Duality Pairs Induced by Auslander and Bass Classes∗
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Abstract

Let \( R \) and \( S \) be any rings and \( R_C S \) a semidualizing bimodule, and let \( A_C(R^{\text{op}}) \) and \( B_C(R) \) be the Auslander and Bass classes respectively. Then both the pairs

\((A_C(R^{\text{op}}), B_C(R))\) and \((B_C(R), A_C(R^{\text{op}}))\)

are coproduct-closed and product-closed duality pairs and both \( A_C(R^{\text{op}}) \) and \( B_C(R) \) are covering and preenveloping; in particular, the former duality pair is perfect. Moreover, if \( B_C(R) \) is enveloping in \( \text{Mod}\, R \), then \( A_C(S) \) is enveloping in \( \text{Mod}\, S \). Then some applications to the Auslander projective dimension of modules are given.

1 Introduction

In relative homological algebra, the theory of covers and envelopes is fundamental and important. Let \( R \) be a ring and \( \text{Mod}\, R \) the category of left \( R \)-modules. Given a subcategory of \( \text{Mod}\, R \), it is always worth studying whether or when it is (pre)covering or (pre)enveloping. This problem has been studied extensively, see [3]–[10] and references therein.

Let \( R \) be a commutative noetherian ring and \( C \) a semidualizing \( R \)-module, and let \( A_C(R) \) and \( B_C(R) \) be the Auslander and Bass classes respectively. By proving that both \( A_C(R) \) and \( B_C(R) \) are Kaplansky classes, Enochs and Holm got in [6, Theorems 3.11 and 3.12] that the pair \((A_C(R), (A_C(R))^{\perp})\) is a perfect cotorsion pair, \( A_C(R) \) is covering and preenveloping and \( B_C(R) \) is preenveloping. Holm and Jørgensen introduced the notion of duality pairs and proved the following remarkable result. Let \( R \) be an arbitrary ring, and let \( \mathcal{X} \) and \( \mathcal{Y} \) be subcategories of \( \text{Mod}\, R \) and \( \text{Mod}\, R^{\text{op}} \) respectively. When \((\mathcal{X}, \mathcal{Y})\) is a duality pair, the following assertions hold true: (1) If \( \mathcal{X} \) is closed under coproducts, then \( \mathcal{X} \) is covering; (2) if \( \mathcal{X} \) is closed under products, then \( \mathcal{X} \) is preenveloping; and (3) if \( R R \in \mathcal{X} \) and \( \mathcal{X} \) is closed under coproducts and extensions, then \((\mathcal{X}, \mathcal{X}^{\perp})\) is a perfect cotorsion pair ([10, Theorem 3.1]). By using it, they generalized the above result of Enochs and Holm to the category of complexes, and Enochs and Iacob investigated in [7] the existence of Gorenstein injective envelopes over commutative noetherian rings.

Let \( R \) and \( S \) be arbitrary rings and \( R_C S \) a semidualizing bimodule, and let \( A_C(R^{\text{op}}) \) be the Auslander class in \( \text{Mod}\, R^{\text{op}} \) and \( B_C(R) \) the Bass class in \( \text{Mod}\, R \). Our first main result is the following

Theorem 1.1. (Theorem 3.3)

∗2010 Mathematics Subject Classification: 18G25, 16E10, 16E30.
†Keywords: Duality pairs, Auslander classes, Bass classes, Semidualizing bimodules, (Pre)covers, (Pre)envelopes, Cotorsion pairs, Auslander projective dimension.
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Both the pairs \((A_C(R^\text{op}), B_C(R))\) and \((B_C(R), A_C(R^\text{op}))\) are coproduct-closed and product-closed duality pairs; and furthermore, the former one is perfect.

(2) \(A_C(R^\text{op})\) is covering and preenveloping in \(\text{Mod } R^\text{op}\) and \(B_C(R)\) is covering and preenveloping in \(\text{Mod } R\).

As a consequence of Theorem 1.1, we get that the pair \((A_C(R^\text{op}), A_C(R^\text{op})^\perp)\) is a hereditary perfect cotorsion pair and \(A_C(R^\text{op})\) is covering and preenveloping in \(\text{Mod } R^\text{op}\). Moreover, we prove the following

**Theorem 1.2.** (Theorem 3.7(2)) If \(B_C(R)\) is enveloping in \(\text{Mod } R\), then \(A_C(S)\) is enveloping in \(\text{Mod } S\).

Then we apply these results and their symmetric results to study the Auslander projective dimension of modules. We obtain some criteria for computing the Auslander projective dimension of modules in \(\text{Mod } S\) (Theorem 4.4). Furthermore, we get the following

**Theorem 1.3.** (Theorem 4.10) If \(RC\) has an ultimately closed projective resolution, then
\[ A_C(S) = C_S^\top = \perp^1 I_C(S), \]
where \(C_S^\top\) is the Tor-orthogonal class of \(C_S\) and \(\perp^1 I_C(S)\) is the left Ext-orthogonal class of the subcategory \(I_C(S)\) of \(\text{Mod } S\) consisting of \(C\)-injective modules.

As a consequence, we have that if \(RC\) has an ultimately closed projective resolution, then the projective dimension of \(C_S\) is at most \(n\) if and only if the Auslander projective dimension of any module in \(\text{Mod } S\) is at most \(n\) (Corollary 4.11).

2 Preliminaries

In this paper, all rings are associative with identities. Let \(R\) be a ring. We use \(\text{Mod } R\) to denote the category of left \(R\)-modules and all subcategories of \(\text{Mod } R\) are full and closed under isomorphisms. For a subcategory \(\mathcal{X}\) of \(\text{Mod } R\), we write
\[
\perp \mathcal{X} := \{ A \in \text{Mod } R \mid \text{Ext}_{R}^{\geq 1}(A, X) = 0 \text{ for any } X \in \mathcal{X} \},
\]
\[
\mathcal{X}^\perp := \{ A \in \text{Mod } R \mid \text{Ext}_{R}^{\geq 1}(X, A) = 0 \text{ for any } X \in \mathcal{X} \},
\]
\[
\perp^1 \mathcal{X} := \{ A \in \text{Mod } R \mid \text{Ext}_{R}^{1}(A, X) = 0 \text{ for any } X \in \mathcal{X} \},
\]
\[
\mathcal{X}^{\perp^1} := \{ A \in \text{Mod } R \mid \text{Ext}_{R}^{1}(X, A) = 0 \text{ for any } X \in \mathcal{X} \}.
\]

For subcategories \(\mathcal{X}, \mathcal{Y}\) of \(\text{Mod } R\), we write \(\mathcal{X} \perp \mathcal{Y}\) if \(\text{Ext}_{R}^{\geq 1}(X, Y) = 0\) for any \(X \in \mathcal{X}\) and \(Y \in \mathcal{Y}\).
Definition 2.1. ([5, 8]) Let \( \mathcal{X} \subseteq \mathcal{Y} \) be subcategories of \( \text{Mod} R \). A homomorphism \( f : X \to Y \) in \( \text{Mod} R \) with \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \) is called an \( \mathcal{X} \)-precover of \( Y \) if \( \text{Hom}_R(X', f) \) is epic for any \( X' \in \mathcal{X} \); and \( f \) is called right minimal if an endomorphism \( h : X \to X \) is an automorphism whenever \( f = fh \).

An \( \mathcal{X} \)-precover \( f : X \to Y \) is called an \( \mathcal{X} \)-cover of \( Y \) if it is right minimal. The subcategory \( \mathcal{X} \) is called (pre)covering in \( \mathcal{Y} \) if any object in \( \mathcal{Y} \) admits an \( \mathcal{X} \)-(pre)cover. Dually, the notions of an \( \mathcal{X} \)-(pre)envelope, a left minimal homomorphism and a (pre)enveloping subcategory are defined.

