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Homological aspects of the dual Auslander transpose

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Abstract. As a dual of the Auslander transpose of modules, we introduce and study the cotranspose of modules with respect to a semidualizing module C. Then using it we introduce n-C-cotorsionfree modules, and show that n-C-cotorsionfree modules possess many dual properties of n-torsionfree modules. In particular, we show that n-C-cotorsionfree modules are useful in characterizing the Bass class and investigating the approximation theory for modules. Moreover, we study n-cotorsionfree modules over Artin algebras and answer negatively an open question of Huang and Huang posed in 2012.

Keywords. Cotranspose, n-C-cotorsionfree modules, n-C-cospherical modules, n-syzygy modules, cograde.

2010 Mathematics Subject Classification. 16E05, 16E30, 16E10.

1 Introduction

It is well known that the Auslander–Reiten theory plays a very important role in representation theory of Artin algebras and homological algebra. One of the most powerful tools in this theory is the Auslander transpose. With the aid of the Auslander transpose, as a special case of n-syzygy modules over left and right Noether rings, Auslander and Bridger [1] introduced n-torsionfree modules and obtained an approximation theory for finitely generated modules when n-syzygy modules and n-torsionfree modules coincide. Ever since then many authors have studied the homological properties of these modules and related modules; see e.g. [1–4, 11–13, 16–18, 20], and so on. Based on these references, two natural questions arise:

Question 1. How to dualize the Auslander transpose of modules appropriately?

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Question 2. Does the notion of n-torsionfree modules have its dual as many notions in classical homological algebra do?

The aim of this paper is to study these two questions, and we will define and investigate the cotranspose of modules and n-cotorsionfree modules.

The paper is organized as follows.

In Section 2 we give some terminology and some preliminary results, and we also introduce the notions of cotorsionless modules and coreflexive modules.

In Section 3 we introduce the cotranspose of modules with respect to a semidualizing bimodule C, and using it we introduce n-C-cotorsionfree modules as a dual of n-(C-)torsionfree modules in [1, 20]. We show that n-C-cotorsionfree modules possess many dual properties of n-(C-)torsionfree modules. For example, we prove that a module is n-C-cotorsionfree if and only if it admits some special proper resolutions of length at least n. Then, as an application, we deduce that the Bass class with respect to C coincides with the intersection of the class of ∞-C-cotorsionfree modules and that of ∞-C-cospherical modules. As another application, we get a dual version of the approximation theorem for finitely generated modules over left and right noetherian rings in [1, Proposition 2.21] and its semiduazlizing version in [20, Theorem A].

In Section 4 we generalize the cograde of finitely generated modules in [14] to general modules, and prove that for a ring R, the i-th cosyzygy of a left R-module M is i-C-cotorsionfree for any $1 \leq i \leq n$ if and only if the cograde of $\mathrm{Ext}^i_R(C, M)$ with respect to C is at least $i-1$ for any $1 \leq i \leq n$. This result can be regarded as a dual version of [1, Proposition 2.26].

In Section 5, we focus on studying some special finitely generated n-C-cotorsionfree modules (called n-cotorsionfree modules) over Artin algebras. In this case, we first show that the ordinary Matlis duality induces a duality between the cotranspose (resp. n-cotorsionfree modules) and the transpose (resp. n-torsionfree modules). Then we obtain an equivalent characterization when $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ forms a cotorsion pair, where $\mathcal{GI}$ denotes the class of finitely generated Gorenstein injective modules and ${}^{\perp}\mathcal{GI}$ is its left orthogonal class. Finally, we give an example to illustrate that the class of ∞-torsionfree modules is not closed under kernels of epimorphisms in general. It answers negatively an open question of Huang and Huang ([11]).

2 Preliminaries

Throughout this paper, R, S are fixed associative rings with units. We use Mod R (resp. Mod S^{op}) to denote the class of left R-modules (resp. right S-modules).

Definition 2.1 ([10]). An $(R\text{-}S)$-bimodule ${}_RC_S$ is called *semidualizing* if

(a1) ${}_RC$ admits a degreewise finite R-projective resolution,

(a2) C_S admits a degreewise finite S-projective resolution,

(b1) the homothety map ${}_RR_R \xrightarrow{R\gamma} \operatorname{Hom}_S(C, C)$ is an isomorphism,

(b2) the homothety map ${}_SS_S \xrightarrow{\gamma_S} \operatorname{Hom}_R(C, C)$ is an isomorphism,

(c1) $\operatorname{Ext}_R^{\geqslant 1}(C, C) = 0$,

(c2) $\operatorname{Ext}_S^{\geqslant 1}(C, C) = 0$.

From now on, ${}_RC_S$ is a semidualizing bimodule. We write $(-)^* = \operatorname{Hom}(-, C)$ and $(-)_* = \operatorname{Hom}(C, -)$. For a module $M \in \operatorname{Mod} R$, we have the following two canonical valuation homomorphisms:

$$\sigma_M : M \to M^{**}$$

defined by $\sigma_M(x)(f) = f(x)$ for any $x \in M$ and $f \in M^*$, and

$$\theta_M : C \otimes_S M_* \to M$$

defined by $\theta_M(x \otimes f) = f(x)$ for any $x \in C$ and $f \in M_*$.

Definition 2.2 ([10]). The *Bass class* $\mathcal{B}_C(R)$ with respect to C consists of all left R-modules M satisfying

(B1) $\operatorname{Ext}_R^{\geqslant 1}(C, M) = 0$,

(B2) $\operatorname{Tor}^S_{\geqslant 1}(C, \operatorname{Hom}_R(C, M)) = 0$,

(B3) θ_M is an isomorphism in $\operatorname{Mod} R$.

Let M be a finitely presented left R-module and let

$$P_1 \xrightarrow{f_0} P_0 \longrightarrow M \longrightarrow 0$$

be a finitely generated projective presentation of M. Then $\operatorname{Tr}_C M := \operatorname{Coker} {f_0}^*$ is called the *(Auslander) transpose with respect to* C (see [12]). When $R = S$ and ${}_RC_S = {}_RR_R$, the Auslander transpose with respect to C is just the *Auslander transpose* ([1]).

Proposition 2.3 ([1, Proposition 2.6] and [12, Lemma 2.1]). *Let M be a finitely presented left R-module. Then there exists an exact sequence*

$$0 \longrightarrow \operatorname{Ext}_S^1(\operatorname{Tr}_C M, C) \longrightarrow M \xrightarrow{\sigma_M} M^{**} \longrightarrow \operatorname{Ext}_S^2(\operatorname{Tr}_C M, C) \longrightarrow 0.$$

Recall that a module $M \in \operatorname{Mod} R$ is called *C-torsionless* if σ_M is a monomorphism, and M is called *C-reflexive* if σ_M is an isomorphism. As the duals of C-torsionless modules and C-reflexive modules, we introduce the following:

Definition 2.4. A module $M \in \operatorname{Mod} R$ is called *C-cotorsionless* if θ_M is an epimorphism, and M is called *C-coreflexive* if θ_M is an isomorphism.

For a module $M \in \operatorname{Mod} R$, we denote by $\operatorname{Add}_R M$ the subclass of $\operatorname{Mod} R$ consisting of all direct summands of direct sums of copies of M.

Lemma 2.5. *The following statements hold.*

(1) *For any $W \in \operatorname{Add}_R C$, W is C-coreflexive, W_* is a projective left S-module and* $\operatorname{Ext}_R^{\geqslant 1}(C, W) = 0$.

(2) *For any injective left R-module I, I is C-coreflexive and* $\operatorname{Tor}^S_{\geqslant 1}(C, I_*) = 0$.

Proof. Statement (1) follows from [10, Lemma 5.1 (b)], and statement (2) follows from [10, Lemma 5.1 (c)]. □

Definition 2.6 ([19]). Let $\mathcal{X}$ be a subclass of $\operatorname{Mod} R$.

(1) A sequence $\mathbb{E}$ in $\operatorname{Mod} R$ is $\operatorname{Hom}_R(\mathcal{X}, -)$*-exact* (resp. $\operatorname{Hom}_R(-, \mathcal{X})$*-exact*) if $\operatorname{Hom}_R(X, \mathbb{E})$ (resp. $\operatorname{Hom}_R(\mathbb{E}, X)$) is exact for any $X \in \mathcal{X}$.

(2) An exact sequence

$$\mathbf{X} := \cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots$$

in $\operatorname{Mod} R$ with $X_i, X^i \in \mathcal{X}$ is *totally $\mathcal{X}$-acyclic* if it is $\operatorname{Hom}_R(\mathcal{X}, -)$-exact and $\operatorname{Hom}_R(-, \mathcal{X})$-exact.

Definition 2.7 ([7]). A module $M \in \operatorname{Mod} R$ is called *Gorenstein injective* if there exists a totally acyclic complex of injective modules

$$\mathbf{I} := \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

in $\operatorname{Mod} R$ such that $M \cong \operatorname{Im}(I_0 \to I^0)$.

