DE GRUYTER Georgian Math. J. 2021; 28(6): 867–882

Research Article

Zhaoyong Huang*

Duality pairs induced by Auslander and Bass classes

https://doi.org/10.1515/gmj-2021-2101
Received January 9, 2020; accepted November 26, 2020

Abstract: Let R and S be arbitrary rings and let ${}_RC_S$ be a semidualizing bimodule, and let $\mathcal{A}_C(R^{\mathrm{op}})$ and $\mathcal{B}_C(R)$ be the Auslander and Bass classes, respectively. Then both pairs

$$(\mathcal{A}_C(R^{\mathrm{op}}), \mathcal{B}_C(R)) \quad \text{and} \quad (\mathcal{B}_C(R), \mathcal{A}_C(R^{\mathrm{op}}))$$

are coproduct-closed and product-closed duality pairs and both $\mathcal{A}_C(R^{\mathrm{op}})$ and $\mathcal{B}_C(R)$ are covering and preenveloping; in particular, the former duality pair is perfect. Moreover, if $\mathcal{B}_C(R)$ is enveloping in $\operatorname{Mod} R$, then $\mathcal{A}_C(S)$ is enveloping in $\operatorname{Mod} S$. Also, some applications to the Auslander projective dimension of modules are given.

Keywords: Duality pairs, Auslander classes, Bass classes, semidualizing bimodules, (pre)covers, (pre)envelopes, cotorsion pairs, Auslander projective dimension

MSC 2010: 18G25, 16E10, 16E30

1 Introduction

In relative homological algebra, the theory of covers and envelopes is fundamental and important. Let R be a ring and let $\operatorname{Mod} R$ be the category of left R-modules. Given a subcategory of $\operatorname{Mod} R$, it is always worth studying whether or when it is (pre)covering or (pre)enveloping. This problem has been studied extensively; see [3–10] and the references therein.

Let R be a commutative noetherian ring, let C be a semidualizing R-module, and let $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ be the Auslander and Bass classes, respectively. By proving that both $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ are Kaplansky classes, Enochs and Holm got in [6, Theorems 3.11 and 3.12] that the pair $(\mathcal{A}_C(R), (\mathcal{A}_C(R))^{\perp})$ is a perfect cotorsion pair, $\mathcal{A}_C(R)$ is covering and preenveloping and $\mathcal{B}_C(R)$ is preenveloping. Holm and Jørgensen introduced the notion of duality pairs and proved the following remarkable result. Let R be an arbitrary ring, and let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of $\operatorname{Mod} R$ and $\operatorname{Mod} R^{\mathrm{op}}$, respectively. When $(\mathscr{X}, \mathscr{Y})$ is a duality pair, the following assertions hold true: (1) If $\mathscr{X}$ is closed under coproducts, then $\mathscr{X}$ is covering; (2) if $\mathscr{X}$ is closed under products, then $\mathscr{X}$ is preenveloping; and (3) if ${}_RR \in \mathscr{X}$ and $\mathscr{X}$ is closed under coproducts and extensions, then $(\mathscr{X}, \mathscr{X}^{\perp})$ is a perfect cotorsion pair [10, Theorem 3.1]. By using it, they generalized the above result of Enochs and Holm to the category of complexes, and Enochs and Iacob investigated in [7] the existence of Gorenstein injective envelopes over commutative noetherian rings.

Let R and S be arbitrary rings, let ${}_RC_S$ be a semidualizing bimodule, let $\mathcal{A}_C(R^{\mathrm{op}})$ be the Auslander class in $\operatorname{Mod} R^{\mathrm{op}}$ and let $\mathcal{B}_C(R)$ be the Bass class in $\operatorname{Mod} R$. Our first main result reads as follows.

Theorem 1.1 (see Theorem 3.3). (1) *Both pairs* $(\mathcal{A}_C(R^{\mathrm{op}}), \mathcal{B}_C(R))$ *and* $(\mathcal{B}_C(R), \mathcal{A}_C(R^{\mathrm{op}}))$ *are coproduct-closed and product-closed duality pairs. Furthermore, the former one is perfect.*
(2) $\mathcal{A}_C(R^{\mathrm{op}})$ *is covering and preenveloping in* $\operatorname{Mod} R^{\mathrm{op}}$ *and* $\mathcal{B}_C(R)$ *is covering and preenveloping in* $\operatorname{Mod} R$.

***Corresponding author: Zhaoyong Huang,** Department of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, P. R. China, e-mail: huangzy@nju.edu.cn

As a consequence of Theorem 1.1, we get that the pair $(\mathcal{A}_C(R^{\mathrm{op}}), \mathcal{A}_C(R^{\mathrm{op}})^{\perp})$ is a hereditary perfect cotorsion pair and $\mathcal{A}_C(R^{\mathrm{op}})$ is covering and preenveloping in $\operatorname{Mod} R^{\mathrm{op}}$, where $\mathcal{A}_C(R^{\mathrm{op}})^{\perp}$ is the right Ext-orthogonal class of $\mathcal{A}_C(R^{\mathrm{op}})$ (Corollary 3.4). This result was proved in [6, Theorem 3.11] when R is a commutative noetherian ring and ${}_RC_S = {}_RC_R$.

By Theorem 1.1 and its symmetric result, we have that $\mathcal{B}_C(R)$ is preenveloping in $\operatorname{Mod} R$ and $\mathcal{A}_C(S)$ is preenveloping in $\operatorname{Mod} S$. Moreover, we prove the following theorem.

Theorem 1.2 (see Theorem 3.7 (2)). *If $\mathcal{B}_C(R)$ is enveloping in* $\operatorname{Mod} R$, *then $\mathcal{A}_C(S)$ is enveloping in* $\operatorname{Mod} S$.

Then we apply these results and their symmetric results to study the Auslander projective dimension of modules. We obtain some criteria for computing the Auslander projective dimension of modules in $\operatorname{Mod} S$ (Theorem 4.4). Furthermore, we get the following theorem.

Theorem 1.3 (see Theorem 4.10). *If ${}_RC$ has an ultimately closed projective resolution, then*

$$\mathcal{A}_C(S) = {C_S}^{\top} = {}^{\perp}\mathcal{I}_C(S),$$

where ${C_S}^{\top}$ is the Tor*-orthogonal class of C_S and ${}^{\perp}\mathcal{I}_C(S)$ is the left* Ext*-orthogonal class of the subcategory $\mathcal{I}_C(S)$ of* $\operatorname{Mod} S$ *consisting of C-injective modules.*

As a consequence, we have that if ${}_RC$ has an ultimately closed projective resolution, then the projective dimension of C_S is at most n if and only if the Auslander projective dimension of any module in $\operatorname{Mod} S$ is at most n (Corollary 4.11).

2 Preliminaries

In this paper, all rings are associative with identities. Let R be a ring. We use $\operatorname{Mod} R$ to denote the category of left R-modules and all subcategories of $\operatorname{Mod} R$ involved are full and closed under isomorphisms. For a subcategory $\mathscr{X}$ of $\operatorname{Mod} R$, we write

$$\begin{aligned} {}^{\perp}\mathscr{X} &:= \{A \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geq 1}(A, X) = 0 \text{ for any } X \in \mathscr{X}\}, \\ \mathscr{X}^{\perp} &:= \{A \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{\geq 1}(X, A) = 0 \text{ for any } X \in \mathscr{X}\}, \\ {}^{\perp_1}\mathscr{X} &:= \{A \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{1}(A, X) = 0 \text{ for any } X \in \mathscr{X}\}, \\ \mathscr{X}^{\perp_1} &:= \{A \in \operatorname{Mod} R \mid \operatorname{Ext}_R^{1}(X, A) = 0 \text{ for any } X \in \mathscr{X}\}. \end{aligned}$$

For subcategories $\mathscr{X}, \mathscr{Y}$ of $\operatorname{Mod} R$, we write $\mathscr{X} \perp \mathscr{Y}$ if $\operatorname{Ext}_R^{\geq 1}(X, Y) = 0$ for any $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.

Definition 2.1 ([5, 8]). Let $\mathscr{X} \subseteq \mathscr{Y}$ be subcategories of $\operatorname{Mod} R$. A homomorphism $f : X \to Y$ in $\operatorname{Mod} R$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$ is called an *$\mathscr{X}$-precover* of Y if $\operatorname{Hom}_R(X', f)$ is epic for any $X' \in \mathscr{X}$; and f is called *right minimal* if an endomorphism $h : X \to X$ is an automorphism whenever $f = fh$. An *$\mathscr{X}$-precover* $f : X \to Y$ is called an *$\mathscr{X}$-cover* of Y if it is right minimal. The subcategory $\mathscr{X}$ is called *(pre)covering* in $\mathscr{Y}$ if any object in $\mathscr{Y}$ admits an $\mathscr{X}$-(pre)cover. Dually, the notions of an *$\mathscr{X}$-(pre)envelope*, a *left minimal homomorphism* and a *(pre)enveloping subcategory* are defined.

