, injective dimension, cotilting bimodules, generalized Gorenstein dimension.

(C_2) $\mathrm{Ext}^i_\Lambda({}_\Lambda\omega, {}_\Lambda\omega)=0$, $\mathrm{Ext}^i_\Lambda(\omega_\Lambda, \omega_\Lambda)=0$ for any $i\geq 1$.
(C_3) $\mathrm{l.\,id}_\Lambda(\omega)<\infty$, $\mathrm{r.\,id}_\Lambda(\omega)<\infty$.

If ${}_\Lambda\omega_\Lambda$ is a cotilting bimodule, then ${}_\Lambda\omega$ and ω_Λ are cotilting modules in the sense of Auslander and Reiten [6]. This can be seen easily by dualizing a result of Miyashita [17, Proposition 1.6]. It follows from Auslander and Reiten [7, Lemma 1.7] that the injective dimensions of ${}_\Lambda\omega$ and of ω_Λ coincide.

Definition 1.2. For an algebra Λ, a bimodule ${}_\Lambda\omega_\Lambda$ is called a faithfully balanced selforthogonal bimodule if it satisfies the above conditions (C_1) and (C_2).

Remark. A faithfully balanced selforthogonal bimodule is called a generalized tilting bimodule by Wakamatsu [18].

It is clear that ${}_\Lambda\Lambda_\Lambda$ is a faithfully balanced selforthogonal bimodule. In fact, we will replace the functor $\mathrm{Hom}_\Lambda(-, {}_\Lambda\Lambda_\Lambda)$ by the functor $\mathrm{Hom}_\Lambda(-, {}_\Lambda\omega_\Lambda)$ where ${}_\Lambda\omega_\Lambda$ is a given faithfully balanced selforthogonal bimodule, and we will extend some known results to this more general setting.

In this paper, we mainly study the fundamental properties of faithfully balanced selforthogonal bimodules and characterize cotilting bimodules. The given characterizations will lead to a better understanding of cotilting bimodules. There is another reason why we study faithfully balanced selforthogonal bimodules. We know that the generalized Nakayama conjecture posed by Auslander and Reiten [5] still remains open, which is equivalent to the following version: if a module M is in mod Λ satisfying the property $\mathrm{Ext}^i_\Lambda(M\oplus\Lambda, M\oplus\Lambda)=0$ for any $i\geq 1$, then M is projective. So giving some characterizations of faithfully balanced selforthogonal bimodules may be useful for comprehending this conjecture.

Faithfully balanced selforthogonal bimodules and cotilting bimodules had been studied extensively (see [6–7], [17–19]). Let ${}_\Lambda\omega_\Lambda$ be a cotilting bimodule. Miyashita in [17] studied the properties of dual modules with respect to ${}_\Lambda\omega_\Lambda$. In [18], Wakamatsu showed that $\mathrm{l.\,id}_\Lambda(\omega)=\mathrm{r.\,id}_\Lambda(\omega)$. In [7], Auslander and Reiten characterized Cohen-Macaulay algebras and Gorenstein algebras in terms of the existence of certain cotilting bimodules and they proved that the tensor product of two Gorenstein algebras over a field is also a Gorenstein algebra. Also in [7], Auslander and Reiten generalized the notion of Gorenstein dimension (see Auslander and Bridger [3]) to that of generalized Gorenstein dimension with respect to a given faithfully balanced selforthogonal bimodule. Auslander and Reiten in [6] gave some properies of algebras of injective dimension at most one and established a one-to-one correspondence between isomorphism classes of basic cotilting modules and covariantly finite coresolving subcategories of the category of modules with finite injective dimension. In addition, Wang and Xu [19] characterized so called $*$-modules and co-$*$-modules which were first introduced and studied by Menini and Orsatti [16] and Colpi [11−12].

The discussion in this paper is based on the above known results. In Section 2 we introduce the transpose $\mathrm{Tr}_\omega A$ of a module A with respect to a faithfully balanced selforthogonal bimodule ${}_\Lambda\omega_\Lambda$. This construction generalizes the Auslander-Bridger transpose $\mathrm{Tr}A$, in the sense that $\mathrm{Tr}A$ is the transpose of A with respect to the faithfully balanced selforthogonal bimodule ${}_\Lambda\Lambda_\Lambda$. It is well known that, denoting by $A^*=\mathrm{Hom}_\Lambda(A,\Lambda)$ and by σ_A the evaluation map $A\to A^{**}$, we have an exact sequence $0\to\mathrm{Ext}^1_\Lambda(\mathrm{Tr}A,\Lambda)\to A\xrightarrow{\sigma_A}A^{**}\to\mathrm{Ext}^2_\Lambda(\mathrm{Tr}A,\Lambda)\to 0$. We prove a corresponding result where the functor $\mathrm{Hom}_\Lambda(-,\Lambda)$ is replaced by $\mathrm{Hom}_\Lambda(-,\omega)$ and $\mathrm{Tr}A$ by $\mathrm{Tr}_\omega A$ (Theorems 2. 3 and 2. 4). This plays a central role in the rest of the paper. In Section 3 we then apply these two exact sequences in order to characterize cotilting bimodules. As a consequence, we can give a partial answer to the strong Nakayama conjecture. In Section 4, we prove that the tensor product of cotilting bimodules over finite dimensional k-algebras, where k is a field, is also a cotilting bimodule. In Section 5, we give criteria for computing generalized Gorenstein dimension. In particular, some results by Auslander and Bridger are generalized.

2. TWO EXACT SEQUENCES AND SOME LEMMAS

From now on, we assume that ${}_\Lambda\omega_\Lambda$ is a faithfully balanced selforthogonal bimodule. In this section, we will obtain two important exact sequences (see Theorem 2. 3 and Theorem 2. 4) which are crucial for the rest of this paper. Also we will give some fundamental properties of faithfully balanced selforthogonal bimodules which will be used later. For a module A in mod Λ (resp. mod Λ^{op}), we put $A^\omega=\mathrm{Hom}_\Lambda(A, {}_\Lambda\omega_\Lambda)$. For a homomorphism f between Λ-modules (resp. Λ^{op}-modules), we put $f^\omega=\mathrm{Hom}_\Lambda(f, {}_\Lambda\omega_\Lambda)$.

Definition 2.1. Suppose $A \in \bmod \Lambda$ (resp. $\bmod \Lambda^{op}$) and suppose $P_1 \xrightarrow{f} P_0 \to A \to 0$ is a minimal projective resolution of A. Then we have an exact sequence $0 \to A^{\omega} \to P_0^{\omega} \xrightarrow{f^{\omega}} P_1^{\omega} \to \operatorname{Coker} f^{\omega} \to 0$. We call $\operatorname{Coker} f^{\omega}$ the transpose (with respect to ${}_{\Lambda}\omega_{\Lambda}$) of A, and denote it by $\mathrm{Tr}_{\omega} A$.

If ${}_{\Lambda}\omega_{\Lambda} = {}_{\Lambda}\Lambda_{\Lambda}$, then the transpose defined above is just the Auslander-Bridger transpose (c. f. [3] and [8]).

Definition 2.2. Let $A \in \bmod \Lambda$ (resp. $\bmod \Lambda^{op}$), and let $\sigma_A : A \to A^{\omega\omega}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^{\omega}$ be the canonical evaluation homomorphism. If σ_A is a monomorphism, then A is called a ω-torsionless module. If σ_A is an isomorphism, then A is called a ω-reflexive module.

For any $T \in \bmod \Lambda$ (resp. $\bmod \Lambda^{op}$), we use $\mathrm{add}_{\Lambda} T$ (resp. $\mathrm{add} T_{\Lambda}$) to denote the full subcategory of $\bmod \Lambda$ (resp. $\bmod \Lambda^{op}$) consisting of all modules isomorphic to the direct summands of finite direct sums of copies of T. It is easy to see that any projective module in $\bmod \Lambda$ (resp. $\bmod \Lambda^{op}$) and any module in $\mathrm{add}_{\Lambda}\omega$ (resp. $\mathrm{add}\,\omega_{\Lambda}$) are ω-reflexive.