3 The cotranspose and n-C-cotorsionfree modules

In this section, we introduce and study the cotranspose of modules and n-cotorsionfree modules with respect to the given semidualizing bimodule ${}_RC_S$.

Let $M \in \operatorname{Mod} R$. We use

$$0 \longrightarrow M \longrightarrow I^0(M) \xrightarrow{f^0} I^1(M) \xrightarrow{f^1} \cdots \xrightarrow{f^{i-1}} I^i(M) \xrightarrow{f^i} \cdots \qquad (3.1)$$

to denote a minimal injective resolution of M in $\operatorname{Mod} R$. For any $n \geq 1$,

$$\operatorname{co}\Omega^n(M) := \operatorname{Im} f^{n-1}$$

is called the *n-th cosyzygy* of M, and in particular, put $\operatorname{co}\Omega^0(M) := M$. A module in $\operatorname{Mod} R$ is called *n-cosyzygy* if it is isomorphic to the n-th cosyzygy of some module in $\operatorname{Mod} R$. We introduce the dual notion of the Auslander transpose of modules as follows.

Definition 3.1. For a module $M \in \operatorname{Mod} R$, $\operatorname{cTr}_C M := \operatorname{Coker} {f^0}_*$ is called the *cotranspose* of M with respect to ${}_RC_S$.

The following result is a dual version of Proposition 2.3.

Proposition 3.2. *Let $M \in \operatorname{Mod} R$. Then there exists an exact sequence*

$$0 \longrightarrow \operatorname{Tor}_2^S(C, \operatorname{cTr}_C M) \longrightarrow C \otimes_S M_* \xrightarrow{\theta_M} M \longrightarrow \operatorname{Tor}_1^S(C, \operatorname{cTr}_C M) \longrightarrow 0.$$

Proof. By applying the functor $(-)_*$ to the minimal injective resolution (3.1) of M, we get an exact sequence

$$0 \longrightarrow M_* \longrightarrow I^0(M)_* \xrightarrow{{f^0}_*} I^1(M)_* \longrightarrow \operatorname{cTr}_C M \longrightarrow 0$$

in $\operatorname{Mod} S$. Let

$$f^0 = \alpha \cdot \pi$$

(where $\pi : I^0(M) \twoheadrightarrow \operatorname{Im} f^0$ and $\alpha : \operatorname{Im} f^0 \rightarrowtail I^1(M)$) and

$${f^0}_* = \alpha' \cdot \pi'$$

(where $\pi' : I^0(M)_* \twoheadrightarrow \operatorname{Im} {f^0}_*$ and $\alpha' : \operatorname{Im} {f^0}_* \rightarrowtail I^1(M)_*$) be the natural epic-monic decompositions of f^0 and ${f^0}_*$ respectively. Since $\operatorname{Tor}_1^S(C, I^0(M)_*) = 0$ and $\theta_{I^0(M)}$ is an isomorphism by Lemma 2.5 (2), we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \to \operatorname{Tor}_1^S(C, \operatorname{Im} {f^0}_*) \to & C \otimes_R M_* & \to & C \otimes_S I^0(M)_* & \xrightarrow{1_C \otimes \pi'} & C \otimes_S \operatorname{Im} {f^0}_* & \to 0 \\ & \downarrow \theta_M & & \downarrow \theta_{I^0(M)} & & \downarrow h & \\ 0 \longrightarrow & M & \longrightarrow & I^0(M) & \xrightarrow{\pi} & \operatorname{Im} f^0 & \longrightarrow 0, \end{array}$$

where h is an induced homomorphism. Then

$$\pi \cdot \theta_{I^0(M)} = h \cdot (1_C \otimes \pi').$$

In addition, by the snake lemma, we have

$$\operatorname{Ker} \theta_M \cong \operatorname{Tor}_1^S(C, \operatorname{Im} {f^0}_*) \quad \text{and} \quad \operatorname{Coker} \theta_M \cong \operatorname{Ker} h.$$

On the other hand, since $\mathrm{Tor}_1^S(C, I^1(M)_*) = 0 = \mathrm{Tor}_2^S(C, I^1(M)_*)$ by Lemma 2.5 (2), by applying the functor $C \otimes_S -$ to the exact sequence

$$0 \longrightarrow \mathrm{Im}\, f^0{}_* \xrightarrow{\alpha'} I^1(M)_* \longrightarrow \mathrm{cTr}_C\, M \longrightarrow 0,$$

we get the exact sequence

$$\begin{array}{ccccc} 0 \longrightarrow \mathrm{Tor}_1^S(C, \mathrm{cTr}_C\, M) \longrightarrow C \otimes_S \mathrm{Im}\, f^0{}_* & & \\ \downarrow {\scriptstyle 1_C \otimes \alpha'} & & \\ C \otimes_S I^1(M)_* \longrightarrow C \otimes_S \mathrm{cTr}_C\, M \longrightarrow 0 \end{array}$$

and the isomorphism

$$\mathrm{Tor}_1^S(C, \mathrm{Im}\, f^0{}_*) \cong \mathrm{Tor}_2^S(C, \mathrm{cTr}_C\, M).$$

Because

$$\begin{array}{ccc} C \otimes_S I^0(M)_* & \xrightarrow{1_C \otimes f^0{}_*} & C \otimes_S I^1(M)_* \\ \downarrow {\scriptstyle \theta_{I^0(M)}} & & \downarrow {\scriptstyle \theta_{I^1(M)}} \\ I^0(M) & \xrightarrow{f^0} & I^1(M) \end{array}$$

is a commutative diagram, we have

$$f^0 \cdot \theta_{I^0(M)} = \theta_{I^1(M)} \cdot (1_C \otimes f^0{}_*).$$

Because $f^0{}_* = \alpha' \cdot \pi'$, we get

$$1_C \otimes f^0{}_* = 1_C \otimes (\alpha' \cdot \pi') = (1_C \otimes \alpha') \cdot (1_C \otimes \pi').$$

Thus we have

$$\begin{aligned} \alpha \cdot h \cdot (1_C \otimes \pi') &= \alpha \cdot \pi \cdot \theta_{I^0(M)} = f^0 \cdot \theta_{I^0(M)} = \theta_{I^1(M)} \cdot (1_C \otimes f^0{}_*) \\ &= \theta_{I^1(M)} \cdot (1_C \otimes \alpha') \cdot (1_C \otimes \pi'). \end{aligned}$$

Because $1_C \otimes \pi'$ is epic, we get $\alpha \cdot h = \theta_{I^1(M)} \cdot (1_C \otimes \alpha')$. Notice that α is monic and $\theta_{I^1(M)}$ is an isomorphism (by Lemma 2.5 (2)), so

$$\mathrm{Coker}\, \theta_M \cong \mathrm{Ker}\, h \cong \mathrm{Ker}(1_C \otimes \alpha') \cong \mathrm{Tor}_1^S(C, \mathrm{cTr}_C\, M).$$

Consequently we obtain the desired exact sequence. □

For any $n \geq 1$, recall from [20] that a finitely presented left R-module M is called *n-C-torsionfree* if $\mathrm{Ext}_S^i(\mathrm{Tr}_C\, M, C) = 0$ for any $1 \leq i \leq n$. When $R = S$ and ${}_RC_S = {}_RR_R$, an n-C-torsionfree module is just an *n-torsionfree module* ([1]). We introduce the dual notion of n-C-torsionfree modules as follows.

Definition 3.3. Let $M \in \operatorname{Mod} R$ and $n \geq 1$. Then M is called *n-C-cotorsionfree* if $\operatorname{Tor}_i^S(C, \operatorname{cTr}_C M) = 0$ for any $1 \leq i \leq n$; and M is called *∞-C-cotorsionfree* if it is n-C-cotorsionfree for all n. In particular, every left R-module is 0-C-cotorsionfree.

It is trivial that a left R-module is n-C-cotorsionfree if it is m-C-cotorsionfree for some $m \geq n$. It is easy to verify that the class of n-C-cotorsionfree R-modules is closed under direct summands and finite direct sums.

Note that for any $M \in \operatorname{Mod} R$, there exists an exact sequence

$$0 \longrightarrow M_* \longrightarrow I^0(M)_* \xrightarrow{f^0{}_*} I^1(M)_* \longrightarrow \operatorname{cTr}_C M \longrightarrow 0.$$

The following corollary is an immediate consequence of Proposition 3.2.

