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ON THE GRADE OF MODULES OVER NOETHERIAN RINGS

Zhaoyong Huang
Department of Mathematics, Nanjing University, Nanjing, China

Let Λ be a left and right noetherian ring and $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules. In this article, we show the following results. (1) For a positive integer k, the condition that the subcategory of $\operatorname{mod}\Lambda$ consisting of i-torsionfree modules coincides with the subcategory of $\operatorname{mod}\Lambda$ consisting of i-syzygy modules for any $1 \leq i \leq k$ is left-right symmetric. (2) If Λ is an ∞-Gorenstein ring and N is in $\operatorname{mod}\Lambda^{op}$ with $\operatorname{grade} N = k < \infty$, then N is pure of grade k if and only if N can be embedded into a finite direct sum of copies of the $(k+1)$st term in a minimal injective resolution of Λ as a right Λ-module. (3) Assume that both the left and right self-injective dimensions of Λ are k. If $\operatorname{grade}\operatorname{Ext}_{\Lambda}^{k}(M, \Lambda) \geq k$ for any $M \in \operatorname{mod}\Lambda$ and $\operatorname{grade}\operatorname{Ext}_{\Lambda}^{i}(N, \Lambda) \geq i$ for any $N \in \operatorname{mod}\Lambda^{op}$ and $1 \leq i \leq k-1$, then the socle of the last term in a minimal injective resolution of Λ as a right Λ-module is nonzero.

Key Words: Flat dimension; k-Gorenstein rings; k-Syzygy modules; k-Torsionfree modules; Pure modules; Socle; (Strong) Grade of modules.

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1. INTRODUCTION

Throughout this article, Λ is a left and right noetherian ring, and $\operatorname{mod}\Lambda$ denotes the category of finitely generated left Λ-modules. It is well known that the properties of grade of modules are useful to characterize rings as well as to study the dual properties of modules (see, for example, Auslander and Bridger, 1969; Auslander and Reiten, 1996; Björk, 1989; Fossum et al., 1975; Huang, 1999, 2003; Huang and Qin, Preprint). In this article, we study the homological properties of modules over noetherian rings, especially over k-Gorenstein rings and related rings, under some grade conditions of modules.

Let M be in $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Lambda^{op}$). For a non-negative integer t, we say that the *grade* of M is equal to t, denoted by $\operatorname{grade} M = t$, if $\operatorname{Ext}_{\Lambda}^{i}(M, \Lambda) = 0$ for any $0 \leq i < t$ and $\operatorname{Ext}_{\Lambda}^{t}(M, \Lambda) \neq 0$. We say the *strong grade* of M is equal to t, denoted by $\operatorname{s.grade} M = t$, if $\operatorname{grade} A = t$ for each nonzero submodule A of M. Moreover, if $\operatorname{Ext}_{\Lambda}^{i}(M, \Lambda) = 0$ for any $i \geq 0$, then we write $\operatorname{grade} M = \infty$ (see Auslander and Bridger, 1969).

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Address correspondence to Zhaoyong Huang, Department of Mathematics, Nanjing University, 22 Hankou Rd., Nanjing 210093, China; Fax: +86-25-83597130; E-mail: huangzy@nju.edu.cn

For a positive integer k, Auslander and Bridger (1969) introduced the notion of k-torsionfree modules. Such a class of modules is natural and interesting in homological algebra. It was showed in Auslander and Bridger (1969, Theorem 2.17) that a k-torsionfree module is k-syzygy; but the converse is not true in general. It was showed in Auslander and Bridger (1969, Proposition 2.26) that an i-syzygy module is i-torsionfree for any $1 \leq i \leq k$ if and only if $\operatorname{grade} \operatorname{Ext}_\Lambda^{i+1}(M, \Lambda) \geq i$ for any $M \in \operatorname{mod} \Lambda$ and $1 \leq i \leq k-1$. We show in Section 2 that this result is left-right symmetric. Under these equivalent conditions we show that $\operatorname{l.fin.dim} \Lambda \leq k$ if and only if Λ satisfies the condition that $N = 0$ for every $N \in \operatorname{mod} \Lambda^{op}$ satisfying $\operatorname{grade} N \geq k+1$. As a corollary, we get that if the left flat dimension of the ith term in a minimal injective resolution of Λ as a left Λ-module is at most i for all i, then the left self-injective dimension of Λ and its small left finitistic dimension are identical, and the difference between the right self-injective dimension of Λ and its small right finitistic dimension is at most one.

Noncommutative Gorenstein rings are already defined. These are rings for which the (left/right) self-injective dimension of the ring is finite. In addition, recall that Λ is called *k-Gorenstein* if the right flat dimension of the ith term in a minimal injective resolution of Λ as a right Λ-module is at most $i-1$ for any $1 \leq i \leq k$; and Λ is called an ∞-Gorenstein ring if it is k-Gorenstein for all k (see Fossum et al., 1975; Fuller and Iwanaga, 1993). Auslander gave some useful equivalent conditions of k-Gorenstein rings in term of the right flat dimension and grade of modules as follows, which shows that the notion of k-Gorenstein rings is left-right symmetric.

Auslander's Theorem (Fossum et al., 1975, Theorem 3.7). *The following statements are equivalent for* Λ:

(1) Λ *is a k-Gorenstein ring*;
(2) *The left flat dimension of the ith term in a minimal injective resolution of* Λ *as a left* Λ*-module is at most* $i-1$ *for any* $1 \leq i \leq k$;
(3) $\operatorname{s.grade} \operatorname{Ext}_\Lambda^i(M, \Lambda) \geq i$ *for any* $M \in \operatorname{mod} \Lambda$ *and* $1 \leq i \leq k$;
(4) $\operatorname{s.grade} \operatorname{Ext}_\Lambda^i(N, \Lambda) \geq i$ *for any* $N \in \operatorname{mod} \Lambda^{op}$ *and* $1 \leq i \leq k$.

In Section 3 we study the purity of modules over k-Gorenstein rings. Let k be a non-negative integer and Λ a $(k+1)$-Gorenstein ring. For a module $M \in \operatorname{mod} \Lambda^{op}$ with $\operatorname{grade} N = k < \infty$, we show that N is pure of grade k if and only if N can be embedded into a finite direct sum of copies of the $(k+1)$st term in a minimal injective resolution of Λ as a right Λ-module.

Assume that both the left and right self-injective dimensions of Λ are k. If $\operatorname{grade} \operatorname{Ext}_\Lambda^k(M, \Lambda) \geq k$ for any $M \in \operatorname{mod} \Lambda$ and $\operatorname{grade} \operatorname{Ext}_\Lambda^i(N, \Lambda) \geq i$ for any $N \in \operatorname{mod} \Lambda^{op}$ and $1 \leq i \leq k-1$, we show in Section 4 that the socle of the last term in a minimal injective resolution of Λ as a right Λ-module is nonzero. As an immediate result, we have that if Λ is k-Gorenstein with both the left and right self-injective dimensions k, then the socle of the last term in a minimal injective resolution of Λ as a right Λ-module is also nonzero.

In the following, we assume that

$$0 \to \Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$$

is a minimal injective resolution of Λ as a right Λ-module, and that

$$0 \to \Lambda \to I'_0 \to I'_1 \to \cdots \to I'_i \to \cdots$$

is a minimal injective resolution of Λ as a left Λ-module. For a left Λ-module M, $\text{l.fd}_\Lambda M$, $\text{l.pd}_\Lambda M$, and $\text{l.id}_\Lambda M$ denote the left flat dimension, the left projective dimension, and the left injective dimension of M, respectively, and for a right Λ-module N, $\text{r.fd}_\Lambda N$, $\text{r.pd}_\Lambda N$, and $\text{r.id}_\Lambda N$ denote the right flat dimension, the right projective dimension, and the right injective dimension of N, respectively.