Definition 2.2. ([8, 9]) Let \( \mathcal{U}, \mathcal{V} \) be subcategories of \( \text{Mod} R \).

1. The pair \( (\mathcal{U}, \mathcal{V}) \) is called a cotorsion pair in \( \text{Mod} R \) if \( \mathcal{U} = \perp_1 \mathcal{V} \) and \( \mathcal{V} = \perp_1 \mathcal{U} \).
2. A cotorsion pair \( (\mathcal{U}, \mathcal{V}) \) is called perfect if \( \mathcal{U} \) is covering and \( \mathcal{V} \) is enveloping in \( \text{Mod} R \).
3. A cotorsion pair \( (\mathcal{U}, \mathcal{V}) \) is called hereditary if one of the following equivalent conditions is satisfied.
   1. \( \mathcal{U} \perp \mathcal{V} \).
   2. \( \mathcal{U} \) is projectively resolving in the sense that \( \mathcal{U} \) contains all projective modules in \( \text{Mod} R \), \( \mathcal{U} \) is closed under extensions and kernels of epimorphisms.
   3. \( \mathcal{V} \) is injectively coresolving in the sense that \( \mathcal{V} \) contains all injective modules in \( \text{Mod} R \), \( \mathcal{V} \) is closed under extensions and cokernels of monomorphisms.

Set \( (-)^+ := \text{Hom}_\mathbb{Z}(-, \mathbb{Q}/\mathbb{Z}) \), where \( \mathbb{Z} \) is the additive group of integers and \( \mathbb{Q} \) is the additive group of rational numbers. The following is the definition of duality pairs (cf. [7, 10]).

Definition 2.3. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be subcategories of \( \text{Mod} R \) and \( \text{Mod} R^{\text{op}} \) respectively.

1. The pair \( (\mathcal{X}, \mathcal{Y}) \) is called a duality pair if the following conditions are satisfied.
   1.1) For a module \( X \in \text{Mod} R \), \( X \in \mathcal{X} \) if and only if \( X^+ \in \mathcal{Y} \).
   1.2) \( \mathcal{Y} \) is closed under direct summands and finite direct sums.
2. A duality pair \( (\mathcal{X}, \mathcal{Y}) \) is called (co)product-closed if \( \mathcal{X} \) is closed under (co)products.
3. A duality pair \( (\mathcal{X}, \mathcal{Y}) \) is called perfect if it is coproduct-closed, \( \mathcal{X} \) is closed under extensions.

We also recall the following remarkable result.

Lemma 2.4. ([7, p.7, Theorem] and [10, Theorem 3.1]) Let \( \mathcal{X} \) and \( \mathcal{Y} \) be subcategories of \( \text{Mod} R \) and \( \text{Mod} R^{\text{op}} \) respectively. If \( (\mathcal{X}, \mathcal{Y}) \) is a duality pair, then the following assertions hold true.

1. If \( (\mathcal{X}, \mathcal{Y}) \) is coproduct-closed, then \( \mathcal{X} \) is covering.
2. If \( (\mathcal{X}, \mathcal{Y}) \) is product-closed, then \( \mathcal{X} \) is preenveloping.
3. If \( (\mathcal{X}, \mathcal{Y}) \) is perfect, then \( (\mathcal{X}, \mathcal{X}^\perp) \) is a perfect cotorsion pair.

Definition 2.5. ([1, 11]). Let \( R \) and \( S \) be rings. An \( (R, S) \)-bimodule \( \mu C_S \) is called semidualizing if the following conditions are satisfied.
(a1) $RC$ admits a degreewise finite $R$-projective resolution.

(a2) $CS$ admits a degreewise finite $S$-projective resolution.

(b1) The homothety map $\mu_{R}: R \rightarrow \text{Hom}_{S^{op}}(C, C)$ is an isomorphism.

(b2) The homothety map $\mu_{S}: S \rightarrow \text{Hom}_{R}(C, C)$ is an isomorphism.

(c1) $\text{Ext}_{R}^{\geq 1}(C, C) = 0$.

(c2) $\text{Ext}_{S^{op}}^{\geq 1}(C, C) = 0$.

Wakamatsu in [18] introduced and studied the so-called generalized tilting modules, which are usually called Wakamatsu tilting modules, see [3, 16]. Note that a bimodule $RC$ is semidualizing if and only if it is Wakamatsu tilting ([20, Corollary 3.2]). Examples of semidualizing bimodules are referred to [11, 19].

3 Duality pairs

In this section, $R$ and $S$ are arbitrary rings and $RC$ is a semidualizing bimodule. We write $(-)^{*} := \text{Hom}(C, -)$ and

$$R^\perp := \{M \in \text{Mod} R \mid \text{Ext}_{R}^{1}(C, M) = 0\} \quad \text{and} \quad S^\perp := \{B \in \text{Mod} S^{op} \mid \text{Ext}_{S^{op}}^{1}(C, B) = 0\},$$

$$R^\top := \{N \in \text{Mod} R^{op} \mid \text{Tor}_{1}^{R}(N, C) = 0\} \quad \text{and} \quad S^\top := \{A \in \text{Mod} S \mid \text{Tor}_{1}^{S}(C, A) = 0\}.$$

**Definition 3.1.** ([11])

(1) The **Auslander class** $A_{C}(R^{op})$ with respect to $C$ consists of all modules $N$ in $\text{Mod} R^{op}$ satisfying the following conditions.

(a1) $N \in R^\top$.

(a2) $N \otimes_{R} C \in C^\perp$.

(a3) The canonical valuation homomorphism

$$\mu_{N} : N \rightarrow (N \otimes_{R} C)^{*}$$

defined by $\mu_{N}(x)(c) = x \otimes c$ for any $x \in N$ and $c \in C$ is an isomorphism in $\text{Mod} R^{op}$.

(2) The **Bass class** $B_{C}(R)$ with respect to $C$ consists of all modules $M$ in $\text{Mod} R$ satisfying the following conditions.

(b1) $M \in R^\perp$.

(b2) $M^{*} \in C^\top$.

(b3) The canonical valuation homomorphism

$$\theta_{M} : C \otimes_{S} M^{*} \rightarrow M$$

defined by $\theta_{M}(c \otimes f) = f(c)$ for any $c \in C$ and $f \in M^{*}$ is an isomorphism in $\text{Mod} R$. 
(3) The **Auslander class** \( \mathcal{A}_C(S) \) in \( \text{Mod} \ S \) and the **Bass class** \( \mathcal{B}_C(S^{\text{op}}) \) in \( \text{Mod} \ S^{\text{op}} \) are defined symmetrically.

The following result is crucial. From its proof, it is known that the conditions in the definitions of \( \mathcal{A}_C(R^{\text{op}}) \) and \( \mathcal{B}_C(R) \) are dual item by item.

**Proposition 3.2.**

1. For a module \( N \in \text{Mod} \ R^{\text{op}} \), \( N \in \mathcal{A}_C(R^{\text{op}}) \) if and only if \( N^+ \in \mathcal{B}_C(R) \).
2. For a module \( M \in \text{Mod} \ R \), \( M \in \mathcal{B}_C(R) \) if and only if \( M^+ \in \mathcal{A}_C(R^{\text{op}}) \).