Definition 2.2 ([8, 9]). Let $\mathscr{U}, \mathscr{V}$ be subcategories of $\operatorname{Mod} R$.
(1) The pair $(\mathscr{U}, \mathscr{V})$ is called a *cotorsion pair* in $\operatorname{Mod} R$ if $\mathscr{U} = {}^{\perp_1}\mathscr{V}$ and $\mathscr{V} = \mathscr{U}^{\perp_1}$.
(2) A cotorsion pair $(\mathscr{U}, \mathscr{V})$ is called *perfect* if $\mathscr{U}$ is covering and $\mathscr{V}$ is enveloping in $\operatorname{Mod} R$.
(3) A cotorsion pair $(\mathscr{U}, \mathscr{V})$ is called *hereditary* if one of the following equivalent conditions is satisfied:
  (3a) $\mathscr{U} \perp \mathscr{V}$.
  (3b) $\mathscr{U}$ is projectively resolving in the sense that $\mathscr{U}$ contains all projective modules in $\operatorname{Mod} R$, and $\mathscr{U}$ is closed under extensions and kernels of epimorphisms.
  (3c) $\mathscr{V}$ is injectively coresolving in the sense that $\mathscr{V}$ contains all injective modules in $\operatorname{Mod} R$, and $\mathscr{V}$ is closed under extensions and cokernels of monomorphisms.

Set $(-)^+ := \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers. The following is the definition of duality pairs (cf. [7, 10]).

Definition 2.3. Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of $\mathrm{Mod}\,R$ and $\mathrm{Mod}\,R^{\mathrm{op}}$, respectively.
(1) The pair $(\mathscr{X}, \mathscr{Y})$ is called a *duality pair* if the following conditions are satisfied:
 (1a) For a module $X \in \mathrm{Mod}\,R$, one has $X \in \mathscr{X}$ if and only if $X^+ \in \mathscr{Y}$.
 (1b) $\mathscr{Y}$ is closed under direct summands and finite direct sums.
(2) A duality pair $(\mathscr{X}, \mathscr{Y})$ is called *(co)product-closed* if $\mathscr{X}$ is closed under (co)products.
(3) A duality pair $(\mathscr{X}, \mathscr{Y})$ is called *perfect* if it is coproduct-closed, if ${}_RR \in \mathscr{X}$ and if $\mathscr{X}$ is closed under extensions.

We also recall the following remarkable result.

Lemma 2.4 ([7, p. 7, Theorem] and [10, Theorem 3.1]). *Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of* $\mathrm{Mod}\,R$ *and* $\mathrm{Mod}\,R^{\mathrm{op}}$, *respectively. If $(\mathscr{X}, \mathscr{Y})$ is a duality pair, then the following assertions hold true:*
(1) *If $(\mathscr{X}, \mathscr{Y})$ is coproduct-closed, then $\mathscr{X}$ is covering.*
(2) *If $(\mathscr{X}, \mathscr{Y})$ is product-closed, then $\mathscr{X}$ is preenveloping.*
(3) *If $(\mathscr{X}, \mathscr{Y})$ is perfect, then $(\mathscr{X}, \mathscr{X}^{\perp})$ is a perfect cotorsion pair.*

Definition 2.5 ([1, 11]). Let R and S be rings. An (R, S)-bimodule ${}_RC_S$ is called *semidualizing* if the following conditions are satisfied:
(a1) ${}_RC$ admits a degreewise finite R-projective resolution.
(a2) C_S admits a degreewise finite S^{op}-projective resolution.
(b1) The homothety map ${}_RR_R \xrightarrow{{}_R\gamma} \mathrm{Hom}_{S^{\mathrm{op}}}(C, C)$ is an isomorphism.
(b2) The homothety map ${}_SS_S \xrightarrow{\gamma_S} \mathrm{Hom}_R(C, C)$ is an isomorphism.
(c1) $\mathrm{Ext}_R^{\geq 1}(C, C) = 0$.
(c2) $\mathrm{Ext}_{S^{\mathrm{op}}}^{\geq 1}(C, C) = 0$.

Wakamatsu [18] introduced and studied the so-called *generalized tilting modules*, which are usually called *Wakamatsu tilting modules*; see [3, 16]. Note that a bimodule ${}_RC_S$ is semidualizing if and only if it is Wakamatsu tilting [20, Corollary 3.2]. For examples of semidualizing bimodules, the reader is referred to [11, 19].

3 Duality pairs

In this section, R and S are arbitrary rings and ${}_RC_S$ is a semidualizing bimodule. We write $(-)_* := \mathrm{Hom}(C, -)$ and

$$
\begin{aligned}
{}_RC^{\perp} &:= \{M \in \mathrm{Mod}\,R \mid \mathrm{Ext}_R^{\geq 1}(C, M) = 0\} \quad \text{and} \quad {C_S}^{\perp} := \{B \in \mathrm{Mod}\,S^{\mathrm{op}} \mid \mathrm{Ext}_{S^{\mathrm{op}}}^{\geq 1}(C, B) = 0\},\\
{}^{\top}{}_RC &:= \{N \in \mathrm{Mod}\,R^{\mathrm{op}} \mid \mathrm{Tor}^R_{\geq 1}(N, C) = 0\} \quad \text{and} \quad {C_S}^{\top} := \{A \in \mathrm{Mod}\,S \mid \mathrm{Tor}^S_{\geq 1}(C, A) = 0\}.
\end{aligned}
$$

Definition 3.1 ([11]). (1) The *Auslander class* $\mathcal{A}_C(R^{\mathrm{op}})$ with respect to C consists of all modules N in $\mathrm{Mod}\,R^{\mathrm{op}}$ satisfying the following conditions:
 (a1) $N \in {}^{\top}{}_RC$.
 (a2) $N \otimes_R C \in {C_S}^{\perp}$.
 (a3) The canonical evaluation homomorphism

$$\mu_N : N \to (N \otimes_R C)_*$$

 defined by $\mu_N(x)(c) = x \otimes c$ for any $x \in N$ and $c \in C$ is an isomorphism in $\mathrm{Mod}\,R^{\mathrm{op}}$.
(2) The *Bass class* $\mathcal{B}_C(R)$ with respect to C consists of all modules M in $\mathrm{Mod}\,R$ satisfying the following conditions:
 (b1) $M \in {}_RC^{\perp}$.
 (b2) $M_* \in {C_S}^{\top}$.

(b3) The canonical evaluation homomorphism

$$\theta_M : C \otimes_S M_* \to M$$

defined by $\theta_M(c \otimes f) = f(c)$ for any $c \in C$ and $f \in M_*$ is an isomorphism in Mod R.

(3) The *Auslander class* $\mathcal{A}_C(S)$ in Mod S and the *Bass class* $\mathcal{B}_C(S^{\mathrm{op}})$ in Mod S^{op} are defined symmetrically.

The following result is crucial. From its proof, it is known that the conditions in the definitions of $\mathcal{A}_C(R^{\mathrm{op}})$ and $\mathcal{B}_C(R)$ are dual item by item.

Proposition 3.2. (1) *For a module $N \in$ Mod R^{op}, one has $N \in \mathcal{A}_C(R^{\mathrm{op}})$ if and only if $N^+ \in \mathcal{B}_C(R)$.*
(2) *For a module $M \in$ Mod R, one has $M \in \mathcal{B}_C(R)$ if and only if $M^+ \in \mathcal{A}_C(R^{\mathrm{op}})$.*

Proof. (1) Let $N \in$ Mod R^{op}. Then we have the following assertions:

(a) We have the equivalences

$$\begin{aligned} N \in {}^{\top}{}_R C &\iff \mathrm{Tor}^R_{\geq 1}(N, C) = 0 \\ &\iff [\mathrm{Tor}^R_{\geq 1}(N, C)]^+ = 0 \\ &\iff \mathrm{Ext}^{\geq 1}_R(C, N^+) = 0 \quad \text{(by [9, Lemma 2.16 (b)])} \\ &\iff N^+ \in {}_R C^{\perp}. \end{aligned}$$

(b) We have the equivalences

$$\begin{aligned} N \otimes_R C \in C_S{}^{\perp} &\iff \mathrm{Ext}^{\geq 1}_{S^{\mathrm{op}}}(C, N \otimes_R C) = 0 \\ &\iff [\mathrm{Ext}^{\geq 1}_{S^{\mathrm{op}}}(C, N \otimes_R C)]^+ = 0 \\ &\iff \mathrm{Tor}^S_{\geq 1}(C, (N \otimes_R C)^+) = 0 \quad \text{(by [9, Lemma 2.16 (d)])} \\ &\iff \mathrm{Tor}^S_{\geq 1}(C, (N^+)_*) = 0 \quad \text{(by [9, Lemma 2.16 (a)])} \\ &\iff (N^+)_* \in C_S{}^{\top}. \end{aligned}$$

(c) By [9, Lemma 2.16 (c)], the canonical evaluation homomorphism

$$\alpha : C \otimes_S (N \otimes_R C)^+ \to [\mathrm{Hom}_{S^{\mathrm{op}}}(C, N \otimes_R C)]^+$$

defined by $\alpha(c \otimes g)(f) = gf(c)$ for any $c \in C$, $g \in (N \otimes_R C)^+$ and $f \in \mathrm{Hom}_{S^{\mathrm{op}}}(C, N \otimes_R C)$ is an isomorphism in Mod R. By [9, Lemma 2.16 (a)], the canonical evaluation homomorphism

$$\beta : (N \otimes_R C)^+ \to \mathrm{Hom}_R(C, N^+)$$

defined by $\beta(g)(c)(x) = g(x \otimes c)$ for any $g \in (N \otimes_R C)^+$, $c \in C$ and $x \in N$ is an isomorphism in Mod S. So,

$$1_C \otimes \beta : C \otimes_S (N \otimes_R C)^+ \to C \otimes_S \mathrm{Hom}_R(C, N^+)$$

defined by $(1_C \otimes \beta)(c \otimes g) = c \otimes \beta(g)$ for any $c \in C$ and $g \in (N \otimes_R C)^+$ is an isomorphism in Mod R.