Theorem 2.3. *For any $A \in \bmod \Lambda$ (resp. $\bmod \Lambda^{op}$) we have the following exact sequence:*

$$0 \to \mathrm{Ext}^1_{\Lambda}(\mathrm{Tr}_{\omega} A, \omega) \to A \xrightarrow{\sigma_A} A^{\omega\omega} \to \mathrm{Ext}^2_{\Lambda}(\mathrm{Tr}_{\omega} A, \omega) \to 0.$$

Proof. Suppose $A \in \bmod \Lambda$ and suppose

$$\begin{tikzcd} P_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & P_0 \arrow[r] & A \arrow[r] & 0 \\ & K \arrow[ur, "i_1"'] & & & \end{tikzcd}$$

is a minimal projective resolution of A. From the exact sequence

$$(2.3.1)\quad \begin{tikzcd} 0 \arrow[r] & A^{\omega} \arrow[r] & P_0^{\omega} \arrow[rr, "f^{\omega}"] \arrow[dr, "\pi_2"'] & & P_1^{\omega} \arrow[r] & \mathrm{Tr}_{\omega} A \arrow[r] & 0 \\ & & & C \arrow[ur, "i_2"'] & & & \end{tikzcd}$$

we have a long exact sequence $0 \to (\mathrm{Tr}_\omega A)^\omega \to P_1^{\omega\omega} \xrightarrow{i_2^\omega} C^\omega \to \mathrm{Ext}_\Lambda^1(\mathrm{Tr}_\omega A, \omega) \to 0 \to \mathrm{Ext}_\Lambda^1(C, \omega) \to \mathrm{Ext}_\Lambda^2(\mathrm{Tr}_\omega A, \omega) \to 0$ and the following exact commutative diagram:

(2.3.2)

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i_1"] \arrow[d, "g"] & P_0 \arrow[r] \arrow[d, "\sigma_{P0}"] & A \arrow[r] \arrow[d, "\sigma_A"] & 0 & \\
0 \arrow[r] & C^\omega \arrow[r, "\pi_2^\omega"] & P_0^{\omega\omega} \arrow[r] & A^{\omega\omega} \arrow[r] & \mathrm{Ext}_\Lambda^1(C, \omega) \arrow[r] & 0
\end{tikzcd}$$

where σ_{P0} is an isomorphism and g is an induced homomorphism. By the snake lemma we have $\mathrm{Ker}\sigma_A \cong \mathrm{Coker} g$ and $\mathrm{Coker}\sigma_A \cong \mathrm{Ext}_\Lambda^1(C, \omega) \cong \mathrm{Ext}_\Lambda^2(\mathrm{Tr}_\omega A, \omega)$.

Consider the following diagram:

$$\begin{tikzcd}
P_1 \arrow[r, "\sigma_{P1}"] \arrow[d, "\pi_1"] & P_1^{\omega\omega} \arrow[r, "i_2^\omega"] & C^\omega \arrow[d, equal] \\
K \arrow[rr, "g"] \arrow[d, "i_1"] & & C^\omega \arrow[d, "\pi_2^\omega"] \\
P_0 \arrow[rr, "\sigma_{P0}"] & & P_0^{\omega\omega}
\end{tikzcd}$$

By Diagram (2.3.2) $\sigma_{P0} \circ i_1 = \pi_2^\omega \circ g$, so $(\sigma_{P0} \circ i_1) \circ \pi_1 = (\pi_2^\omega \circ g) \circ \pi_1$ and hence $\sigma_{P0} \circ f = \pi_2^\omega \circ g \circ \pi_1$. Since $\sigma_{P0} \circ f = f^{\omega\omega} \circ \sigma_{P1}$ and $f^{\omega\omega} = \pi_2^\omega \circ i_2^\omega$, it follows that $\pi_2^\omega \circ i_2^\omega \circ \sigma_{P1} = \pi_2^\omega \circ g \circ \pi_1$. Since π_2^ω is a monomorphism, $i_2^\omega \circ \sigma_{P1} = g \cdot \pi_1$. Hence $\mathrm{Im}(i_2^\omega \circ \sigma_{P1}) \subseteq \mathrm{Im} g$ and there is an induced commutative diagram:

$$\begin{tikzcd}
 & P_1 \arrow[r, "\sigma_{P1}"] \arrow[d, "\pi_1"] & P_1^{\omega\omega} \arrow[r, "i_2^\omega"] & C^\omega \arrow[r] \arrow[d, equal] & \mathrm{Ext}_\Lambda^1(\mathrm{Tr}_\omega A, \omega) \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & K \arrow[rr, "g"] \arrow[d] & & C^\omega \arrow[r] & \mathrm{Coker} g \arrow[r] & 0 \\
 & 0 & & & &
\end{tikzcd}$$

It follows from the snake lemma that h is an isomorphism. So $\mathrm{Ker}\sigma_A \cong \mathrm{Coker} g \cong \mathrm{Ext}_\Lambda^1(\mathrm{Tr}_\omega A, \omega)$ and we obtain the required exact sequence. $\square$

Remark. Theorem 2. 3 is a generalization of a result by Auslander [2, Proposition 6. 3] (also c. f. Auslander, Reiten and Smalø [8, Chapter IV, Proposition 3. 2]) .

From the proof of Theorem 2. 3 we have the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & P_1 & \xrightarrow{f} & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\sigma_{P_1}} & & \downarrow{\sigma_{P_0}} & & \downarrow & & \\ 0 \longrightarrow & (\mathrm{Tr}_\omega A)^\omega & \longrightarrow & P_1^{\omega\omega} & \xrightarrow{f^{\omega\omega}} & P_0^{\omega\omega} & \longrightarrow & \mathrm{Coker} f^{\omega\omega} & \longrightarrow 0 \end{array}$$

It is easy to see that $A \cong \mathrm{Coker} f^{\omega\omega}$. Noting that P_1^ω and P_0^ω are ω-reflexive and there is an exact sequence (2. 3. 1), it is not difficult to see that the proof of the following theorem is analogous to that of Theorem 2. 3. So we omit it.

Theorem 2. 4. *For any* $A \in \mathrm{mod}\,\Lambda$ (*resp.* $\mathrm{mod}\,\Lambda^{op}$), *we have the following exact sequence*:

$$0 \to \mathrm{Ext}^1_\Lambda(A,\omega) \to \mathrm{Tr}_\omega A \xrightarrow{\sigma_{\mathrm{Tr}_\omega A}} (\mathrm{Tr}_\omega A)^{\omega\omega} \to \mathrm{Ext}^2_\Lambda(A,\omega) \to 0.$$

Corollary 2. 5. *The following statements are equivalent.*

(1) *Any* A *in* $\mathrm{mod}\,\Lambda$ (*resp.* $\mathrm{mod}\,\Lambda^{op}$) *with* $\mathrm{Ext}^1_\Lambda(A,\omega)=0=\mathrm{Ext}^2_\Lambda(A,\omega)$ *is* ω*-reflexive.*

(2) *Any* ω*-reflexive module* B *in* $\mathrm{mod}\,\Lambda^{op}$ (*resp.* $\mathrm{mod}\,\Lambda$) *satisfies*

$\mathrm{Ext}^1_\Lambda(B,\omega) = 0 = \mathrm{Ext}^2_\Lambda(B,\omega)$.

Proof. (1)⇒(2) Suppose $B \in \mathrm{mod}\,\Lambda^{op}$ (resp. $\mathrm{mod}\,\Lambda$) is ω-reflexive. By Theorem 2. 3 and Theorem 2. 4, there are the following two exact sequences:

$$0 \to \mathrm{Ext}^1_\Lambda(\mathrm{Tr}_\omega B,\omega) \to B \xrightarrow{\sigma_B} B^{\omega\omega} \to \mathrm{Ext}^2_\Lambda(\mathrm{Tr}_\omega B,\omega) \to 0$$

$$0 \to \mathrm{Ext}^1_\Lambda(B,\omega) \to \mathrm{Tr}_\omega B \xrightarrow{\sigma_{\mathrm{Tr}_\omega B}} (\mathrm{Tr}_\omega B)^{\omega\omega} \to \mathrm{Ext}^2_\Lambda(B,\omega) \to 0$$

Since B is ω-reflexive, from the first exact sequence we know that $\mathrm{Tr}_\omega B$ satisfies $\mathrm{Ext}^1_\Lambda(\mathrm{Tr}_\omega B,\omega)=0=\mathrm{Ext}^2_\Lambda(\mathrm{Tr}_\omega B,\omega)$. So $\mathrm{Tr}_\omega B$ is ω-reflexive by hypothesis (1). It follows from the second exact sequence that $\mathrm{Ext}^1_\Lambda(B,\omega)=0=\mathrm{Ext}^2_\Lambda(B,\omega)$.