2. *k*-TORSIONFREE MODULES AND *k*-SYZYGY MODULES

Let M be in $\text{mod}\,\Lambda$ (resp. $\text{mod}\,\Lambda^{op}$), and let $\sigma_M : M \to M^{**}$ via $\sigma_M(x)(f) = f(x)$ for any $x \in M$ and $f \in M^*$ be the canonical evaluation homomorphism, where $(\ )^* = \text{Hom}_\Lambda(-, \Lambda)$. M is called a *torsionless module* if σ_M is a monomorphism; and M is called a *reflexive module* if σ_M is an isomorphism. For a positive integer k, we call M a *k-syzygy module* if there is an exact sequence

$$0 \to M \to Q_0 \to Q_1 \to \cdots \to Q_{k-1}$$

with all Q_i finitely generated projective. On the other hand, assume that

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

is a resolution of M with P_0 and P_1 finitely generated projective. Then we get an exact sequence

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to X \to 0,$$

where $X = \text{Coker}\, f^*$. M is called a *k-torsionfree module* if $\text{Ext}_\Lambda^i(X, \Lambda) = 0$ for any $1 \leq i \leq k$ (see Auslander and Bridger, 1969). Because X is unique up to projective summands, the notion of k-torsionfree modules is well defined. By Hoshino (1993, Lemma 1.5), we have the exact sequence

$$0 \to \text{Ext}_\Lambda^1(X, \Lambda) \to M \xrightarrow{\sigma_M} M^{**} \to \text{Ext}_\Lambda^2(X, \Lambda) \to 0.$$

So M is 1-torsionfree if and only if it is torsionless if and only if it is 1-syzygy; and that M is 2-torsionfree if and only if it is reflexive. We use $\mathcal{T}^k(\text{mod}\,\Lambda)$ (resp. $\Omega^k(\text{mod}\,\Lambda)$) to denote the full subcategory of $\text{mod}\,\Lambda$ consisting of k-torsionfree modules (resp. k-syzygy modules). It is known from Auslander and Bridger (1969, Theorem 2.17), that $\mathcal{T}^k(\text{mod}\,\Lambda) \subseteq \Omega^k(\text{mod}\,\Lambda)$. So, it is natural to ask when they are identical. Auslander and Bridger (1969) proved the following result.

Proposition 2.1 (Auslander and Bridger, 1969, Proposition 2.26 or Auslander and Reiten, 1996, Proposition 1.6). *For a positive integer k, the following statements are equivalent*:

(1) $\text{grade}\,\text{Ext}_\Lambda^{i+1}(M, \Lambda) \geq i$ *for any* $M \in \text{mod}\,\Lambda$ *and* $1 \leq i \leq k-1$;
(2) $\Omega^i(\text{mod}\,\Lambda) = \mathcal{T}^i(\text{mod}\,\Lambda)$ *for any* $1 \leq i \leq k$.

In this section we show this result is left-right symmetric. To get our result we need two lemmas.

Lemma 2.2 (Hoshino, 1993, Lemma 1.6). *The following statements are equivalent*:

(1) $[\mathrm{Ext}^2_\Lambda(M, \Lambda)]^* = 0$ *for any* $M \in \mathrm{mod}\,\Lambda$;
(2) $[\mathrm{Ext}^2_\Lambda(N, \Lambda)]^* = 0$ *for any* $N \in \mathrm{mod}\,\Lambda^{op}$;
(3) M^* *is reflexive for any* $M \in \mathrm{mod}\,\Lambda$;
(4) N^* *is reflexive for any* $N \in \mathrm{mod}\,\Lambda^{op}$.

Lemma 2.3 (Hoshino, 1993, Lemma 6.2). *Let n be a non-negative integer and $X \in \mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Lambda^{op}$). If* $\mathrm{grade}\,X \geq n$ *and* $\mathrm{grade}\,\mathrm{Ext}^n_\Lambda(X, \Lambda) \geq n+1$, *then* $\mathrm{Ext}^n_\Lambda(X, \Lambda) = 0$.

Theorem 2.4. *For a positive integer k, the following statements are equivalent*:

(1) $\mathrm{grade}\,\mathrm{Ext}^{i+1}_\Lambda(M, \Lambda) \geq i$ *for any* $M \in \mathrm{mod}\,\Lambda$ *and* $1 \leq i \leq k-1$;
(2) $\Omega^i(\mathrm{mod}\,\Lambda) = \mathcal{T}^i(\mathrm{mod}\,\Lambda)$ *for any* $1 \leq i \leq k$;
(3) $\mathrm{grade}\,\mathrm{Ext}^{i+1}_\Lambda(N, \Lambda) \geq i$ *for any* $N \in \mathrm{mod}\,\Lambda^{op}$ *and* $1 \leq i \leq k-1$;
(4) $\Omega^i(\mathrm{mod}\,\Lambda^{op}) = \mathcal{T}^i(\mathrm{mod}\,\Lambda^{op})$ *for any* $1 \leq i \leq k$.

Proof. By Proposition 2.1 and its dual statement, we get the equivalence of (1) and (2) and that of (3) and (4). In the following, we prove (3) implies (1) by induction on k. The case $k = 1$ is trivial. The case $k = 2$ follows from Lemma 2.2. Now suppose $k \geq 3$.

Let $M \in \mathrm{mod}\,\Lambda$ and

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

a projective resolution of M in $\mathrm{mod}\,\Lambda$. Put $M_i = \mathrm{Coker}\,(P_i \to P_{i-1})$ (where $M_1 = M$) and $X_i = \mathrm{Coker}(P^*_{i-1} \to P^*_i)$ for any $i \geq 1$. By the induction hypothesis, we have $\mathrm{grade}\,\mathrm{Ext}^{i+1}_\Lambda(M, \Lambda) \geq i$ for any $1 \leq i \leq k-2$ and $\mathrm{grade}\,\mathrm{Ext}^k_\Lambda(M, \Lambda) \geq k-2$. So it suffices to prove $\mathrm{Ext}^{k-2}_\Lambda(\mathrm{Ext}^k_\Lambda(M, \Lambda), \Lambda) = 0$.

By Proposition 2.1, $\Omega^i(\mathrm{mod}\,\Lambda) = \mathcal{T}^i(\mathrm{mod}\,\Lambda)$ for any $1 \leq i \leq k-1$. Since $M_t \in \Omega^{k-1}(\mathrm{mod}\,\Lambda)$ for any $t \geq k$, $M_t \in \mathcal{T}^{k-1}(\mathrm{mod}\,\Lambda)$ for any $t \geq k$. It follows that $\mathrm{Ext}^i_\Lambda(X_t, \Lambda) = 0$ for any $1 \leq i \leq k-1$ and $t \geq k$.

On the other hand, by Huang (2003, Lemma 2) we have the exact sequence

$$0 \to \mathrm{Ext}^k_\Lambda(M, \Lambda) \to X_k \to P^*_{k+1} \to X_{k+1} \to 0.$$

Put $K = \mathrm{Im}(X_k \to P^*_{k+1})$. From the exactness of $0 \to K \to P^*_{k+1} \to X_{k+1} \to 0$, we know that $\mathrm{Ext}^i_\Lambda(K, \Lambda) = 0$ for any $1 \leq i \leq k-2$ and $\mathrm{Ext}^{k-1}_\Lambda(K, \Lambda) \cong \mathrm{Ext}^k_\Lambda(X_{k+1}, \Lambda)$. Moreover, from the exactness of $0 \to \mathrm{Ext}^k_\Lambda(M, \Lambda) \to X_k \to K \to 0$ we know that $\mathrm{Ext}^{k-1}_\Lambda(K, \Lambda) \cong \mathrm{Ext}^{k-2}_\Lambda(\mathrm{Ext}^k_\Lambda(M, \Lambda), \Lambda)$. So $\mathrm{Ext}^{k-2}_\Lambda(\mathrm{Ext}^k_\Lambda(M, \Lambda), \Lambda) \cong \mathrm{Ext}^k_\Lambda(X_{k+1}, \Lambda)$. By (3) we then have $\mathrm{grade}\,\mathrm{Ext}^{k-2}_\Lambda(\mathrm{Ext}^k_\Lambda(M, \Lambda), \Lambda) = \mathrm{grade}\,\mathrm{Ext}^k_\Lambda(X_{k+1}, \Lambda) \geq k-1$. It follows from Lemma 2.3 that $\mathrm{Ext}^{k-2}_\Lambda(\mathrm{Ext}^k_\Lambda(M, \Lambda), \Lambda) = 0$. Dually, we have (1) implies (3). $\square$

Corollary 2.5. *The following statements are equivalent*:

(1) $\operatorname{grade}\operatorname{Ext}_\Lambda^{i+1}(M,\Lambda) \geq i$ *for any* $M \in \operatorname{mod}\Lambda$ *and* $i \geq 1$;
(2) $\Omega^i(\operatorname{mod}\Lambda) = \mathcal{T}^i(\operatorname{mod}\Lambda)$ *for any* $i \geq 1$;
(3) $\operatorname{grade}\operatorname{Ext}_\Lambda^{i+1}(N,\Lambda) \geq i$ *for any* $N \in \operatorname{mod}\Lambda^{op}$ *and* $i \geq 1$;
(4) $\Omega^i(\operatorname{mod}\Lambda^{op}) = \mathcal{T}^i(\operatorname{mod}\Lambda^{op})$ *for any* $i \geq 1$.

