Auslander-Reiten Triangles in Homotopy Categories *†

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Abstract

Let A be an artin algebra. We show that the bounded homotopy category of finitely generated right A-modules has Auslander-Reiten triangles. Two applications are given: (1) we provide an alternative proof of a theorem of Happel in [H2]; (2) we prove that over a Gorenstein algebra, the bounded homotopy category of finitely generated Gorenstein projective (resp. injective) modules admits Auslander-Reiten triangles, which improves a main result in [G].

1 Introduction

Throughout this paper, A is an artin algebra over a fixed commutative artin ring R and $D := \mathrm{Hom}_R(-, E(R/J))$ is the usual duality, where J is the Jacobson radical of R and $E(R/J)$ is the injective envelope of R/J. We denote by $\mathrm{mod}\,A$ the category of finitely generated right A-modules. As usual, we write $\mathrm{proj}\,A$ (resp. $\mathrm{inj}\,A$) the category of finitely generated projective (resp. injective) right A-modules.

Auslander-Reiten sequences, also known as almost split sequences, are one of the central tools in the representation theory of artin algebras. Auslander-Reiten triangles can also be defined by almost split morphisms in Hom-finite Krull-Schmidt triangulated R-categories. Happel proved in [H3] that the bounded homotopy category $K^b(\mathrm{proj}\,A)$ of finitely generated projective right A-modules has right Auslander-Reiten triangles if and only if the left self-injective dimension of A is finite; and dually, the bounded homotopy category $K^b(\mathrm{inj}\,A)$ of finitely generated injective right A-modules has left Auslander-Reiten triangles if and only if the right self-injective dimension of A is finite.

There is a close relation between Auslander-Reiten triangles and Serre functors ([RV]): a Hom-finite Krull-Schmidt triangulated R-category has right (resp. left) Auslander-Reiten triangles if and only if it has a right (resp. left) Serre functor. A Serre functor by definition is a right Serre functor which is an equivalence. In [BJ], Backelin and Jaramillo proved that the bounded homotopy category $K^b(\mathrm{mod}\,A)$ of $\mathrm{mod}\,A$ has a right Serre duality. Their method is based on the construction of a t-structure in $K^b(\mathrm{mod}\,A)$, and their proof is somewhat complicated although they obtained some more results. We use the terminology of Auslander-Reiten triangles to prove that the right Serre functor in $K^b(\mathrm{mod}\,A)$ is always an equivalence (Theorem 3.4). Our result is based on the fact that right (left)

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minimal almost split morphisms are stable under quotients. It seems more elementary. In particular, we determine Auslander-Reiten triangles admitting special ending (resp. starting) terms (Proposition 4.2).

As in abelian categories, the existence of Auslander-Reiten triangles in subcategories were investigated by Jørgensen in [J]. Note that $K^b(\mathrm{proj}\, A)$ and $K^b(\mathrm{inj}\, A)$ can be embedded in $K^b(\mathrm{mod}\, A)$ naturally. By using the obtained result about the existence of Auslander-Reiten triangles in $K^b(\mathrm{mod}\, A)$, we reprove the Happel's theorem mentioned above (Theorem 4.4). The steps we take seem more "categorization" and can be easily treated dually. The advantage here is that the Auslander-Reiten triangles we treat always lie in $K^b(\mathrm{mod}\, A)$. Similarly, we prove that over a Gorenstein algebra, the bounded homotopy category of finitely generated Gorenstein projective (resp. injective) modules admits Auslander-Reiten triangles, which improves a main result in [G].

2 Preliminaries

Recall that a right A-module M is called *Gorenstein projective* if there exists an exact sequence $T^\bullet$ of projective modules which remains exact when applying the functor $\mathrm{Hom}_A(-, P)$ for any $P \in \mathrm{proj}\, A$ such that M is isomorphic to some kenrel of $T^\bullet$. Dually, The notion of *Gorenstein injective modules* is defined. We denote the category of all finitely generated Gorenstein projective (resp. injective) modules by $\mathrm{Gproj}\, A$ (resp. $\mathrm{Ginj}\, A$). Note that we have $\mathrm{proj}\, A \subset \mathrm{Gproj}\, A$ and $\mathrm{inj}\, A \subset \mathrm{Ginj}\, A$.