Corollary 3.4. *Let $M \in \operatorname{Mod} R$. Then we have:*

(1) *M is 1-C-cotorsionfree if and only if it is C-cotorsionless.*

(2) *M is 2-C-cotorsionfree if and only if it is C-coreflexive.*

(3) *For any $n \geq 3$, M is n-C-cotorsionfree if and only if it is C-coreflexive and $\operatorname{Tor}_i^S(C, M_*) = 0$ for any $1 \leq i \leq n-2$.*

Proposition 3.5. *Let $0 \to L \to M \to N \to 0$ be a $\operatorname{Hom}_R(C, -)$-exact exact sequence in $\operatorname{Mod} R$ with L being n-C-cotorsionfree. Then M is n-C-cotorsionfree if and only if so is N.*

Proof. By assumption we have an exact sequence $0 \to L_* \to M_* \to N_* \to 0$ in $\operatorname{Mod} S$. Then we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & C \otimes_S L_* & \longrightarrow & C \otimes_S M_* & \longrightarrow & C \otimes_S N_* & \longrightarrow & 0 \\
 & & \downarrow \theta_L & & \downarrow \theta_M & & \downarrow \theta_N & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

and the following exact sequence:

$$\operatorname{Tor}_i^S(C, L_*) \to \operatorname{Tor}_i^S(C, M_*) \to \operatorname{Tor}_i^S(C, N_*) \to \operatorname{Tor}_{i-1}^S(C, L_*)$$

for any $i \geq 2$. Now the assertion follows easily from the snake lemma and Corollary 3.4. □

Let $\mathcal{X}$ be a subclass of $\operatorname{Mod} R$ and $M \in \operatorname{Mod} R$. Following Enochs–Jenda [7], a homomorphism $\phi : X \to M$ in $\operatorname{Mod} R$ with $X \in \mathcal{X}$ is called an *$\mathcal{X}$-precover* of M if $\operatorname{Hom}_R(X', \phi) : \operatorname{Hom}_R(X', X) \to \operatorname{Hom}_R(X', M)$ is epic for any $X' \in \mathcal{X}$.

An $\mathcal{X}$-precover $\phi : X \to M$ is an $\mathcal{X}$*-cover* if every endomorphism $g : X \to X$ such that $\phi g = \phi$ is an isomorphism. Dually the notion of an $\mathcal{X}$*-(pre)envelope* of M is defined. Recall from [9] that an exact sequence (of finite or infinite length)

$$\cdots \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

in Mod R is called an $\mathcal{X}$*-resolution* of M if each $X_i \in \mathcal{X}$. Such an $\mathcal{X}$-resolution is called *proper* if $X_i \twoheadrightarrow \mathrm{Im}(X_i \to X_{i-1})$ is an $\mathcal{X}$-precover of $\mathrm{Im}(X_i \to X_{i-1})$ (note: $X_{-1} = M$ for any $i \geq 0$). Dually, the notion of an $\mathcal{X}$*-coresolution* of M is defined. The $\mathcal{X}$*-injective dimension* $\mathcal{X}\text{-id}_R(M)$ of M is defined as the infimum of the set of all n such that there exists an $\mathcal{X}$-coresolution

$$0 \to M \to X^0 \to X^1 \to \cdots \to X^n \to 0$$

of M in Mod R.

Lemma 3.6. *Let $M \in$ Mod R. Then we have:*

(1) *M is 1-C-cotorsionfree if and only if M admits an epic* $\mathrm{Add}_R\, C$*-precover in* Mod R.

(2) *M is 2-C-cotorsionfree if and only if there exists a proper* $\mathrm{Add}_R\, C$*-resolution* $W_1 \to W_0 \to M \to 0$ *of M in* Mod R.

Proof. (1) Let M be 1-C-cotorsionfree. Then θ_M is epic by Corollary 3.4. Since there exists an epimorphism $S^{(X)} \twoheadrightarrow M_*$, we get an epimorphism

$$C^{(X)} \twoheadrightarrow C \otimes_S M_*,$$

which induces an epimorphism $C^{(X)} \twoheadrightarrow M$ because θ_M is epic. By [10, Proposition 5.3], every module in Mod R admits an $\mathrm{Add}_R\, C$-precover. It follows that M admits an epic $\mathrm{Add}_R\, C$-precover.

Conversely, let $W_0 \twoheadrightarrow M$ be an epic $\mathrm{Add}_R\, C$-precover of M. Because θ_{W_0} is an isomorphism by Lemma 2.5 (2), from the commutative diagram with exact rows

$$\begin{array}{ccccc} C \otimes_S W_{0*} & \longrightarrow & C \otimes_S M_* & \longrightarrow & 0 \\ \downarrow{\scriptstyle \theta_{W_0}} & & \downarrow{\scriptstyle \theta_M} & & \\ W_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

we get that θ_M is epic and M is 1-C-cotorsionfree.

(2) Let M be 2-C-cotorsionfree. By (1), there exists an exact sequence

$$0 \to N \to W_0 \to M \to 0$$

in Mod R with $W_0 \twoheadrightarrow M$ an $\mathrm{Add}_R\, C$-precover of M. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & C\otimes_S N_* & \longrightarrow & C\otimes_S W_{0*} & \longrightarrow & C\otimes_S M_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_N} & & \downarrow{\scriptstyle \theta_{W_0}} & & \downarrow{\scriptstyle \theta_M} & & \\ 0 & \longrightarrow & N & \longrightarrow & W_0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Because both θ_{W_0}, θ_M are isomorphisms by Lemma 2.5 (1) and Corollary 3.4 (2), θ_N is epic by the snake lemma, and hence N is 1-C-cotorsionfree by Corollary 3.4 (1). It follows from (1) that N admits an epic $\mathrm{Add}_R\, C$-precover $W_1 \twoheadrightarrow N$. Then the spliced sequence $W_1 \to W_0 \to M \to 0$ is as desired.

Conversely, let $W_1 \to W_0 \to M \to 0$ be a proper $\mathrm{Add}_R\, C$-resolution of M. Put $N = \mathrm{Ker}(W_0 \to M)$. Then N is 1-C-cotorsionfree by (1), and so θ_N is epic by Corollary 3.4 (1). Now the commutative diagram above implies that θ_M is an isomorphism. Thus M is 2-C-cotorsionfree by Corollary 3.4 (2). □

In the following result we give an equivalent characterization of n-C-cotorsionfree modules in terms of proper $\mathrm{Add}_R\, C$-resolutions of modules. It is dual to [20, Corollary 3.3].

Proposition 3.7. *Let $M \in \mathrm{Mod}\, R$ and $n \geq 1$. Then M is n-C-cotorsionfree if and only if there exists a proper* $\mathrm{Add}_R\, C$*-resolution*

$$W_{n-1} \to \cdots \to W_1 \to W_0 \to M \to 0$$

of M in $\mathrm{Mod}\, R$.

Proof. We proceed by induction on n. Note that the case for $n \leq 2$ follows from Lemma 3.6. Now suppose $n \geq 3$.

If M is n-C-cotorsionfree, then θ_M is an isomorphism and $\mathrm{Tor}_i^S(C, M_*) = 0$ for any $1 \leq i \leq n-2$ by Corollary 3.4 (3). In addition, by Lemma 3.6 (1) there exists an exact sequence $0 \to N \to W_0 \to M \to 0$ in $\mathrm{Mod}\, R$ with $W_0 \in \mathrm{Add}_R\, C$ such that $0 \to N_* \to W_{0*} \to M_* \to 0$ is also exact with W_{0*} projective. Then $\mathrm{Tor}_i^S(C, N_*) \cong \mathrm{Tor}_{i+1}^S(C, M_*) = 0$ for $1 \leq i \leq n-3$, and we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C\otimes_S N_* & \longrightarrow & C\otimes_S W_{0*} & \longrightarrow & C\otimes_S M_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_N} & & \downarrow{\scriptstyle \theta_{W_0}} & & \downarrow{\scriptstyle \theta_M} & & \\ 0 & \longrightarrow & N & \longrightarrow & W_0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Because θ_{W_0} is an isomorphism by Lemma 2.5 (1), θ_N is also an isomorphism. Thus N is $(n-1)$-C-cotorsionfree by Corollary 3.4 (3) and therefore the assertion follows from the induction hypothesis.

Conversely, assume that there exists a proper $\mathrm{Add}_R\, C$-resolution

$$W_{n-1} \to \cdots \to W_1 \to W_0 \to M \to 0$$

of M in Mod R. Put $N = \mathrm{Im}(W_1 \to W_0)$. Then $0 \to N_* \to W_{0*} \to M_* \to 0$ is exact with W_{0*} projective. Because N is $(n-1)$-C-cotorsionfree by the induction hypothesis, θ_N is an isomorphism and $\mathrm{Tor}_i^S(C, N_*) = 0$ for any $1 \leq i \leq n-3$ by Corollary 3.4 (3).