**Proof.** (1) Let \( N \in \text{Mod} \ R^{\text{op}} \). Then we have the following

\[
N \in ^\top R C \\
\Leftrightarrow \text{Tor}^R_{\geq 1}(N, C) = 0 \\
\Leftrightarrow [\text{Tor}^S_{\geq 1}(N, C)]^+ = 0 \\
\Leftrightarrow \text{Ext}^1_R(C, N^+) = 0 \text{ (by [9, Lemma 2.16(b)])} \\
\Rightarrow N^+ \in R C^\perp.
\]

(b)

\[
N \otimes_R C \in C_S^\perp \\
\Leftrightarrow \text{Ext}^1_{S^{\text{op}}}(C, N \otimes_R C) = 0 \\
\Leftrightarrow [\text{Ext}^1_{S^{\text{op}}}(C, N \otimes_R C)]^+ = 0 \\
\Leftrightarrow \text{Tor}^S_{\geq 1}(C, (N \otimes_R C)^+) = 0 \text{ (by [9, Lemma 2.16(d)])} \\
\Leftrightarrow \text{Tor}^S_{\geq 1}(C, (N^+)_-) = 0 \text{ (by [9, Lemma 2.16(a)])} \\
\Rightarrow (N^+)_- \in C_S^\top.
\]

(c) By [9, Lemma 2.16(c)], the canonical valuation homomorphism

\[
\alpha : C \otimes_S (N \otimes_R C)^+ \to [\text{Hom}_{S^{\text{op}}}(C, N \otimes_R C)]^+
\]

defined by \( \alpha(c \otimes g)(f) = g f(c) \) for any \( c \in C, \ g \in (N \otimes_R C)^+ \) and \( f \in \text{Hom}_{S^{\text{op}}}(C, N \otimes_R C) \) is an isomorphism in \( \text{Mod} R \). By [9, Lemma 2.16(a)], the canonical valuation homomorphism

\[
\beta : (N \otimes_R C)^+ \to \text{Hom}_R(C, N^+)
\]

defined by \( \beta(g)(c)(x) = g(x \otimes c) \) for any \( g \in (N \otimes_R C)^+ \), \( c \in C \) and \( x \in N \) is an isomorphism in \( \text{Mod} S \). So

\[
1_C \otimes \beta : C \otimes_S (N \otimes_R C)^+ \to C \otimes_S \text{Hom}_R(C, N^+)
\]

defined via \( (1_C \otimes \beta)(c \otimes g) = c \otimes \beta(g) \) for any \( c \in C \) and \( g \in (N \otimes_R C)^+ \) is an isomorphism in \( \text{Mod} R \).
Consider the following diagram

\[
\begin{array}{ccc}
C \otimes_S (N \otimes_R C)^+ & \xrightarrow{\alpha} & [\text{Hom}_{S^{op}}(C, N \otimes_R C)]^+ \\
\downarrow^{1_C \otimes \beta} & & \downarrow^{(\mu_N)^+} \\
C \otimes_S \text{Hom}_R(C, N^+) & \xrightarrow{\theta_{N^+}} & N^+,
\end{array}
\]

where

\[(\mu_N)^+: [\text{Hom}_{S^{op}}(C, N \otimes_R C)]^+ \to N^+
\]
defined via \((\mu_N)^+(f') = f' \mu_N\) for any \(f' \in [\text{Hom}_{S^{op}}(C, N \otimes_R C)]^+\) is a natural homomorphism in \text{Mod} \: R, and

\[
\theta_{N^+} : C \otimes_S \text{Hom}_R(C, N^+) \to N^+
\]
defined by \(\theta_{N^+}(c \otimes f^\circ) = f^\circ(c)\) for any \(c \in C\) and \(f^\circ \in \text{Hom}_R(C, N^+)\) is a canonical valuation homomorphism in \text{Mod} \: R. Then for any \(c \in C, g \in (N \otimes_R C)^+\) and \(x \in N\), we have

\[
(\mu_N)^+ \alpha(c \otimes g)(x) = \alpha(c \otimes g)\mu_N(x) = g\mu_N(x)(c) = g(x \otimes c)
\]

\[
\theta_{N^+}(1_C \otimes \beta)(c \otimes g)(x) = \theta_{N^+}(c \otimes \beta(g))(x) = \beta(g)(c)(x) = g(x \otimes c),
\]

Thus

\[
(\mu_N)^+ \alpha = \theta_{N^+}(1_C \otimes \beta),
\]

and therefore \(\mu_N\) is an isomorphism \(\iff\) \((\mu_N)^+\) is an isomorphism \(\iff\) \(\theta_{N^+}\) is an isomorphism.

We conclude that \(N \in \mathcal{A}_C(R^{op}) \iff N^+ \in \mathcal{B}_C(R)\).

(2) Let \(M \in \text{Mod} \: R\). Then we have the following

(a)

\[
M \in R^C
\]

\[\iff\]

\[\text{Ext}_{R^1}^1(C, M) = 0
\]

\[\iff\]

\[\text{Ext}_{R^1}^1(C, M)^+ = 0
\]

\[\iff\]

\[\text{Tor}_{R^1}^0(M^+, C) = 0 \text{ (by [9, Lemma 2.16(d)])}
\]

\[\iff\]

\[M^+ \in \overline{\text{R}^C}.
\]

(b)

\[
M_* \in C_S^1
\]

\[\iff\]

\[\text{Tor}_{S^1}^0(C, M_*) = 0
\]

\[\iff\]

\[\text{Tor}_{S^1}^0(C, M_*)^+ = 0
\]

\[\iff\]

\[\text{Ext}_{S^{op}}^0(C, (M_*)^+) = 0 \text{ (by [9, Lemma 2.16(b)])}
\]

\[\iff\]

\[\text{Ext}_{S^{op}}^0(C, M^+ \otimes_R C) = 0 \text{ (by [9, Lemma 2.16(c)])}
\]

\[\iff\]

\[M^+ \otimes_R C \in C_S^1.
\]

(c) By [9, Lemma 2.16(a)], the canonical valuation homomorphism

\[
\tau : [C \otimes_S \text{Hom}_R(C, M)]^+ \to \text{Hom}_{S^{op}}(C, [\text{Hom}_R(C, M)]^+)
\]
defined by \( \tau(g')(c)(f) = g'(c \otimes f) \) for any \( g' \in [C \otimes_S \text{Hom}_R(C, M)]^+ \), \( c \in C \) and \( f \in \text{Hom}_R(C, M) \) is an isomorphism in \( \text{Mod} R^\text{op} \). By [9, Lemma 2.16(c)], the canonical valuation homomorphism

\[
\sigma : M^+ \otimes_R C \to [\text{Hom}_R(C, M)]^+
\]
defined by \( \sigma(g \otimes c)(f) = gf(c) \) for any \( g \in M^+ \), \( c \in C \) and \( f \in \text{Hom}_R(C, M) \) is an isomorphism in \( \text{Mod} S^\text{op} \). So

\[
\text{Hom}_{S^\text{op}}(C, \sigma) : \text{Hom}_{S^\text{op}}(C, M^+ \otimes_R C) \to \text{Hom}_{S^\text{op}}(C, [\text{Hom}_R(C, M)]^+)
\]
defined via \( \text{Hom}_{S^\text{op}}(C, \sigma)(g) = \sigma g'' \) for any \( g'' \in \text{Hom}_{S^\text{op}}(C, M^+ \otimes_R C) \) is an isomorphism in \( \text{Mod} R^\text{op} \).