Let $f : M \to N$ be a morphism in $\mathrm{mod}\, A$. According to [AR1], f is called *right almost split* if it is not a retraction, and any morphism $g : L \to N$ which is not a retraction factors through f; it is called *right minimal* if any morphism h satisfying $f = f \cdot h$ is an automorphism of M; and it is called *right minimal almost split* if it is both right almost split and right minimal. The left versions are defined dually. An exact sequence

$$0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} L \to 0$$

in $\mathrm{mod}\, A$ is called an *almost split sequence* if either α is left minimal almost split or β is right minimal almost split. This also means that M and L must be indecomposable. We say $\mathrm{mod}\, A$ has *almost split sequences* if for any indecomposable non-injective module M in $\mathrm{mod}\, A$ there is an almost split sequence starting at M, and for any indecomposable non-projective module $N \in \mathrm{mod}\, A$ there is an almost split sequence ending at N. By [AR1], $\mathrm{mod}\, A$ always has almost split sequences.

Let $\mathcal{T}$ be a Hom-finite Krull-Schmidt triangulated R-category. The notion of almost split triangles in $\mathcal{T}$ was introduced by Happel in [H1]. A triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to X[1]$$

in $\mathcal{T}$ is called an *Auslander-Reiten triangle* (or *almost split triangle*) if either α is left minimal almost split or β is right minimal almost split (see [H2], where the notions source morphisms and sink morphisms were used). A triangulated category is said to *have right* (*resp. left*) *Auslander-Reiten* *triangles* if for any indecomposable object M there is an Auslander-Reiten triangle ending (resp. starting) at M.

Let $F : \mathcal{T} \to \mathcal{T}$ be a triangulated functor. According to [RV], F is called a *right Serre functor* if for any X, Y in $\mathcal{T}$, there is an isomorphism $D\operatorname{Hom}_{\mathcal{T}}(X, Y) \cong \operatorname{Hom}_{\mathcal{T}}(Y, FX)$ which is natural in X and Y. This F is unique up to natural isomorphism. A left Serre functor is defined dually.

Theorem 2.1. ([BJ, Corollary 2.5 and Proposition 4.6])

Let A be an Artin algebra. Then there is a right serre functor $S : K^b(\operatorname{mod} A) \to K^b(\operatorname{mod} A)$, equivalently, $K^b(\operatorname{mod} A)$ has right Auslander-Reiten triangles.

3 AR-triangles in homotopy categories

Let $\mathcal{A}$ be an additive R-category. By an $\mathcal{A}$-module, we mean a contravariant R-linear functor from $\mathcal{A}$ to the category of R-modules. We denote by $\operatorname{mod}\mathcal{A}$ the category of finitely presented $\mathcal{A}$-modules. Note that, in general, $\operatorname{mod}\mathcal{A}$ is not an abelian category. It is an abelian category if and only if $\mathcal{A}$ has pseudo-kernels ([A]). We call $\mathcal{A}$ a *dualizing R-category* if D gives a duality between $\operatorname{mod}\mathcal{A}$ and $\operatorname{mod}\mathcal{A}^{op}$. Note that, in this case, $\operatorname{mod}\mathcal{A}$ is always an abelian category, and hence the bounded complex category $C^b(\operatorname{mod}\mathcal{A})$ of $\operatorname{mod}\mathcal{A}$ is also an abelian category. We begin with a main theorem in [BJR].

Lemma 3.1. ([BJR, Theorem 4.3]) *Let $\mathcal{A}$ be a dualizing R-category. Then $C^b(\operatorname{mod}\mathcal{A})$ has almost split sequences.*

Note that $\operatorname{mod} A$ is equivalent to $\operatorname{mod}(\operatorname{proj} A)$ as additive R-categories ([A]). This means that $C^b(\operatorname{mod} A)$ always has almost split sequences. As a consequence, for any indecomposable non-projective (resp. non-injective) object X in $C^b(\operatorname{mod} A)$, there is always an almost split sequence ending (resp. starting) at X.