Consider the following diagram:

$$\begin{array}{ccc} C \otimes_S (N \otimes_R C)^+ & \xrightarrow{\alpha} & [\mathrm{Hom}_{S^{\mathrm{op}}}(C, N \otimes_R C)]^+ \\ \downarrow{\scriptstyle 1_C \otimes \beta} & & \downarrow{\scriptstyle (\mu_N)^+} \\ C \otimes_S \mathrm{Hom}_R(C, N^+) & \xrightarrow{\theta_{N^+}} & N^+, \end{array}$$

where

$$(\mu_N)^+ : [\mathrm{Hom}_{S^{\mathrm{op}}}(C, N \otimes_R C)]^+ \to N^+$$

defined by $(\mu_N)^+(f') = f'\mu_N$ for any $f' \in [\mathrm{Hom}_{S^{\mathrm{op}}}(C, N \otimes_R C)]^+$ is a natural homomorphism in Mod R, and

$$\theta_{N^+} : C \otimes_S \mathrm{Hom}_R(C, N^+) \to N^+$$

defined by $\theta_{N^+}(c\otimes f'')=f''(c)$ for any $c\in C$ and $f''\in \mathrm{Hom}_R(C,N^+)$ is a canonical evaluation homomorphism in $\mathrm{Mod}\,R$. Then, for any $c\in C$, $g\in (N\otimes_R C)^+$ and $x\in N$, we have

$$(\mu_N)^+\alpha(c\otimes g)(x)=\alpha(c\otimes g)\mu_N(x)=g\mu_N(x)(c)=g(x\otimes c),$$
$$\theta_{N^+}(1_C\otimes\beta)(c\otimes g)(x)=\theta_{N^+}(c\otimes\beta(g))(x)=\beta(g)(c)(x)=g(x\otimes c).$$

Thus

$$(\mu_N)^+\alpha=\theta_{N^+}(1_C\otimes\beta).$$

Therefore, μ_N is an isomorphism, which is equivalent to the fact that $(\mu_N)^+$ is an isomorphism, which in turn is equivalent to θ_{N^+} being an isomorphism.

We conclude that $N\in\mathcal{A}_C(R^{\mathrm{op}})$, which is equivalent to $N^+\in\mathcal{B}_C(R)$.

(2) Let $M\in\mathrm{Mod}\,R$. Then we have the following assertions.

(a) We have the equivalences

$$\begin{aligned} M\in {}_RC^{\perp} &\iff \mathrm{Ext}_R^{\geq 1}(C,M)=0\\ &\iff [\mathrm{Ext}_R^{\geq 1}(C,M)]^+=0\\ &\iff \mathrm{Tor}^R_{\geq 1}(M^+,C)=0 \quad \text{(by [9, Lemma 2.16 (d)])}\\ &\iff M^+\in {}^{\top}{}_RC. \end{aligned}$$

(b) We have the equivalences

$$\begin{aligned} M_*\in {C_S}^{\top} &\iff \mathrm{Tor}^S_{\geq 1}(C,M_*)=0\\ &\iff [\mathrm{Tor}^S_{\geq 1}(C,M_*)]^+=0\\ &\iff \mathrm{Ext}^{\geq 1}_{S^{\mathrm{op}}}(C,(M_*)^+)=0 \qquad \text{(by [9, Lemma 2.16 (b)])}\\ &\iff \mathrm{Ext}^{\geq 1}_{S^{\mathrm{op}}}(C,M^+\otimes_R C)=0 \quad \text{(by [9, Lemma 2.16 (c)])}\\ &\iff M^+\otimes_R C\in {C_S}^{\perp}. \end{aligned}$$

(c) By [9, Lemma 2.16 (a)], the canonical evaluation homomorphism

$$\tau:[C\otimes_S \mathrm{Hom}_R(C,M)]^+\to \mathrm{Hom}_{S^{\mathrm{op}}}(C,[\mathrm{Hom}_R(C,M)]^+)$$

defined by $\tau(g')(c)(f)=g'(c\otimes f)$ for any $g'\in[C\otimes_S\mathrm{Hom}_R(C,M)]^+$, $c\in C$ and $f\in\mathrm{Hom}_R(C,M)$ is an isomorphism in $\mathrm{Mod}\,R^{\mathrm{op}}$. By [9, Lemma 2.16(c)], the canonical evaluation homomorphism

$$\sigma: M^+\otimes_R C\to[\mathrm{Hom}_R(C,M)]^+$$

defined by $\sigma(g\otimes c)(f)=gf(c)$ for any $g\in M^+$, $c\in C$ and $f\in\mathrm{Hom}_R(C,M)$ is an isomorphism in $\mathrm{Mod}\,S^{\mathrm{op}}$. So,

$$\mathrm{Hom}_{S^{\mathrm{op}}}(C,\sigma):\mathrm{Hom}_{S^{\mathrm{op}}}(C,M^+\otimes_R C)\to\mathrm{Hom}_{S^{\mathrm{op}}}(C,[\mathrm{Hom}_R(C,M)]^+)$$

defined by $\mathrm{Hom}_{S^{\mathrm{op}}}(C,\sigma)(g'')=\sigma g''$ for any $g''\in\mathrm{Hom}_{S^{\mathrm{op}}}(C,M^+\otimes_R C)$ is an isomorphism in $\mathrm{Mod}\,R^{\mathrm{op}}$.

Consider the following diagram:

$$\begin{array}{ccc} M^+ & \xrightarrow{(\theta_M)^+} & [C\otimes_S\mathrm{Hom}_R(C,M)]^+\\ \big\downarrow{\scriptstyle \mu_{M^+}} & & \big\downarrow{\scriptstyle\tau}\\ \mathrm{Hom}_{S^{\mathrm{op}}}(C,M^+\otimes_R C) & \xrightarrow{\mathrm{Hom}_{S^{\mathrm{op}}}(C,\sigma)} & \mathrm{Hom}_{S^{\mathrm{op}}}(C,[\mathrm{Hom}_R(C,M)]^+), \end{array}$$

where

$$(\theta_M)^+: M^+\to[C\otimes_S\mathrm{Hom}_R(C,M)]^+$$

defined by $(\theta_M)^+(g)=g\theta_M$ for any $g\in M^+$ is a natural homomorphism in $\mathrm{Mod}\,R^{\mathrm{op}}$, and

$$\mu_{M^+}: M^+\to\mathrm{Hom}_{S^{\mathrm{op}}}(C,M^+\otimes_R C)$$

defined by $\mu_{M^+}(g)(c)=g\otimes c$ for any $g\in M^+$ and $c\in C$ is a canonical evaluation homomorphism in $\mathrm{Mod}\,R^{\mathrm{op}}$.

Then, for any $g \in M^+$, $c \in C$ and $f \in \mathrm{Hom}_R(C, M)$, we have

$$\tau(\theta_M)^+(g)(c)(f) = (\theta_M)^+(g)(c \otimes f) = g\theta_M(c \otimes f) = gf(c),$$
$$\mathrm{Hom}_{S^{\mathrm{op}}}(C, \sigma)\mu_{M^+}(g)(c)(f) = \sigma\mu_{M^+}(g)(c)(f) = \sigma(g \otimes c)(f) = gf(c).$$

Thus

$$\tau(\theta_M)^+ = \mathrm{Hom}_{S^{\mathrm{op}}}(C, \sigma)\mu_{M^+}.$$

Therefore, θ_M is an isomorphism, which is equivalent to the fact that $(\theta_M)^+$ is an isomorphism, which in turn is equivalent to μ_{M^+} being an isomorphism.

We conclude that $M \in \mathcal{B}_C(R)$, which is equivalent to $M^+ \in \mathcal{A}_C(R^{\mathrm{op}})$. □

As a consequence, we get the following theorem.

Theorem 3.3. (1) *The pair* $(\mathcal{A}_C(R^{\mathrm{op}}), \mathcal{B}_C(R))$ *is a perfect coproduct-closed and product-closed duality pair and* $\mathcal{A}_C(R^{\mathrm{op}})$ *is covering and preenveloping in* Mod R^{op}.
(2) *The pair* $(\mathcal{B}_C(R), \mathcal{A}_C(R^{\mathrm{op}}))$ *is a coproduct-closed and product-closed duality pair and* $\mathcal{B}_C(R)$ *is covering and preenveloping in* Mod R.

Proof. It follows from [11, Proposition 4.2 (a)] that both $\mathcal{A}_C(R^{\mathrm{op}})$ and $\mathcal{B}_C(R)$ are closed under direct summands, coproducts and products. So, by Lemma 2.4 (1) and (2), and by Proposition 3.2, we have that both the pairs

$$(\mathcal{A}_C(R^{\mathrm{op}}), \mathcal{B}_C(R)) \quad \text{and} \quad (\mathcal{B}_C(R), \mathcal{A}_C(R^{\mathrm{op}}))$$

are coproduct-closed and product-closed duality pairs, $\mathcal{A}_C(R^{\mathrm{op}})$ is covering and preenveloping in Mod R^{op} and $\mathcal{B}_C(R)$ is covering and preenveloping in Mod R. Moreover, $\mathcal{A}_C(R^{\mathrm{op}})$ is projectively resolving by [11, Theorem 6.2], so the duality pair $(\mathcal{A}_C(R^{\mathrm{op}}), \mathcal{B}_C(R))$ is perfect. □

We write

$$\mathcal{A}_C(R^{\mathrm{op}})^{\perp} := \{Y \in \mathrm{Mod}\, R^{\mathrm{op}} \mid \mathrm{Ext}^{\geq 1}_{R^{\mathrm{op}}}(N, Y) = 0 \text{ for any } N \in \mathcal{A}_C(R^{\mathrm{op}})\}.$$

The following corollary was proved in [6, Theorem 3.11] when R is a commutative noetherian ring and ${}_RC_S = {}_RC_R$.