(2)⇒(1) is shown by the same argument. □

Lemma 2. 6. *Let Λ be a ring (not necessary an artin algebra). If $0\to A\to H\xrightarrow{f}B$ is an exact sequence in* mod Λ *(resp.* mod Λ^{op}*) with H ω-reflexive and B ω-torsionless, and if $C=$* Cokerf^{ω}*, then $A\cong C^{\omega}$. Moreover, if f is an epimorphism, then C is ω-torsionless.*

Proof. From the exact sequence $0\to A\to H\xrightarrow{f}B$ we have the exact sequence $B^{\omega}\xrightarrow{f^{\omega}}H^{\omega}\to C\to 0$ and the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "g"] & H \arrow[r, "f"] \arrow[d, "\sigma_H"] & B \arrow[d, "\sigma_B"] \\
0 \arrow[r] & C^{\omega} \arrow[r] & H^{\omega\omega} \arrow[r, "f^{\omega\omega}"] & B^{\omega\omega}
\end{tikzcd}$$

We know that σ_H is an isomorphism and σ_B is a monomorphism, it is easy to see that the induced homomorphism g is an isomorphism and $A\cong C^{\omega}$.

If f is an epimorphism, then $C\subseteq A^{\omega}$. Since A^{ω} is ω-torsionless by Faith [13, Proposition 23. 5], C is also ω-torsionless. $\square$

Observe that a special instance of the above lemma was already discussed by Jones and Teply in [15, Lemma 3]. They considered the case ${}_{\Lambda}\omega_{\Lambda}={}_{\Lambda}\Lambda_{\Lambda}$ and H is finitely generated free, claiming that in this situation C is always torsionless. However, their statement is not correct. In fact, if f is not surjective, C need not be torsionless, as shown by the following example.

Example. 2. 7. Let Λ be an algebra which is given by the quiver: $1\to 2\to 3$. We use P_i and id_{P_i} to denote the indecomposable projective module corresponding to the vertex i and the identity homomorphism of $P_i (i=1, 2, 3)$, respectively. Take $H=\Lambda=(P_1\oplus P_2\oplus P_3)$, $B=P_1\oplus P_1\oplus P_3$ and $f=\mathrm{id}_{P_1}\oplus\iota\oplus\mathrm{id}_{P_3}$, where $\iota: P_2\to P_1$ is the canonical embedding. It is not difficult to check that $C=\mathrm{Coker}f^*$ is not torsionless.

Lemma 2. 8. *For any ω-torsionless module A in* mod Λ *(resp.* mod Λ^{op}*), there is a ω-torsionless module C in* mod Λ^{op} *(resp.* mod Λ*) and a projective module P in* mod Λ *(resp.* mod Λ^{op}*) such that there are the following exact sequences:*

$$0 \to C^{\omega} \to P \to A \to 0$$
$$0 \to A^{\omega} \to P^{\omega} \to C \to 0$$
$$0 \to A \xrightarrow{\sigma_A} A^{\omega\omega} \to \mathrm{Ext}^1_{\Lambda}(C,\omega) \to 0$$
$$0 \to C \xrightarrow{\sigma_C} C^{\omega\omega} \to \mathrm{Ext}^1_{\Lambda}(A,\omega) \to 0$$

Proof. By using Lemma 2.6, we find that this lemma in fact has been proven in the proof of Theorem 2.3. $\square$

Definition 2.9. An exact sequence $A_n \to \cdots \to A_1 \to A_0$ in mod Λ (resp. mod Λ^{op}) is said to be dual exact (with respect to ω) if $A_0^{\omega} \to A_1^{\omega} \to \cdots \to A_n^{\omega}$ is exact in mod Λ^{op} (resp. mod Λ).

Lemma 2.10. *For $A \in$ mod Λ (resp. mod Λ^{op}) and a positive integer n, the following statements are equivalent.*

(1) $\mathrm{Ext}^i_{\Lambda}(A,\omega)=0$ *for any* $1 \leq i \leq n$.

(2) *Any exact sequence* $0 \to K \to P_{n-1} \to \cdots \to P_0 \to A \to 0$ *with all* P_i *projective is dual exact (with respect to ω).*

(3) *Any exact sequence* $P_{n+1} \to P_n \to \cdots \to P_0 \to A \to 0$ *with all* P_i *projective is dual exact (with respect to ω).*

Proof. (1)⇒(2) The case for $n=1$ is clear. Suppose $n \geq 2$ and suppose $0 \to K \to P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to A \to 0$ is an exact sequence with all P_i projective. Then $\mathrm{Ext}^1_{\Lambda}(\mathrm{Im}d_i,\omega) \cong \mathrm{Ext}^{i+1}_{\Lambda}(A,\omega)=0$ for any $1 \leq i \leq n-1$, and hence it is easy to see that $0 \to A^{\omega} \to P_0^{\omega} \to P_1^{\omega} \to \cdots \to P_{n-1}^{\omega} \to K^{\omega} \to 0$ is exact.

(2)⇒(3) It is trivial.

(3)⇒(1) Suppose $n=1$ and suppose the exact sequence

$$\begin{tikzcd} P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[dr, "\pi"'] & P_0 \arrow[r] & A \arrow[r] & 0 \\ & & K \arrow[u, "i"'] & & \end{tikzcd}$$

with all P_i projective is dual exact. Consider the following exact commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A^{\omega} \arrow[r] \arrow[d, equal] & P_0^{\omega} \arrow[r, "i^{\omega}"] \arrow[d, equal] & K^{\omega} \arrow[r] \arrow[d, "\pi^{\omega}"] & \mathrm{Ext}^1_{\Lambda}(A,\omega) \arrow[r] & 0 \\ 0 \arrow[r] & A^{\omega} \arrow[r] & P_0^{\omega} \arrow[r, "d_1^{\omega}"] & P_1^{\omega} \arrow[r, "d_2^{\omega}"] & P_2^{\omega} & \end{tikzcd}$$

Since $0 \to K^{\omega} \xrightarrow{\pi^{\omega}} P_1^{\omega} \xrightarrow{d_2^{\omega}} P_2^{\omega}$ is also exact and π^{ω} is a monomorphism, $\mathrm{Im}i^{\omega} \cong \mathrm{Im}(\pi^{\omega} \cdot i^{\omega}) = \mathrm{Im}d_1^{\omega} = \mathrm{Ker}d_2^{\omega} = \mathrm{Im}\pi^{\omega} \cong K^{\omega}$. So i^{ω} an epimorphism and hence $\mathrm{Ext}_{\Lambda}^1(A, \omega) = 0$. By using induction on n, we can get our conclusion. □

Lemma 2.11. *For a positive integer n, the following statements are equivalent.*

(1) $\mathrm{l.id}_{\Lambda}(\omega) \leq n$ (*resp.* $\mathrm{r.id}_{\Lambda}(\omega) \leq n$).

(2) $\mathrm{Ext}_{\Lambda}^n(B, \omega) = 0$ *for any B ω-torsionless in* mod Λ (*resp.* mod Λ^{op}).

Proof. Consider an exact sequence $0 \to B \to P \to A \to 0$ with every term in mod Λ (resp. mod Λ^{op}) and P projective and B ω-torsionless. Since $\mathrm{Ext}_{\Lambda}^n(B, \omega) \cong \mathrm{Ext}_{\Lambda}^{n+1}(A, \omega)$, it is easy to get the desired equivalence. □

Lemma 2.12. *For any $A \in$* mod Λ (*resp.* mod Λ^{op}), *the following statements are equivalent.*

(1) A^{ω} *is ω-reflexive.*

(2) $A^{\omega\omega\omega}$ *is ω-reflexive.*

Proof. (1)⇒(2) By Faith [13, Proposition 23.5], $(\sigma_{A^{\omega}})^{\omega} \cdot \sigma_{A^{\omega\omega\omega}} = 1_{A^{\omega\omega\omega}}$. So if $\sigma_{A^{\omega}}$ is an isomorphism then $\sigma_{A^{\omega\omega\omega}}$ is also an isomorphism, which means that if A^{ω} is ω-reflexive then $A^{\omega\omega\omega}$ is also ω-reflexive.

(2)⇒(1) It is clear that $\mathrm{Ker}(\sigma_A)^{\omega} \cong (\mathrm{Coker}\sigma_A)^{\omega}$. By Faith [13, Proposition 23.5]$(\sigma_A)^{\omega} \cdot \sigma_{A^{\omega}} = 1_{A^{\omega}}$. So $(\mathrm{Coker}\sigma_A)^{\omega} \cong \mathrm{Ker}(\sigma_A)^{\omega} \cong \mathrm{Coker}\sigma_{A^{\omega}}$. By using the same trick as above, we have $(\mathrm{Coker}\sigma_{A^{\omega}})^{\omega} \cong \mathrm{Coker}\sigma_{A^{\omega\omega\omega}}$. But $(\mathrm{Coker}\sigma_{A^{\omega}})^{\omega} \cong (\mathrm{Coker}\sigma_A)^{\omega\omega}$, so $\mathrm{Coker}\sigma_{A^{\omega\omega\omega}} \cong (\mathrm{Coker}\sigma_A)^{\omega\omega}$. Now if $A^{\omega\omega\omega}$ is ω-reflexive, then $(\mathrm{Coker}\sigma_A)^{\omega\omega} \cong \mathrm{Coker}\sigma_{A^{\omega\omega\omega}} = 0$, and thus $(\mathrm{Coker}\sigma_A)^{\omega} = 0$. Then by the above argument we know that $\mathrm{Coker}\sigma_{A^{\omega}} = 0$. So A^{ω} is ω-reflexive. □

Lemma 2.13. *The following statements are equivalent.*

(1) A^{ω} *is ω-reflexive for any A in* mod Λ.