Let $\mathcal{B}$ be an additive category and $\mathcal{C}$ an additive full subcategory of $\mathcal{B}$ closed under summands. Then we can form the factor category $\mathcal{B}/\mathcal{C}$. The objects in $\mathcal{B}/\mathcal{C}$ are the same as in $\mathcal{B}$, and the morphisms are the morphisms in $\mathcal{B}$ modulo morphisms factor through an object of $\mathcal{C}$. There is a natural factor functor $\pi : \mathcal{B} \to \mathcal{B}/\mathcal{C}$. It is an additive functor. For both objects and morphisms, we denote their images under π by adding $\widetilde{}$ above. The following lemma is a well known fact. For the reader's convenience we give a quick proof here.

Lemma 3.2. *Let $\mathcal{B}$ be a Hom-finite Krull-Schmidt R-category and $\mathcal{C}$ an additive full subcategory of $\mathcal{B}$ closed under summands. If $f : M \to N$ is a right (resp. left) minimal almost split morphism in $\mathcal{B}$, then $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ is also a right (resp. left) minimal almost split morphism in $\mathcal{B}/\mathcal{C}$.*

Proof. Obviously, N is indecomposable; in particular, it has no nonzero summands in $\mathcal{C}$. By [AR2, Lemma 1.1(c)], $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ is not a retraction. Let $\widetilde{g} : \widetilde{L} \to \widetilde{N}$ be not a retraction. Then it is induced by a morphism $g : L \to N$ which is also not a retraction. So g factors through f since f is right almost split, and hence $\widetilde{g}$ factors through $\widetilde{f}$. If $\widetilde{f}$ is not minimal, then $\widetilde{f}$ has a direct summand of the form $\widetilde{W} \to 0$, and so f has a direct summand of the form $W \to 0$ or $W \to C$ with $0 \neq C \in \mathcal{C}$. Note that the former one gives a contradiction to the minimality of f, and the latter one gives a contradiction to the indecomposableness of N. $\square$

A complex X is called *contractible* if it is isomorphic to zero in $K^b(\operatorname{mod} A)$, that is, it is splitting exact. Note that a chain map of complexes is homotopic to zero if and only if it factors through some contractible complex. So $K^b(\operatorname{mod} A)$ is exactly the factor category of $C^b(\operatorname{mod} A)$ by modulo contractible complexes ([H2, p.28]). We also need the following

Lemma 3.3. *A complex X is a projective (resp. injective) object in $C^b(\operatorname{mod} A)$ if and only if it is a contractible complex consisting of projective (resp. injective) modules in* $\operatorname{mod} A$.

Proof. See for example [EJ2, Theorem 1.4.7]. $\square$

Now we can prove the following result, which improves Theorem 2.1.

Theorem 3.4. *$K^b(\operatorname{mod} A)$ has Auslander-Reiten triangles.*

Proof. Let $0 \neq \widetilde{X} \in K^b(\operatorname{mod} A)$ be indecomposable. Then it is induced by an indecomposable object X in $C^b(\operatorname{mod} A)$. Note that X is neither projective nor injective by Lemma 3.3. It follows from Lemma 3.1 that there is a minimal right almost split morphism $f : Y \to X$ in $C^b(\operatorname{mod} A)$. Then by Lemma 3.2, its image $\widetilde{f} : \widetilde{Y} \to \widetilde{X}$ is also right minimal almost split. Complete it to a triangle

$$L \to \widetilde{Y} \to \widetilde{X} \to L[1]$$

in $K^b(\operatorname{mod} A)$. Then by definition, it is an Auslander-Reiten triangle in $K^b(\operatorname{mod} A)$. Dually, there is an Auslander-Reiten triangle starting at $\widetilde{X}$. $\square$

4 Applications

In this section, we will reprove the Happel's theorem by using the main result in Section 3. Our proof is based on the restriction of Auslander-Reiten triangles in subcategories. Also one can see that using this technique, over a Gorenstein algebra the existence of Auslander-Reiten triangles in the bounded homotopy category of Gorenstein projective modules is valid. This improves a result by Gao in [G] where only the existence of right Auslander-Reiten triangles is proved and the condition CM-finiteness is necessary.