Corollary 3.4. *The pair* $(\mathcal{A}_C(R^{\mathrm{op}}), \mathcal{A}_C(R^{\mathrm{op}})^{\perp})$ *is a hereditary perfect cotorsion pair and* $\mathcal{A}_C(R^{\mathrm{op}})$ *is covering and preenveloping in* Mod R^{op}.

Proof. It follows from Theorem 3.3 (1) and Lemma 2.4 (3). □

The following two results are the symmetric versions of Theorem 3.3 and Corollary 3.4, respectively.

Theorem 3.5. (1) *The pair* $(\mathcal{A}_C(S), \mathcal{B}_C(S^{\mathrm{op}}))$ *is a perfect coproduct-closed and product-closed duality pair and* $\mathcal{A}_C(S)$ *is covering and preenveloping in* Mod S.
(2) *The pair* $(\mathcal{B}_C(S^{\mathrm{op}}), \mathcal{A}_C(S))$ *is a coproduct-closed and product-closed duality pair and* $\mathcal{B}_C(S^{\mathrm{op}})$ *is covering and preenveloping in* Mod S^{op}.

We write

$$\mathcal{A}_C(S)^{\perp} := \{X \in \mathrm{Mod}\, S \mid \mathrm{Ext}^{\geq 1}_S(N', X) = 0 \text{ for any } N' \in \mathcal{A}_C(S)\}.$$

Corollary 3.6. *The pair* $(\mathcal{A}_C(S), \mathcal{A}_C(S)^{\perp})$ *is a hereditary perfect cotorsion pair and* $\mathcal{A}_C(S)$ *is covering and preenveloping in* Mod S.

Holm and White proved in [11, Proposition 4.1] that there exist the following (Foxby) equivalences of categories:

$$\mathcal{A}_C(S) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_S-}{\underset{\sim}{\rightleftarrows}}} \mathcal{B}_C(R),$$

$$\mathcal{A}_C(R^{\mathrm{op}}) \underset{\mathrm{Hom}_{S^{\mathrm{op}}}(C,-)}{\overset{-\otimes_R C}{\underset{\sim}{\rightleftarrows}}} \mathcal{B}_C(S^{\mathrm{op}}).$$

Compare this result with Theorems 3.3 and 3.5.

By Theorems 3.3 (2) and 3.5 (1), $\mathcal{B}_C(R)$ is preenveloping in Mod R and $\mathcal{A}_C(S)$ is preenveloping in Mod S. In the following result, we construct an $\mathcal{A}_C(S)$-preenvelope of a given module in Mod S from a $\mathcal{B}_C(R)$-preenvelope of some module in Mod R.

Theorem 3.7. (1) *Let* $N \in \operatorname{Mod} S$ *and let* $f : C \otimes_S N \to B$ *be a* $\mathcal{B}_C(R)$*-preenvelope of* $C \otimes_S N$ *in* Mod R. *Then we have the following assertions:*

(1a) $f_* \mu_N : N \to B_*$ *is an* $\mathcal{A}_C(S)$*-preenvelope of* N *in* Mod S.

(1b) *If* f *is a* $\mathcal{B}_C(R)$*-envelope of* $C \otimes_S N$, *then* $f_* \mu_N$ *is an* $\mathcal{A}_C(S)$*-envelope of* N.

(2) *If* $\mathcal{B}_C(R)$ *is enveloping in* Mod R, *then* $\mathcal{A}_C(S)$ *is enveloping in* Mod S.

Proof. (1a) Let $N \in \operatorname{Mod} S$ and let

$$f : C \otimes_S N \to B$$

be a $\mathcal{B}_C(R)$-preenvelope in Mod R. By [11, Proposition 4.1], we have $B_* \in \mathcal{A}_C(S)$. Let $g \in \operatorname{Hom}_S(N, A)$ with $A \in \mathcal{A}_C(S)$. By [11, Proposition 4.1] again, we have $C \otimes_S A \in \mathcal{B}_C(R)$. So there exists $h \in \operatorname{Hom}_R(B, C \otimes_S A)$ such that $1_C \otimes g = hf$, that is, the diagram

$$\begin{array}{ccc} C \otimes_S N & \xrightarrow{f} & B \\ {\scriptstyle 1_C \otimes g}\downarrow & \swarrow {\scriptstyle h} & \\ C \otimes_S A & & \end{array}$$

commutes. From the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{g} & A \\ {\scriptstyle \mu_N}\downarrow & & \downarrow{\scriptstyle \mu_A} \\ (C \otimes_S N)_* & \xrightarrow{(1_C \otimes g)_*} & (C \otimes_S A)_*, \end{array}$$

we get $\mu_A g = (1_C \otimes g)_* \mu_N$. Since μ_A is an isomorphism, we have

$$g = {\mu_A}^{-1}(1_C \otimes g)_* \mu_N = ({\mu_A}^{-1} h_*)(f_* \mu_N),$$

that is, the diagram

$$\begin{array}{ccc} N & \xrightarrow{f_* \mu_N} & B \\ {\scriptstyle g}\downarrow & \swarrow {\scriptstyle {\mu_A}^{-1} h_*} & \\ A & & \end{array}$$

commutes. Thus $f_* \mu_N : N \to B_*$ is an $\mathcal{A}_C(S)$-preenvelope of N.

(1b) By (1a), it suffices to prove that if f is left minimal, then so is $f_* \mu_N$.

Let f be left minimal and let $h \in \operatorname{Hom}_S(B_*, B_*)$ such that $f_* \mu_N = h(f_* \mu_N)$. Then we have

$$\begin{aligned} (1_C \otimes f_*)(1_C \otimes \mu_N) &= 1_C \otimes (f_* \mu_N) = 1_C \otimes (h(f_* \mu_N)) \\ &= (1_C \otimes h)(1_C \otimes f_*)(1_C \otimes \mu_N). \end{aligned} \tag{3.1}$$

From the commutative diagram

$$\begin{array}{ccc} C \otimes_S (C \otimes_S N)_* & \xrightarrow{1_C \otimes f_*} & C \otimes_S B_* \\ {\scriptstyle \theta_{C \otimes_S N}}\downarrow & & \downarrow{\scriptstyle \theta_B} \\ C \otimes_S N & \xrightarrow{f} & B, \end{array}$$

we get

$$f \theta_{C \otimes_S N} = \theta_B (1_C \otimes f_*). \tag{3.2}$$

So we have

$$
\begin{aligned}
f &= f1_{C\otimes_S N} \\
&= f(\theta_{C\otimes_S N}(1_C\otimes\mu_N)) && \text{(by [21, Proposition 2.2 (1)])} \\
&= \theta_B(1_C\otimes f_*)(1_C\otimes\mu_N) && \text{(by (3.2))} \\
&= \theta_B(1_C\otimes h)(1_C\otimes f_*)(1_C\otimes\mu_N) && \text{(by (3.1))} \\
&= \theta_B(1_C\otimes h)({\theta_B}^{-1}\theta_B)(1_C\otimes f_*)(1_C\otimes\mu_N) && \text{(because } \theta_B \text{ is an isomorphism)} \\
&= \theta_B(1_C\otimes h){\theta_B}^{-1}f\theta_{C\otimes_S N}(1_C\otimes\mu_N) && \text{(by (3.2))} \\
&= \theta_B(1_C\otimes h){\theta_B}^{-1}f1_{C\otimes_S N} && \text{(by [21, Proposition 2.2 (1)])} \\
&= \theta_B(1_C\otimes h){\theta_B}^{-1}f.
\end{aligned}
$$

Because f is left minimal, $\theta_B(1_C\otimes h){\theta_B}^{-1}$ is an isomorphism, which implies that $1_C\otimes h$ and $(1_C\otimes h)_*$ are also isomorphisms. From the commutative diagram

$$
\begin{array}{ccc}
B_* & \xrightarrow{\ h\ } & B_* \\
{\scriptstyle \mu_{B_*}}\downarrow & & \downarrow{\scriptstyle \mu_{B_*}} \\
(C\otimes_S B_*)_* & \xrightarrow{(1_C\otimes h)_*} & (C\otimes_S B_*)_*,
\end{array}
$$

we get

$$(1_C\otimes h)_*\mu_{B_*} = \mu_{B_*}h.$$

Because $B_*\in\mathcal{A}_C(S)$ by [11, Proposition 4.1], we obtain that μ_{B_*} is an isomorphism. It follows that h is also an isomorphism and $f_*\mu_N$ is left minimal.

(2): It follows from assertion (1b) immediately. □

We do not know whether a $\mathcal{B}_C(R)$-preenvelope of a given module in Mod R can be constructed from an $\mathcal{A}_C(S)$-preenvelope of some module in Mod S, and we do not know whether the converse of Theorem 3.7 (2) holds true.

By Theorems 3.3 (2) and 3.5 (1), $\mathcal{B}_C(R)$ is covering in Mod R and $\mathcal{A}_C(S)$ is covering in Mod S. In the following result, we construct a $\mathcal{B}_C(R)$-cover of a given module in Mod R from an $\mathcal{A}_C(S)$-cover of some module in Mod S.

Proposition 3.8. *Let* $M\in$ Mod R *and let*

$$g: A\to M_*$$

be an $\mathcal{A}_C(S)$*-cover of* M_* *in* Mod S. *Then*

$$\theta_M(1_C\otimes g): C\otimes_S A\to M$$

is a $\mathcal{B}_C(R)$*-cover of* M *in* Mod R.