(2) B^{ω} *is ω-reflexive for any B in* mod Λ^{op}.

Proof. (1)⇒(2) Suppose $B \in$ mod Λ^{op}. By (1), $B^{\omega\omega\omega}$ is ω-reflexive. So by Lemma 2.12 B^{ω} is ω-reflexive.

(2)⇒(1) is similar to the above argument. □

3. COTILTING BIMODULES

In this section we will characterize cotilting bimodules in terms of their injective dimension. We are going to show all statements only for $\mathrm{l.\,id}_\Lambda(\omega)$, symmetric statements hold for $\mathrm{r.\,id}_\Lambda(\omega)$.

Proposition 3.1. *The following statements are equivalent.*

(1) $\mathrm{l.\,id}_\Lambda(\omega)=0$.

(2) *Every module in* mod Λ^{op} *is* ω*-reflexive.*

Proof. It is immediate from Theorem 2.3. $\square$

Corollary 3.2. (*see Jans* [14]) *The following statements are equivalent.*

(1) Λ *is left self-injective.*

(2) *Every module in* mod Λ^{op} *is reflexive* (*with respect to* Λ).

The following theorem contains a result by Auslander and Reiten [6, Proposition 2.2].

Theorem 3.3. *The following statements are equivalent.*

(1) $\mathrm{l.\,id}_\Lambda(\omega)\leq 1$.

(2) *Every* ω*-torsionless module in* mod Λ^{op} *is* ω*-reflexive.*

(3) *Every module* B *in* mod Λ^{op} *with* $\mathrm{Ext}^1_\Lambda(B,\omega)=0$ *is* ω*-reflexive.*

Proof. (1)$\Leftrightarrow$(2) By Lemma 2.8, we know that condition (2) is satisfied if and only if $\mathrm{Ext}^1_\Lambda(A,\omega)=0$ for all ω-torsionless modules A in mod Λ. By Lemma 2.11, we get the equivalence of (1) and (2).

(2)$\Rightarrow$(3) Suppose $B\in$ mod Λ^{op} with $\mathrm{Ext}^1_\Lambda(B,\omega)=0$, and suppose $0\to K\to P\to B\to 0$ is an exact sequence in mod Λ^{op} with P projective and K ω-torsionless. Then $0\to B^\omega\to P^\omega\to K^\omega\to 0$ is exact. Since K^ω is ω-torsionless in mod Λ, it follows from Lemma 2.11 and the equivalence of (2) and (1) that $\mathrm{Ext}^1_\Lambda(K^\omega,\omega)=0$. Thus we obtain the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\sigma_K"] & P \arrow[r] \arrow[d, "\sigma_P"] & B \arrow[r] \arrow[d, "\sigma_B"] & 0 \\
0 \arrow[r] & K^{\omega\omega} \arrow[r] & P^{\omega\omega} \arrow[r] & B^{\omega\omega} \arrow[r] & 0
\end{tikzcd}$$

where σ_P is an isomorphism. By (2), K is ω-reflexive and so σ_K is an isomorphism and hence σ_B is also an isomorphism. Therefore B is ω-reflexive.

(3)$\Rightarrow$(1) Suppose A is ω-torsionless in mod Λ. By Theorem 2.3 and Theorem 2.4 we have the following two exact sequences:

$$0 \to \mathrm{Ext}^1_\Lambda(\mathrm{Tr}_\omega A, \omega) \to A \xrightarrow{\sigma_A} A^{\omega\omega} \to \mathrm{Ext}^2_\Lambda(\mathrm{Tr}_\omega A, \omega) \to 0$$

$$0 \to \mathrm{Ext}^1_\Lambda(A, \omega) \to \mathrm{Tr}_\omega A \xrightarrow{\sigma_{\mathrm{Tr}_\omega A}} (\mathrm{Tr}_\omega A)^{\omega\omega} \to \mathrm{Ext}^2_\Lambda(A, \omega) \to 0$$

Since A is ω-torsionless, it follows from the first exact sequence that $\mathrm{Ext}^1_\Lambda(\mathrm{Tr}_\omega A, \omega)=0$. Then by (3), $\mathrm{Tr}_\omega A$ is ω-reflexive. So from the second exact sequence we know that $\mathrm{Ext}^1_\Lambda(A, \omega)=0$. Hence $\mathrm{l.id}_\Lambda(\omega)\leq 1$ by Lemma 2.11. $\square$

In the following result we will give some properties of the functor $\mathrm{Hom}_\Lambda(-, \omega)$ in the case $\mathrm{l.id}_\Lambda(\omega)\leq 1$.

Proposition 3.4. *The following statements are equivalent.*

(1) $\mathrm{l.id}_\Lambda(\omega)\leq 1$.

(2) *Short exact sequences in* mod Λ *where every term is ω-torsionless are carried to short exact sequences by the functor* $\mathrm{Hom}_\Lambda(-, {}_\Lambda\omega_\Lambda)$.

(3) $(-)^{\omega\omega}$ *preserves the epimorphisms in* mod Λ^{op}.

Proof. (1)$\Rightarrow$(2) From Lemma 2.11 we know that every ω-torsionless module C in mod Λ satisfies $\mathrm{Ext}^1_\Lambda(C, \omega)=0$. Hence any short exact sequence in mod Λ with ω-torsionless end-term has the desired property.

(2)$\Rightarrow$(3) Suppose $B \xrightarrow{f} C \to 0$ is an epimorphism in mod Λ^{op}. Then $0 \to C^{\omega} \xrightarrow{f^{\omega}} B^{\omega}$ is exact in mod Λ with C^{ω} and B^{ω} ω-torsionless. Since $\mathrm{Coker}f^{\omega}$ is a submodule of $(\mathrm{Ker}f)^{\omega}$ and $(\mathrm{Ker}f)^{\omega}$ is ω-torsionless, $\mathrm{Coker}f^{\omega}$ is also ω-torsionless. It follows from (2) that $B^{\omega\omega} \xrightarrow{f^{\omega\omega}} C^{\omega\omega} \to 0$ is exact.

(3)$\Rightarrow$(1) Suppose M is ω-torsionless in mod Λ^{op} and suppose $P \to M \to 0$ is an epimorphism with P projective. By (3) we have the following exact commutative diagram:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\sigma_P"] & M \arrow[r] \arrow[d, "\sigma_M"] & 0 \\
P^{\omega\omega} \arrow[r] & M^{\omega\omega} \arrow[r] & 0
\end{tikzcd}$$

Since σ_P is an isomorphism, σ_M is an epimorphism. But M is ω-torsionless, so M is ω-reflexive. It follows from Theorem 3.3 that $\mathrm{l.id}_\Lambda(\omega)\leq 1$. □

Proposition 3.5. *Suppose* $\mathrm{l.id}_\Lambda(\omega)\leq 2$. *If* N *is in* mod Λ^{op} *with* $\mathrm{Ext}^1_\Lambda(N,\omega)=0=\mathrm{Ext}^2_\Lambda(N,\omega)$, *then* N *is* ω-*reflexive.*

Proof. Suppose $M\in$ mod Λ is ω-reflexive and suppose $P_1\to P_0\to M^\omega\to 0$ is a minimal projective resolution of M^ω in mod Λ^{op}. Then we have an exact sequence $0\to M\cong M^{\omega\omega}\to P_0^\omega\to P_1^\omega\to \mathrm{Tr}_\omega M^\omega\to 0$. Since $\mathrm{l.id}_\Lambda(\omega)\leqslant 2$, $\mathrm{Ext}^i_\Lambda(M,\omega)\cong\mathrm{Ext}^{i+2}_\Lambda(\mathrm{Tr}_\omega M^\omega,\omega)=0$ for any $i\geqslant 1$. Then from Corollary 2.5 we know that our conclusion holds. □

Theorem 3.6. *Suppose* $\mathrm{r.id}_\Lambda(\omega)\leq 2$. *The following statements are equivalent.*

(1) $\mathrm{l.id}_\Lambda(\omega)\leq 2$.