Remark. (1) We remark that if $\mathrm{r.fd}_\Lambda I_i \leq i+1$ for any $0 \leq i \leq k-2$ (especially, if Λ is $(k-1)$-Gorenstein), then the condition (1) in Theorem 2.4 is satisfied by Auslander and Reiten (1996, Proposition 2.2).

(2) Consider the following grade conditions.

(a_k) $\operatorname{s.grade}\operatorname{Ext}_\Lambda^i(M,\Lambda) \geq i$ for any $M \in \operatorname{mod}\Lambda$ and $1 \leq i \leq k$.
(b_k) $\operatorname{grade}\operatorname{Ext}_\Lambda^i(M,\Lambda) \geq i$ for any $M \in \operatorname{mod}\Lambda$ and $1 \leq i \leq k$.
(c_k) $\operatorname{s.grade}\operatorname{Ext}_\Lambda^{i+1}(M,\Lambda) \geq i$ for any $M \in \operatorname{mod}\Lambda$ and $1 \leq i \leq k$.
(d_k) $\operatorname{grade}\operatorname{Ext}_\Lambda^{i+1}(M,\Lambda) \geq i$ for any $M \in \operatorname{mod}\Lambda$ and $1 \leq i \leq k$.

About the symmetry of these conditions, the following facts are known: Both (a_k) and (d_k) are left-right symmetric by Auslander's Theorem and Theorem 2.4, respectively. Neither (b_k) nor (c_k) are left-right symmetric by Huang (1999, Theorem 3.3 and p. 1460 "Remark"). On the other hand, we clearly have the implications $(a_k) \Rightarrow (b_k) \Rightarrow (d_k)$ and $(a_k) \Rightarrow (c_k) \Rightarrow (d_k)$. However, by the above argument, none of these implications can be reversed.

Corollary 2.6. *Let Λ be a left and a right artinian ring. If one of the equivalent conditions in Theorem* 2.4 *is satisfied for Λ, then* $\mathrm{l.id}_\Lambda\Lambda \leq k$ *if and only if* $\mathrm{r.id}_\Lambda\Lambda \leq k$. *In particular, if one of the equivalent conditions in Corollary* 2.5 *is satisfied for Λ, then* $\mathrm{l.id}_\Lambda\Lambda = \mathrm{r.id}_\Lambda\Lambda$.

Proof. By assumption, we have $\operatorname{grade}\operatorname{Ext}_\Lambda^k(M,\Lambda) \geq k-1$ for any $M \in \operatorname{mod}\Lambda$. If $\mathrm{r.id}_\Lambda\Lambda \leq k$, then $\mathrm{l.id}_\Lambda\Lambda \leq 2k-1$ by Huang (2003, Theorem). It follows from Zaks (1969, Lemma A) that $\mathrm{l.id}_\Lambda\Lambda = \mathrm{r.id}_\Lambda\Lambda(\leq k)$. The converse is proved dually. The latter assertion follows easily from the former one. □

Recall that the *small left finitistic dimension of* Λ, written $\mathrm{l.fin.dim}\,\Lambda$, is defined to be $\sup\{\mathrm{l.pd}_\Lambda M \mid M$ is in $\operatorname{mod}\Lambda$ with $\mathrm{l.pd}_\Lambda M < \infty\}$. The *small right finitistic dimension* of Λ is defined dually and is denoted by $\mathrm{r.fin.dim}\,\Lambda$. As an application of Theorem 2.4, we will prove the following theorem.

Theorem 2.7. *Let k be a non-negative integer. If one of the equivalent conditions in Theorem* 2.4 *is satisfied for Λ, then the following statements are equivalent*:

(1) $\mathrm{l.fin.dim}\,\Lambda \leq k$;
(2) $N = 0$ *for every* $N \in \operatorname{mod}\Lambda^{op}$ *satisfying* $\operatorname{grade} N \geq k+1$.

To prove this theorem, we need some lemmas.

Let n be a positive integer and

$$P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \to A \to 0$$

a projective resolution of A in $\operatorname{mod}\Lambda$ (resp. $\operatorname{mod}\Lambda^{op}$). Put $X = \operatorname{Coker} d_n^*$.

Lemma 2.8 (Hoshino, 1993, Lemma 1.5). *Let A and X be as above. If $\mathrm{Ext}^i_\Lambda(A, \Lambda) = 0$ for any $1 \le i \le n-1$, then we have the following exact sequence*:

$$0 \to \mathrm{Ext}^n_\Lambda(X, \Lambda) \to A \xrightarrow{\sigma_A} A^{**} \to \mathrm{Ext}^{n+1}_\Lambda(X, \Lambda) \to 0.$$

In particular, when $n = 1$, we obtain the exact sequence

$$0 \to \mathrm{Ext}^1_\Lambda(X, \Lambda) \to A \xrightarrow{\sigma_A} A^{**} \to \mathrm{Ext}^2_\Lambda(X, \Lambda) \to 0,$$

where $X = \mathrm{Coker}\, d_1^*$.

Lemma 2.9. $\mathrm{l.fin.dim}\,\Lambda = 0$ *if and only if $N = 0$ for every $N \in \mathrm{mod}\,\Lambda^{op}$ satisfying $N^* = 0$.*

Proof. By Bass (1960, Corollary 5.6 and Theorem 5.4). □

Lemma 2.10. $\mathrm{l.fin.dim}\,\Lambda \le 1$ *if and only if $N = 0$ for every $N \in \mathrm{mod}\,\Lambda^{op}$ satisfying $N^* = 0 = \mathrm{Ext}^1_\Lambda(N, \Lambda)$.*

Proof. The necessity. Let N be in $\mathrm{mod}\,\Lambda^{op}$ with $N^* = 0 = \mathrm{Ext}^1_\Lambda(N, \Lambda)$ and

$$0 \to K \to P \to N \to 0$$

an exact sequence in $\mathrm{mod}\,\Lambda^{op}$ with P projective. Then $K^*(\cong P^*)$ is projective. By Bass (1960, Proposition 5.3), K is projective. It then follows from Lemma 2.3 that we have an exact sequence

$$0 \to \mathrm{Ext}^1_\Lambda(\mathrm{Ext}^1_\Lambda(N, \Lambda), \Lambda) \to N \xrightarrow{\sigma_N} N^{**}.$$

But $N^* = 0 = \mathrm{Ext}^1_\Lambda(N, \Lambda)$, so $N = 0$.

The sufficiency. It suffices to show that if there is an exact sequence

$$0 \to P_1 \to P_0 \to M \to 0$$

in $\mathrm{mod}\,\Lambda$ with P_0 and P_1 projective and M torsionless, then M is projective. Put $N = \mathrm{Coker}(P_0^* \to P_1^*)$. Then $N^* = 0$ since P_0 and P_1 are reflexive. By Lemma 2.8 we have that $\mathrm{Ext}^1_\Lambda(N, \Lambda) \cong \mathrm{Ker}\,\sigma_M = 0$. Then $N = 0$ by assumption. So M^* and M^{**} are projective. On the other hand, we have an exact sequence

$$0 \to P_1^{**} \to P_0^{**} \to M^{**} \to 0.$$

Thus M is reflexive, and therefore it is projective. □

The next lemma finishes the proof of Theorem 2.7 in one direction.