Consider the following diagram

\[
\begin{array}{ccc}
M^+ & \xrightarrow{(\theta_M)^+} & [C \otimes_S \text{Hom}_R(C, M)]^+
\\[-1em]
\downarrow{\mu_{M^+}} & & \downarrow{\tau}
\\[-1em]
\text{Hom}_{S^\text{op}}(C, M^+ \otimes_R C) & \xrightarrow{\text{Hom}_{S^\text{op}}(C, \sigma)} & \text{Hom}_{S^\text{op}}(C, [\text{Hom}_R(C, M)]^+)
\end{array}
\]

where

\[
(\theta_M)^+ : M^+ \to [C \otimes_S \text{Hom}_R(C, M)]^+
\]
defined via \( (\theta_M)^+(g) = g \theta_M \) for any \( g \in M^+ \) is a natural homorphism in \( \text{Mod} R^\text{op} \), and

\[
\mu_{M^+} : M^+ \to \text{Hom}_{S^\text{op}}(C, M^+ \otimes_R C)
\]
defined by \( \mu_{M^+}(g)(c) = g \otimes c \) for any \( g \in M^+ \) and \( c \in C \) is a canonical valuation homomorphism in \( \text{Mod} R^\text{op} \). Then for any \( g \in M^+ \), \( c \in C \) and \( f \in \text{Hom}_R(C, M) \), we have

\[
\tau(\theta_M)^+(g)(c)(f) = (\theta_M)^+(g)(c \otimes f) = g \theta_M(c \otimes f) = gf(c),
\]

\[
\text{Hom}_{S^\text{op}}(C, \sigma)\mu_{M^+}(g)(c)(f) = \sigma \mu_{M^+}(g)(c)(f) = \sigma(g \otimes c)(f) = gf(c),
\]

Thus

\[
\tau(\theta_M)^+ = \text{Hom}_{S^\text{op}}(C, \sigma)\mu_{M^+},
\]

and therefore \( \theta_M \) is an isomorphism \( \iff \) \( (\theta_M)^+ \) is an isomorphism \( \iff \mu_{M^+} \) is an isomorphism.

We conclude that \( M \in \mathcal{B}_C(R) \iff M^+ \in \mathcal{A}_C(R^\text{op}) \).

As a consequence, we get the following

**Theorem 3.3.**

1. The pair

\[
(\mathcal{A}_C(R^\text{op}), \mathcal{B}_C(R))
\]

is a perfect coproduct-closed and product-closed duality pair and \( \mathcal{A}_C(R^\text{op}) \) is covering and preenveloping in \( \text{Mod} R^\text{op} \).

2. The pair

\[
(\mathcal{B}_C(R), \mathcal{A}_C(R^\text{op}))
\]

is a coproduct-closed and product-closed duality pair and \( \mathcal{B}_C(R) \) is covering and preenveloping in \( \text{Mod} R \).

\[\square\]
Proof. It follows from [11, Proposition 4.2(a)] that both \( \mathcal{A}_C(R^{op}) \) and \( \mathcal{B}_C(R) \) are closed under direct summands, coproducts and products. So by Lemma 2.4(1)(2) and Proposition 3.2, we have that both the pairs

\[(\mathcal{A}_C(R^{op}), \mathcal{B}_C(R)) \text{ and } (\mathcal{B}_C(R), \mathcal{A}_C(R^{op}))\]

are coproduct-closed and product-closed duality pairs, \( \mathcal{A}_C(R^{op}) \) is covering and preenveloping in \( \text{Mod } R^{op} \) and \( \mathcal{B}_C(R) \) is covering and preenveloping in \( \text{Mod } R \). Moreover, \( \mathcal{A}_C(R^{op}) \) is projectively resolving by [11, Theorem 6.2], so the duality pair \( (\mathcal{A}_C(R^{op}), \mathcal{B}_C(R)) \) is perfect. \( \square \)

We write

\[\mathcal{A}_C(R^{op}) = \{ Y \in \text{Mod } R^{op} \mid \text{Ext}^{\geq 1}_{R^{op}}(N, Y) = 0 \text{ for any } N \in \mathcal{A}_C(R^{op})\} \]

The following corollary was proved in [6, Theorem 3.11] when \( R \) is a commutative noetherian ring and \( r \mathcal{C}_S = r \mathcal{C}_R \).

**Corollary 3.4.** The pair

\[(\mathcal{A}_C(R^{op}), \mathcal{A}_C(R^{op})^\perp)\]

is a hereditary perfect cotorsion pair and \( \mathcal{A}_C(R^{op}) \) is covering and preenveloping in \( \text{Mod } R^{op} \).

**Proof.** It follows from Theorem 3.3(1) and Lemma 2.4(3). \( \square \)

The following two results are the symmetric versions of Theorem 3.3 and Corollary 3.4 respectively.

**Theorem 3.5.**

1. The pair

\[(\mathcal{A}_C(S), \mathcal{B}_C(S^{op}))\]

is a perfect coproduct-closed and product-closed duality pair and \( \mathcal{A}_C(S) \) is covering and preenveloping in \( \text{Mod } S \).

2. The pair

\[(\mathcal{B}_C(S^{op}), \mathcal{A}_C(S))\]

is a coproduct-closed and product-closed duality pair and \( \mathcal{B}_C(S^{op}) \) is covering and preenveloping in \( \text{Mod } S^{op} \).

We write

\[\mathcal{A}_C(S)^\perp = \{ X \in \text{Mod } S \mid \text{Ext}^{\geq 1}_S(N', X) = 0 \text{ for any } N' \in \mathcal{A}_C(S)\} \]

**Corollary 3.6.** The pair

\[(\mathcal{A}_C(S), \mathcal{A}_C(S)^\perp)\]

is a hereditary perfect cotorsion pair and \( \mathcal{A}_C(S) \) is covering and preenveloping in \( \text{Mod } S \).
Holm and White proved in [11, Proposition 4.1] that there exist the following (Foxby) equivalences of categories

\[
\mathcal{A}_C(S) \xrightarrow{\sim} \mathcal{B}_C(R),
\]

\[
\mathcal{A}_C(R^{op}) \xrightarrow{\sim} \mathcal{B}_C(S^{op}).
\]

Compare this result with Theorems 3.3 and 3.5.

By Theorems 3.3(2) and 3.5(1), \(B_C(R)\) is preenveloping in \(\text{Mod} R\) and \(A_C(S)\) is preenveloping in \(\text{Mod} S\). In the following result, we construct an \(A_C(S)\)-preenvelope of a given module in \(\text{Mod} S\) from a \(B_C(R)\)-preenvelope of some module in \(\text{Mod} R\).

**Theorem 3.7.**

(1) Let \(N \in \text{Mod} S\) and \(f : C \otimes S N \to B\) be a \(B_C(R)\)-preenvelope of \(C \otimes S N\) in \(\text{Mod} R\). Then we have

(1.1) \(f_* \mu_N : N \to B_*\)

is an \(A_C(S)\)-preenvelope of \(N\) in \(\text{Mod} S\).

(1.2) If \(f\) is a \(B_C(R)\)-envelope of \(C \otimes S N\), then \(f_* \mu_N\) is an \(A_C(S)\)-envelope of \(N\).

(2) If \(B_C(R)\) is enveloping in \(\text{Mod} R\), then \(A_C(S)\) is enveloping in \(\text{Mod} S\).

**Proof.** (1.1) Let \(N \in \text{Mod} S\) and \(f : C \otimes S N \to B\) be a \(B_C(R)\)-preenvelope in \(\text{Mod} R\). By [11, Proposition 4.1], we have \(B_* \in \mathcal{A}_C(S)\). Let \(g \in \text{Hom}_S(N, A)\) with \(A \in \mathcal{A}_C(S)\). By [11, Proposition 4.1] again, we have \(C \otimes S A \in \mathcal{B}_C(R)\). So there exists \(h \in \text{Hom}_R(B, C \otimes S A)\) such that \(1_C \otimes g = hf\), that is, the following diagram

\[
\begin{array}{ccc}
C \otimes S N & \xrightarrow{f} & B \\
\downarrow{1_C \otimes g} & & \downarrow{h} \\
C \otimes S A & & \\
\end{array}
\]

commutes. From the following commutative diagram

\[
\begin{array}{ccc}
N & \xrightarrow{g} & A \\
\downarrow{\mu_N} & & \downarrow{\mu_A} \\
(C \otimes S N)_* \xrightarrow{(1_C \otimes g)_*} (C \otimes S A)_* \\
\end{array}
\]

we get \(\mu_A g = (1_C \otimes g)_* \mu_N\). Because \(\mu_A\) is an isomorphism, we have

\[
g = \mu_A^{-1}(1_C \otimes g)_* \mu_N = (\mu_A^{-1} h_*)(f_* \mu_N).
\]
that is, the following diagram

\[
\begin{array}{ccc}
N & \xrightarrow{f_*} & B \\
\downarrow{\mu_N} & & \downarrow{h_*} \\
A & \xrightarrow{\mu_A^{-1}} & 1
\end{array}
\]

commutes. Thus \( f_*\mu_N : N \to B_* \) is an \( \mathcal{A}_C(S) \)-preenvelope of \( N \).