(2) *If* N *in* mod Λ^{op} *satisfies* $\mathrm{Ext}^1_\Lambda(N,\omega)=0=\mathrm{Ext}^2_\Lambda(N,\omega)$, *then* N *is* ω-*reflexive.*

(3) *A module* N *in* mod Λ^{op} *is* ω-*reflexive if and only if* $\mathrm{Ext}^1_\Lambda(N,\omega)=0=\mathrm{Ext}^2_\Lambda(N,\omega)$.

Proof. (1)⇒(2) By Proposition 3.5.

(2)⇒(1) Suppose $M\in$ mod Λ and suppose $P_1\xrightarrow{f}P_0\to M\to 0$ is a minimal projective resolution of M in mod Λ. Then we have an exact sequence $0\to M^\omega\to P_0^\omega\to P_1^\omega\to \mathrm{Tr}_\omega M\to 0$. Since $\mathrm{r.id}_\Lambda(\omega)\leq 2$, $\mathrm{Ext}^i_\Lambda(M^\omega,\omega)\cong\mathrm{Ext}^{i+2}_\Lambda(\mathrm{Tr}_\omega M,\omega)=0$ for any $i\geqslant 1$. By (2), M^ω is ω-reflexive. Then by Lemma 2.13, $(\mathrm{Tr}_\omega M)^\omega$ is ω-reflexive. It follows from (2) and Corollary 2.5 that $\mathrm{Ext}^1_\Lambda((\mathrm{Tr}_\omega M)^\omega,\omega)=0$. Since $\mathrm{Ker}f\cong(\mathrm{Tr}_\omega M)^\omega$ by Lemma 2.6, $\mathrm{Ext}^3_\Lambda(M,\omega)\cong\mathrm{Ext}^1_\Lambda(\mathrm{Ker}f,\omega)\cong\mathrm{Ext}^1_\Lambda((\mathrm{Tr}_\omega M)^\omega,\omega)=0$ and $\mathrm{l.id}_\Lambda(\omega)\leq 2$.

(3)⇒(2) It is trivial.

(2)⇒(3) It suffices to prove that any ω-reflexive module N in mod Λ^{op} satisfies $\mathrm{Ext}^1_\Lambda(N,\omega)=0=\mathrm{Ext}^2_\Lambda(N,\omega)$. Since $\mathrm{r.id}_\Lambda(\omega)\leq 2$, any M in mod Λ with $\mathrm{Ext}^1_\Lambda(M,\omega)=0=\mathrm{Ext}^2_\Lambda(M,\omega)$ is ω-reflexive by the symmetric statement of Proposition 3.5. Then our conclusion follows from Corollary 2.5. □

Theorem 3.7. *Suppose* $n\geqslant 2$ *is a positive integer and* $\mathrm{l.id}_\Lambda(\omega)\leq n$. *If* M *is* ω-*torsionless in* mod Λ^{op} *with* $\mathrm{Ext}^i_\Lambda(M,\omega)=0$ *for any* $1\leq i\leq n$, *then M is ω-reflexive. Moreover, if* l. $\mathrm{id}_\Lambda(\omega)=$ r. $\mathrm{id}_\Lambda(\omega)\leq n$, *then any G in* mod Λ^{op} *(resp.* mod Λ) *with* $\mathrm{Ext}^i_\Lambda(G,\omega)=0$ *for any* $1\leq i\leq n$, *is ω-reflexive.*

Proof. Suppose l. $\mathrm{id}_\Lambda(\omega)\leq n$ and M is ω-torsionless in mod Λ^{op} with $\mathrm{Ext}^i_\Lambda(M,\omega)=0$ for any $1\leq i\leq n$. By Lemma 2. 8 there is a module N ω-torsionless in mod Λ such that the following sequences are exact.

(3. 7. 1) $$0\to N^\omega\to P\to M\to 0$$

(3. 7. 2) $$0\to M\xrightarrow{\sigma_M}M^{\omega\omega}\to \mathrm{Ext}^1_\Lambda(N,\omega)\to 0$$

(3. 7. 3) $$0\to N\xrightarrow{\sigma_N}N^{\omega\omega}\to \mathrm{Ext}^1_\Lambda(M,\omega)\to 0$$

where P is projective in mod Λ^{op}.

Since $\mathrm{Ext}^1_\Lambda(M,\omega)=0$, from the exact sequence (3. 7. 3) we know that N is ω-reflexive. Since $\mathrm{Ext}^i_\Lambda(M,\omega)=0$ for any $1\leq i\leq n$, by the exact sequence (3. 7. 1) we have $\mathrm{Ext}^i_\Lambda(N^\omega,\omega)=0$ for any $1\leq i\leq n-1$.

Consider the following exact sequence:

$$0\to K\to P_{n-2}\to\cdots\to P_1\to P_0\to N^\omega\to 0$$

where all P_i are projective in mod Λ^{op}. By Lemma 2. 10, $0\to N\cong N^{\omega\omega}\to P_0^\omega\to P_1^\omega\to\cdots\to P_{n-2}^\omega\to K^\omega\to 0$ is exact. Since K^ω is ω-torsionless, $\mathrm{Ext}^n_\Lambda(K^\omega,\omega)=0$ by Lemma 2. 11. So $\mathrm{Ext}^1_\Lambda(N,\omega)\cong\mathrm{Ext}^n_\Lambda(K^\omega,\omega)=0$ and hence M is ω-reflexive by the exactness of the sequence (3. 7. 2).

Now if l. $\mathrm{id}_\Lambda(\omega)=$ r. $\mathrm{id}_\Lambda(\omega)\leq n$ and suppose $G\in$ mod Λ^{op} with $\mathrm{Ext}^i_\Lambda(G,\omega)=0$ for any $1\leq i\leq n$. We have an exact sequence $0\to H\to P\to G\to 0$ with P projective and H ω-torsionless in mod Λ^{op}. Since r. $\mathrm{id}_\Lambda(\omega)\leq n$, it is easy to see that $\mathrm{Ext}^i_\Lambda(H,\omega)=0$ for any $1\leq i\leq n$. From the above argument we know that H is ω-reflexive.

Consider the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & P & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\sigma_H} & & \downarrow{\scriptstyle\sigma_P} & & \downarrow{\scriptstyle\sigma_G} & & \\ 0 & \longrightarrow & H^{\omega\omega} & \longrightarrow & P^{\omega\omega} & \longrightarrow & G^{\omega\omega} & & \end{array}$$

Since σ_H and σ_P are isomorphisms, σ_G is a monomorphism and G is a ω-torsionless module. From the above argument we know that G is a ω-reflexive module. □

We know that l. $\mathrm{id}_\Lambda(\omega)=$ r. $\mathrm{id}_\Lambda(\omega)$ when ${}_\Lambda\omega_\Lambda$ is a cotilting bimodule, so from Theorem 3. 3 and Theorem 3. 7 we have the following conclusion, which has been proven with different methods by Miyashita in [17. Theorem 6. 1].

Theorem 3. 8. *Suppose* ${}_\Lambda\omega_\Lambda$ *is a cotilting bimodule. If a module M in* mod Λ^{op} (*resp.* mod Λ) *satisfies* $\mathrm{Ext}^i_\Lambda(M,\omega)=0$ *for any* $i\geqslant 1$, *then M is* ω-*reflexive.*

Corollary 3. 9. *Suppose* ${}_\Lambda\omega_\Lambda$ *is a cotilting bimodule. If a module M in* mod Λ^{op} (*resp.* mod Λ) *satisfies* $\mathrm{Ext}^i_\Lambda(M,\omega)=0$ *for any* $i\geq 0$, *then* $M=0$.

Proof. By Theorem 3. 8. □

Following Colby and Fuller [10], we say that the strong Nakayama conjecure is true for Λ if the condition of $\mathrm{Ext}^i_\Lambda(M,\Lambda)=0$ for any $i\geqslant 0$ implies $M=0$. By results of Auslander and Reiten [5], we know that the verification of this conjecture would imply the generalized Nakayama conjecture and hence also Nakayama's conjecture. It follows from a result of Colby and Fuller [10, Theorem 2] that the strong Nakayama conjecture is true for Gorenstein algebras. The following corollary yields a new proof of this fact.

Theorem 3. 10. *If an algebra* Λ *has a cotilting bimodule* ${}_\Lambda\omega_\Lambda$ *with* ${}_\Lambda\omega$ *flat, then the strong Nakayama conjecture holds over* Λ.