Lemma 2.11. *Let k be a non-negative integer. If* $\mathrm{l.fin.dim}\,\Lambda \le k$, *then $N = 0$ for every $N \in \mathrm{mod}\,\Lambda^{op}$ satisfying* $\mathrm{grade}\, N \ge k+1$.

Proof. The cases $k = 0$ and $k = 1$ have been proved in Lemmas 2.9 and 2.10, respectively. Now suppose $k \geq 2$.

Let N be in $\operatorname{mod} \Lambda^{op}$ with $\operatorname{grade} N \geq k+1$ and

$$\cdots \to Q_{k+1} \to Q_k \to \cdots \to Q_1 \to Q_0 \to N \to 0$$

a projective resolution of N in $\operatorname{mod} \Lambda^{op}$. Put $Y = \operatorname{Coker}(Q_k^* \to Q_{k+1}^*)$. Then $\mathrm{l.pd}_\Lambda Y \leq k+1$. By assumption $\mathrm{l.fin.dim}\, \Lambda \leq k$, hence $\mathrm{l.pd}_\Lambda Y \leq k$. On the other hand, by Lemma 2.8 we have that $N \cong \operatorname{Ext}_\Lambda^{k+1}(Y, \Lambda)$. It follows that $N = 0$. $\square$

We now recall some notions from Jans (1963). A monomorphism $X^{**} \xrightarrow{\rho^*} Y^*$ in $\operatorname{mod} \Lambda$ (resp. $\operatorname{mod} \Lambda^{op}$) is called a *double dual embedding* if it is the dual of an epimorphism $Y \xrightarrow{\rho} X^*$ in $\operatorname{mod} \Lambda^{op}$ (resp. $\operatorname{mod} \Lambda$). For a positive integer k, a torsionless module T_k in $\operatorname{mod} \Lambda$ (resp. $\operatorname{mod} \Lambda^{op}$) is said to be of *D-class k* if it can be fitted into a diagram of the form

$$\begin{tikzcd}
 & & & & 0 \arrow[r] & T_{k-1}^{**} \arrow[r] & P_{k-1} \arrow[r] & T_k \arrow[r] & 0 \\
 & & & \cdots \arrow[r] & P_{k-2} \arrow[r] & T_{k-1} \arrow[r] \arrow[u, "\sigma_{T_{k-1}}"'] & 0 & & \\
 & & & \cdots & & & & & \\
 & & 0 \arrow[r] & T_2^{**} \arrow[r] & \cdots & & & & \\
0 \arrow[r] & T_1^{**} \arrow[r] & P_1 \arrow[r] & T_2 \arrow[r] \arrow[u, "\sigma_{T_2}"'] & 0 & & & &
\end{tikzcd}$$

where each P_i is projective in $\operatorname{mod} \Lambda$ (resp. $\operatorname{mod} \Lambda^{op}$) and the horizontal monomorphisms are double dual embeddings. Any torsionless module is said to be of *D-class 1*.

Lemma 2.12 (Jans, 1963, Theorem 4.2). *For a positive integer k, the following statements are equivalent*:

(1) $\mathrm{l.fin.dim}\, \Lambda \leq k$;
(2) *The only modules of D-class k in* $\operatorname{mod} \Lambda^{op}$ *with projective duals are the projective modules.*

Lemma 2.13. *Assume that* Λ *satisfies one of the equivalent conditions in Theorem* 2.4 *and let k be a non-negative integer. If* Λ *in addition satisfies the condition that* $N = 0$ *for every* $N \in \operatorname{mod} \Lambda^{op}$ *satisfying* $\operatorname{grade} N \geq k+1$, *then* $\mathrm{l.fin.dim}\, \Lambda \leq k$.

Proof. By Lemmas 2.9 and 2.10, we only need to prove the case $k \geq 2$.

Let T_k be a module of D-class k in $\operatorname{mod} \Lambda^{op}$ with T_k^* projective (where $k \geq 2$). From the proof of Jans (1963, Theorem 2.1) we know that there are exact sequences

$$0 \to T_i^{**} \xrightarrow{\rho_i^*} P_i^* \xrightarrow{\pi_i^*} T_{i+1} \to 0 \qquad \text{(i)}^*$$

$$0 \to T_{i+1}^* \xrightarrow{\sigma_{P_i}^{-1}\pi_i^*} P_i \xrightarrow{\rho_i} T_i^* \to 0 \tag{i}$$

for any $1 \leq i \leq k-1$, where all P_i are projective in $\bmod \Lambda$ and all T_i are of D-class i in $\bmod \Lambda^{op}$. Then we have an exact sequence

$$0 \to T_{k-1}^* \to P_{k-2} \to \cdots \to P_2 \to P_1 \to T_1^* \to 0.$$

It is trivial that T_1^* is 2-syzygy. So T_{k-1}^* is k-syzygy and hence it is k-torsionfree by Theorem 2.4.

The exact sequence $(k-1)$ induces an exact sequence

$$0 \to T_{k-1}^{**} \xrightarrow{\rho_{k-1}^*} P_{k-1}^* \xrightarrow{\pi_{k-1}^*(\sigma_{P_{k-1}}^{-1})^*} T_k^{**} \to N \to 0, \tag{\dagger}$$

where $N = \mathrm{Coker}(\pi_{k-1}^*(\sigma_{P_{k-1}}^{-1})^*)$. Since T_k^* is projective and T_{k-1}^* is k-torsionfree, $\mathrm{Ext}_\Lambda^i(N, \Lambda) = 0$ for any $1 \leq i \leq k$. On the other hand, notice that T_k^* and P_{k-1} are reflexive, it then follows that $N^* = 0$ and so $\mathrm{grade}\, N \geq k+1$. Thus, by assumption, we have that $N = 0$, and the exact sequence (†) splits.

By Anderson and Fuller (1992, Proposition 20.14), we have $(\sigma_{P_{k-1}}^{-1})^* = (\sigma_{P_{k-1}}^*)^{-1} = \sigma_{P_{k-1}^*}$. We in addition note that $\sigma_{T_k}\pi_{k-1} = \pi_{k-1}^{**}\sigma_{P_{k-1}^*}$. So we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & T_{k-1}^{**} \arrow[r, "\rho_{k-1}^*"] \arrow[d, equal] & P_{k-1}^* \arrow[r, "\pi_{k-1}"] \arrow[d, equal] & T_k \arrow[r] \arrow[d, "\sigma_{T_k}"] & 0 \\
0 \arrow[r] & T_{k-1}^{**} \arrow[r, "\rho_{k-1}^*"] & P_{k-1}^* \arrow[r, "\pi_{k-1}^{**}(\sigma_{P_{k-1}}^{-1})^*"] & T_k^{**} \arrow[r] & 0.
\end{tikzcd}$$

Then it is trivial that σ_{T_k} is an isomorphism, and so T_k is projective. It follows from Lemma 2.12 that $\mathrm{l.fin.dim}\, \Lambda \leq k$. □

Now Theorem 2.7 follows from Lemmas 2.11 and 2.13.

It was showed in Huang and Qin (Preprint, Theorem 2.12) that for a $(k+1)$-Gorenstein ring Λ, if $\mathrm{l.fin.dim}\, \Lambda = k$, then $\mathrm{l.id}_\Lambda \Lambda \leq k$. The following corollary generalizes this result.

Corollary 2.14. *Let k be a non-negative integer. If* $\mathrm{l.fd}_\Lambda I_i' \leq i+1$ *for any* $0 \leq i \leq k$, *then we have*:

(1) $\mathrm{l.id}_\Lambda \Lambda = k$ *if and only if* $\mathrm{l.fin.dim}\, \Lambda = k$;
(2) $k \leq \mathrm{r.id}_\Lambda \Lambda \leq k+1$ *if* $\mathrm{r.fin.dim}\, \Lambda = k$.

In particular, if Λ *is* $(k+1)$*-Gorenstein, then* $\mathrm{l.id}_\Lambda \Lambda = k$ *if and only if* $\mathrm{l.fin.dim}\, \Lambda = k$, *and* $\mathrm{r.id}_\Lambda \Lambda = k$ *if and only if* $\mathrm{r.fin.dim}\, \Lambda = k$.