(1.2) By (1.1), it suffices to prove that if \( f \) is left minimal, then so is \( f_*\mu_N \).

Let \( f \) be left minimal and \( h \in \text{Hom}_S(B_*, B_* \ast) \) such that \( f_*\mu_N = h(f_*\mu_N) \). Then we have

\[
(1_C \otimes f_*)(1_C \otimes \mu_N) = 1_C \otimes (f_*\mu_N) = 1_C \otimes (h(f_*\mu_N)) = (1_C \otimes h)(1_C \otimes f_*)(1_C \otimes \mu_N).
\tag{3.1}
\]

From the following commutative diagram

\[
\begin{array}{ccc}
C \otimes_S (C \otimes_S N) & \xrightarrow{1_C \otimes f_*} & C \otimes_S B_* \\
\downarrow{\theta_{C \otimes S N}} & & \downarrow{\theta_B} \\
C \otimes_S N & \xrightarrow{f} & B_*,
\end{array}
\]

we get

\[
f \theta_{C \otimes S N} = \theta_B(1_C \otimes f_*).
\tag{3.2}
\]

So we have

\[
f = f 1_{C \otimes S N}
\]

\[
= f(\theta_{C \otimes S N}(1_C \otimes \mu_N)) \quad \text{(by [21, Proposition 2.2(1)])}
\]

\[
= \theta_B(1_C \otimes f_*)(1_C \otimes \mu_N) \quad \text{(by (3.2))}
\]

\[
= \theta_B(1_C \otimes h)(1_C \otimes f_*)(1_C \otimes \mu_N) \quad \text{(by (3.1))}
\]

\[
= \theta_B(1_C \otimes h)(\theta_B^{-1} \theta_B)(1_C \otimes f_*)(1_C \otimes \mu_N) \quad \text{(because \( \theta_B \) is an isomorphism)}
\]

\[
= \theta_B(1_C \otimes h)\theta_B^{-1} f \theta_{C \otimes S N}(1_C \otimes \mu_N) \quad \text{(by (3.2))}
\]

\[
= \theta_B(1_C \otimes h)\theta_B^{-1} f 1_{C \otimes S N} \quad \text{(by [21, Proposition 2.2(1)])}
\]

\[
= \theta_B(1_C \otimes h)\theta_B^{-1} f.
\]

Because \( f \) is left minimal, \( \theta_B(1_C \otimes h)\theta_B^{-1} \) is an isomorphism, which implies that \( 1_C \otimes h \) and \( (1_C \otimes h)_* \) are also isomorphisms. From the following commutative diagram

\[
\begin{array}{ccc}
B_* & \xrightarrow{h} & B_* \\
\downarrow{\mu_B} & & \downarrow{\mu_B} \\
(C \otimes_S B)_* & \xrightarrow{(1_C \otimes h)} & (C \otimes_S B)_* 
\end{array}
\]

we get

\[
(1_C \otimes h)_* \mu_{B_*} = \mu_B h.
\]

Because \( B_* \in \mathcal{A}_C(S) \) by [11, Proposition 4.1], \( \mu_{B_*} \) is an isomorphism. It follows that \( h \) is also an isomorphism and \( f_*\mu_N \) is left minimal.

(2) It follows from the assertion (1.2) immediately. \( \square \)
We do not know whether a $B_C(R)$-preenvelope of given module in $\text{Mod } R$ can be constructed from an $A_C(S)$-preenvelope of some module in $\text{Mod } S$, and do not know whether the converse of Theorem 3.7(2) holds true.

By Theorems 3.3(2) and 3.5(1), $B_C(R)$ is covering in $\text{Mod } R$ and $A_C(S)$ is covering in $\text{Mod } S$. In the following result, we construct a $B_C(R)$-cover of a given module in $\text{Mod } R$ from an $A_C(S)$-cover of some module in $\text{Mod } S$.

**Proposition 3.8.** Let $M \in \text{Mod } R$ and $g : A \to M_*$ be an $A_C(S)$-cover of $M_*$ in $\text{Mod } S$. Then $\theta_M(1_C \otimes g) : C \otimes S A \to M$ is a $B_C(R)$-cover of $M$ in $\text{Mod } R$.

**Proof.** Let $M \in \text{Mod } R$ and $g : A \to M_*$ be an $A_C(S)$-cover of $M_*$ in $\text{Mod } S$. By [11, Proposition 4.1], we have $C \otimes S A \in B_C(R)$. Let $f \in \text{Hom}_R(B, M)$ with $B \in B_C(R)$. By [11, Proposition 4.1] again, we have $B_* \in A_C(S)$. So there exists $h \in \text{Hom}_S(B_*, A)$ such that $f_* = gh$, that is, the following diagram

$$
\begin{array}{c}
B_* \\
\downarrow h \\
A \xrightarrow{g} M_*
\end{array}
$$

commutes. From the following commutative diagram

$$
\begin{array}{ccc}
C \otimes S B_* & \xrightarrow{1_C \otimes f_*} & C \otimes S M_* \\
\downarrow \theta_B & & \downarrow \theta_M \\
B & \xrightarrow{f} & M
\end{array}
$$

we get $f\theta_B = \theta_M(1_C \otimes f_*)$. Because $\theta_B$ is an isomorphism, we have

$$f = \theta_M(1_C \otimes f_*)\theta_B^{-1} = \theta_M(1_C \otimes (gh))\theta_B^{-1} = (\theta_M(1_C \otimes g))(1_C \otimes h)\theta_B^{-1},$$

that is, the following diagram

$$
\begin{array}{c}
(1_C \otimes h)\theta_B^{-1} \\
\downarrow f \\
C \otimes S A \xrightarrow{\theta_M(1_C \otimes g)} M
\end{array}
$$

commutes. Thus $\theta_M(1_C \otimes g) : C \otimes S A \to M$ is a $B_C(R)$-precover of $M$.

In the following, it suffices to prove that $\theta_M(1_C \otimes g)$ is right minimal.