Although $K^b(\operatorname{mod} A)$ has Auslander-Reiten triangles, it is difficult to compute the Auslander-Reiten translation. By [RV, Theorem I.2.4], $K^b(\operatorname{mod} A)$ has Auslander-Reiten triangles if and only if it has a Serre functor. Denote the Serre functor of $K^b(\operatorname{mod} A)$ by S. Then by [RV, Proposition I.2.3], we have that for any indecomposable object X, the left end term of an Auslander-Reiten triangle ending at X is $S \cdot [-1]$, the other end term is its quasi-inverse $S^{-1} \cdot [1]$. Thanks to this result, we only need to compute the Serre dual object for an indecomposable object X.

Lemma 4.1. *Let X and Y be in $C^b(\operatorname{mod} A)$.*

(1) If X is degreewise projective, then we have a natural isomorphism

$$D\operatorname{Hom}_A(X,Y) \cong \operatorname{Hom}_A(Y, X\otimes_A DA).$$

(2) If Y is degreewise injective, then we have a natural isomorphism

$$D\operatorname{Hom}_A(X,Y) \cong \operatorname{Hom}_A(\operatorname{Hom}_A(DA,Y),X).$$

Proof. (1) Note that for any X, Y in $\operatorname{mod} A$, we have a natural morphism

$$\delta_{Y,X} : Y\otimes_A \operatorname{Hom}_A(X,A) \to \operatorname{Hom}_A(X,Y),\ \ y\otimes f \longmapsto (x \longmapsto yf(x)).$$

So we have a natural morphism

$$\begin{aligned}\eta_{X,Y} : D\operatorname{Hom}_A(X,Y) &\stackrel{D\delta_{Y,X}}{\longrightarrow} D(Y\otimes_A \operatorname{Hom}_A(X,A))\\ &\cong \operatorname{Hom}_R(Y\otimes_A \operatorname{Hom}_A(X,A), E(R/J))\\ &\cong \operatorname{Hom}_A(Y, \operatorname{Hom}_R(\operatorname{Hom}_A(X,A), E(R/J))\\ &\cong \operatorname{Hom}_A(Y, X\otimes_A DA).\end{aligned}$$

It is known that if X is projective, then $\delta_{Y,X}$ is an isomorphism. Thus $D\delta_{Y,X}$ is also an isomorphism. Therefore we have a natural isomorphism

$$\eta_{X,Y} : D\operatorname{Hom}_A(X,Y) \cong \operatorname{Hom}_A(Y, X\otimes_A DA).$$

Now the isomorphism can be extended to the desired situation.

(2) Let Y be injective. Then $Y \cong DA\otimes_A P$ for some $P \in \operatorname{proj} A$. Then we have isomorphisms

$$\begin{aligned}&\operatorname{Hom}_A(\operatorname{Hom}_A(DA,Y),X)\\ \cong\ &\operatorname{Hom}_A(\operatorname{Hom}_A(DA, DA\otimes_A P),X)\\ \cong\ &\operatorname{Hom}_A(P,X)\\ \cong\ &D\operatorname{Hom}_A(X, P\otimes_A DA)\ \text{(by (1))}\\ \cong\ &D\operatorname{Hom}_A(X,Y).\end{aligned}$$