Proof. Let $M\in$ Mod R and let $g: A\to M_*$ be an $\mathcal{A}_C(S)$-cover of M_* in Mod S. By [11, Proposition 4.1], we have $C\otimes_S A\in\mathcal{B}_C(R)$. Let $f\in\mathrm{Hom}_R(B, M)$ with $B\in\mathcal{B}_C(R)$. By [11, Proposition 4.1] again, we have $B_*\in\mathcal{A}_C(S)$. So there exists $h\in\mathrm{Hom}_S(B_*, A)$ such that $f_* = gh$, that is, the diagram

$$
\begin{array}{ccc}
 & & B_* \\
 & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle f_*} \\
A & \xrightarrow{\ g\ } & M_*
\end{array}
$$

commutes. From the commutative diagram

$$
\begin{array}{ccc}
C\otimes_S B_* & \xrightarrow{1_C\otimes f_*} & C\otimes_S M_* \\
{\scriptstyle \theta_B}\downarrow & & \downarrow{\scriptstyle \theta_M} \\
B & \xrightarrow{\ f\ } & M,
\end{array}
$$

we get $f\theta_B = \theta_M(1_C \otimes f_*)$. Because θ_B is an isomorphism, we have

$$f = \theta_M(1_C \otimes f_*)\theta_B{}^{-1} = \theta_M(1_C \otimes (gh))\theta_B{}^{-1} = (\theta_M(1_C \otimes g))((1_C \otimes h)\theta_B{}^{-1}),$$

that is, the diagram

$$\begin{array}{ccc} & & B \\ & \swarrow^{(1_C\otimes h))\theta_B{}^{-1}} & \downarrow f \\ C \otimes_S A & \xrightarrow[\theta_M(1_C\otimes g)]{} & M \end{array}$$

commutes. Thus $\theta_M(1_C \otimes g) : C \otimes_S A \to M$ is a $\mathcal{B}_C(R)$-precover of M.

In the following, it suffices to prove that $\theta_M(1_C \otimes g)$ is right minimal.

Let $h \in \mathrm{Hom}_R(C \otimes_S A, C \otimes_S A)$ such that $\theta_M(1_C \otimes g) = (\theta_M(1_C \otimes g))h$. Then we have

$$(\theta_M)_*(1_C \otimes g)_* = (\theta_M(1_C \otimes g))_* = ((\theta_M(1_C \otimes g))h)_* = (\theta_M)_*(1_C \otimes g)_* h_*. \tag{3.3}$$

From the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & M_* \\ \downarrow \mu_A & & \downarrow \mu_{M_*} \\ (C \otimes_S A)_* & \xrightarrow{(1_C\otimes g)_*} & (C \otimes_S M_*)_*, \end{array}$$

we get

$$\mu_{M_*} g = (1_C \otimes g)_* \mu_A. \tag{3.4}$$

So we have

$$\begin{aligned} g &= 1_{M_*} g \\ &= (\theta_M)_* \mu_{M_*} g && \text{(by [21, Proposition 2.2 (1)])} \\ &= (\theta_M)_*(1_C \otimes g)_* \mu_A && \text{(by (3.4))} \\ &= (\theta_M)_*(1_C \otimes g)_* h_* \mu_A && \text{(by (3.3))} \\ &= (\theta_M)_*(1_C \otimes g)_* \mu_A \mu_A{}^{-1} h_* \mu_A && \text{(because } \mu_A \text{ is an isomorphism)} \\ &= (\theta_M)_* \mu_{M_*} g \mu_A{}^{-1} h_* \mu_A && \text{(by (3.4))} \\ &= 1_{M_*} g \mu_A{}^{-1} h_* \mu_A && \text{(by [21, Proposition 2.2 (1)])} \\ &= g \mu_A{}^{-1} h_* \mu_A. \end{aligned}$$

Because g is right minimal, $\mu_A{}^{-1} h_* \mu_A$ is an isomorphism, which implies that h_* and $1_C \otimes h_*$ are also isomorphisms. From the commutative diagram

$$\begin{array}{ccc} C \otimes_S (C \otimes_S A)_* & \xrightarrow{1_C\otimes h_*} & C \otimes_S (C \otimes_S A)_* \\ \downarrow \theta_{C\otimes_S A} & & \downarrow \theta_{C\otimes_S A} \\ C \otimes_S A & \xrightarrow{h} & C \otimes_S A, \end{array}$$

we get

$$h\theta_{C\otimes_S A} = \theta_{C\otimes_S A}(1_C \otimes h_*).$$

Because $C \otimes_S A \in \mathcal{B}_C(R)$ by [11, Proposition 4.1], we obtain that $\theta_{C\otimes_S A}$ is an isomorphism. It follows that h is also an isomorphism and $\theta_M(1_C \otimes g)$ is right minimal. □

We do not know whether an $\mathcal{A}_C(S)$-cover of a given module in Mod S can be constructed from a $\mathcal{B}_C(R)$-cover of some module in Mod R.

4 The Auslander projective dimension of modules

For a subcategory $\mathscr{X}$ of Mod S and $N \in \operatorname{Mod} S$, the *$\mathscr{X}$-projective dimension* $\mathscr{X}\text{-pd}_S\, N$ of N is defined as

$$\inf\{n \mid \text{there exists an exact sequence } 0 \to X_n \to \cdots \to X_1 \to X_0 \to N \to 0 \text{ in } \operatorname{Mod} S \text{ with all } X_i \in \mathscr{X}\},$$

and we set $\mathscr{X}\text{-pd}_S\, N$ infinite if no such integer exists. We call $\mathcal{A}_C(S)\text{-pd}_S\, N$ the *Auslander projective dimension* of N. For any $n \geq 0$, we use $\Omega^n(N)$ to denote the n-th syzygy of N (note: $\Omega^0(N) = N$).

Lemma 4.1. *Let $N \in \operatorname{Mod} S$ and $n \geq 0$. If $\mathcal{A}_C(S)\text{-pd}_S\, N \leq n$ and*

$$0 \to K_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to N \to 0$$

is an exact sequence in Mod S *with all A_i in $\mathcal{A}_C(S)$, then $K_n \in \mathcal{A}_C(S)$; in particular, $\Omega^n(N) \in \mathcal{A}_C(S)$.*

Proof. Because $\mathcal{A}_C(S)$ is projectively resolving and is closed under direct summands and coproducts by [11, Theorem 6.2 and Proposition 4.2 (a)], the assertion follows from [2, Lemma 3.12]. □

We use $\mathcal{A}_C(S)\text{-pd}^{<\infty}$ to denote the subcategory of Mod S consisting of modules with finite Auslander projective dimension.

Proposition 4.2. *The category $\mathcal{A}_C(S)\text{-pd}^{<\infty}$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms.*

Proof. Let

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

be an exact sequence in Mod S and let $n \geq 0$. If $\max\{\mathcal{A}_C(S)\text{-}\operatorname{pd}_S N_1, \mathcal{A}_C(S)\text{-}\operatorname{pd}_S N_3\} = n$, then, by Lemma 4.1, there exist exact sequences

$$0 \to \Omega^n(N_1) \to P_1^{n-1} \to \cdots \to P_1^1 \to P_1^0 \to N_1 \to 0,$$
$$0 \to \Omega^n(N_3) \to P_3^{n-1} \to \cdots \to P_3^1 \to P_3^0 \to N_3 \to 0$$

in Mod S with all P_i^j projective and $\Omega^n(N_1), \Omega^n(N_3) \in \mathcal{A}_C(S)$. Then we get exact sequences

$$0 \to K_n \to P_1^{n-1} \oplus P_3^{n-1} \to \cdots \to P_1^1 \oplus P_3^1 \to P_1^0 \oplus P_3^0 \to N_2 \to 0,$$
$$0 \to \Omega^n(N_1) \to K_n \to \Omega^n(N_3) \to 0$$

in Mod S. By [11, Theorem 6.2], we have $K_n \in \mathcal{A}_C(S)$ and $\mathcal{A}_C(S)\text{-pd}_S\, N_2 \leq n$.

If

$$\max\{\mathcal{A}_C(S)\text{-}\operatorname{pd}_S N_1, \mathcal{A}_C(S)\text{-}\operatorname{pd}_S N_2\} = n,$$

then, by Corollary 3.6 and Lemma 4.1, there exist $\operatorname{Hom}_S(\mathcal{A}_C(S), -)$-exact exact sequences

$$0 \to A_1^n \to A_1^{n-1} \to \cdots \to A_1^1 \to A_1^0 \to N_1 \to 0,$$
$$0 \to A_2^n \to A_2^{n-1} \to \cdots \to A_2^1 \to A_2^0 \to N_2 \to 0$$

in Mod S with all A_i^j in $\mathcal{A}_C(S)$. By [12, Theorem 3.6], we get an exact sequence

$$0 \to A_1^n \to A_1^{n-1} \oplus A_2^n \to \cdots \to A_1^0 \oplus A_2^1 \to A_2^0 \to N_3 \to 0$$

in Mod S, and so $\mathcal{A}_C(S)\text{-pd}_S\, N_3 \leq n+1$.