Let $h \in \text{Hom}_R(C \otimes S A, C \otimes S A)$ such that $\theta_M(1_C \otimes g) = (\theta_M(1_C \otimes g))h$. Then we have

$$
(\theta_M)_*(1_C \otimes g)_* = (\theta_M(1_C \otimes g))_* = ((\theta_M(1_C \otimes g))h)_* = (\theta_M)_*(1_C \otimes g)_* h_*.
$$

(3.3)
From the following commutative diagram

\[
\begin{array}{c}
A \xrightarrow{g} M_* \\
\mu_A \downarrow \quad \downarrow \mu_{M_*} \\
(C \otimes_S A)_* \xrightarrow{(1_C \otimes g)} (C \otimes_S M)_*,
\end{array}
\]

we get

\[
\mu_{M_*} g = (1_C \otimes g)_* \mu_A.
\]

So we have

\[
g = 1_{M_*} g = (\theta_M)_* \mu_{M_*} g \quad \text{(by [21, Proposition 2.2(1)])}
\]

\[
= (\theta_M)_* (1_C \otimes g)_* \mu_A \quad \text{(by (3.4))}
\]

\[
= (\theta_M)_* (1_C \otimes g)_* h_* \mu_A \quad \text{(by (3.3))}
\]

\[
= (\theta_M)_* (1_C \otimes g)_* \mu_A \mu_A^{-1} h_* \mu_A \quad \text{(because } \mu_A \text{ is an isomorphism)}
\]

\[
= (\theta_M)_* \mu_{M_*} g \mu_A^{-1} h_* \mu_A \quad \text{(by (3.4))}
\]

\[
= 1_{M_*} g \mu_A^{-1} h_* \mu_A \quad \text{(by [21, Proposition 2.2(1)])}
\]

\[
= g \mu_A^{-1} h_* \mu_A.
\]

Because \( g \) is right minimal, \( \mu_A^{-1} h_* \mu_A \) is an isomorphism, which implies that \( h_* \) and \( 1_C \otimes h_* \) are also isomorphisms. From the following commutative diagram

\[
\begin{array}{c}
C \otimes_S (C \otimes_S A)_* \xrightarrow{1_C \otimes h_*} C \otimes_S (C \otimes_S A)_* \\
\theta_{C \otimes S A} \downarrow \quad \downarrow \theta_{C \otimes S A} \\
C \otimes_S A \xrightarrow{h} C \otimes_S A,
\end{array}
\]

we get

\[
h \theta_{C \otimes S A} = \theta_{C \otimes S A} (1_C \otimes h_*).
\]

Because \( C \otimes_S A \in \mathcal{B}_C(R) \) by [11, Proposition 4.1], \( \theta_{C \otimes S A} \) is an isomorphism. It follows that \( h \) is also an isomorphism and \( \theta_M(1_C \otimes g) \) is right minimal. \( \square \)

We do not know whether an \( \mathcal{A}_C(S) \)-cover of a given module in \( \text{Mod} S \) can be constructed from a \( \mathcal{B}_C(R) \)-cover of some module in \( \text{Mod} R \).

### 4 The Auslander projective dimension of modules

For a subcategory \( \mathcal{X} \) of \( \text{Mod} S \) and \( N \in \text{Mod} S \), the \( \mathcal{X} \)-projective dimension \( \mathcal{X} \)-pd\(S\) \( N \) of \( N \) is defined as \( \inf \{ n \mid \text{there exists an exact sequence} \}

\[
0 \to X_n \to \cdots \to X_1 \to X_0 \to N \to 0
\]

in \( \text{Mod} S \) with all \( X_i \in \mathcal{X} \), and we set \( \mathcal{X} \)-pd\(S\) \( N \) infinite if no such integer exists. We call \( \mathcal{A}_C(S) \)-pd\(S\) \( N \) the **Auslander projective dimension** of \( N \). For any \( n \geq 0 \), we use \( \Omega^n(N) \) to denote the \( n \)-th syzygy of \( N \) (note: \( \Omega^0(N) = N \)).
Proposition 4.2. $\text{pd}^<\infty$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms.

Proof. Let

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

be an exact sequence in $\text{Mod} S$ and $n \geq 0$. If $\max\{A_C(S)-\text{pd}_S N_1, A_C(S)-\text{pd}_S N_3\} \leq n$, then by Corollary 3.6 and Lemma 4.1, there exist exact sequences

$$0 \to \Omega^n(N_1) \to P_1^{n-1} \to \cdots \to P_1^1 \to P_1^0 \to N_1 \to 0,$$

$$0 \to \Omega^n(N_3) \to P_3^{n-1} \to \cdots \to P_3^1 \to P_3^0 \to N_3 \to 0$$

in $\text{Mod} S$ with all $P_i^j$ projective and $\Omega^n(N_1), \Omega^n(N_3) \in A_C(S)$. Then we get exact sequences

$$0 \to K_n \to P_1^{n-1} \oplus P_3^{n-1} \to \cdots \to P_1^1 \oplus P_3^1 \to P_1^0 \oplus P_3^0 \to N_2 \to 0,$$

$$0 \to \Omega^n(N_1) \to K_n \to \Omega^n(N_3) \to 0$$

in $\text{Mod} S$. By [11, Theorem 6.2], we have $K_n \in A_C(S)$ and $A_C(S)-\text{pd}_S N_2 \leq n$.

If $\max\{A_C(S)-\text{pd}_S N_1, A_C(S)-\text{pd}_S N_2\} \leq n$, then by Corollary 3.6 and Lemma 4.1, there exist $\text{Hom}_S(A_C(S), -)$-exact exact sequences

$$0 \to A_1^n \to A_1^{n-1} \to \cdots \to A_1^1 \to A_1^0 \to N_1 \to 0,$$

$$0 \to A_2^n \to A_2^{n-1} \to \cdots \to A_2^1 \to A_2^0 \to N_2 \to 0$$

in $\text{Mod} S$ with all $A_i^j$ in $A_C(S)$. By [12, Theorem 3.6], we get an exact sequence

$$0 \to A_1^n \to A_1^{n-1} \oplus A_2^n \to \cdots \to A_1^0 \oplus A_2^0 \to A_2^0 \to N_3 \to 0$$

in $\text{Mod} S$, and so $A_C(S)-\text{pd}_S N_3 \leq n + 1$.

If $\max\{A_C(S)-\text{pd}_S N_2, A_C(S)-\text{pd}_S N_3\} \leq n$, then by Corollary 3.6 and Lemma 4.1, there exist $\text{Hom}_S(A_C(S), -)$-exact exact sequences

$$0 \to A_2^n \to A_2^{n-1} \to \cdots \to A_2^1 \to A_2^0 \to N_2 \to 0,$$

$$0 \to A_3^n \to A_3^{n-1} \to \cdots \to A_3^1 \to A_3^0 \to N_1 \to 0$$
in \text{Mod}S with all \(A_i^j\) in \(\mathcal{A}_C(S)\). By [12, Theorem 3.2], we get exact sequences
\[
0 \rightarrow A_1^n \rightarrow A_2^{n-1} \oplus A_3^n \rightarrow \cdots \rightarrow A_1^1 \oplus A_3^2 \rightarrow A \rightarrow N_1 \rightarrow 0,
\]
\[
0 \rightarrow A \rightarrow A_2^0 \oplus A_3^1 \rightarrow A_3^0 \rightarrow 0
\]
in \text{Mod}S. By [11, Theorem 6.2], we have \(A \in \mathcal{A}_C(S)\), and so \(\mathcal{A}_C(S)\)-pd \(N_1 \leq n\).
\[
\square
\]
We write
\[
\mathcal{I}_C(S) := \{ I_* | I \text{ is injective in } \text{Mod}R \}.
\]
The modules in \(\mathcal{I}_C(S)\) is called \(C\)-injective ([11]). Let \(Q\) be an injective cogenerator for \(\text{Mod}R\). Then
\[
\mathcal{I}_C(S) = \text{Prod}_S Q_*
\]
by [15, Proposition 2.4(2)], where Prod\(S Q_*\) is the subcategory of \(\text{Mod}S\) consisting of direct summands of products of copies of \(Q_*\). By [9, Lemma 2.16(b)], we have the following isomorphism of functors
\[
\text{Hom}_R(\text{Tor}_i^S(C, -, Q)) \cong \text{Ext}_S^i(-, Q_*)
\]
for any \(i \geq 1\). This gives the following

**Lemma 4.3.** \(C_S^{-1} = -\mathcal{I}_C(S)\).

For a subcategory \(\mathcal{X}\) of \(\text{Mod}S\), a sequence in \(\text{Mod}S\) is called \(\text{Hom}_S(-, \mathcal{X})\)-exact if it is exact after applying the functor \(\text{Hom}_S(-, X)\) for any \(X \in \mathcal{X}\). Now we give some criteria for computing the Auslander projective dimension of modules.