Proof. Let ${}_\Lambda\omega_\Lambda$ be a cotilting bimodule with ${}_\Lambda\omega$ flat and let M be in mod Λ with $\mathrm{Ext}^i_\Lambda(M,\Lambda)=0$ for any $i\geq 0$. By Auslander and Bridger [3, Theorem 2. 8], for any $i\geq 0$, we have an exact sequence:

$$\mathrm{Ext}^i_\Lambda(M,\Lambda)\otimes_\Lambda\omega\to\mathrm{Ext}^i_\Lambda(M,\omega)\to\mathrm{Tor}^\Lambda_1(\mathrm{Tr}_\Lambda\Omega^i(M),\omega)$$

Since ${}_\Lambda\omega$ is flat, the third term of the above exact sequence is always zero. Consequently $\mathrm{Ext}^i_\Lambda(M,\omega)=0$ for any $i\geq 0$, which implies $M=0$ by Corollary 3. 9. This finishes the proof. □

4. TENSOR PRODUCT OF COTILTING BIMODULES

The notions of Cohen-Macaulay rings and Gorenstein rings, as well as Cohen-Macaulay modules, whose importance is well established in commutative Noetherian ring theory, were extended to artin algebras by Auslander and Reiten in [6], and were developed further by them in [7]. The following definition is recalled from Auslander and Reiten [7]. An algebra Λ is called a Cohen-Macaulay algebra if there is a pair of adjoint functors (G,F) between mod Λ and mod Λ, inducing inverse equivalences:

$$\mathscr{I}^{\infty}(\Lambda) \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{P}^{\infty}(\Lambda)$$

where $\mathscr{I}^{\infty}(\Lambda)$ and $\mathscr{P}^{\infty}(\Lambda)$ are the full subcategories of mod Λ consisting of the modules of finite injective dimension and the modules of finite projective dimension, respectively. For a subcategory $\mathscr{X}$ of mod Λ (resp. mod Λ^{op}), we use $\widehat{\mathscr{X}}$ to denote the category consisting of the C in mod Λ (resp. mod Λ^{op}) such that there is an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to C \to 0$ with all X_i in $\mathscr{X}$. Also from Auslander and Reiten [7] we recall the following facts and definition.

Facts and Definition 4. 1. *If Λ is a Cohen-Macaulay algebra and (G, F) is an associated pair of adjoint functors, then F is left exact and given by $F = \operatorname{Hom}_\Lambda({}_\Lambda\omega_\Lambda, -)$, G is right exact and given by $G = {}_\Lambda\omega_\Lambda \otimes -$. The bimodule ${}_\Lambda\omega_\Lambda$ is called a dualizing module. So Λ is a Cohen-Macaulay algebra if and only if Λ has a dualizing module. Λ is called a Gorenstein algebra if ${}_\Lambda\Lambda_\Lambda$ is a dualizing module, which is equivalent to* $\text{l.}\ \mathrm{id}_\Lambda(\Lambda) = \text{r.}\ \mathrm{id}_\Lambda(\Lambda) < \infty$.

Moreover, for an algebra Λ, a bimodule ${}_\Lambda\omega_\Lambda$ is a dualizing module if and only if it is a cotilting bimodule and satisfies the condition:

(C_4) *For a left Λ-module ${}_\Lambda T$, if* $\text{l.}\ \mathrm{id}_\Lambda(T) < \infty$ *then* ${}_\Lambda T \in \widehat{\operatorname{add}_\Lambda \omega}$. *For a right Λ-module T'_Λ, if* $\text{r.}\ \mathrm{id}_\Lambda(T') < \infty$ *then* $T'_\Lambda \in \widehat{\operatorname{add} \omega_\Lambda}$.

Let Λ and Γ be finite dimensional algebras over a field k. Auslander and Reiten [7] posed an open question: Is the tensor product $\Lambda \otimes_k \Gamma$ a Cohen-Macaulay algebra provided that Λ and Γ are both Cohen-Macaulay? They gave an affirmative answer to this question in the case of Λ and Γ being Gorenstein algebras (see Auslander and Reiten [7, Proposition 2. 2]). According to 4. 1, the above open question is related to the following question: If Λ and Γ have dualizing modules ${}_\Lambda U_\Lambda$ and ${}_\Gamma V_\Gamma$ respectively, is it true that $\Lambda \otimes_k \Gamma$ has a dualizing module ${}_{\Lambda\otimes_k\Gamma}(U \otimes_k V)_{\Lambda\otimes_k\Gamma}$? In this section, we will discuss this question. We use $D(-)$ to denote $\operatorname{Hom}_k(-, k)$.

The following lemma is well-known (see Cartan and Eilenberg[9, Chapter XI, Proposition 1.2.3 and Theorem 3.1]).

Lemma 4.2. *Let* A, $M\in \text{mod}\,\Lambda$ *and* B, $N\in \text{mod}\,\Gamma$. *Then we have the following isomorphisms.*

(1) $\text{Hom}_{\Lambda\otimes_k\Gamma}(A\otimes_k B, M\otimes_k N)\cong \text{Hom}_\Lambda(A,M)\otimes_k \text{Hom}_\Gamma(B,N)$.

(2) *For any* $n\geq 1$, $\text{Ext}^n_{\Lambda\otimes_k\Gamma}(A\otimes_k B, M\otimes_k N)\cong \bigoplus_{r+s=n}\text{Ext}^r_\Lambda(A,M)\otimes_k \text{Ext}^s_\Gamma(B,N)$.

Lemma 4.3. *Let* $M\in \text{mod}\,\Lambda$ *and* $N\in\text{mod}\,\Gamma$. *Then* $\text{l.id}_{\Lambda\otimes_k\Gamma}(M\otimes_k N)=\text{l.id}_\Lambda(M)+\text{l.id}_\Gamma(N)$.

Proof. By Lemma 4.2(1) and Cartan and Eilenberg [9. Chapter XI, Theorem 3.2] we have

$$\begin{aligned}&\text{l.id}_{\Lambda\otimes_k\Gamma}(M\otimes_k N)\\ &=\text{r.pd}_{\Lambda\otimes_k\Gamma}(D(M\otimes_k N))\\ &=\text{r.pd}_{\Lambda\otimes_k\Gamma}(D(M)\otimes_k D(N))\\ &=\text{r.pd}_\Lambda(D(M))+\text{r.pd}_\Gamma(D(N))\\ &=\text{l.id}_\Lambda(M)+\text{l.id}_\Gamma(N).\ \square\end{aligned}$$

Lemma 4.4. *If* ${}_\Lambda M$ *and* ${}_\Gamma N$ *satisfy the condition* $\text{Ext}^n_\Lambda(M,M)=0=\text{Ext}^n_\Gamma(N,N)$ *for any* $n\geqslant 1$, *then* $\text{Ext}^n_{\Lambda\otimes_k\Gamma}(M\otimes_k N, M\otimes_k N)=0$ *for any* $n\geqslant 1$.

Proof. For any $n\geq 1$, by Lemma 4.2(2) we have

$$\begin{aligned}&\text{Ext}^n_{\Lambda\otimes_k\Gamma}(M\otimes_k N, M\otimes_k N)\\ &\cong\bigoplus_{r+s=n}\text{Ext}^r_\Lambda(M,M)\otimes_k \text{Ext}^s_\Gamma(N,N)\\ &=0\ (\text{since } r\geq 1 \text{ or } s\geq 1).\ \square\end{aligned}$$

Remark 4.5. For the case of right modules, we have symmetric conclusions of above lemmas.

The following corollary is a result by Auslander and Reiten [7, Proposition 2.2].

Corollary 4.6. (1) $\text{l.id}_{\Lambda\otimes_k\Gamma}(\Lambda\otimes_k\Gamma)=\text{l.id}_\Lambda(\Lambda)+\text{l.id}_\Gamma(\Gamma)$ *and* $\text{r.id}_{\Lambda\otimes_k\Gamma}(\Lambda\otimes_k\Gamma)=\text{r.id}_\Lambda(\Lambda)+\text{r.id}_\Gamma(\Gamma)$.