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & C \otimes_S N_* & \longrightarrow & C \otimes_S W_{0*} & \longrightarrow & C \otimes_S M_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_N} & & \downarrow{\scriptstyle \theta_{W_0}} & & \downarrow{\scriptstyle \theta_M} & & \\ 0 & \longrightarrow & N & \longrightarrow & W_0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Because θ_{W_0} is an isomorphism by Lemma 2.5 (1), θ_M is an isomorphism and $0 \to C \otimes_S N_* \to C \otimes_S W_{0*} \to C \otimes_S M_* \to 0$ is exact. So $\mathrm{Tor}_1^S(C, M_*) = 0$ and $\mathrm{Tor}_{i+1}^S(C, M_*) \cong \mathrm{Tor}_i^S(C, N_*) = 0$ for any $1 \leq i \leq n-3$, that is,

$$\mathrm{Tor}_i^S(C, M_*) = 0 \quad \text{for any } 1 \leq i \leq n-2.$$

Thus M is n-C-cotorsionfree by Corollary 3.4 (3). □

As an immediate consequence of Proposition 3.7 we have the following

Corollary 3.8. *For a module $M \in$ Mod R, the following statements are equivalent.*

(1) *M is 1-C-cotorsionfree (that is, M is C-cotorsionless).*

(2) *There exists an exact sequence*

$$0 \to N \to W \to M \to 0$$

in Mod R *with $W \in \mathrm{Add}_R\, C$ and $\mathrm{Ext}_R^1(C, N) = 0$.*

(3) *There exists an epimorphism $W \twoheadrightarrow M$ in* Mod R *with $W \in \mathrm{Add}_R\, C$.*

It follows from Proposition 3.7 that a module $M \in$ Mod R is ∞-C-cotorsionfree if and only if M has an exact proper $\mathrm{Add}_R\, C$-resolution

$$\cdots \to W_2 \to W_1 \to W_0 \to M \to 0$$

in Mod R. A module $M \in$ Mod R is called *n-C-cospherical* if $\mathrm{Ext}_R^i(C, M) = 0$ for and $1 \leq i \leq n$, and M is called *∞-C-cospherical* if it is n-C-cospherical for all n. The following result shows that the Bass class with respect to C coincides with the intersection of the class of ∞-C-cotorsionfree modules and that of ∞-C-cospherical modules.

Theorem 3.9. *For a module $M \in \operatorname{Mod} R$, the following statements are equivalent.*

(1) *M is ∞-C-cotorsionfree and ∞-C-cospherical.*

(2) *$M \in \mathcal{B}_C(R)$.*

Proof. By Proposition 3.7 and [10, Theorem 6.1]. □

Auslander and Bridger obtained in [1, Proposition 2.21] an approximation theorem for finitely generated modules over left and right noetherian rings. Takahashi in [20, Theorem A] got a semiduazlizing version of this result. We dualize [20, Theorem A] as follows.

Theorem 3.10. *Let $M \in \operatorname{Mod} R$ and $n \geq 1$. Then the following statements are equivalent.*

(1) $\operatorname{co}\Omega^n(M)$ *is n-C-cotorsionfree.*

(2) *There exists an exact sequence*

$$0 \to M \to X \to Y \to 0$$

in $\operatorname{Mod} R$ *such that X is n-C-cospherical and* $\operatorname{Add}_R C\text{-id}_R\, Y \leq n-1$.

Proof. (1) $\Rightarrow$ (2) By Proposition 3.7 and Corollary 3.8, the fact that $\operatorname{co}\Omega^n(M)$ is n-C-cotorsionfree implies that there exists an exact sequence in $\operatorname{Mod} R$

$$0 \to N_0 \to W_0 \to \operatorname{co}\Omega^n(M) \to 0$$

with $W_0 \in \operatorname{Add}_R C$, N_0 being $(n-1)$-C-cotorsionfree and $\operatorname{Ext}^1_R(C, N_0) = 0$. We get the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \operatorname{co}\Omega^{n-1}(M) & = & \operatorname{co}\Omega^{n-1}(M) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_0 & \longrightarrow & X_0 & \longrightarrow & I^{n-1}(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_0 & \longrightarrow & W_0 & \longrightarrow & \operatorname{co}\Omega^n(M) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

If $n = 1$, then the middle column is the desired sequence.

Let now $n \geq 2$. Since $I^{n-1}(M) \in \mathcal{B}_C(R)$, $I^{n-1}(M)$ is ∞-C-cotorsionfree by Theorem 3.9. Note that N_0 is $(n-1)$-C-cotorsionfree and $\operatorname{Ext}^1_R(C, N_0) = 0$. By Proposition 3.5, X_0 is $(n-1)$-C-cotorsionfree. Therefore there exists an exact sequence $0 \to Z_0 \to U_0 \to X_0 \to 0$ in $\operatorname{Mod} R$ with $U_0 \in \operatorname{Add}_R C$, Z_0 being $(n-2)$-C-cotorsionfree and $\operatorname{Ext}^1_R(C, Z_0) = 0$ by Proposition 3.7. We construct the pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Z_0 & = & Z_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y_0 & \longrightarrow & U_0 & \longrightarrow & W_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \operatorname{co}\Omega^{n-1}(M) & \longrightarrow & X_0 & \longrightarrow & W_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

such that $\operatorname{Add}_R C$-$\operatorname{id}_R Y_0 \leq 1$ and $\operatorname{Ext}^i_R(C, Z_0) = 0$ for $i = 1, 2$ because we have $\operatorname{Ext}^1_R(C, X_0) = 0$. Using the leftmost column in this diagram, we also have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \operatorname{co}\Omega^{n-2}(M) & = & \operatorname{co}\Omega^{n-2}(M) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_0 & \longrightarrow & X_1 & \longrightarrow & I^{n-2}(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_0 & \longrightarrow & Y_0 & \longrightarrow & \operatorname{co}\Omega^{n-1}(M) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

It follows from the middle row in the above diagram that $\operatorname{Ext}^i_R(C, X_1) = 0$ for $i = 1, 2$. Therefore, if $n = 2$, then the middle column in the above diagram is the desired exact sequence.

Let now $n \geq 3$. Since Z_0 is $(n-2)$-C-cotorsionfree and $\operatorname{Ext}^1_R(C, Z_0) = 0$, it follows that X_1 is $(n-2)$-C-cotorsionfree by Proposition 3.5. We have an exact sequence $0 \to Z_1 \to U_1 \to X_1 \to 0$ in $\mathrm{Mod}\, R$ with $U_1 \in \mathrm{Add}_R\, C$, Z_1 being $(n-3)$-C-cotorsionfree and $\mathrm{Ext}^1_R(C, Z_1) = 0$ by Proposition 3.7 again. Iterating the above construction of pullback diagrams, we eventually obtain the desired exact sequence.

(2) $\Rightarrow$ (1) Since $\mathrm{Add}_R\, C$-$\mathrm{id}_R\, Y \le n-1$, there exists an exact sequence

$$0 \longrightarrow Y \xrightarrow{d_0} W^0 \xrightarrow{d_1} W^1 \longrightarrow \cdots \xrightarrow{d_{n-1}} W^{n-1} \longrightarrow 0$$

in $\mathrm{Mod}\, R$ with all $W^i \in \mathrm{Add}_R\, C$. Set $Y_i = \mathrm{Im}\, d_i$ for each i. Then we have the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & I^0(M) & \longrightarrow & \mathrm{co}\,\Omega^1(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & H_0 & \longrightarrow & \mathrm{co}\,\Omega^1(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y & = & Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

We now conclude that $H_0 \cong Y \oplus I^0(M)$. Adding $I^0(M)$ to the exact sequence $0 \to Y \to W^0 \to Y_1 \to 0$, we get an exact sequence

$$0 \to Y \oplus I^0(M) \to W^0 \oplus I^0(M) \to Y_1 \to 0.$$

Thus the following two pushout diagrams are obtained:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y \oplus I^0(M) & \longrightarrow & \mathrm{co}\,\Omega^1(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & W^0 \oplus I^0(M) & \longrightarrow & X_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y_1 & = & Y_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{co}\Omega^1(M) & \longrightarrow & I^1(M) & \longrightarrow & \operatorname{co}\Omega^2(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X_1 & \longrightarrow & H_1 & \longrightarrow & \operatorname{co}\Omega^2(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y_1 & = & Y_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Repeating the procedure in this way yields the exact sequence

$$0 \to X_i \to W^i \oplus I^i(M) \to X_{i+1} \to 0$$

for any $0 \leq i \leq n-1$, where $X_0 = X$. Since $\operatorname{Ext}^i_R(C, X_0) = 0$ for any $1 \leq i \leq n$ by assumption, we get $\operatorname{Ext}^j_R(C, X_i) = 0$ for any $1 \leq j \leq n-i$. Then there exists an exact sequence

$$0 \to X_{i*} \to (W^i \oplus I^i(M))_* \to X_{i+1*} \to 0$$

for any $0 \leq i \leq n-1$. By Lemma 2.5, each $\theta_{W^i \oplus I^i(M)}$ is an isomorphism. Now we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
C \otimes_S (W^0 \oplus I^0(M))_* & \longrightarrow & C \otimes_S X_{1*} & \longrightarrow & 0 \\
\downarrow {\scriptstyle \theta_{W^0 \oplus I^0(M)}} & & \downarrow {\scriptstyle \theta_{X_1}} & & \\
W^0 \oplus I^0(M) & \longrightarrow & X_1 & \longrightarrow & 0.
\end{array}$$

It follows that θ_{X_1} is epic and so X_1 is 1-C-cotorsionfree by Corollary 3.4 (1). Also, there exists the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & C \otimes_S X_{1*} & \longrightarrow & C \otimes_S (W^1 \oplus I^1(M))_* & \longrightarrow & C \otimes_S X_{2*} & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \theta_{X_1}} & & \downarrow {\scriptstyle \theta_{W^1 \oplus I^1(M)}} & & \downarrow {\scriptstyle \theta_{X_2}} & & \\
0 & \longrightarrow & X_1 & \longrightarrow & W^1 \oplus I^1(M) & \longrightarrow & X_2 & \longrightarrow & 0.
\end{array}$$

So θ_{X_2} is an isomorphism and hence X_2 is 2-C-cotorsionfree by Corollary 3.4 (2).