Proof. It is well known that $\mathrm{l.id}_\Lambda \Lambda \geq \mathrm{l.fin.dim}\, \Lambda$ and $\mathrm{r.id}_\Lambda \Lambda \geq \mathrm{r.fin.dim}\, \Lambda$.

Assume that $\mathrm{l.fd}_\Lambda I_i' \leq i+1$ for any $0 \leq i \leq k$. Then, by the dual statement of Auslander and Reiten (1996, Theorem 4.7), we have that $\mathrm{grade}\, \mathrm{Ext}_\Lambda^i(M, \Lambda) \geq$

i and s.grade $\mathrm{Ext}_{\Lambda}^{i+1}(N, \Lambda) \geq i$ for any $M \in \mathrm{mod}\, \Lambda$, $N \in \mathrm{mod}\, \Lambda^{op}$ and $1 \leq i \leq k+1$. If l.fin.dim $\Lambda = k$, then $\mathrm{Ext}_{\Lambda}^{k+1}(M, \Lambda) = 0$ by Theorem 2.7, hence l.id$_{\Lambda}\Lambda \leq k$. On the other hand, if r.fin.dim $\Lambda = k$, then $\mathrm{Ext}_{\Lambda}^{k+2}(M, \Lambda) = 0$ by the dual statement of Theorem 2.7, hence r.id$_{\Lambda}\Lambda \leq k+1$.

In particular, if Λ is $(k+1)$-Gorenstein, then, by Auslander's Theorem and Theorem 2.7, we get our conclusion similarly. □

Corollary 2.15. *If* l.fd$_{\Lambda}I_i' \leq i+1$ *for any* $i \geq 0$, *then* l.id$_{\Lambda}\Lambda =$ l.fin.dim Λ *and* r.fin.dim $\Lambda \leq$ r.id$_{\Lambda}\Lambda \leq$ r.fin.dim $\Lambda + 1$. *In particular, if* Λ *is an* ∞*-Gorenstein ring, then* l.id$_{\Lambda}\Lambda =$ l.fin.dim Λ *and* r.id$_{\Lambda}\Lambda =$ r.fin.dim Λ.

3. FLAT DIMENSION AND GRADE OF MODULES

Recall that Λ has *dominant dimension* at least k, written dom.dim $\Lambda \geq k$, if I_i is flat for any $0 \leq i \leq k-1$. We write dom.dim $\Lambda = \infty$ if I_i is flat for all i. In addition, we denote $K_i = \mathrm{Ker}(I_i \to I_{i+1})$ for any $i \geq 0$.

Proposition 3.1. dom.dim $\Lambda = \infty$ *if and only if* s.grade $\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda) = \infty$ *for any* $M \in \mathrm{mod}\, \Lambda$.

Proof. The Sufficiency. Assume that s.grade $\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda) = \infty$ for any $M \in \mathrm{mod}\, \Lambda$.

We will prove that $\mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), I_i) = 0$ for any $M \in \mathrm{mod}\, \Lambda$ and $i \geq 0$ by using induction on i. We first claim that $\mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), I_0) = 0$. Otherwise, there is a nonzero homomorphism $f: \mathrm{Ext}_{\Lambda}^{1}(M, \Lambda) \to I_0$. Then $\mathrm{Im}\, f \cap \Lambda \neq 0$ since Λ is essential in I_0. So there is a submodule $f^{-1}(\mathrm{Im}\, f \cap \Lambda)$ of $\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda)$ such that $\mathrm{Hom}_{\Lambda}(f^{-1}(\mathrm{Im}\, f \cap \Lambda), \Lambda) \neq 0$, which contradicts that s.grade $\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda) = \infty$. Thus we conclude that $\mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), I_0) = 0$. Now suppose $i \geq 1$. Consider the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), K_{i-1}) \to \mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), I_{i-1}) \\ &\to \mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), K_i) \to \mathrm{Ext}_{\Lambda}^{i}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), \Lambda) \to 0 \end{aligned}$$

for any $i \geq 1$. Since s.grade $\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda) = \infty$, $\mathrm{Ext}_{\Lambda}^{i}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), \Lambda) = 0$. On the other hand, by the induction hypothesis we have $\mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), I_{i-1}) = 0$. So, by the above exact sequence, we have $\mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), K_i) = 0$. By using an argument similar to the proof of the case $i = 0$ we then get that $\mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), I_i) = 0$. The assertion is proved.

By Cartan and Eilenberg (1999, Chapter VI, Proposition 5.3), we have that $\mathrm{Tor}_1^{\Lambda}(I_i, M) \cong \mathrm{Hom}_{\Lambda}(\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda), I_i) = 0$ for any $M \in \mathrm{mod}\, \Lambda$, which implies that I_i is flat for any $i \geq 0$.

The Necessity. If dom.dim $\Lambda = \infty$, then Λ is an ∞-Gorenstein ring and s.grade $\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda) \geq 1$ for any $M \in \mathrm{mod}\, \Lambda$. Let X be a submodule of $\mathrm{Ext}_{\Lambda}^{1}(M, \Lambda)$. Then grade $X \geq 1$. Consider the exact sequence

$$0 \to \mathrm{Hom}_{\Lambda}(X, K_i) \to \mathrm{Hom}_{\Lambda}(X, I_i) \to \mathrm{Hom}_{\Lambda}(X, K_{i+1}) \to \mathrm{Ext}_{\Lambda}^{i+1}(X, \Lambda) \to 0$$

for any $i \geq 0$. Since dom.dim $\Lambda = \infty$, I_i is flat for any $i \geq 0$. By Berrick and Keating (2000, Theorem 5.2.7), each I_i is a direct limit of free modules in mod Λ^{op}.

So $\mathrm{Hom}_\Lambda(X, I_i) = 0$ and hence $\mathrm{Hom}_\Lambda(X, K_i) = 0$. By the exactness of the above sequence, we have $\mathrm{Ext}^i_\Lambda(X, \Lambda) = 0$ for any $i \geq 0$ and grade $X = \infty$. $\square$

The generalized Nakayama conjecture has an equivalent version as follows: grade $S < \infty$ for any simple module S in mod Λ^{op} over an artin algebra Λ (see Auslander and Reiten, 1975); and the strong Nakayama conjecture says that grade $N = \infty$ implies $N = 0$ for any N in mod Λ^{op} (see Colby and Fuller, 1990). It is clear that the generalized Nakayama conjecture is a special case of the strong Nakayama conjecture. The following result shows that these conjectures are equivalent if the right flat dimension of each I_i is finite.

Proposition 3.2. *If* $\mathrm{r.fd}_\Lambda I_i < \infty$ *for all i, then the following statements are equivalent*:

(1) grade $N = \infty$ *implies* $N = 0$ *for any* N *in* mod Λ^{op};
(2) grade $S < \infty$ *for any simple module* S *in* mod Λ^{op};
(3) $\bigoplus_{i\geq 0} I_i$ *is an injective cogenerator for the category of right* Λ*-modules.*

Proof. (1) $\Rightarrow$ (2) It is trivial.

(2) $\Rightarrow$ (3) Let S be a simple module in mod Λ^{op}. By (2), grade $S < \infty$ and there is a non-negative integer t such that $\mathrm{Ext}^t_\Lambda(S, \Lambda) \neq 0$.

Consider the exact sequences

$$0 \to \mathrm{Hom}_\Lambda(S, K_i) \to \mathrm{Hom}_\Lambda(S, I_i)$$

and

$$\mathrm{Hom}_\Lambda(S, K_i) \to \mathrm{Ext}^i_\Lambda(S, \Lambda) \to 0,$$

where $K_i = \mathrm{Ker}(I_i \to I_{i+1})$ for any $i \geq 0$. Since $\mathrm{Ext}^t_\Lambda(S, \Lambda) \neq 0$, $\mathrm{Hom}_\Lambda(S, K_t) \neq 0$. So $\mathrm{Hom}_\Lambda(S, I_t) \neq 0$ and $\mathrm{Hom}_\Lambda(S, \bigoplus_{i\geq 0} I_i) \neq 0$. It then follows from Anderson and Fuller (1992, Proposition 18.15) that $\bigoplus_{i\geq 0} I_i$ is an injective cogenerator for the category of right Λ-modules.