Similarly, the isomorphism can be extended to the desired situation. $\square$

Proposition 4.2. *Let X and Y be in $C^b(\operatorname{mod} A)$.*

(1) If X is degreewise projective, then there is an Auslander-Reiten triangle in $K^b(\operatorname{mod} A)$

$$X[-1]\otimes_A DA \to M \to X \to X\otimes_A DA.$$

(2) If Y is degreewise injective, then there is an Auslander-Reiten triangle in $K^b(\operatorname{mod} A)$

$$Y \to N \to \operatorname{Hom}_A(DA, Y[1]) \to Y[1].$$

Proof. We only prove (1), and the proof of (2) is similar. Let $X \in K^b(\operatorname{proj} A)$. By Lemma 4.1, we have $D\operatorname{Hom}_A(X,Y) \cong \operatorname{Hom}_A(Y, X \otimes_A DA)$. Thus we have isomorphisms

$$D\operatorname{Hom}_{K^b(\operatorname{mod} A)}(X,Y) \cong DH^0\operatorname{Hom}_A(X,Y) \cong H^0 D\operatorname{Hom}_A(X,Y) \cong$$

$$H^0\operatorname{Hom}_A(Y, X \otimes_A DA) \cong \operatorname{Hom}_{K^b(\operatorname{mod} A)}(Y, X \otimes_A DA).$$

This holds for any $Y \in K^b(\operatorname{mod} A)$. Then by the Yoneda's lemma, the Serre dual object for X is $X \otimes_A DA$. Now by [RV, Proposition I.2.3], we have the desired triangle. $\square$

Let $\mathcal{B}$ be an additive category and $\mathcal{C}$ a full subcategory of $\mathcal{B}$. Recall that a morphism $f: B \to C$ with $B \in \mathcal{B}$ and $C \in \mathcal{C}$ is called a *$\mathcal{C}$-preenvelope* if the natural map $\operatorname{Hom}_{\mathcal{B}}(C, C') \to \operatorname{Hom}_{\mathcal{B}}(B, C') \to 0$ is exact for any $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope C is called a *$\mathcal{C}$-envelope* if it is left minimal. Dually, the notion of (pre)covers is defined ([AR3, E]). The following proposition involves Auslander-Reiten triangles in subcategories.

Lemma 4.3. ([J, Theorem 3.1 and Theorem 3.2]) *Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ a full subcategory of $\mathcal{T}$ closed under extensions.*

(1) Let $X \to Y \to Z \to X[1]$ be an Auslander-Reiten triangle in $\mathcal{T}$ with $Z \in \mathcal{C}$. If there is an object $A' \in \mathcal{C}$ with a nonzero morphism $Z \to A'[1]$, then the following are equivalent.

- *X has a $\mathcal{C}$-cover of the form $A \to X$.*
- *There is an Auslander-Reiten triangle $A \to B \to Z \to A[1]$ in $\mathcal{C}$.*

(2) Let $X \to Y \to Z \to X[1]$ be an Auslander-Reiten triangle in $\mathcal{T}$ with $X \in \mathcal{C}$. If there is an object $Z' \in \mathcal{C}$ with a nonzero morphism $Z' \to X[1]$, then the following are equivalent.

- *Z has a $\mathcal{C}$-envelope of the form $Z \to N$.*
- *There is an Auslander-Reiten triangle $X \to M \to N \to X[1]$ in $\mathcal{C}$.*

As an application of Theorem 3.4, we now are in a position to reprove the following Happel's theorem. Our argument is very different from the original one.

Theorem 4.4. ([H2, Section 3.4])

(1) $K^b(\operatorname{proj} A)$ has right Auslander-Reiten triangles if and only if $\operatorname{id} A_{A^{op}} < \infty$.

(2) $K^b(\operatorname{inj} A)$ has left Auslander-Reiten triangles if and only if $\operatorname{id} A_A < \infty$.

Proof. We only prove (2), and (1) is its dual.