Furthermore, there exists the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \longrightarrow \mathrm{Tor}_1^S(C, X_{3*}) & \longrightarrow & C \otimes_S X_{2*} & \longrightarrow & C \otimes_S (W^2 \oplus I^2(M))_* & \longrightarrow & C \otimes_S X_{3*} & \longrightarrow & 0 \\
 & & \downarrow \theta_{X_2} & & \downarrow \theta_{W^2 \oplus I^2(M)} & & \downarrow \theta_{X_3} & & \\
0 & \longrightarrow & X_2 & \longrightarrow & W^2 \oplus I^2(M) & \longrightarrow & X_3 & \longrightarrow & 0.
\end{array}$$

So θ_{X_3} is an isomorphism and $\mathrm{Tor}_1^S(C, X_{3*}) = 0$, and hence X_3 is 3-C-cotorsionfree by Corollary 3.4 (3). Repeating a similar argument, we eventually get that $\mathrm{co}\,\Omega^n(M) \cong X_n$ is n-C-cotorsionfree. □

The following result is an addendum to Theorem 3.10.

Proposition 3.11. *Let* $M \in \mathrm{Mod}\, R$ *and* $n \geq 1$. *If* $\mathrm{co}\,\Omega^n(M)$ *is* ∞-C*-cotorsionfree, then there exists an exact sequence* $0 \to M \to X \to Y \to 0$ *in* $\mathrm{Mod}\, R$ *with* X *being* ∞*-cotorsionfree and* $\mathrm{Add}_R\, C\text{-}\mathrm{id}_R\, Y \leq n-1$.

Proof. We proceed by induction on n.

Let $n = 1$. Since $\mathrm{co}\,\Omega^1(M)$ is ∞-C-cotorsionfree by assumption, there exists an exact sequence

$$0 \to N_1 \to W_1 \to \mathrm{co}\,\Omega^1(M) \to 0$$

in $\mathrm{Mod}\, R$ with $W_1 \in \mathrm{Add}_R\, C$, N_1 being ∞-C-cotorsionfree and $\mathrm{Ext}_R^1(C, N_1) = 0$ by Proposition 3.7. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N_1 & = & N_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & W_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & I^0(M) & \longrightarrow & \mathrm{co}\,\Omega^1(M) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

It follows from Proposition 3.5 that the middle row in the above diagram is the desired sequence.

Now suppose $n \geq 2$. By the induction hypothesis, there exists an exact sequence

$$0 \to \operatorname{co}\Omega^1(M) \to X' \to Y' \to 0$$

in $\operatorname{Mod} R$ with X' being ∞-C-cotorsionfree and $\operatorname{Add}_R C$-$\operatorname{id}_R Y' \leq n-2$. We also have an exact sequence

$$0 \to X'' \to W' \to X' \to 0$$

in $\operatorname{Mod} R$ with $W' \in \operatorname{Add}_R C$, X'' being ∞-C-cotorsionfree and $\operatorname{Ext}^1_R(C, X'') = 0$ by Proposition 3.7. We have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & X'' & = & X'' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y & \longrightarrow & W' & \longrightarrow & Y' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \operatorname{co}\Omega^1(M) & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Then $\operatorname{Add}_R C$-$\operatorname{id}_R Y \leq n-1$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X'' & = & X'' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & I^0(M) & \longrightarrow & \operatorname{co}\Omega^1(M) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Note that the middle column in this diagram is $\operatorname{Hom}_R(C, -)$-exact. Hence X is ∞-C-cotorsionfree by Proposition 3.5. Therefore the middle row in this diagram is as desired. □

4 Cograde and cotorsionfreeness

In this section, for a module $M \in \operatorname{Mod} R$ and a positive integer n, we will give a criterion in terms of the properties of the cograde of modules for judging when $\operatorname{co}\Omega^i(M)$ is i-C-cotorsionfree for any $1 \leq i \leq n$.

Let $M \in \operatorname{Mod} R$ and $n \geq 1$. From the exact sequence

$$0 \longrightarrow \operatorname{co}\Omega^{n-1}(M) \xrightarrow{\lambda^{n-1}} I^{n-1}(M) \xrightarrow{p^n} \operatorname{co}\Omega^n(M) \longrightarrow 0 \tag{4.1}$$

we get the following exact sequence:

$$\begin{array}{ccccccc} 0 \longrightarrow & (\operatorname{co}\Omega^{n-1}(M))_* & \xrightarrow{{\lambda^{n-1}}_*} & I^{n-1}(M)_* & \xrightarrow{{p^n}_*} & (\operatorname{co}\Omega^n(M))_* & \\ & & & & & \downarrow & \\ & & & & & \operatorname{Ext}^n_S(C, M) & \longrightarrow 0. \end{array}$$

Set $\operatorname{Im} {p^n}_* = N$, and decompose this sequence into two short exact sequences:

$$0 \longrightarrow (\operatorname{co}\Omega^{n-1}(M))_* \xrightarrow{{\lambda^{n-1}}_*} I^{n-1}(M)_* \xrightarrow{\beta} N \longrightarrow 0 \tag{4.2}$$

and

$$0 \longrightarrow N \xrightarrow{\alpha} (\operatorname{co}\Omega^n(M))_* \longrightarrow \operatorname{Ext}^n_R(C, M) \longrightarrow 0.$$

Then we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & C \otimes_S (\operatorname{co}\Omega^{n-1}(M))_* & \xrightarrow{1_C \otimes {\lambda^{n-1}}_*} & C \otimes_S I^{n-1}(M)_* & \xrightarrow{1_C \otimes \beta} & C \otimes_S N & \longrightarrow 0 \\ & \downarrow{\scriptstyle \theta_{\operatorname{co}\Omega^{n-1}(M)}} & & \downarrow{\scriptstyle \theta_{I^{n-1}(M)}} & & \downarrow{\scriptstyle g} & \\ 0 \longrightarrow & \operatorname{co}\Omega^{n-1}(M) & \xrightarrow{\lambda^{n-1}} & I^{n-1}(M) & \xrightarrow{p^n} & \operatorname{co}\Omega^n(M) & \to 0. \end{array} \tag{4.3}$$

Then it is straightforward to check that there exists the following commutative diagram with exact rows:

$$\begin{array}{ccccc} C \otimes_S N & \xrightarrow{1_C \otimes \alpha} & C \otimes_S (\operatorname{co}\Omega^n(M))_* & \longrightarrow C \otimes_S \operatorname{Ext}^n_R(C, M) & \longrightarrow 0 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \theta_{\operatorname{co}\Omega^n(M)}} & & \\ \operatorname{co}\Omega^n(M) & = & \operatorname{co}\Omega^n(M). & & \end{array} \tag{4.4}$$

Lemma 4.1. *For a module $M \in \operatorname{Mod} R$, we have:*

(1) $\operatorname{co}\Omega^1(M)$ *is* 1-C*-cotorsionfree.*

(2) *For any* $n \geq 2$, $\operatorname{Ker}\theta_{\operatorname{co}\Omega^n(M)} \cong C \otimes_S \operatorname{Ext}_R^n(C, M)$.

Proof. (1) Since $I^0(M)$ is ∞-C-cotorsionfree by Theorem 3.9, the assertion follows from Corollary 3.8.

(2) If $n \geq 2$, then $\theta_{\operatorname{co}\Omega^{n-1}(M)}$ is an epimorphism by (1). Because $\theta_{I^{n-1}(M)}$ is an isomorphism by Theorem 3.9, g is an isomorphism in the above two diagrams. So $\operatorname{Ker}\theta_{\operatorname{co}\Omega^n(M)} \cong C \otimes_S \operatorname{Ext}_R^n(C, M)$. □

The notion of the cograde of finitely generated modules has been introduced in [14, Corollary 3.11]. The following definition generalizes it to a general setting.

Definition 4.2. For a module $N \in \operatorname{Mod} S$, the *cograde* of N with respect to C is defined by $\operatorname{cograde}_C N := \inf\{i : \operatorname{Tor}_i^S(C, N) \neq 0\}$.

We are now in a position to give the main result in this section, which can be regarded as a dual version of [1, Proposition 2.26].