If

$$\max\{\mathcal{A}_C(S)\text{-}\operatorname{pd}_S N_2, \mathcal{A}_C(S)\text{-}\operatorname{pd}_S N_3\} = n,$$

then, by Corollary 3.6 and Lemma 4.1, there exist $\operatorname{Hom}_S(\mathcal{A}_C(S), -)$-exact exact sequences

$$0 \to A_2^n \to A_2^{n-1} \to \cdots \to A_2^1 \to A_2^0 \to N_2 \to 0,$$
$$0 \to A_3^n \to A_3^{n-1} \to \cdots \to A_3^1 \to A_3^0 \to N_3 \to 0$$

in Mod S with all A_i^j in $\mathcal{A}_C(S)$. By [12, Theorem 3.2], we get exact sequences

$$0 \to A_2^n \to A_2^{n-1} \oplus A_3^n \to \cdots \to A_2^1 \oplus A_3^2 \to A \to N_1 \to 0,$$
$$0 \to A \to A_2^0 \oplus A_3^1 \to A_3^0 \to 0$$

in Mod S. By [11, Theorem 6.2], we have $A \in \mathcal{A}_C(S)$, and so $\mathcal{A}_C(S)\text{-pd}_S\, N_1 \le n$. □

We write

$$\mathcal{I}_C(S) := \{I_* \mid I \text{ is injective in Mod } R\}.$$

The module in $\mathcal{I}_C(S)$ is called *C-injective* [11]. Let Q be an injective cogenerator for Mod R. Then

$$\mathcal{I}_C(S) = \text{Prod}_S\, Q_*$$

by [15, Proposition 2.4 (2)], where $\text{Prod}_S\, Q_*$ is the subcategory of Mod S consisting of direct summands of products of copies of Q_*. By [9, Lemma 2.16 (b)], we have the following isomorphism of functors:

$$\text{Hom}_R(\text{Tor}_i^S(C, -), Q) \cong \text{Ext}_S^i(-, Q_*)$$

for any $i \ge 1$. This gives the following lemma.

Lemma 4.3. *One has* ${C_S}^{\top} = {}^{\perp}\mathcal{I}_C(S)$.

For a subcategory $\mathscr{X}$ of Mod S, a sequence in Mod S is called $\text{Hom}_S(-, \mathscr{X})$*-exact* if it is exact after applying the functor $\text{Hom}_S(-, X)$ for any $X \in \mathscr{X}$. Now we give some criteria for computing the Auslander projective dimension of modules.

Theorem 4.4. *Let $N \in \text{Mod}\, S$ with $\mathcal{A}_C(S)\text{-pd}_S\, N < \infty$ and $n \ge 0$. Then the following statements are equivalent:*
(1) $\mathcal{A}_C(S)\text{-pd}_S\, N \le n$.
(2) $\Omega^n(N) \in \mathcal{A}_C(S)$.
(3) $\text{Tor}_{\ge n+1}^S(C, N) = 0$.
(4) *There exists an exact sequence*

$$0 \to H \to A \to N \to 0$$

in Mod S *with* $A \in \mathcal{A}_C(S)$ *and* $\mathcal{I}_C(S)\text{-pd}_S\, H \le n - 1$.
(5) *There exists a (*$\text{Hom}_S(-, \mathcal{I}_C(S))$*-exact) exact sequence*

$$0 \to N \to H' \to A' \to 0$$

in Mod S *with* $A' \in \mathcal{A}_C(S)$ *and* $\mathcal{I}_C(S)\text{-pd}_S\, H' \le n$.

Proof. By Lemma 4.1 and the dimension shifting, we have (1) $\Longleftrightarrow$ (2) and (2) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (2): Because $\text{Tor}_{\ge n+1}^S(C, N) = 0$ by (3), we have $\Omega^n(N) \in {C_S}^{\top}$, and so $\Omega^n(N) \in {}^{\perp}\mathcal{I}_C(S)$ by Lemma 4.3. Note that all projective modules in Mod S are in $\mathcal{A}_C(S)$ by [11, Theorem 6.2]. Because $\mathcal{A}_C(S)\text{-pd}_S\, N < \infty$ by assumption, we have $\mathcal{A}_C(S)\text{-pd}_S\, \Omega^n(N) < \infty$ by Proposition 4.2.

Assume that $\mathcal{A}_C(S)\text{-pd}_S\, \Omega^n(N) = m(< \infty)$ and

$$0 \to A_m \to \cdots \to A_1 \to A_0 \to \Omega^n(N) \to 0 \tag{4.1}$$

is an exact sequence in Mod S with all A_j in $\mathcal{A}_C(S)$. Because $\mathcal{A}_C(S) \subseteq {C_S}^{\top} = {}^{\perp}\mathcal{I}_C(S)$ by Lemma 4.3, the exact sequence (4.1) is $\text{Hom}_S(-, \mathcal{I}_C(S))$-exact. By [17, Theorem 3.11 (1)], we have the following $\text{Hom}_S(-, \mathcal{I}_C(S))$-exact exact sequence:

$$0 \to A_j \to U_j^0 \to U_j^1 \to \cdots \to U_j^i \to \cdots$$

in Mod S with all U_j^i in $\mathcal{I}_C(S)$ for any $0 \le j \le m$ and $i \ge 0$. It follows from [12, Corollary 3.5] that there exist the following two exact sequences:

$$0 \to \Omega^n(N) \to U \to \bigoplus_{i=0}^m U_i^{i+1} \to \bigoplus_{i=0}^m U_i^{i+2} \to \bigoplus_{i=0}^m U_i^{i+3} \to \cdots,$$
$$0 \to U_m^0 \to U_m^1 \oplus U_{m-1}^0 \to \cdots \to \bigoplus_{i=2}^m U_i^{i-2} \to \bigoplus_{i=1}^m U_i^{i-1} \to \bigoplus_{i=0}^m U_i^i \to U \to 0,$$

and the former one is $\mathrm{Hom}_S(-, \mathcal{I}_C(S))$-exact. Because $\mathcal{I}_C(S)$ is closed under finite direct sums and cokernels of monomorphisms by [11, Proposition 5.1 (c) and Corollary 6.4], we have $U \in \mathcal{I}_C(S)$. By [17, Theorem 3.11 (1)] again, we have $\Omega^n(N) \in \mathcal{A}_C(S)$.

(1) $\Longrightarrow$ (4): By [11, Theorem 6.2], $\mathcal{A}_C(S)$ is closed under extensions. By [17, Theorem 3.11 (1)], we have that $\mathcal{I}_C(S)$ is an $\mathcal{I}_C(S)$-coproper cogenerator for $\mathcal{A}_C(S)$ in the sense of [13]. Then the assertion follows from [13, Theorem 4.7].

(4) $\Longrightarrow$ (5): Let

$$0 \to H \to A \to N \to 0$$

be an exact sequence in Mod S with $A \in \mathcal{A}_C(S)$ and $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H \le n-1$. By [17, Theorem 3.11 (1)], there exists a $\mathrm{Hom}_S(-, \mathcal{I}_C(S))$-exact exact sequence

$$0 \to A \to U \to A' \to 0$$

in Mod S with $U \in \mathcal{I}_C(S)$ and $A' \in \mathcal{A}_C(S)$. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & A & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & H & \dashrightarrow & U & \dashrightarrow & H' & \dashrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & A' & = = = & A' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

By the middle row in this diagram, we have $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H' \le n$. Because the middle column in the above diagram is $\mathrm{Hom}_S(-, \mathcal{I}_C(S))$-exact, the rightmost column is also $\mathrm{Hom}_S(-, \mathcal{I}_C(S))$-exact by [12, Lemma 2.4 (2)] and it is the desired exact sequence.

(5) $\Longrightarrow$ (1): Let

$$0 \to N \to H' \to A' \to 0$$

be an exact sequence in Mod S with $A' \in \mathcal{A}_C(S)$ and $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H' \le n$. Then there exists an exact sequence

$$0 \to U_n \to \cdots \to U_1 \to U_0 \to H' \to 0$$

in Mod S with all U_i in $\mathcal{I}_C(S)$. Set $H := \mathrm{Ker}(U_0 \to H')$. Then $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H \le n-1$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = = = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & A & \dashrightarrow & U_0 & \dashrightarrow & A' & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & H' & \longrightarrow & A' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Applying [11, Theorem 6.2] to the middle row in this diagram yields $A \in \mathcal{A}_C(S)$. Thus $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, N \le n$ by the leftmost column in the above diagram. □

The only place where the assumption $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, N < \infty$ in Theorem 4.4 is used is in showing that (3) implies (2). By Theorem 4.4, it is easy to get the following standard observation.

Corollary 4.5. *Let*

$$0 \to L \to M \to K \to 0$$

be an exact sequence in Mod S.

(1) *We have*

$$\mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, K \leq \max\{\mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, M, \mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, L + 1\},$$

and the equality holds true if $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, M \neq \mathcal{A}_C(S)$-$\mathrm{pd}_S\, L$.

(2) *We have*

$$\mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, L \leq \max\{\mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, M, \mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, K - 1\},$$

and the equality holds true if $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, M \neq \mathcal{A}_C(S)$-$\mathrm{pd}_S\, K$.

(3) *We have*

$$\mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, M \leq \max\{\mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, L, \mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, K\},$$

and the equality holds true if $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, K \neq \mathcal{A}_C(S)$-$\mathrm{pd}_S\, L + 1$.

The following corollary is an addendum to the implications (1) $\Longrightarrow$ (4) and (1) $\Longrightarrow$ (5) in Theorem 4.4.

Corollary 4.6. *Let* $N \in$ Mod S *with* $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, N = n$ $(< \infty)$. *Then there exist exact sequences*

$$0 \to H \to A \to N \to 0,$$
$$0 \to N \to H' \to A' \to 0$$

in Mod S *with* $A, A' \in \mathcal{A}_C(S)$, $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H = n - 1$ *and* $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H' = n$.