**Theorem 4.4.** Let \(N \in \text{Mod}S\) with \(\mathcal{A}_C(S)\)-pd \(S N < \infty\) and \(n \geq 0\). Then the following statements are equivalent.

1. \(\mathcal{A}_C(S)\)-pd \(S N \leq n\).
2. \(\Omega^n(N) \in \mathcal{A}_C(S)\).
3. \(\text{Tor}_i^S(C, N) = 0\).
4. There exists an exact sequence
\[
0 \rightarrow H \rightarrow A \rightarrow N \rightarrow 0
\]
in \(\text{Mod}S\) with \(A \in \mathcal{A}_C(S)\) and \(\mathcal{I}_C(S)\)-pd \(S H \leq n - 1\).
5. There exists a \(\text{(Hom}_S(-, \mathcal{I}_C(S))\)-exact) exact sequence
\[
0 \rightarrow N \rightarrow H' \rightarrow A' \rightarrow 0
\]
in \(\text{Mod}S\) with \(A' \in \mathcal{A}_C(S)\) and \(\mathcal{I}_C(S)\)-pd \(S H' \leq n\).
**Proof.** By Lemma 4.1 and the dimension shifting, we have (1) ⇔ (2) ⇒ (3).

(3) ⇒ (2) Because $\text{Tor}_n^S(C, N) = 0$ by (3), we have $\Omega^n(N) \in C_S^\perp$, and so $\Omega^n(N) \in ^\perp C_S(S)$ by Lemma 4.3. Note that all projective modules in $\text{Mod} S$ are in $A_C(S)$ by [11, Theorem 6.2]. Because $A_C(S)-\text{pd}_S N < \infty$ by assumption, we have $A_C(S)-\text{pd}_S \Omega^n(N) < \infty$ by Proposition 4.2.

Assume that $A_C(S)-\text{pd}_S \Omega^n(N) = m(< \infty)$ and

$$0 \rightarrow A_m \rightarrow \cdots \rightarrow A_1 \rightarrow A_0 \rightarrow \Omega^n(N) \rightarrow 0 \tag{4.1}$$

is an exact sequence in $\text{Mod} S$ with all $A_j$ in $A_C(S)$. Because $A_C(S) \subseteq C_S^\perp = ^\perp C_S(S)$ by Lemma 4.3, the exact sequence (4.1) is $\text{Hom}_S(-, I_C(S))$-exact. By [17, Theorem 3.11(1)], we have the following $\text{Hom}_S(-, I_C(S))$-exact exact sequence

$$0 \rightarrow A_j \rightarrow U_j^0 \rightarrow U_j^1 \rightarrow \cdots \rightarrow U_j^i \rightarrow \cdots$$

in $\text{Mod} S$ with all $U_j^i$ in $I_C(S)$ for any $0 \leq j \leq m$ and $i \geq 0$. It follows from [12, Corollary 3.5] that there exist the following two exact sequences

$$0 \rightarrow \Omega^n(N) \rightarrow U \rightarrow \bigoplus_{i=0}^{m} U_i^{i+1} \rightarrow \bigoplus_{i=0}^{m} U_i^{i+2} \rightarrow \bigoplus_{i=0}^{m} U_i^{i+3} \rightarrow \cdots,$$

$$0 \rightarrow U_m^0 \rightarrow U_m^1 \oplus U_{m-1}^0 \rightarrow \cdots \rightarrow \bigoplus_{i=2}^{m} U_i^{i-2} \rightarrow \bigoplus_{i=1}^{m} U_i^{i-1} \rightarrow \bigoplus_{i=0}^{m} U_i^1 \rightarrow U \rightarrow 0,$$

and the former one is $\text{Hom}_S(-, I_C(S))$-exact. Because $I_C(S)$ is closed under finite direct sums and cokernels of monomorphisms by [11, Proposition 5.1(c) and Corollary 6.4], we have $U \in I_C(S)$. By [17, Theorem 3.11(1)] again, we have $\Omega^n(N) \in A_C(S)$.

(1) ⇒ (4) By [11, Theorem 6.2], $A_C(S)$ is closed under extensions. By [17, Theorem 3.11(1)], we have that $I_C(S)$ is an $I_C(S)$-coproper cogenerator for $A_C(S)$ in the sense of [13]. Then the assertion follows from [13, Theorem 4.7].

(4) ⇒ (5) Let

$$0 \rightarrow H \rightarrow A \rightarrow N \rightarrow 0$$

be an exact sequence in $\text{Mod} S$ with $A \in A_C(S)$ and $I_C(S)-\text{pd}_S H \leq n - 1$. By [17, Theorem 3.11(1)], there exists a $\text{Hom}_S(-, I_C(S))$-exact exact sequence

$$0 \rightarrow A \rightarrow U \rightarrow A' \rightarrow 0$$

in $\text{Mod} S$ with $U \in I_C(S)$ and $A' \in A_C(S)$. Consider the following push-out diagram

```
          0               0
            ↓               ↓
          0 → H → A → N → 0
            ↓               ↓
          0 → H → U → H' → 0
            ↓               ↓
        A = A' = A
            ↓               ↓
          0               0.
```
By the middle row in this diagram, we have $\mathcal{I}_C(S)\text{-pd}_S H' \leq n$. Because the middle column in the above diagram is $\text{Hom}_S(-, \mathcal{I}_C(S))$-exact, the rightmost column is also $\text{Hom}_S(-, \mathcal{I}_C(S))$-exact by [12, Lemma 2.4(2)] and it is the desired exact sequence.

(5) $\Rightarrow$ (1) Let

$$0 \rightarrow N \rightarrow H' \rightarrow A' \rightarrow 0$$

be an exact sequence in $\text{Mod}S$ with $A' \in \mathcal{A}_C(S)$ and $\mathcal{I}_C(S)\text{-pd}_S H' \leq n$. Then there exists an exact sequence

$$0 \rightarrow U_n \rightarrow \cdots \rightarrow U_1 \rightarrow U_0 \rightarrow H' \rightarrow 0$$

in $\text{Mod}S$ with all $U_i$ in $\mathcal{I}_C(S)$. Set $H := \text{Ker}(U_0 \rightarrow H')$. Then $\mathcal{I}_C(S)\text{-pd}_S H \leq n - 1$. Consider the following pull-back diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
H & \rightarrow & H \\
\downarrow & & \downarrow \\
0 & \rightarrow & A' \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & H' \rightarrow A' \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

Applying [11, Theorem 6.2] to the middle row in this diagram yields $A \in \mathcal{A}_C(S)$. Thus $\mathcal{A}_C(S)\text{-pd}_S N \leq n$ by the leftmost column in the above diagram.

The only place where the assumption $\mathcal{A}_C(S)\text{-pd}_S N < \infty$ in Theorem 4.4 is used is in showing (3) $\Rightarrow$ (2). By Theorem 4.4, it is easy to get the following standard observation.

**Corollary 4.5.** Let

$$0 \rightarrow L \rightarrow M \rightarrow K \rightarrow 0$$

be an exact sequence in $\text{Mod}S$. Then we have

1. $\mathcal{A}_C(S)\text{-pd}_S K \leq \max\{\mathcal{A}_C(S)\text{-pd}_S M, \mathcal{A}_C(S)\text{-pd}_S L + 1\}$, and the equality holds true if $\mathcal{A}_C(S)\text{-pd}_S M \neq \mathcal{A}_C(S)\text{-pd}_S L$.