Theorem 4.3. *Let* $M \in \operatorname{Mod} R$ *and* $n \geq 1$. *Then* $\operatorname{co}\Omega^i(M)$ *is* i-C*-cotorsionfree for any* $1 \leq i \leq n$ *if and only if* $\operatorname{cograde}_C \operatorname{Ext}_R^i(C, M) \geq i - 1$ *for any* $1 \leq i \leq n$.

Proof. We proceed by induction on n. If $n = 1$, then the assertion follows from Lemma 4.1 (1).

Let $n = 2$. Then $\operatorname{co}\Omega^2(M)$ is 2-C-cotorsionfree if and only if $\theta_{\operatorname{co}\Omega^2(M)}$ is an isomorphism by Corollary 3.4 (2). Note that $\theta_{\operatorname{co}\Omega^2(M)}$ is epic by Lemma 4.1 (1). So $\operatorname{co}\Omega^2(M)$ is 2-C-cotorsionfree if and only if $\theta_{\operatorname{co}\Omega^2(M)}$ is monic. But

$$\operatorname{Ker}\theta_{\operatorname{co}\Omega^2(M)} \cong C \otimes_S \operatorname{Ext}_R^2(C, M)$$

by Lemma 4.1 (2). So $\operatorname{co}\Omega^2(M)$ is 2-C-cotorsionfree if and only if

$$C \otimes_S \operatorname{Ext}_S^2(C, M) = 0,$$

that is,

$$\operatorname{cograde}_C \operatorname{Ext}_S^2(C, M) \geq 1.$$

Now suppose $n \geq 3$. If $\operatorname{co}\Omega^i(M)$ is i-C-cotorsionfree for $1 \leq i \leq n$, then by the induction hypothesis, it suffices to show that $\operatorname{cograde}_C \operatorname{Ext}_S^n(C, M) \geq n - 1$. By Lemma 4.1 (2),

$$C \otimes_S \operatorname{Ext}_R^n(C, M) \cong \operatorname{Ker}\theta_{\operatorname{co}\Omega^n(M)} = 0.$$

From the exact sequence (4.1) we get the following exact sequence:

$$\begin{array}{ccccc} \mathrm{Tor}_1^S(C,(\mathrm{co}\,\Omega^n(M))_*) & \longrightarrow & \mathrm{Tor}_1^S(C,\mathrm{Ext}_R^n(C,M)) & \longrightarrow & C\otimes_S N \\ & & & & \downarrow{\scriptstyle 1_C\otimes\alpha} \\ & & & & C\otimes_S(\mathrm{co}\,\Omega^n(M))_* \\ & & & & \downarrow \\ & & & & C\otimes_S \mathrm{Ext}_R^n(C,M) \\ & & & & \downarrow \\ & & & & 0. \end{array}$$

Because both $\theta_{\mathrm{co}\,\Omega^{n-1}(M)}$ and $\theta_{I^{n-1}(M)}$ are isomorphisms, the homomorphism g in (4.3) is also an isomorphism. Then from (4.4) we know that $1_C\otimes\alpha$ is monic. In addition, by Corollary 3.4 (3) we have

$$\mathrm{Tor}_i^S(C,(\mathrm{co}\,\Omega^n(M))_*)=0 \quad \text{for any } 1\le i\le n-2.$$

So $\mathrm{Tor}_1^S(C,\mathrm{Ext}_R^n(C,M))=0$, and hence

$$\mathrm{cograde}_C\,\mathrm{Ext}_R^n(C,M)\ge 2.$$

From the exact sequence (4.1) we get the following exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathrm{co}\,\Omega^{n-1}(M))_* & \xrightarrow{\lambda^{n-1}{}_*} & I^{n-1}(M)_* & \xrightarrow{p^n{}_*} & (\mathrm{co}\,\Omega^n(M))_* \\ & & & & & & \downarrow \\ & & & & & & \mathrm{Ext}_S^n(C,M) \\ & & & & & & \downarrow \\ & & & & & & 0. \end{array} \tag{4.5}$$

By Theorem 3.9 and Corollary 3.4 (3), $\mathrm{Tor}_i^S(C,I^{n-1}(M)_*)=0$ for any $i\ge 1$. Again by Corollary 3.4 (3) we have

$$\mathrm{Tor}_i^S(C,(\mathrm{co}\,\Omega^{n-1}(M))_*)=0 \quad \text{for any } 1\le i\le n-3.$$

So by the dimension shifting, $\mathrm{Tor}_i^S(C,\mathrm{Ext}_R^n(C,M))=0$ for any $3\le i\le n-2$, and hence

$$\mathrm{cograde}_C\,\mathrm{Ext}_R^n(C,M)\ge n-1.$$

Conversely, if $\text{cograde}_C \operatorname{Ext}^i_R(C, M) \geq i - 1$ for any $1 \leq i \leq n$, then by the induction hypothesis, it suffices to show that $\operatorname{co}\Omega^n(M)$ is n-C-cotorsionfree. Since $\operatorname{co}\Omega^{n-1}(M)$ is $(n-1)$-C-cotorsionfree by the induction hypothesis, it follows that $\theta_{\operatorname{co}\Omega^{n-1}(M)}$ is an isomorphism. Notice that $\theta_{I^{n-1}(M)}$ is also an isomorphism, so is the homomorphism g in (4.3). Because $\text{cograde}_C \operatorname{Ext}^n_R(C, M) \geq n-1$ by assumption, $1_C \otimes \alpha$ in (4.4) is an isomorphism. It implies that $\theta_{\operatorname{co}\Omega^n(M)}$ is also an isomorphism and $\operatorname{co}\Omega^n(M)$ is C-coreflexive. On the other hand, similar to the above argument, using the dimension shifting, from the exact sequence (4.5) we get that $\operatorname{Tor}^S_i(C, (\operatorname{co}\Omega^n(M))_*) = 0$ for any $1 \leq i \leq n-2$. Then we conclude that $\operatorname{co}\Omega^n(M)$ is n-C-cotorsionfree by Corollary 3.4 (3). □

5 Special cotorsionfree modules over Artin algebras

Throughout this section, Λ is an Artin R-algebra over a commutative Artin ring R. Let $\operatorname{mod}\Lambda$ be the class of finitely generated left Λ-modules. We denote by D the ordinary Matlis duality between $\operatorname{mod}\Lambda^{\text{op}}$ and $\operatorname{mod}\Lambda$, that is,

$$D(-) := \operatorname{Hom}_R(-, I^0(R/J(R))),$$

where $J(R)$ is the Jacobson radical of R and $I^0(R/J(R))$ is the injective envelope of $R/J(R)$. It is easy to verify that the (Λ, Λ)-bimodule $D(\Lambda)$ is semidualizing. We use $\operatorname{add} D(\Lambda)$ to denote the subclass of $\operatorname{mod}\Lambda$ consisting of modules isomorphic to direct summands of finite direct sums of copies of $D(\Lambda)$. We use the abbreviation $\operatorname{cTr}(-)$ for $\operatorname{cTr}_{D(\Lambda)}(-)$. Let $A \in \operatorname{mod}\Lambda$ and $n \geq 1$. Then A is called

- *n-cotorsionfree* if $\operatorname{Tor}^\Lambda_i(D(\Lambda), \operatorname{cTr} A) = 0$ for any $1 \leq i \leq n$,
- *∞-cotorsionfree* if it is n-cotorsionfree for all n;

in particular, every module in $\operatorname{mod}\Lambda$ is 0-cotorsionfree. In addition, A is called

- *n-cospherical* if $\operatorname{Ext}^i_\Lambda(D(\Lambda), A) = 0$ for any $1 \leq i \leq n$,
- *∞-cospherical* if it is n-cospherical for all n.

Put $(-)^* := \operatorname{Hom}_\Lambda(-, \Lambda)$. The following result establishes the dual relation between the cotranspose (resp. n-cotorsionfree modules) and the transpose (resp. n-torsionfree modules).

Proposition 5.1. *Let $A \in \operatorname{mod}\Lambda$ and $n \geq 1$. Then we have:*

(1) $\operatorname{Tr} A \cong \operatorname{cTr} D(A)$,

(2) $\operatorname{cTr} A \cong \operatorname{Tr} D(A)$,

(3) *A is n-torsionfree if and only if $D(A)$ is n-cotorsionfree,*

(4) *A is n-cotorsionfree if and only if $D(A)$ is n-torsionfree.*

Proof. Because (2) and (4) are duals of (1) and (3) respectively, it suffices to prove (1) and (3).

(1) Let

$$P_1 \to P_0 \to A \to 0$$

be a minimal projective presentation of A in $\bmod \Lambda$. Then we have the exact sequence

$$0 \to A^* \to P_0^* \to P_1^* \to \operatorname{Tr} A \to 0,$$

and a minimal injective presentation

$$0 \to D(A) \to D(P_0) \to D(P_1)$$

of $D(A)$. Now we obtain another exact sequence:

$$\begin{array}{l} 0 \longrightarrow \operatorname{Hom}_\Lambda(D(\Lambda), D(A)) \longrightarrow \operatorname{Hom}_\Lambda(D(\Lambda), D(P_0)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad \operatorname{Hom}_\Lambda(D(\Lambda), D(P_1)) \longrightarrow \operatorname{cTr} D(A) \longrightarrow 0. \end{array}$$

Since $P_i^* \cong \operatorname{Hom}_\Lambda(D(\Lambda), D(P_i))$ for $i = 1, 2$, we get $\operatorname{Tr} A \cong \operatorname{cTr} D(A)$.