Let $0 \neq Y \in K^b(\operatorname{inj} A)$ be indecomposable. Then by Proposition 4.2, there is an Auslander-Reiten triangle

$$Y \to L \to \operatorname{Hom}_A(DA, Y)[1] \to Y[1]$$

in $K^b(\mathrm{mod}\, A)$. Since $(Y[1])[-1] \stackrel{\mathrm{Id}_Y}{\to} Y$ is not homotopic to zero, by Lemma 4.3 we have that $K^b(\mathrm{inj}\, A)$ has left Auslander-Reiten triangles if and only if $\mathrm{Hom}_A(DA, Y)$ has a $K^b(\mathrm{inj}\, A)$-envelope for any $Y \in K^b(\mathrm{inj}\, A)$. Now it suffices to prove that $\mathrm{id}\, A_A < \infty$ if and only if $\mathrm{Hom}_A(DA, Y)$ has a $K^b(\mathrm{inj}\, A)$-envelope for any $Y \in K^b(\mathrm{inj}\, A)$.

If $\mathrm{id}\, A_A < \infty$, then the injective dimension of any module in $\mathrm{proj}\, A$ is finite. Since $\mathrm{Hom}_A(DA, Y)$ consists of modules in $\mathrm{proj}\, A$, then by using induction on the width of $\mathrm{Hom}_A(DA, Y)$, one can get an injective resolution $f : \mathrm{Hom}_A(DA, Y) \to L$, where f is a quasi-isomorphism and $L \in K^b(\mathrm{inj}\, A)$. If there is a chain map $\alpha : \mathrm{Hom}_A(DA, Y) \to I$ with $I \in K^b(\mathrm{inj}\, A)$, then $f^* : \mathrm{Hom}_A(L, I) \to \mathrm{Hom}_A(\mathrm{Hom}_A(DA, Y), I)$ is a quasi-isomorphism and $H^0(f^*)$ is an isomorphism. Hence there is some $\beta : L \to I$ such that α homotopic to $\beta \cdot f$. If there is some g satisfying $g \cdot f$ homotopic to f, then it is a quasi-isomorphism, and hence a homotopic equivalence. It follows that L is a $K^b(\mathrm{inj}\, A)$-envelope of $\mathrm{Hom}_A(DA, Y)$.

Now suppose that $\mathrm{Hom}_A(DA, Y)$ has a $K^b(\mathrm{inj}\, A)$-envelope for any $Y \in K^b(\mathrm{inj}\, A)$. Let $Y = DA$. Then $\mathrm{Hom}_A(DA, Y) \cong A$. Let $A \to I$ be the $K^b(\mathrm{inj}\, A)$-envelope of A. Complete it to a triangle

$$A \stackrel{\alpha}{\to} I \to L \to A[1]$$

in $K^b(\mathrm{mod}\, A)$. Then we have that $\mathrm{Hom}_{K^b(\mathrm{mod}\, A)}(L, Z) \cong 0$ for any $Z \in K^b(\mathrm{inj}\, A)$ by the Wakamatsu lemma (see for example [J, Lemma 2.1]); in particular, $\mathrm{Hom}_{K^b(\mathrm{mod}\, A)}(L, (DA)[i]) = 0$ for all i, which implies that L is exact. As a consequence, α is a quasi-isomorphism. Since any injective resolution of A is homotopically equivalent to I. It follows that $\mathrm{id}\, A_A < \infty$. $\square$

Remark 4.5. *Note that the functor* $- \otimes_A DA : K^b(\mathrm{proj}\, A) \to K^b(\mathrm{inj}\, A)$ *is an equivalence. Hence the Gorenstein symmetry conjecture, which states that* $\mathrm{id}\, A_A = \mathrm{id}\, A_{A^{op}}$ *for any artin algebra* A*, can be reformulated as follows.*

•$K^b(\mathrm{proj}\, A)$ *has right Auslander-Reiten triangles if and only if it has left Auslander-Reiten triangles. Dually,*

•$K^b(\mathrm{inj}\, A)$ *has right Auslander-Reiten triangles if and only if it has left Auslander-Reiten triangles.*

Corollary 4.6. *The following are equivalent.*

(1) A *is a Gorenstein algebra.*

(2) $K^b(\mathrm{proj}\, A)$ *has Auslander-Reiten triangles.*

(3) $K^b(\mathrm{inj}\, A)$ *has Auslander-Reiten triangles.*

Let A and B be artin algebras. According to [R], A and B are derived equivalent if and only if $K^b(\mathrm{proj}\, A)$ and $K^b(\mathrm{proj}\, B)$ are equivalent as triangulated categories.