Proof. Let $N \in$ Mod S with $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, N = n$ $(< \infty)$. By Theorem 4.4, there exists an exact sequence

$$0 \to H \to A \to N \to 0$$

in Mod S with $A \in \mathcal{A}_C(S)$ and ($\mathcal{A}_C(S)$-$\mathrm{pd}_S\, H \leq$) $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H \leq n - 1$. By Theorem 4.4 again, we have

$$\sup\{i \geq 0 \mid \mathrm{Tor}_i^S(C, N) \neq 0\} = n.$$

So $\sup\{i \geq 0 \mid \mathrm{Tor}_i^S(C, H) \neq 0\} = n - 1$, and hence $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, H = n - 1$ by Theorem 4.4. It follows that $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H = n - 1$.

By Theorem 4.4, there exists an exact sequence

$$0 \to N \to H' \to A' \to 0$$

in Mod S with $A' \in \mathcal{A}_C(S)$ and ($\mathcal{A}_C(S)$-$\mathrm{pd}_S\, H \leq$) $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H' \leq n$. By Corollary 4.5 (3), we have

$$\mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, H = \mathcal{A}_C(S)\text{-}\mathrm{pd}_S\, N = n,$$

and so $\mathcal{I}_C(S)$-$\mathrm{pd}_S\, H' = n$. □

Let $N \in$ Mod S. Bican, El Bashir and Enochs proved in [4] that N has a flat cover. We use

$$\cdots \xrightarrow{f_{n+1}} F_n(N) \xrightarrow{f_n} \cdots \xrightarrow{f_2} F_1(N) \xrightarrow{f_1} F_0(N) \xrightarrow{f_0} N \to 0 \tag{4.2}$$

to denote a minimal flat resolution of N in Mod S, where each $F_i(N) \to \mathrm{Im}\, f_i$ is a flat cover of $\mathrm{Im}\, f_i$.

Lemma 4.7. *Let* $N \in$ Mod S *and* $n \geq 0$. *If* $\mathrm{Tor}_{1 \leq i \leq n}^S(C, N) = 0$, *then we have the following assertions:*

(1) *There exists an exact sequence*

$$0 \to \mathrm{Ext}_R^{n+1}(C, \mathrm{Ker}(1_C \otimes f_{n+1})) \to N \xrightarrow{\mu_N} (C \otimes_S N)_* \to \mathrm{Ext}_R^{n+2}(C, \mathrm{Ker}(1_C \otimes f_{n+1})) \to 0$$

in Mod S.

(2) $\mathrm{Ext}_R^{1 \leq i \leq n}(C, \mathrm{Ker}(1_C \otimes f_{n+1})) = 0$.

Proof. (1): The case for $n = 0$ follows from [17, Proposition 3.2]. Now suppose $n \geq 1$. If $\mathrm{Tor}^S_{1\leq i\leq n}(C, N) = 0$, then the exact sequence (4.2) yields the following exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{Ker}(1_C \otimes f_{n+1}) \longrightarrow C \otimes_S F_{n+1}(N) \xrightarrow{1_C\otimes f_{n+1}} C \otimes_S F_n(N) \xrightarrow{1_C\otimes f_n} \cdots \\ \xrightarrow{1_C\otimes f_2} C \otimes_S F_1(N) \xrightarrow{1_C\otimes f_1} C \otimes_S F_0(N) \xrightarrow{1_C\otimes f_0} C \otimes_S N \longrightarrow 0 \end{aligned} \tag{4.3}$$

in $\mathrm{Mod}\, R$. Because all $C \otimes_S F_i(N)$ are in ${}_RC^{\perp}$ by [17, Lemma 2.3 (1)], we have

$$\begin{aligned} \mathrm{Ext}^1_R(C, \mathrm{Ker}(1_C \otimes f_1)) &\cong \mathrm{Ext}^{n+1}_R(C, \mathrm{Ker}(1_C \otimes f_n)), \\ \mathrm{Ext}^2_R(C, \mathrm{Ker}(1_C \otimes f_1)) &\cong \mathrm{Ext}^{n+2}_R(C, \mathrm{Ker}(1_C \otimes f_n)). \end{aligned}$$

Now the assertion follows from [17, Proposition 3.2].

(2): Applying the functor $(-)_*$ to the exact sequence (4.3), we get the following commutative diagram:

$$\begin{array}{ccccccccc} & & F_{n+1}(N) & \xrightarrow{f_{n+1}} & F_n(N) & \xrightarrow{f_n} \cdots \xrightarrow{f_1} & F_0(N) \\ & & \downarrow{\scriptstyle \mu_{F_{n+1}(N)}} & & \downarrow{\scriptstyle \mu_{F_n(N)}} & & \downarrow{\scriptstyle \mu_{F_0(N)}} \\ 0 \longrightarrow (\mathrm{Ker}(1_C \otimes f_{n+1}))_* & \longrightarrow & (C \otimes_S F_{n+1}(N))_* & \xrightarrow{(1_C\otimes f_{n+1})_*} & (C \otimes_S F_n(N))_* & \xrightarrow{(1_C\otimes f_n)_*} \cdots \xrightarrow{(1_C\otimes f_1)_*} & (C \otimes_S F_0(N))_*. \end{array}$$

All columns are isomorphisms by [11, Lemma 4.1]. So the bottom row in this diagram is exact. Because all $C \otimes_S F_i(N)$ are in ${}_RC^{\perp}$, we have $\mathrm{Ext}^{1\leq i\leq n}_R(C, \mathrm{Ker}(1_C \otimes f_{n+1})) = 0$. □

Let $X \in \mathrm{Mod}\, R$ and let

$$\cdots \xrightarrow{1_C\otimes g_{n+1}} P_n \xrightarrow{1_C\otimes g_n} \cdots \xrightarrow{1_C\otimes g_2} P_1 \xrightarrow{1_C\otimes g_1} P_0 \xrightarrow{1_C\otimes g_0} X \longrightarrow 0$$

be a projective resolution of X in $\mathrm{Mod}\, R$. If there exists $n \geq 1$ such that $\mathrm{Im}\, g_n \cong \oplus_j W_j$, where each W_j is isomorphic to a direct summand of some $\mathrm{Im}\, g_{i_j}$ with $i_j < n$, then we say that *X has an ultimately closed projective resolution at n*; and we say that *X has an ultimately closed projective resolution* if it has an ultimately closed projective resolution at some n (see [14]). It is trivial that if $\mathrm{pd}_R X$ (the projective dimension of X) is less than or equal to n, then X has an ultimately closed projective resolution at $n + 1$. Let R be an Artin algebra. If either R is of finite representation type or the square of the radical of R is zero, then any finitely generated left R-module has an ultimately closed projective resolution [14, p. 341]. Following [21], a module $N \in \mathrm{Mod}\, S$ is called *C-adstatic* if μ_N is an isomorphism.

Proposition 4.8. *Let $N \in \mathrm{Mod}\, S$ and $n \geq 1$. If $\mathrm{Tor}^S_{1\leq i\leq n}(C, N) = 0$, then N is C-adstatic provided that one of the following conditions is satisfied:*
(1) $\mathrm{pd}_R C \leq n$.
(2) *${}_RC$ has an ultimately closed projective resolution at n.*

Proof. (1): It follows directly from Lemma 4.7 (1).

(2): Let

$$\cdots \xrightarrow{1_C\otimes g_{n+1}} P_n \xrightarrow{1_C\otimes g_n} \cdots \xrightarrow{1_C\otimes g_2} P_1 \xrightarrow{1_C\otimes g_1} P_0 \xrightarrow{1_C\otimes g_0} C \longrightarrow 0$$

be a projective resolution of C in $\mathrm{Mod}\, R$ ultimately closed at n. Then $\mathrm{Im}\, g_n \cong \oplus_j W_j$ such that each W_j is isomorphic to a direct summand of some $\mathrm{Im}\, g_{i_j}$ with $i_j < n$. Let $N \in \mathrm{Mod}\, S$ with $\mathrm{Tor}^S_{1\leq i\leq n}(C, N) = 0$. By Lemma 4.7 (2), we have

$$\mathrm{Ext}^1_R(\mathrm{Im}\, g_{i_j}, \mathrm{Ker}(1_C \otimes f_{n+1})) \cong \mathrm{Ext}^{i_j+1}_R(C, \mathrm{Ker}(1_C \otimes f_{n+1})) = 0.$$

Because W_j is isomorphic to a direct summand of some $\mathrm{Im}\, g_{i_j}$, we have $\mathrm{Ext}^1_R(W_j, \mathrm{Ker}(1_C \otimes f_{n+1})) = 0$ for any j, which implies

$$\begin{aligned} \mathrm{Ext}^{n+1}_R(C, \mathrm{Ker}(1_C \otimes f_{n+1})) &\cong \mathrm{Ext}^1_R(\mathrm{Im}\, g_n, \mathrm{Ker}(1_C \otimes f_{n+1})) \\ &\cong \mathrm{Ext}^1_R(\oplus_j W_j, \mathrm{Ker}(1_C \otimes f_{n+1})) \\ &\cong \Pi_j\, \mathrm{Ext}^1_R(W_j, \mathrm{Ker}(1_C \otimes f_{n+1})) \\ &= 0. \end{aligned}$$

Then, by Lemma 4.7 (2), we conclude that $\mathrm{Ext}_R^{1\le i\le n+1}(C, \mathrm{Ker}(1_C \otimes f_{n+1})) = 0$. Similarly to the above argument, we get $\mathrm{Ext}_R^{n+2}(C, \mathrm{Ker}(1_C \otimes f_{n+1})) = 0$. It follows from Lemma 4.7 (1) that μ_N is an isomorphism and N is C-adstatic. □

Corollary 4.9. *For any $n \ge 1$, a module $N \in \mathrm{Mod}\, S$ satisfying $\mathrm{Tor}^S_{0\le i\le n}(C, N) = 0$ implies $N = 0$ provided that one of the following conditions is satisfied:*
(1) $\mathrm{pd}_R\, C \le n$.
(2) ${}_RC$ *has an ultimately closed projective resolution at n.*

Proof. Let $N \in \mathrm{Mod}\, S$ with

$$\mathrm{Tor}^S_{0\le i\le n}(C, N) = 0.$$

By Proposition 4.8, we have that N is C-adstatic and $N \cong (C \otimes_S N)_* = 0$. □

We are now in a position to give the following theorem.