(3) For any $i \geq 1$, we have

$$\begin{aligned} \operatorname{Ext}^i_\Lambda(\operatorname{Tr} A, \Lambda) &\cong \operatorname{Ext}^i_\Lambda(\operatorname{Tr} A, \operatorname{Hom}_\Lambda(D(\Lambda), D(\Lambda))) \\ &\cong \operatorname{Hom}_\Lambda(\operatorname{Tor}_i^\Lambda(\operatorname{Tr} A, D(\Lambda)), D(\Lambda)), \end{aligned}$$

the second isomorphism by [6, Chapter VI, Proposition 5.1]. Note that $D(\Lambda)$ is an injective cogenerator for $\operatorname{Mod} \Lambda$. So, for any $i \geq 1$ we have that $\operatorname{Ext}^i_\Lambda(\operatorname{Tr} A, \Lambda) = 0$ if and only if $\operatorname{Tor}_i^\Lambda(\operatorname{Tr} A, D(\Lambda)) = 0$ and if and only if $\operatorname{Tor}_i^\Lambda(\operatorname{cTr} D(A), D(\Lambda)) = 0$ by Proposition 5.1 (1). It follows that A is n-torsionfree if and only if $D(A)$ is n-cotorsionfree. □

Note that a module in $\bmod \Lambda^{\mathrm{op}}$ is Gorenstein flat (see [7] for the definition) if and only if it is Gorenstein projective by [5, Proposition 1.3]. So the ordinary Matlis duality D between $\bmod \Lambda^{\mathrm{op}}$ and $\bmod \Lambda$ induces a duality between Gorenstein projective modules in $\bmod \Lambda^{\mathrm{op}}$ and Gorenstein injective modules in $\bmod \Lambda$ (cf. [9, Theorem 3.6]). Then by [7, Proposition 10.2.6] and Proposition 5.1, we immediately have the following

Corollary 5.2. *For a module $A \in \bmod \Lambda$, the following statements are equivalent.*

(1) *A is ∞-cotorsionfree and ∞-cospherical.*

(2) *There exists a totally* add $D(\Lambda)$*-acyclic complex* $\mathbf{I}$ *(as in Definition* 2.7*) such that $A \cong \operatorname{Im}(I_0 \to I^0)$.*

(3) *A is Gorenstein injective.*

Recall that Λ is called *Gorenstein* if $\mathrm{id}_{\Lambda}\,\Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}}\,\Lambda < \infty$, where $\mathrm{id}_{\Lambda}\,\Lambda$ and $\mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda$ are the left and right self-injective dimensions of Λ respectively.

Corollary 5.3. *The following statements are equivalent for any $n \geq 0$.*

(1) Λ *is Gorenstein with* $\mathrm{id}_{\Lambda}\,\Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}}\,\Lambda \leq n$.

(2) *The n-cosyzygy of a module in* $\mathrm{mod}\,\Lambda$ *and that of a module in* $\mathrm{mod}\,\Lambda^{\mathrm{op}}$ *are ∞-cotorsionfree.*

(3) *Every module in* $\mathrm{mod}\,\Lambda$ *and every module in* $\mathrm{mod}\,\Lambda^{\mathrm{op}}$ *are quotient modules of a left Λ-module and a right Λ-module with injective dimension at most n respectively.*

Proof. By Corollary 5.2 and [11, Theorem 1.4 and Lemma 3.8], using the duality functor D we get the assertion. □

The following example illustrates that the condition "∞-cotorsionfree" in (2) of Corollary 5.3 cannot be replaced by "n-cotorsionfree".

Example 5.4. Let Λ be a finite-dimensional algebra over an algebraically closed field given by the quiver

modulo the ideal generated by $\{\alpha^2, \beta^2, \alpha\beta, \beta\alpha\}$. Then Λ is not Gorenstein, but for any $A \in \mathrm{mod}\,\Lambda$, $\mathrm{co}\,\Omega^1(A)$ is 1-cotorsionfree.

Corollary 5.5. *If both R and Λ are local, then the following statements are equivalent.*

(1) Λ *is Gorenstein.*

(2) Λ *is self-injective.*

(3) *For* $A \in \mathrm{mod}\,\Lambda$ *and* $B \in \mathrm{mod}\,\Lambda^{\mathrm{op}}$, $D(\Lambda) \otimes_{\Lambda} \mathrm{cTr}\,A$ *and* $\mathrm{cTr}\,B \otimes_{\Lambda} D(\Lambda)$ *are Gorenstein injective.*

(4) *For* $A \in \mathrm{mod}\,\Lambda$ *and* $B \in \mathrm{mod}\,\Lambda^{\mathrm{op}}$, $D(\Lambda) \otimes_{\Lambda} \mathrm{cTr}\,A$ *and* $\mathrm{cTr}\,B \otimes_{\Lambda} D(\Lambda)$ *are ∞-cotorsionfree.*

Proof. The implication (1) ⇒ (2) follows from [15, Corollary 2.15], and (3) ⇒ (4) follows from Corollary 5.2.

Note that $D(\Lambda) \otimes_{\Lambda} \mathrm{cTr}\,A$ (resp. $\mathrm{cTr}\,B \otimes_{\Lambda} D(\Lambda)$) is isomorphic to the 2-cosyzygy of A (resp. B). So both implications (4) ⇒ (1) and (2) ⇒ (3) follow from Corollaries 5.3 and 5.2. □

We will denote by $\mathcal{GI}$ the class of finitely generated Gorenstein injective left Λ-modules. We write

$$^{\perp}\mathcal{GI} = \{M \in \operatorname{mod}\Lambda : \operatorname{Ext}_{\Lambda}^{\geq 1}(M, X) = 0 \text{ for any } X \in \mathcal{GI}\}$$

and

$$(^{\perp}\mathcal{GI})^{\perp} = \{M \in \operatorname{mod}\Lambda : \operatorname{Ext}_{\Lambda}^{\geq 1}(M, Y) = 0 \text{ for any } Y \in {}^{\perp}\mathcal{GI}\}.$$

Lemma 5.6. *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence in* $\operatorname{mod}\Lambda$. *If* $L, M \in {}^{\perp}\mathcal{GI}$, *then* $N \in {}^{\perp}\mathcal{GI}$.

Proof. By dimension shifting, we have $\operatorname{Ext}_{\Lambda}^{\geq 2}(N, A) = 0$ for any $A \in \mathcal{GI}$. Now let $X \in \mathcal{GI}$. It suffices to prove $\operatorname{Ext}_{\Lambda}^{1}(N, X) = 0$. By Corollary 5.2 there exists an exact sequence $0 \to K \to I_0 \to X \to 0$ in $\operatorname{mod}\Lambda$ with $I_0 \in \operatorname{add} D(\Lambda)$ and $K \in \mathcal{GI}$. So $\operatorname{Ext}_{\Lambda}^{1}(N, X) \cong \operatorname{Ext}_{\Lambda}^{2}(N, K) = 0$. □

Let $\mathcal{X}$ be a full subcategory of an abelian category $\mathcal{A}$. We write

$$^{\perp}\mathcal{X} = \{M \in \mathcal{A} : \operatorname{Ext}_{\mathcal{A}}^{\geq 1}(M, X) = 0 \text{ for any } X \in \mathcal{X}\}$$

and

$$\mathcal{X}^{\perp} = \{M \in \mathcal{A} : \operatorname{Ext}_{\mathcal{A}}^{\geq 1}(X, M) = 0 \text{ for any } X \in \mathcal{X}\}.$$

Recall that a pair of subcategories $(\mathcal{X}, \mathcal{Y})$ of an abelian category $\mathcal{A}$ is called a *cotorsion pair* if $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$. We denote by $\operatorname{GInj}(\Lambda)$ the subclass of $\operatorname{Mod}\Lambda$ consisting of Gorenstein injective modules, and write

$$^{\perp}\operatorname{GInj}(\Lambda) = \{M \in \operatorname{Mod}\Lambda : \operatorname{Ext}_{\Lambda}^{\geq 1}(M, X) = 0 \text{ for any } X \in \operatorname{GInj}(\Lambda)\}.$$

It is known that $(^{\perp}\operatorname{GInj}(\Lambda), \operatorname{GInj}(\Lambda))$ forms a cotorsion pair in $\operatorname{Mod}\Lambda$ (see [5]). The following result gives an equivalent characterization when $(^{\perp}\mathcal{GI}, \mathcal{GI})$ forms a cotorsion pair in $\operatorname{mod}\Lambda$.