(3) $\Rightarrow$ (1) Let $N \in$ mod Λ^{op} with grade $N = \infty$ and

$$\cdots \to Q_i \to \cdots \to Q_1 \to Q_0 \to N \to 0$$

a projective resolution of N in mod Λ^{op}. Put $X_i = \mathrm{Coker}(Q^*_{i-1} \to Q^*_i)$ for any $i \geq 1$. By Lemma 2.8, we have $N \cong \mathrm{Ext}^i_\Lambda(X_i, \Lambda)$ for any $i \geq 1$.

Without loss of generality, we assume that $\mathrm{r.fd}_\Lambda I_i = n_i (<\infty)$ for any $i \geq 0$. It follows from Cartan and Eilenberg (1999, Chapter VI, Proposition 5.3) that $\mathrm{Hom}_\Lambda(N, I_i) \cong \mathrm{Hom}_\Lambda(\mathrm{Ext}^{n_i+1}_\Lambda(X_{n_i+1}, \Lambda), I_i) \cong \mathrm{Tor}^\Lambda_{n_i+1}(I_i, X_{n_i+1}) = 0$ for any $i \geq 0$. So $\mathrm{Hom}_\Lambda(N, \bigoplus_{i\geq 0} I_i) = 0$. However, $\bigoplus_{i\geq 0} I_i$ is an injective cogenerator for the category of right Λ-modules. We then conclude that $N = 0$. $\square$

The famous Nakayama conjecture says that an artin algebra Λ is self-injective if dom.dim $\Lambda = \infty$. From Auslander and Reiten (1975) we know that the generalized Nakayama conjecture implies the Nakayama conjecture. By Propositions 3.1 and 3.2, we give here a simple proof of this implication. Assume that the generalized

Nakayama conjecture is true. If dom.dim $\Lambda = \infty$, then s.grade $\mathrm{Ext}^1_\Lambda(M, \Lambda) = \infty$ for any $M \in \mathrm{mod}\,\Lambda$ by Proposition 3.1. It follows from Proposition 3.2 that $\mathrm{Ext}^1_\Lambda(M, \Lambda) = 0$ for any $M \in \mathrm{mod}\,\Lambda$, and Λ is self-injective.

In the rest of this section, we study the properties of pure modules. Let k be a non-negative integer. A nonzero module M in $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{mod}\,\Lambda^{op}$) is said to be *pure* of grade k if grade $A = k$ for each nonzero submodule A of M. The notion of pure modules here coincides with that given in Björk (1989) when Λ is Auslander–Gorenstein.

Lemma 3.3. *Let Λ be a 1-Gorenstein ring. Then a module N in* $\mathrm{mod}\,\Lambda^{op}$ *is pure of grade* 0 *if and only if it is torsionless.*

Proof. If N is torsionless, then each nonzero submodule A of N is also torsionless and so $A^* \neq 0$, that is, grade $A = 0$. Conversely, assume that N is pure of grade 0. By Lemma 2.8, there is a module $A \in \mathrm{mod}\,\Lambda$ such that $\mathrm{Ker}\,\sigma_N \cong \mathrm{Ext}^1_\Lambda(M, \Lambda)$. Since Λ is 1-Gorenstein, $[\mathrm{Ext}^1_\Lambda(M, \Lambda)]^* = 0$. So $\mathrm{Ext}^1_\Lambda(M, \Lambda) = 0$ since N is pure of grade 0. It follows that σ_N is a monomorphism and N is torsionless. □

Remark. The proof of Lemma 3.3 in fact proves the following more general result. If $[\mathrm{Ext}^1_\Lambda(M, \Lambda)]^* = 0$ for any $M \in \mathrm{mod}\,\Lambda$, then a module N in $\mathrm{mod}\,\Lambda^{op}$ is pure of grade 0 if and only if it is torsionless. So, by the dual statement of Auslander and Reiten (1996, Theorem 4.7) we have that if Λ is a noetherian algebra with $\mathrm{l.fd}_\Lambda I'_0 \leq 1$, then a module N in $\mathrm{mod}\,\Lambda^{op}$ is pure of grade 0 if and only if it is torsionless.

Lemma 3.4. *Let k be a positive integer and $m \geq -1$ an integer. Then the following statements are equivalent*:

(1) $\mathrm{r.fd}_\Lambda \bigoplus_{i=0}^{k-1} I_i \leq k + m$;
(2) s.grade $\mathrm{Ext}^{k+m+1}_\Lambda(M, \Lambda) \geq k$ *for any* $M \in \mathrm{mod}\,\Lambda$.

Proof. It was proved in Huang (1999, Theorem 2.8) in case m is a non-negative integer. When $m = -1$, the proof is similar to that of Huang (1999, Theorem 2.8), so we omit it. □

The following lemma is a dual statement of Lemma 3.4.

Lemma 3.5. *Let k be a positive integer and $m \geq -1$ an integer. Then the following statements are equivalent*:

(1) $\mathrm{l.fd}_\Lambda \bigoplus_{i=0}^{k-1} I'_i \leq k + m$;
(2) s.grade $\mathrm{Ext}^{k+m+1}_\Lambda(N, \Lambda) \geq k$ *for any* $N \in \mathrm{mod}\,\Lambda^{op}$.

We are now in a position to give the main result in this section.

Theorem 3.6. *Let k be a non-negative integer and N in* $\mathrm{mod}\,\Lambda^{op}$ *with* grade $N = k < \infty$.

(1) *If* $\mathrm{l.fd}_\Lambda \bigoplus_{i=0}^{k} I'_i \leq k$ *and N is pure of grade k, then N can be embedded into a finite direct sum of copies of I_k.*

(2) *If* $\mathrm{r.fd}_\Lambda \bigoplus_{i=0}^{k-1} I_i \leq k-1$, $\mathrm{r.fd}_\Lambda I_k \leq k$, *and* N *can be embedded into a finite direct sum of copies of* I_k, *then* N *is pure of grade* k.

Proof. The case $k = 0$ follows from Lemma 3.3. Now suppose $k \geq 1$.

(1) Also put $K_i = \mathrm{Ker}(I_i \to I_{i+1})$ for any $i \geq 0$. Since N is pure of grade k, $\mathrm{grade}\, X = k$ for any submodule X of N. Then it is not difficult to see that $\mathrm{Hom}_\Lambda(N, I_i) = 0$ for any $0 \leq i \leq k-1$ and so $\mathrm{Ext}_\Lambda^k(N, \Lambda) \cong \mathrm{Hom}_\Lambda(N, K_k)$.

Let $\eta_1, \eta_2, \ldots, \eta_n$ be a set of generators of $\mathrm{Hom}_\Lambda(N, K_k)$ in $\mathrm{End}_\Lambda(K_k)$. Put $\eta = (\eta_1, \eta_2, \ldots, \eta_n)' : N \to K_k^{(n)}$, $U = \mathrm{Ker}\, \eta$ and $V = \mathrm{Im}\, \eta$. We use $\pi : N \to V$ to denote the natural epimorphism. Since $\mathrm{grade}\, N = k$ and $\mathrm{grade}\, U \geq k$ by assumption, $\mathrm{grade}\, V \geq k$. By using an argument similar to the above, we then have $\mathrm{Ext}_\Lambda^k(V, \Lambda) \cong \mathrm{Hom}_\Lambda(V, K_k)$. In addition, it is not difficult to see that $\mathrm{Hom}_\Lambda(\eta, K_k)$ is an epimorphism, so $\mathrm{Hom}_\Lambda(\pi, K_k) : \mathrm{Hom}_\Lambda(V, K_k) \to \mathrm{Hom}_\Lambda(N, K_k)$ is also an epimorphism and hence an isomorphism.