Corollary 4.7. *Let* A *and* B *be artin algebras. If* A *and* B *are derived equivalent, then* A *is Gorenstein if and only if* B *is Gorenstein.*

In general, Theorem 3.4 only tells us the validity of Auslander-Reiten triangles in $K^b(\mathrm{mod}\, A)$. When consider some subcategory $\mathcal{C}$ of $K^b(\mathrm{mod}\, A)$, one usually relies on the restriction as in Lemma 4.3. However, it is often difficult to compute the serre dual objects for $\mathcal{A}$. For example, the isomorphism in Lemma 4.2 for projective modules can not be extended to Gorenstein projective version unless A is self-injective, see [ASS, Lemma 2.12]. In the following, we only consider the subcategory $K^b(\mathrm{Gproj}\, A)$ (resp. $K^b(\mathrm{Ginj}\, A)$).

It was proved in [EJ1] that over a Gorenstein algebra any finitely generated module admits a finitely generated Gorenstein projective precover. That is, for any $M \in$ mod A, there is a complex G_M consisting of modules in $\mathrm{Gproj}\, A$ and a chain map $G_M \to M$ which is a quasi-isomorphism after applying the functor $\mathrm{Hom}_A(G', -)$ for any $G' \in \mathrm{Gproj}\, A$. Since M has finite Gorenstein projective dimension, G_M can be selected to be in $K^b(\mathrm{Gproj}\, A)$ by [Ho, Proposition 2.18]. The dual version for finitely generated Gorenstein injective modules is also valid.

Theorem 4.8. *Let A be a Gorenstein algebra. Then $K^b(\mathrm{Gproj}\, A)$ has Auslander-Reiten triangles.*

Proof. The proof is similar to that of Theorem 4.4. First, we prove that $K^b(\mathrm{Gproj}\, A)$ has right Auslander-Reiten triangles. Let $0 \neq X \in K^b(\mathrm{Gproj}\, A)$ be indecomposable. Then by Theorem 3.4, we have an Auslander-Reiten triangle

$$Y \to L \to X \to Y[1]$$

in $K^b(\mathrm{mod}\, A)$. We only need to prove that Y has a $K^b(\mathrm{Gproj}\, A)$-cover. In fact, we will prove that any $Y \in K^b(\mathrm{mod}\, A)$ has a $K^b(\mathrm{Gproj}\, A)$-cover. Note that for any $M \in \mathrm{mod}\, A$, there is a chain map $G_M \to M$ with $G_M \in K^b(\mathrm{Gproj}\, A)$, which is a quasi-isomorphism after applying the functor $\mathrm{Hom}_A(G', -)$ for any $G' \in \mathrm{Gproj}\, A$ as above. By using induction on the width of Y, we have that there is a chain map $f_Y : G_Y \to Y$ with $G_Y \in K^b(\mathrm{Gproj}\, A)$, which is also a quasi-isomorphism after applying the functor $\mathrm{Hom}_A(G', -)$ for any $G' \in \mathrm{Gproj}\, A$. Hence $\mathrm{Hom}_A(G', f_Y)$ is also a quasi-isomorphism for any $G' \in K^b(\mathrm{Gproj}\, A)$. It is easy to see that G_Y is a $K^b(\mathrm{Gproj}\, A)$-cover of Y. If we consider the category $K^b(\mathrm{Ginj}\, A)$, we then obtain that $K^b(\mathrm{Ginj}\, A)$ admits left Auslander-Reiten triangles. Note that $- \otimes_A DA : K^b(\mathrm{Gproj}\, A) \to K^b(\mathrm{Ginj}\, A)$ is an equivalence by [B]. This completes the proof. □

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