(2) *$\Lambda \otimes_k \Gamma$ is Gorenstein if and only if Λ and Γ are Gorenstein.*

(3) *$\Lambda \otimes_k \Gamma$ is self-injective if and only if Λ and Γ are self-injective.*

Proof. By Lemma 4.3 and Remark 4.5. $\square$

Theorem 4.7. *Let ${}_\Lambda M_\Lambda$ and ${}_\Gamma N_\Gamma$ be cotilting bimodules. Then ${}_{\Lambda\otimes_k\Gamma}(M \otimes_k N)_{\Lambda\otimes_k\Gamma}$ is also a cotiliting bimodule.*

Proof. By Lemma 4.2, Lemma 4.3, Lemma 4.4 and Remark 4.5. $\square$

Proposition 4.8. *Let ${}_\Lambda M_\Lambda$ and ${}_\Gamma N_\Gamma$ be cotilting bimodules.*

(1) *If ${}_\Lambda A \in \widehat{\mathrm{add}_\Lambda M}$ and ${}_\Gamma B \in \widehat{\mathrm{add}_\Gamma N}$, then ${}_{\Lambda\otimes_k\Gamma}(A \otimes_k B) \in \widehat{\mathrm{add}_{\Lambda\otimes_k\Gamma}(M \otimes_k N)}$.*

(2) *If $C_\Lambda \in \widehat{\mathrm{add} M_\Lambda}$ and $D_\Gamma \in \widehat{\mathrm{add} N_\Gamma}$, then $(C \otimes_k D)_{\Lambda\otimes_k\Gamma} \in \widehat{\mathrm{add}(M \otimes_k N)_{\Lambda\otimes_k\Gamma}}$.*

Proof. (1) Since ${}_\Lambda A \in \widehat{\mathrm{add}_\Lambda M}$ and ${}_\Gamma B \in \widehat{\mathrm{add}_\Gamma N}$, there are exact sequences $0 \to M_m \to \cdots \to M_0 \to A \to 0$ with all M_i in $\mathrm{add}_\Lambda M$, and $0 \to N_n \to \cdots \to N_0 \to B \to 0$ with all N_i in $\mathrm{add}_\Lambda N$. Then we have the following exact commutative diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_m \otimes_k N_n \arrow[r] \arrow[d] & \cdots \arrow[r] & M_0 \otimes_k N_n \arrow[r] \arrow[d] & A \otimes_k N_n \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & M_m \otimes_k N_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & M_0 \otimes_k N_0 \arrow[r] \arrow[d] & A \otimes_k N_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_m \otimes_k B \arrow[r] \arrow[d] & \cdots \arrow[r] & M_0 \otimes_k B \arrow[r] \arrow[d] & A \otimes_k B \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}
$$

It is clear that ${}_{\Lambda\otimes_k\Gamma}(M_i \otimes_k B) \in \widehat{\mathrm{add}_{\Lambda\otimes_k\Gamma}(M \otimes_k N)}$ for any $1 \leq i \leq m$. By Theorem 4.7, ${}_{\Lambda\otimes_k\Gamma}(M \otimes_k N)_{\Lambda\otimes_k\Gamma}$ is also a cotilting bimodule. It follows from Auslander and Reiten [6, Theorem 5.5] that $\widehat{\mathrm{add}_{\Lambda\otimes_k\Gamma}(M \otimes_k N)}$

(is coresolving and hence) is closed under cokernels of monomorphisms. It is easy to see from the last row of the above diagram that ${}_{\Lambda\otimes_k\Gamma}(A\otimes_k B)\in \widehat{\mathrm{add}_{\Lambda\otimes_k\Gamma}(M\otimes_k N)}$.

(2) The dual of (1). $\square$

Remark. Let Λ and Γ be Cohen-Macaulay algebras with dualizing modules ${}_\Lambda U_\Lambda$ and ${}_\Gamma V_\Gamma$, respectively. Then from Theorem 4.7 and Proposition 4.8 we know that the bimodule ${}_{\Lambda\otimes_k\Gamma}(U\otimes_k V)_{\Lambda\otimes_k\Gamma}$ is a cotilting bimodule and satisfies partially the condition (C_4): those left $\Lambda\otimes_k\Gamma$-modules with finite injective dimension which have the form $A\otimes_k B$ (where $A\in \mathrm{mod}\ \Lambda$ and $B\in \mathrm{mod}\ \Gamma$) are in $\widehat{\mathrm{add}_{\Lambda\otimes_k\Gamma}(U\otimes_k V)}$ and those right $\Lambda\otimes_k\Gamma$-modules with finite injective dimension which have the form $A'\otimes_k B'$ (where $A'\in \mathrm{mod}\ \Lambda^{op}$ and $B'\in \mathrm{mod}\ \Gamma^{op}$) are in $\widehat{\mathrm{add}(U\otimes_k V)_{\Lambda\otimes_k\Gamma}}$.

5. GENERALIZED GORENSTEIN DIMENSION

Definition 5.1. (see Auslander and Reiten [7, P. 238]) A module M in mod Λ is said to have generalized Gorenstein dimension zero (with respect to ω), denoted by $\text{G-dim}_\omega(M)=0$, if the following conditions hold:

(1) M is ω-reflexive.

(2) $\mathrm{Ext}^i_\Lambda(M,\omega)=0=\mathrm{Ext}^i_\Lambda(M^\omega,\omega)$ for any $i\geq 1$.

Definition 5.2. For any $n\geq 1$, M in mod Λ is said to have generalized Gorenstein dimension at most n (with respect to ω), denoted by $\text{G-dim}_\omega(M)\leq n$, if there is an exact sequence $0\to M_n\to \dots \to M_1\to M_0\to M\to 0$ in mod Λ with $\text{G-dim}_\omega(M_i)=0$ for any $0\leq i\leq n$.

Corollary 5.3. *For any M in* mod Λ, $\text{G-dim}_\omega(M)\leq \mathrm{l.pd}_\Lambda(M)$.

Proof. It follows directly from Definition 5.2. $\square$

If ${}_\Lambda\omega_\Lambda = {}_\Lambda\Lambda_\Lambda$, then G-dim_ω is just the Gorenstein dimension introduced by Auslander and Bridger [3]. In the following, we will give criteria for computing generalized Gorenstein dimension which extend corresponding results by Auslander and Bridger on (ordinary) Gorenstein dimension. We use $\mathrm{CM}_\omega(\Lambda)$ (resp. $\mathrm{CM}_\omega(\Lambda^{op})$) to denote the full subcategory of mod Λ (resp. mod Λ^{op}) consisting of the modules M with

$\mathrm{Ext}^i_\Lambda(M,\omega)=0$ for any $i \geq 1$.

Definition 5. 4. Let n be a positive integer. $A \in \mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Lambda^{op}$) is called an n-th ω-syzygy module if there is an exact sequence $0 \to A \to \omega_0 \to \omega_1 \to \cdots \to \omega_{n-1}$ with all ω_i in $\mathrm{add}_\Lambda\omega$ (resp. $\mathrm{add}\omega_\Lambda$). We use $\Omega^n_\omega(\Lambda)$ (resp. $\Omega^n_\omega(\Lambda^{op})$) to denote the full subcategory of $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Lambda^{op}$) consisting of n-th ω-syzygy modules.

Lemma 5. 5. *Suppose ${}_\Lambda\omega_\Lambda$ is a cotilting bimodule. Then the following statements hold.*

(1) $\mathrm{CM}_\omega(\Lambda) = \Omega^n_\omega(\Lambda)$ *and* $\mathrm{CM}_\omega(\Lambda^{op}) = \Omega^n_\omega(\Lambda^{op})$ *for any* $n \geq \mathrm{l.\,id}_\Lambda(\omega) = \mathrm{r.\,id}_\Lambda(\omega)$.

(2) *If* $A \in \mathrm{CM}_\omega(\Lambda)$, *then* $\mathrm{Tr}_\omega A \in \mathrm{CM}_\omega(\Lambda^{op})$. *If* $A \in \mathrm{CM}_\omega(\Lambda^{op})$, *then* $\mathrm{Tr}_\omega A \in \mathrm{CM}_\omega(\Lambda)$.

Proof. (1) Suppose $n \geq \mathrm{l.\,id}_\Lambda(\omega)$ and $C \in \Omega^n_\omega(\Lambda)$. Then we have an exact sequence $0 \to C \to U_0 \xrightarrow{d_1} U_1 \to \cdots \xrightarrow{d_{n-1}} U_{n-1}$ with all U_i in $\mathrm{add}_\Lambda\omega$. It is clear that $\mathrm{Ext}^i_\Lambda(C,\omega) \cong \mathrm{Ext}^{i+n}_\Lambda(\mathrm{Coker}d_{n-1},\omega) = 0$ for any $i \geq 1$. So $C \in \mathrm{CM}_\omega(\Lambda)$.

Conversely, suppose $C \in \mathrm{CM}_\omega(\Lambda)$ and suppose $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to C^\omega \to 0$ is a projective resolution of C^ω in $\mathrm{mod}\,\Lambda^{op}$. By Miyashita [17, Theorem 6. 1], $C^\omega \in \mathrm{CM}_\omega(\Lambda^{op})$. By Theorem 3. 8, $C \cong C^{\omega\omega}$. So we have an exact sequence $0 \to C \to P_0^\omega \to P_1^\omega \to \cdots \to P_n^\omega \to \cdots$ with all P_i^ω in $\mathrm{add}_\Lambda\omega$. Similarly, we prove the second conclusion.