On the other hand, we have an exact sequence

$$0 = \mathrm{Ext}_\Lambda^{k-1}(U, \Lambda) \to \mathrm{Ext}_\Lambda^k(V, \Lambda) \xrightarrow{\mathrm{Ext}_\Lambda^k(\pi, \Lambda)} \mathrm{Ext}_\Lambda^k(N, \Lambda) \to \mathrm{Ext}_\Lambda^k(U, \Lambda) \to \mathrm{Ext}_\Lambda^{k+1}(V, \Lambda).$$

So $\mathrm{Ext}_\Lambda^k(\pi, \Lambda)$ is an isomorphism and $\mathrm{Ext}_\Lambda^k(U, \Lambda)$ is isomorphic to a submodule of $\mathrm{Ext}_\Lambda^{k+1}(V, \Lambda)$. Since $\mathrm{l.fd}_\Lambda \bigoplus_{i=0}^{k} I_i' \leq k$, $\mathrm{grade}\, \mathrm{Ext}_\Lambda^k(U, \Lambda) \geq k+1$ by Lemma 3.5. It then follows from Lemma 2.3 that $\mathrm{Ext}_\Lambda^k(U, \Lambda) = 0$ and $\mathrm{grade}\, U \geq k+1$, which implies that $U = 0$ and η is a monomorphism.

(2) Since $\mathrm{grade}\, N = k$, by Lemma 2.8, there is an $X \in \mathrm{mod}\, \Lambda$ such that $N \cong \mathrm{Ext}_\Lambda^k(X, \Lambda)$. On the other hand, notice that $\mathrm{r.fd}_\Lambda \bigoplus_{i=0}^{k-1} I_i \leq k-1$, so $\mathrm{s.grade}\, N = \mathrm{s.grade}\, \mathrm{Ext}_\Lambda^k(X, \Lambda) \geq k$ by Lemma 3.4.

Let U be a submodule of N. Then $\mathrm{grade}\, U \geq k$. If $\mathrm{grade}\, U \geq k+1$, then, again by Lemma 2.8, there is a $Y \in \mathrm{mod}\, \Lambda$ such that $U \cong \mathrm{Ext}_\Lambda^{k+1}(Y, \Lambda)$. It follows from Cartan and Eilenberg (1999, Chapter VI, Proposition 5.3) that $\mathrm{Hom}_\Lambda(U, I_k) \cong \mathrm{Hom}_\Lambda(\mathrm{Ext}_\Lambda^{k+1}(Y, \Lambda), I_k) \cong \mathrm{Tor}_{k+1}^\Lambda(I_k, Y) = 0$ since $\mathrm{r.fd}_\Lambda I_k \leq k$. However, U can be embedded into a finite direct sum of copies of I_k, so $U = 0$. This completes the proof. $\square$

Especially, we have the following theorem.

Theorem 3.7. *Let* k *be a non-negative integer and* Λ *a* $(k+1)$*-Gorenstein ring. Then the following statements are equivalent for a module* $N \in \mathrm{mod}\, \Lambda^{op}$ *with* $\mathrm{grade}\, N = k < \infty$:

(1) N *is pure of grade* k;
(2) N *can be embedded into a finite direct sum of copies of* I_k.

Corollary 3.8 (Hoshino, 1993, Theorem 6.3). *Let* I *be an indecomposable injective right* Λ*-module with* $\mathrm{r.fd}_\Lambda I = k < \infty$. *If* $\mathrm{r.fd}_\Lambda \bigoplus_{i=0}^{k-1} I_i \leq k-1$ *and* $\mathrm{l.fd}_\Lambda \bigoplus_{i=0}^{k} I_i' \leq k$, *then* I *appears as a direct summand of* I_k.

Proof. We first prove the case $k = 0$. Let $0 \neq X$ be a finitely generated submodule of I. Then I is the injective envelope of X. By Lazard (1969, Theorem 1.2), X is torsionless. So X can be embedded into a finite direct sum of copies of Λ, and hence I can be embedded into a finite direct sum of copies of I_0, which yields that I is isomorphic to a direct summand of I_0.

Now suppose $k \geq 1$. Put $E = \bigoplus_{i=0}^{k-1} I_i$. Then $\text{r.fd}_\Lambda E \leq k-1$. Notice that $\text{r.fd}_\Lambda I = k$, so E does not cogenerate I and hence there is a submodule X of I such that $\text{Hom}_\Lambda(X, E) = 0$. We may assume that X is finitely generated. Let Y be a submodule of X. Clearly, we have $\text{Hom}_\Lambda(Y, E) = 0$. It follows that $\text{grade} Y \geq k$. If $\text{grade}\, Y \geq k+1$, then by Lemma 2.8, there is a module $M \in \text{mod}\,\Lambda$ such that $Y \cong \text{Ext}_\Lambda^{k+1}(M, \Lambda)$. Thus, by Cartan and Eilenberg (1999, Chapter VI, Proposition 5.3), we have $\text{Hom}_\Lambda(Y, I) \cong \text{Hom}_\Lambda(\text{Ext}_\Lambda^{k+1}(M, \Lambda), I) \cong \text{Tor}_{k+1}^\Lambda(I, M) = 0$ since $\text{r.fd}_\Lambda I = k$, which is a contradiction. Therefore, X is pure of grade k. By Theorem 3.6, X, and hence I, can be embedded into a finite direct sum of copies of I_k. This completes the proof. □

Recall that Λ is called an *Auslander–Gorenstein ring* if it is an ∞-Gorenstein ring with finite left and right self-injective dimensions.

Corollary 3.9. *Let Λ be an Auslander–Gorenstein ring and M in $\text{mod}\,\Lambda$ with $\text{grade}\, M = k$. Then $\text{Ext}_\Lambda^k(M, \Lambda)$ can be embedded into a finite direct sum of I_k.*

Proof. By Huang and Qin (Preprint, Corollary 3.7), $\text{Ext}_\Lambda^k(M, \Lambda)$ is pure of grade k. It follows from Theorem 3.7 that $\text{Ext}_\Lambda^k(M, \Lambda)$ can be embedded into a finite direct sum of I_k. □

Proposition 3.10. *Let Λ be an Auslander–Gorenstein ring with $\text{l.id}_\Lambda \Lambda = \text{r.id}_\Lambda \Lambda = k$. Then $\text{Ext}_\Lambda^k(M, \Lambda)$ is pure of grade k for any $M \in \text{mod}\,\Lambda$ with $\text{Ext}_\Lambda^k(M, \Lambda) \neq 0$.*

Proof. Let M be in $\text{mod}\,\Lambda$ with $\text{Ext}_\Lambda^k(M, \Lambda) \neq 0$. Since Λ is an ∞-Gorenstein ring, $\text{s.grade}\,\text{Ext}_\Lambda^k(M, \Lambda) \geq k$. Let Y be a submodule of $\text{Ext}_\Lambda^k(M, \Lambda)$ in $\text{mod}\,\Lambda^{op}$. Then $\text{grade}\, Y \geq k$. If $\text{grade}\, Y \geq k+1$, then $\text{grade}\, Y = \infty$ since $\text{r.id}_\Lambda \Lambda = k$. It follows from Colby and Fuller (1990, Theorem 2) that $Y = 0$. This completes the proof. □

By Iwanaga and Sato (1996, Proposition 1), we have that if $\text{l.id}_\Lambda \Lambda = \text{r.id}_\Lambda \Lambda = k$, then $\text{r.fd}_\Lambda I_k = k$. So a k-Gorenstein ring with $\text{l.id}_\Lambda \Lambda = \text{r.id}_\Lambda \Lambda = k$ is just an Auslander–Gorenstein ring with $\text{l.id}_\Lambda \Lambda = \text{r.id}_\Lambda \Lambda = k$. By Theorem 3.7 and Proposition 3.10, we then get the following corollary.

Corollary 3.11 (Iwanaga and Sato, 1996, Corollary 5)**.** *Let Λ be a k-Gorenstein ring with $\text{l.id}_\Lambda \Lambda = \text{r.id}_\Lambda \Lambda = k$ and M in $\text{mod}\,\Lambda$. Then $\text{Ext}_\Lambda^k(M, \Lambda)$ can be embedded into a finite direct sum of I_k. Moreover, if $\text{Ext}_\Lambda^k(M, \Lambda) \neq 0$, then $\text{Ext}_\Lambda^k(M, \Lambda)$ is pure of grade k.*

Recall from Anderson and Fuller (1992) that a nonzero right Λ-module H is called *uniform* if each of its nonzero submodules is essential in H.