Theorem 5.7. *The following statements are equivalent.*

(1) $(^{\perp}\mathcal{GI})^{\perp} = \mathcal{GI}$ *(that is,* $(^{\perp}\mathcal{GI}, \mathcal{GI})$ *forms a cotorsion pair).*

(2) *Every module in* $(^{\perp}\mathcal{GI})^{\perp}$ *is 1-cotorsionfree.*

(3) *Every module in* $(^{\perp}\mathcal{GI})^{\perp}$ *is* ∞*-cotorsionfree.*

Proof. The implication (1) ⇒ (2) follows from Corollary 5.2.

(2) ⇒ (3) Let $A \in (^{\perp}\mathcal{GI})^{\perp}$. Then A is 1-cotorsionfree by assumption. So there exists a $\operatorname{Hom}_{\Lambda}(\operatorname{add} D(\Lambda), -)$-exact exact sequence $0 \to K \to I_0 \to A \to 0$ in

mod Λ with $I_0 \in \operatorname{add} D(\Lambda)$ by Proposition 3.7. We claim that $K \in ({}^{\perp}\mathcal{GI})^{\perp}$. Let $Y \in {}^{\perp}\mathcal{GI}$. Then $\operatorname{Ext}^i_\Lambda(Y, K) \cong \operatorname{Ext}^{i-1}_\Lambda(Y, A)$ for $i \geq 2$. Note that $\operatorname{co}\Omega^1(Y) \in {}^{\perp}\mathcal{GI}$ by Lemma 5.6. Then from the exact sequence

$$\operatorname{Ext}^1_\Lambda(I^0(Y), K) \to \operatorname{Ext}^1_\Lambda(Y, K) \to \operatorname{Ext}^2_\Lambda(\operatorname{co}\Omega^1(Y), K)$$

we get $\operatorname{Ext}^1_\Lambda(Y, K) = 0$. The claim follows. So K is 1-cotorsionfree by assumption, and hence A is 2-cotorsionfree by Proposition 3.7. By replacing A by K in the above argument, we get that K is 2-cotorsionfree and then A is 3-cotorsionfree. Continuing this process, we finally have that A is ∞-cotorsionfree.

(3) $\Rightarrow$ (1) Obviously $\mathcal{GI} \subseteq ({}^{\perp}\mathcal{GI})^{\perp}$. Now let $A \in ({}^{\perp}\mathcal{GI})^{\perp}$. It suffices to prove that A is Gorenstein injective. Because $D(\Lambda) \in {}^{\perp}\mathcal{GI}$, $\operatorname{Ext}^{\geq 1}_\Lambda(D(\Lambda), A) = 0$ and A is ∞-cospherical. Note that A is ∞-cotorsionfree by assumption. It follows from Corollary 5.2 that A is Gorenstein injective. □

Proposition 5.8. *Let R be a commutative local Artin ring and let F be a free R-module with* $\operatorname{rank}(F) = 2n$. *If there exists an endomorphism f of F such that $f^2 = 0$ and* $\operatorname{rank}(\operatorname{Im} f) = \operatorname{rank}(\operatorname{Im} f^*) = n$, *then* $(\operatorname{Im} f)^\vee$ *is ∞-cotorsionfree, where* $(-)^\vee = \operatorname{Hom}_R(-, I^0(R/J(R)))$.

Proof. Since $f^2 = 0$, there exists a complex

$$0 \to \operatorname{Im} f \to F \xrightarrow{f} F \xrightarrow{f} \cdots.$$

Now consider the short exact sequence

$$0 \to \operatorname{Ker} f \to F \to \operatorname{Im} f \to 0.$$

Because $\operatorname{rank}(\operatorname{Im} f) = \operatorname{rank}(F)/2$ by assumption, we have

$$\operatorname{rank}(\operatorname{Im} f) = \operatorname{rank}(\operatorname{Ker} f).$$

Observing that $\operatorname{Im} f \subseteq \operatorname{Ker} f$, we get $\operatorname{Im} f = \operatorname{Ker} f$. Thus the above complex is exact. In a similar way, we also get $\operatorname{Im} f^* = \operatorname{Ker} f^*$. Hence $\operatorname{Im} f$ is ∞-torsionfree by [1, Theorem 2.17]. So $(\operatorname{Im} f)^\vee$ is ∞-cotorsionfree by Proposition 5.1. □

We give an example to illustrate Proposition 5.8.

Example 5.9. Let k be a field and $S = k[[X]]$ and $R = S/(X^2)$, and let $F = R^2$ and $f : R^2 \to R^2$ be a map given by the matrix

$$\begin{pmatrix} x & 0 \\ x & x \end{pmatrix}. \tag{5.1}$$

Then $(\operatorname{Im} f)^\vee$ is a non-injective ∞-cotorsionfree module.

Proof. Note that R has a basis consisting of the following two elements: 1 and x, where x denotes the residue class of the variable X modulo the ideal $\langle X^2 \rangle$. It is easy to check that $\operatorname{rank}(F) = 4$ and $f^2 = 0$. Since $\operatorname{Im} f$ is generated by the two elements $f(1,0) = (x,x)$ and $f(0,1) = (0,x)$, it is clear that $\operatorname{rank}(\operatorname{Im} f) = 2$. Similarly, the map f^* is given by the transpose of the matrix defining f. One can see that $\operatorname{rank}(\operatorname{Im} f^*) = 2$. Notice that $\operatorname{Im} f$ is not isomorphic to a direct summand of R^2. So $\operatorname{Im} f$ is not projective. Consequently one gets the assertion by Proposition 5.8. □

Huang and Huang raised in [11] an open question: Is the class of ∞-torsionfree modules closed under kernels of epimorphisms? We will give an example to show that for any $n \geq 2$, neither the class of n-torsionfree modules nor that of ∞-torsionfree modules is closed under kernels of epimorphisms in general. Nevertheless, the class of 1-torsionfree modules is closed under kernels of epimorphisms, since every submodule of a 1-torsionfree module is also 1-torsionfree. The following example is due to Jorgensen and Şega (see [13]).

Example 5.10. Let $R = \mathbb{Q}[V,X,Y,Z]/I$, where $\mathbb{Q}$ is the field of rational numbers and

$$I = \langle V^2, Z^2, XY, VX + 2XZ, VY + YZ, VX + Y^2, VY - X^2 \rangle.$$

Let $f : R^2 \to R^2$ denote the map given by the matrix

$$\begin{pmatrix} v & 2x \\ y & z \end{pmatrix}, \tag{5.2}$$

where v, x, y, z denote the residue classes of the variables modulo I. Take

$$M = \operatorname{Coker} f \quad \text{and} \quad N = \operatorname{Im} f.$$

Then there exists an exact sequence

$$0 \to N \to R^2 \to M \to 0$$

such that M is ∞-torsionfree and N is not n-torsionfree for any $n \geq 2$.

Proof. From [13, Lemma 1.5] we know that M is ∞-torsionfree. By [13, Lemma 1.4] we have a free presentation

$$R^2 \xrightarrow{g} R^2 \to N \to 0$$

of N, where g is given by the following matrix:

$$\begin{pmatrix} v & x \\ y & z \end{pmatrix}. \tag{5.3}$$

Then $\operatorname{Im} g^*$ is generated by the following elements:

$$\begin{aligned} &g^*(1,0) = (v,x), && g^*(z,0) = (vz, -2^{-1}vx), && g^*(0,1) = (y,z), \\ &g^*(0,v) = (vy, vz), && g^*(v,0) = (0, vx), && g^*(0,x) = (0, -2^{-1}vx), \\ &g^*(x,0) = (vx, vy), && g^*(0,y) = (-vx, -vy), && g^*(y,0) = (vy, 0), \\ &g^*(0,z) = (-vy, 0). \end{aligned}$$

One can use a computer algebra software, like Singular (see [8]), to verify that $\operatorname{Ext}^1_R(\operatorname{Im} g^*, R) \neq 0$. Thus we have $\operatorname{Ext}^2_R(\operatorname{Tr} N, R) \neq 0$, and therefore N is not n-torsionfree for any $n \geq 2$. The computation of $\operatorname{Ext}^1_R(\operatorname{Im} g^*, R)$ by Singular is as follows.

LIB "homolog.lib";
ring $S = 0, (V, X, Y, Z)$, dp;
ideal $I = V2, Z2, XY, VX + 2XZ, VY + YZ, VX + Y2, VY - X2$;
qring $R = \operatorname{std}(I)$; // define the ring R
module $F = [V, X], [2VZ, -VX], [Y, Z], [VY, VZ], [0, VX], [VX, VY], [VY, 0]$;
module $H = 1$;
module $E = \operatorname{Ext}(1, \operatorname{syz}(F), \operatorname{syz}(H))$; // compute $\operatorname{Ext}^1_R(\operatorname{Im} g^*, R)$

The output says that the dimension of $\operatorname{Ext}^1_R(\operatorname{Im} g^*, R)$ as a vector space is 3. □

By Example 5.10 and Proposition 5.1, we have that the class of ∞-cotorsionfree modules is not closed under cokernels of monomorphisms in general.

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