Theorem 4.10. *If ${}_RC$ has an ultimately closed projective resolution, then*

$$\mathcal{A}_C(S) = {C_S}^{\top} = {}^{\perp}\mathcal{I}_C(S).$$

Proof. By the definition of $\mathcal{A}_C(S)$ and by Lemma 4.3, we have $\mathcal{A}_C(S) \subseteq {C_S}^{\top} = {}^{\perp}\mathcal{I}_C(S)$.

Now let $N \in {}^{\perp}\mathcal{I}_C(S)$ and let $f : C \otimes_S N \to B$ be a $\mathcal{B}_C(R)$-preenvelope of $C \otimes_S N$ in $\mathrm{Mod}\, R$ as in Theorem 3.7. Because $\mathcal{B}_C(R)$ is injectively coresolving in $\mathrm{Mod}\, R$ by [11, Theorem 6.2], f is monic. By Proposition 4.8, μ_N is an isomorphism. Then, by Theorem 3.7 (1), we have a monic $\mathcal{A}_C(S)$-preenvelope

$$f^0 : N \rightarrowtail A^0$$

of N, where $f^0 = f_*\mu_N$ and $A^0 = B_*$. So we have a $\mathrm{Hom}_S(-, \mathcal{A}_C(S))$-exact exact sequence

$$0 \to N \xrightarrow{f^0} A^0 \to N^1 \to 0$$

in $\mathrm{Mod}\, S$, where $N^1 = \mathrm{Coker}\, f^0$. Because $A^0 \in {}^{\perp}\mathcal{I}_C(S)$, we have $N^1 \in {}^{\perp}\mathcal{I}_C(S)$. Similarly to the above argument, we get a $\mathrm{Hom}_S(-, \mathcal{A}_C(S))$-exact exact sequence

$$0 \to N^1 \xrightarrow{f^1} A^1 \to N^2 \to 0$$

in $\mathrm{Mod}\, S$ with $A^1 \in \mathcal{A}_C(S)$ and $N^2 \in {}^{\perp}\mathcal{I}_C(S)$. Repeating this procedure, we get a $\mathrm{Hom}_S(-, \mathcal{A}_C(S))$-exact exact sequence

$$0 \to N \xrightarrow{f^0} A^0 \xrightarrow{f^1} A^1 \xrightarrow{f^2} \cdots \xrightarrow{f^i} A^i \xrightarrow{f^{i+1}} \cdots$$

in $\mathrm{Mod}\, S$ with all A^i in $\mathcal{A}_C(S)$. Since $\mathcal{I}_C(S) \subseteq \mathcal{A}_C(S)$ by [11, Corollary 6.1], this exact sequence is $\mathrm{Hom}_S(-, \mathcal{I}_C(S))$-exact. By [17, Theorem 3.11 (1)], there exists a $\mathrm{Hom}_S(-, \mathcal{A}_C(S))$-exact exact sequence

$$0 \to A^i \to U_0^i \to U_1^i \to \cdots \to U_j^i \to \cdots$$

in $\mathrm{Mod}\, S$ with all U_j^i in $\mathcal{I}_C(S)$ for any $i, j \ge 0$. Then, by [12, Corollary 3.9], we get the following $\mathrm{Hom}_S(-, \mathcal{A}_C(S))$-exact exact sequence:

$$0 \to N \to U_0^0 \to U_1^0 \oplus U_0^1 \to \cdots \to \bigoplus_{i=0}^{n} U_{n-i}^i \to \cdots$$

in $\mathrm{Mod}\, S$ with all terms in $\mathcal{I}_C(S)$. It follows from [17, Theorem 3.11 (1)] that $N \in \mathcal{A}_C(S)$. □

We use $\mathrm{pd}_{S^{op}}\, C$ and $\mathrm{fd}_{S^{op}}\, C$ to denote the projective and flat dimensions of C_S, respectively.

Corollary 4.11. *If ${}_RC$ has an ultimately closed projective resolution, then the following statements are equivalent for any $n \ge 0$:*
(1) $\mathrm{pd}_{S^{op}}\, C \le n$.
(2) $\mathcal{A}_C(S)$-$\mathrm{pd}_S\, N \le n$ *for any* $N \in \mathrm{Mod}\, S$.

Proof. Assume that ${}_RC$ has an ultimately closed projective resolution. By Theorem 4.10, we have $\mathcal{A}_C(S) = {C_S}^{\top}$. Then it is easy to see that C_S is flat (equivalently, projective) if and only if $\mathcal{A}_C(S) = \operatorname{Mod} S$, so the assertion for the case $n = 0$ follows. Now let $N \in \operatorname{Mod} S$ and $n \geq 1$.

(2) $\Longrightarrow$ (1): By (2) and Theorem 4.4, we have $\Omega^n(N) \in \mathcal{A}_C(S)$ ($\subseteq {C_S}^{\top}$). Then, by dimension shifting, we have $\operatorname{Tor}^S_{\geq n+1}(C, N) = 0$, and so $\operatorname{pd}_{S^{\mathrm{op}}} C = \operatorname{fd}_{S^{\mathrm{op}}} C \leq n$.

(1) $\Longrightarrow$ (2): If $\operatorname{pd}_{S^{\mathrm{op}}} C \leq n$, then we have $\Omega^n(N) \in {C_S}^{\top}$ by dimension shifting. By Theorem 4.10, we have $\Omega^n(N) \in \mathcal{A}_C(S)$ and $\mathcal{A}_C(S)\text{-}\operatorname{pd}_S N \leq n$. □

Acknowledgment: The author thanks the referee for useful suggestions.

Funding: This research was partially supported by NSFC (Grant no. 11971225).

References

[1] T. Araya, R. Takahashi and Y. Yoshino, Homological invariants associated to semi-dualizing bimodules, *J. Math. Kyoto Univ.* **45** (2005), no. 2, 287–306.
[2] M. Auslander and M. Bridger, *Stable Module Theory*, Mem. Amer. Math. Soc. 94, American Mathematical Society, Providence, 1969.
[3] A. Beligiannis and I. Reiten, Homological and homotopical aspects of torsion theories, *Mem. Amer. Math. Soc.* **188** (2007), no. 883, 1–207.
[4] L. Bican, R. El Bashir and E. Enochs, All modules have flat covers, *Bull. Lond. Math. Soc.* **33** (2001), no. 4, 385–390.
[5] E. E. Enochs, Injective and flat covers, envelopes and resolvents, *Israel J. Math.* **39** (1981), no. 3, 189–209.
[6] E. E. Enochs and H. Holm, Cotorsion pairs associated with Auslander categories, *Israel J. Math.* **174** (2009), 253–268.
[7] E. E. Enochs and A. Iacob, Gorenstein injective covers and envelopes over Noetherian rings, *Proc. Amer. Math. Soc.* **143** (2015), no. 1, 5–12.
[8] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, De Gruyter Exp. Math. 30, Walter de Gruyter, Berlin, 2000.
[9] R. Göbel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules. Vol. 1 and 2*, De Gruyter Exp. Math 41, Walter de Gruyter, Berlin, 2012.
[10] H. Holm and P. Jørgensen, Cotorsion pairs induced by duality pairs, *J. Commut. Algebra* **1** (2009), no. 4, 621–633.
[11] H. Holm and D. White, Foxby equivalence over associative rings, *J. Math. Kyoto Univ.* **47** (2007), no. 4, 781–808.
[12] Z. Huang, Proper resolutions and Gorenstein categories, *J. Algebra* **393** (2013), 142–169.
[13] Z. Huang, Homological dimensions relative to preresolving subcategories, *Kyoto J. Math.* **54** (2014), no. 4, 727–757.
[14] J. P. Jans, Some generalizations of finite projective dimension, *Illinois J. Math.* **5** (1961), 334–344.
[15] Z. Liu, Z. Huang and A. Xu, Gorenstein projective dimension relative to a semidualizing bimodule, *Comm. Algebra* **41** (2013), no. 1, 1–18.
[16] F. Mantese and I. Reiten, Wakamatsu tilting modules, *J. Algebra* **278** (2004), no. 2, 532–552.
[17] X. Tang and Z. Huang, Homological aspects of the adjoint cotranspose, *Colloq. Math.* **150** (2017), no. 2, 293–311.
[18] T. Wakamatsu, On modules with trivial self-extensions, *J. Algebra* **114** (1988), no. 1, 106–114.
[19] T. Wakamatsu, Stable equivalence for self-injective algebras and a generalization of tilting modules, *J. Algebra* **134** (1990), no. 2, 298–325.
[20] T. Wakamatsu, Tilting modules and Auslander's Gorenstein property, *J. Algebra* **275** (2004), no. 1, 3–39.
[21] R. Wisbauer, Static modules and equivalences, in: *Interactions Between Ring Theory and Representations of Algebras (Murcia)*, Lecture Notes Pure Appl. Math. 210, Dekker, New York (2000), 423–449.