Corollary 3.12. *Let Λ be an Auslander–Gorenstein ring with $\text{l.id}_\Lambda \Lambda = \text{r.id}_\Lambda \Lambda = k$ and N a uniform module in $\text{mod}\,\Lambda^{op}$. If N is pure of grade k, then N can be embedded into I_k, but cannot be embedded into $\bigoplus_{i=0}^{k-1} I_i$ (if $k \geq 1$).*

Proof. Let N be a uniform module in $\operatorname{mod}\Lambda^{op}$. Then $E(N)$ (the envelope of N) is indecomposable. If N is pure of grade k, then, by Theorem 3.7, N and $E(N)$ can be embedded into a finite direct sum of I_k. So $E(N)$ is isomorphic to a direct summand of I_k and hence N can be embedded into I_k. On the other hand, if $k \geq 1$, then, by Iwanaga and Sato (1996, Corollary 7), we have that I_k and $\bigoplus_{i=0}^{k-1} I_i$ have no isomorphic direct summands in common. It follows from the above argument that $E(N)$, and hence N, cannot be embedded into $\bigoplus_{i=0}^{k-1} I_i$. □

4. THE SOCLE OF THE LAST TERM IN A MINIMAL INJECTIVE RESOLUTION

For a right Λ-module X, the unique largest semisimple submodule of X is called the *socle* of X, and denoted by $\mathrm{Soc}(X)$ (see Anderson and Fuller, 1992). In this section we show that under some grade conditions of modules the socle of I_k is nonzero. In fact, we will prove the following theorem.

Theorem 4.1. *Assume that* $\mathrm{l.id}_\Lambda\Lambda = \mathrm{r.id}_\Lambda\Lambda = k$. *If* $\operatorname{grade}\operatorname{Ext}_\Lambda^k(M, \Lambda) \geq k$ *for any* $M \in \operatorname{mod}\Lambda$ *and* $\operatorname{grade}\operatorname{Ext}_\Lambda^i(N, \Lambda) \geq i$ *for any* $N \in \operatorname{mod}\Lambda^{op}$ *with* $1 \leq i \leq k-1$, *then* $Soc(I_k) \neq 0$.

Proof. The case $k \leq 2$ was proved in Hoshino (1993, Theorem 4.5). Now suppose $k \geq 3$.

Since $\mathrm{l.id}_\Lambda\Lambda = k$, there is a module $M \in \operatorname{mod}\Lambda$ such that $\operatorname{Ext}_\Lambda^k(M, \Lambda) \neq 0$. Let N be a maximal submodule of $\operatorname{Ext}_\Lambda^k(M, \Lambda)$ and

$$0 \to N \to \operatorname{Ext}_\Lambda^k(M, \Lambda) \to S \to 0$$

an exact sequence in $\operatorname{mod}\Lambda^{op}$. Then S is simple.

Since $\operatorname{grade}\operatorname{Ext}_\Lambda^i(N, \Lambda) \geq i$ for any $N \in \operatorname{mod}\Lambda^{op}$ and $1 \leq i \leq k-1$ by assumption, $\mathrm{r.fd}_\Lambda I_i \leq i+1$ for any $0 \leq i \leq k-2$ by Auslander and Reiten (1996, Theorem 0.1) and the remark following it. It follows from Cartan and Eilenberg (1999, Chapter VI, Proposition 5.3) that $\operatorname{Hom}_\Lambda(\operatorname{Ext}_\Lambda^k(M, \Lambda), I_i) \cong \operatorname{Tor}_k^\Lambda(I_i, M) = 0$ for any $0 \leq i \leq k-2$. We then have $\operatorname{Hom}_\Lambda(N, I_i) = 0 = \operatorname{Hom}_\Lambda(S, I_i)$ for any $0 \leq i \leq k-2$, and therefore $\operatorname{grade} N \geq k-1$ and $\operatorname{grade} S \geq k-1$.

On the other hand, $\operatorname{grade}\operatorname{Ext}_\Lambda^k(M, \Lambda) \geq k$ by assumption, so we have $\operatorname{Ext}_\Lambda^{k-1}(S, \Lambda) \cong \operatorname{Ext}_\Lambda^{k-2}(N, \Lambda) = 0$ and hence $\operatorname{grade} S \geq k$. If $\operatorname{grade} S \geq k+1$, then $\operatorname{grade} S = \infty$ since $\mathrm{r.id}_\Lambda\Lambda = k$. It follows from Lemma 2.8 that S is reflexive and $S = 0$, which is a contradiction. So we conclude that $\operatorname{grade} S = k$ and $\operatorname{Ext}_\Lambda^k(S, \Lambda) \neq 0$. Thus we have $\operatorname{Hom}_\Lambda(S, I_k) \neq 0$, which implies that S is isomorphic to a simple submodule of I_k and $\mathrm{Soc}(I_k) \neq 0$. □

Corollary 4.2. *Assume that Λ is a noetherian algebra with* $\mathrm{l.id}_\Lambda\Lambda = \mathrm{r.id}_\Lambda\Lambda = k$. *If* $\mathrm{r.fd}_\Lambda I_i \leq i+1$ *for any* $0 \leq i \leq k-2$ *and* $\mathrm{l.fd}_\Lambda I'_i \leq i+1$ *for any* $0 \leq i \leq k-1$, *then* $\mathrm{Soc}(I_k) \neq 0$.

Proof. Since $\mathrm{r.fd}_\Lambda I_i \leq i+1$ for any $0 \leq i \leq k-2$, by Auslander and Reiten (1996, Theorem 4.7) we have $\operatorname{grade}\operatorname{Ext}_\Lambda^i(N, \Lambda) \geq i$ for any $N \in \operatorname{mod}\Lambda^{op}$ and $1 \leq i \leq k-1$. On the other hand, since $\mathrm{l.fd}_\Lambda I'_i \leq i+1$ for any $0 \leq i \leq k-1$, by the dual statement of Auslander and Reiten (1996, Theorem 4.7), we have grade $\mathrm{Ext}^i_\Lambda(M, \Lambda) \geq i$ for any $M \in \mathrm{mod}\,\Lambda$ and $1 \leq i \leq k$. We now get our conclusion by Theorem 4.1. $\square$

From Auslander's Theorem and Theorem 4.1, we obtain the following corollary.

Corollary 4.3 (Fuller and Iwanaga, 1993, Proposition 1.1). *Let Λ be an Auslander–Gorenstein ring with* $\mathrm{l.id}_\Lambda \Lambda = \mathrm{r.id}_\Lambda \Lambda = k$. *Then* $\mathrm{Soc}(I_k) \neq 0$.

Example 4.4. There are rings satisfying the assumption in Theorem 4.1 but not satisfying the assumption in Corollary 4.3. Let K be a field and Λ the finite dimensional path algebra over K given by the quiver

$$\begin{tikzcd}
1 \arrow[dr, "\alpha"] & & 4 \\
& 3 \arrow[ur, "\beta"] \arrow[dr] & \\
2 \arrow[ur] & & 5
\end{tikzcd}$$

modulo the ideal generated by $\beta\alpha$. Then $\mathrm{l.fd}_\Lambda I'_0 = \mathrm{l.fd}_\Lambda I'_1 = \mathrm{r.fd}_\Lambda I_0 = \mathrm{r.fd}_\Lambda I_1 = 1$, $\mathrm{l.fd}_\Lambda I'_2 = \mathrm{r.fd}_\Lambda I_2 = 2$, and $\mathrm{l.id}_\Lambda \Lambda = \mathrm{r.id}_\Lambda \Lambda = 2$. By Auslander and Reiten (1996, Theorem 4.7) and its dual statement, for any $i \geq 1$, we have grade $\mathrm{Ext}^i_\Lambda(M, \Lambda) \geq i$ for any $M \in \mathrm{mod}\,\Lambda$ and grade $\mathrm{Ext}^i_\Lambda(N, \Lambda) \geq i$ for any $N \in \mathrm{mod}\,\Lambda^{op}$. So Λ satisfies the assumption in Theorem 4.1 (in fact Λ also satisfies the assumption in Corollary 4.2). But Λ is clearly not Auslander–Gorenstein.

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