(2) is proven as in Auslander and Reiten [7, Proposition 3. 1 (d)]. $\square$

The following lemma extends Auslander and Bridger [3, Proposition 3. 8].

Lemma 5. 6. *For any $M \in \mathrm{mod}\,\Lambda$, the following statements are equivalent.*

(1) $\mathrm{G\text{-}dim}_\omega(M) = 0$.

(2) $\mathrm{Ext}^i_\Lambda(M,\omega) = 0 = \mathrm{Ext}^i_\Lambda(\mathrm{Tr}_\omega M,\omega)$ *for any* $i \geq 1$.

Proof. From Definition 2. 1 we have an exact sequence $0 \to M^\omega \to U_0 \to U_1 \to \mathrm{Tr}_\omega M \to 0$ with U_0 and U_1 in $\mathrm{add}\omega_\Lambda$. Then our claim is trivial by Lemma 5. 5. $\square$

Lemma 5.7. *For any A and B in* mod Λ, G-$\dim_\omega(A) = 0$ *and* G-$\dim_\omega(B) = 0$ *if and only if* G-$\dim_\omega(A \oplus B) = 0$.

Proof. It is easy by Definition 5.1. $\square$

The following lemma extends Auslander and Bridger [3, Lemma 3.10].

Lemma 5.8. *Suppose* $0 \to A \to B \to C \to 0$ *is an exact sequence in* mod Λ *with* G-$\dim_\omega(C) = 0$. *Then* G-$\dim_\omega(A) = 0$ *if and only if* G-$\dim_\omega(B) = 0$.

Proof. Since G-$\dim_\omega(C) = 0$, $\mathrm{Ext}^i_\Lambda(C, \omega) = 0$ for any $i \geq 1$. It follows from the exact sequence $0 \to A \to B \to C \to 0$ that for any $i \geq 1$, we have an isomorphism:

(5.8.1) $$\mathrm{Ext}^i_\Lambda(A, \omega) \cong \mathrm{Ext}^i_\Lambda(B, \omega).$$

As in Auslander and Bridger [3, Lemma 3.9], we can show that there is a long exact sequence $0 \to C^\omega \to B^\omega \to A^\omega \to \mathrm{Tr}_\omega C \to \mathrm{Tr}_\omega B \to \mathrm{Tr}_\omega A \to 0$. Since $\mathrm{Ext}^1_\Lambda(C, \omega) = 0$, we have in fact deduced the exactness of the sequences $0 \to C^\omega \to B^\omega \to A^\omega \to 0$ and $0 \to \mathrm{Tr}_\omega C \to \mathrm{Tr}_\omega B \to \mathrm{Tr}_\omega A \to 0$. So we have a long exact sequence:

(5.8.2)

$$0 \to (\mathrm{Tr}_\omega A)^\omega \to (\mathrm{Tr}_\omega B)^\omega \to (\mathrm{Tr}_\omega C)^\omega \to \mathrm{Ext}^1_\Lambda(\mathrm{Tr}_\omega A, \omega) \to \mathrm{Ext}^1_\Lambda(\mathrm{Tr}_\omega B, \omega) \to \mathrm{Ext}^1_\Lambda(\mathrm{Tr}_\omega C, \omega) \to \dots \to \mathrm{Ext}^i_\Lambda(\mathrm{Tr}_\omega A, \omega) \to \mathrm{Ext}^i_\Lambda(\mathrm{Tr}_\omega B, \omega) \to \mathrm{Ext}^i_\Lambda(\mathrm{Tr}_\omega C, \omega) \to \dots$$

Further, we show as in Auslander and Bridger [3, Lemma 3.9] that $0 \to (\mathrm{Tr}_\omega A)^\omega \to (\mathrm{Tr}_\omega B)^\omega \to (\mathrm{Tr}_\omega C)^\omega \to 0$ is exact. Since G-$\dim_\omega(C) = 0$, $\mathrm{Ext}^i_\Lambda(\mathrm{Tr}_\omega C, \omega) = 0$ for any $i \geq 1$ by Lemma 5.6. Then it follows from the long exact sequence (5.8.2) that for any $i \geq 1$ we have the following isomorphisms:

(5.8.3) $$\mathrm{Ext}^i_\Lambda(\mathrm{Tr}_\omega A, \omega) \cong \mathrm{Ext}^i_\Lambda(\mathrm{Tr}_\omega B, \omega).$$

By Lemma 5.6, we know from the isomorphisms (5.8.1) and (5.8.3) that G-$\dim_\omega(A)=0$ if and only if G-$\dim_\omega(B)=0$. $\square$

The following theorem is analogous to a result by Auslander and Bridger [3, Theorem 3.13].

Theorem 5.9. *For any $n \geq 1$ and M in* mod Λ, *the following statements are equivalent*:

(1) $\text{G-dim}_\omega(M) \leq n$.

(2) $\text{G-dim}_\omega(\Omega^n(M)) = 0$, *where* $\Omega^n(M)$ *is the n-th syzygy of* M.

Proof. (2) $\Rightarrow$ (1) It is trivial by Definition 5. 2.

(1) $\Rightarrow$ (2) It is clear that $\text{G-dim}_\omega(P) = 0$ for any projective module P in mod Λ. Then by Lemma 5. 7 and Lemma 5. 8, the subcategory of mod Λ consisting of the modules M with $\text{G-dim}_\omega(M) = 0$ satisfies the assumptions of Auslander and Bridger [3, Lemma 3. 12]. It follows from Definition 5. 2 and Auslander and Bridger [3, Lemma 3. 12] that (1) implies (2). $\square$

ACKNOWLEDGEMENT

The author is grateful to the referee for his careful reading and the valuable and very detailed comments and suggestions in shaping this paper into its present version.

REFERENCES

1. Anderson F. W. and Fuller K. R. , *Rings and Categories of Modules (second edition)*, Graduate Texts in Math. 13, Springer-Verlag, Berlin-Heidelberg-New York, 1992.
2. Auslander M. , *Coherent functors*, Proceedings Conference on Categorical Algebra, La Jolla, Springer-Verlag, Berlin-Heidelberg-New York, (1966), 189—231.
3. Auslander M. and Bridger M. , *Stable module theory*, Memoirs Amer. Math. Soc. 94(1969).
4. Auslander M. and Buchweitz R. O. , *Maximal Cohen-Macaulay approximations*, Soc. Math. France 38(1989), 5—37.
5. Auslander M. and Reiten I. , *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc. 52(1975), 69—74.
6. Auslander M. and Reiten I. , *Applications to contravariantly finite subcategories*, Advances in Math. 86(1991), 111—152.
7. Auslander M. and Reiten I. , *Cohen-Macaulay and Gorenstein artin algebras*, Progress in Math. (Birkhauser, Basel) 95(1991), 221—245.
8. Auslander M. , Reiten I. and Smalø S. O. , *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, Cambridge, 1995.
9. Cartan H. and Eilenberg S. , *Homological Algebra*, Princeton University Press, Princeton, 1956.
10. Colby R. R. and Fuller K. R. , *A note on Nakayama conjecture*, Tsukuba Jour. Math. 14(1990), 343—352.
11. Colpi R. , *Some remarks on equivalences between categories of modules*, Comm. in Algebra 18(1990), 1935—1951.
12. Colpi R. , *Tilting modules and *-modules*, Comm. in Algebra 21(1993), 1095—1102.

13. Faith C., *Algebra* II, *Ring Theory*, Grundlehren der Mathematischen Wissenchaften 191 (A Series of Comprehensive Studies in Mathmatics), Springer-Verlag, Berlin-Heidelberg-New York, 1976.
14. Jans J. P., *Duality in noetherian rings*, Proc. Amer. Math. Soc. 12(1961), 829—835.
15. Jones M. F. and Teply M. L., *Coherent rings of finite weak globe dimension*, Comm. in Algebra 10(1982), 493—503.
16. Menini C. and Orsatti A., *Reprensentation equivalences between categories of modules and applications*, Rend. Sem. Mat. Univ. Padova 82(1989), 203—231.
17. Miyashita Y., *Tilting modules of finite projective dimension*, Math. Zeit. 193 (1986), 113—146.
18. Wakamatsu T., *On modules with trivial self-extensions*, Jour. Algebra 114 (1988), 106—114.
19. Wang M. Y. and Xu Y. H., **-modoles, co-*-modules and generalized cotilting modules over Noetherian rings (in Chinese)*, Science in China (Series A) 25(1995), 1147—1152.

Received: March 1998

Final Revised Version: April 1999