2. $\mathcal{A}_C(S)\text{-pd}_S L \leq \max\{\mathcal{A}_C(S)\text{-pd}_S M, \mathcal{A}_C(S)\text{-pd}_S K - 1\}$, and the equality holds true if $\mathcal{A}_C(S)\text{-pd}_S M \neq \mathcal{A}_C(S)\text{-pd}_S K$.

3. $\mathcal{A}_C(S)\text{-pd}_S M \leq \max\{\mathcal{A}_C(S)\text{-pd}_S L, \mathcal{A}_C(S)\text{-pd}_S K\}$, and the equality holds true if $\mathcal{A}_C(S)\text{-pd}_S K \neq \mathcal{A}_C(S)\text{-pd}_S L + 1$.

The following corollary is an addendum to the implications (1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) in Theorem 4.4.

**Corollary 4.6.** Let $N \in \text{Mod}S$ with $\mathcal{A}_C(S)\text{-pd}_S N = n(\leq \infty)$. Then there exist exact sequences

$$0 \rightarrow H \rightarrow A \rightarrow N \rightarrow 0,$$
Lemma 4.7. All columns are isomorphisms by [11, Lemma 4.1]. So the bottom row in this diagram is exact. Because
now the assertion follows from [17, Proposition 3.2].

in Mod $S$ with $A, A' \in \mathcal{A}(S), \mathcal{I}(S) \cdot \text{pd}_S H = n - 1$ and $\mathcal{I}(S) \cdot \text{pd}_S H' = n.$

Proof. Let $N \in \text{Mod} S$ with $\mathcal{A}(S) \cdot \text{pd}_S N = n(< \infty).$ By Theorem 4.4, there exists an exact sequence

$0 \to H \to A \to N \to 0$

in Mod $S$ with $A \in \mathcal{A}(S)$ and $(\mathcal{A}(S) \cdot \text{pd}_S H \leq) \mathcal{I}(S) \cdot \text{pd}_S H \leq n - 1.$ By Theorem 4.4 again, we have $\sup\{i \geq 0 \mid \text{Tor}_i^S(C, N) \neq 0\} = n.$ So $\sup\{i \geq 0 \mid \text{Tor}_i^S(C, H) \neq 0\} = n - 1,$ and hence $\mathcal{A}(S) \cdot \text{pd}_S H = n - 1$ by Theorem 4.4. It follows that $\mathcal{I}(S) \cdot \text{pd}_S H = n - 1.$

By Theorem 4.4, there exists an exact sequence

$0 \to N \to H' \to A' \to 0$

in Mod $S$ with $A' \in \mathcal{A}(S)$ and $(\mathcal{A}(S) \cdot \text{pd}_S H \leq) \mathcal{I}(S) \cdot \text{pd}_S H' \leq n.$ By Corollary 4.5(3), we have $\mathcal{A}(S) \cdot \text{pd}_S H = \mathcal{A}(S) \cdot \text{pd}_S N = n,$ and so $\mathcal{I}(S) \cdot \text{pd}_S H' = n.$

Let $N \in \text{Mod} S.$ Bican, El Bashir and Enochs proved in [4] that $N$ has a flat cover. We use

$\cdots \xrightarrow{f_{n+1}} F_i(N) \xrightarrow{f} \cdots \xrightarrow{f} F_1(N) \xrightarrow{f} F_0(N) \xrightarrow{f_0} N \to 0$ (4.2)

to denote a minimal flat resolution of $N$ in Mod $S,$ where each $F_i(N) \to \text{Im} f_i$ is a flat cover of $\text{Im} f_i.$

Lemma 4.7. Let $N \in \text{Mod} S$ and $n \geq 0.$ If $\text{Tor}_{1 \leq i \leq n}^S(C, N) = 0,$ then we have

(1) There exists an exact sequence

$0 \to \text{Ext}_{1 \leq i \leq n}^n(C, \text{Ker}(1_C \otimes f_{n+1})) \to N \xrightarrow{\mu_{n+1}} (C \otimes_S N)_* \to \text{Ext}_{1 \leq i \leq n}^{n+2}(C, \text{Ker}(1_C \otimes f_{n+1})) \to 0$

in Mod $S.$

(2) $\text{Ext}_{1 \leq i \leq n}^n(C, \text{Ker}(1_C \otimes f_{n+1})) = 0.$

Proof. (1) The case for $n = 0$ follows from [17, Proposition 3.2]. Now suppose $n \geq 1.$ If $\text{Tor}_{1 \leq i \leq n}^S(C, N) = 0,$ then the exact sequence (4.2) yields the following exact sequence

$0 \to \text{Ker}(1_C \otimes f_{n+1}) \to C \otimes_S F_{n+1}(N) \xrightarrow{1_C \otimes f_{n+1}} C \otimes_S F_n(N) \xrightarrow{1_C \otimes f_n} \cdots \xrightarrow{1_C \otimes f_1} C \otimes_S F_1(N) \xrightarrow{1_C \otimes f_1} C \otimes_S F_0(N) \xrightarrow{1_C \otimes f_0} C \otimes_S N \to 0$ (4.3)

in Mod $R.$ Because all $C \otimes_S F_i(N)$ are in $R C_\perp$ by [17, Lemma 2.3(1)], we have

$\text{Ext}_R^n(C, \text{Ker}(1_C \otimes f_1)) \cong \text{Ext}_{1 \leq i \leq n}^n(C, \text{Ker}(1_C \otimes f_n)),

\text{Ext}_R^n(C, \text{Ker}(1_C \otimes f_1)) \cong \text{Ext}_{1 \leq i \leq n}^{n+2}(C, \text{Ker}(1_C \otimes f_n)).$

Now the assertion follows from [17, Proposition 3.2].

(2) Applying the functor $(-)_*$ to the exact sequence (4.3) we get the following commutative diagram

$\begin{array}{cccccc}
F_{n+1}(N) & \xrightarrow{f_{n+1}} & F_n(N) & \xrightarrow{f_n} & \cdots & \xrightarrow{f_1} F_0(N) \\
\mu_{F_{n+1}(N)} & & \mu_{F_n(N)} & & \mu_{F_0(N)} \\
0 \xrightarrow{(\text{Ker}(1_C \otimes f_{n+1}))_*} (C \otimes_S F_{n+1}(N))_* & \xrightarrow{(1_C \otimes f_{n+1})_*} & (C \otimes_S F_n(N))_* & \xrightarrow{(1_C \otimes f_n)_*} & \cdots & \xrightarrow{(1_C \otimes f_1)_*} (C \otimes_S F_0(N))_*
\end{array}$

All columns are isomorphisms by [11, Lemma 4.1]. So the bottom row in this diagram is exact. Because all $C \otimes_S F_i(N)$ are in $R C_\perp,$ we have $\text{Ext}_{1 \leq i \leq n}^n(C, \text{Ker}(1_C \otimes f_{n+1})) = 0.$
Proposition 4.8. Let $X \in \text{Mod} R$ and let

$$
\cdots \xrightarrow{g_{n+1}} P_n \xrightarrow{g_n} \cdots \xrightarrow{g_2} P_1 \xrightarrow{g_1} P_0 \xrightarrow{g_0} X \to 0
$$

be a projective resolution of $X$ in $\text{Mod} R$. If there exists $n \geq 1$ such that $\text{Im} g_n \cong \oplus_j W_j$, where each $W_j$ is isomorphic to a direct summand of some $\text{Im} g_{i_j}$ with $i_j < n$, then we say that $X$ has an ultimately closed projective resolution at $n$; and we say that $X$ has an ultimately closed projective resolution if it has an ultimately closed projective resolution at some $n$ ([14]). It is trivial that if $\text{pd}_R X$ (the projective dimension of $X$) $\leq n$, then $X$ has an ultimately closed projective resolution at $n+1$. Let $R$ be an artin algebra. If either $R$ is of finite representation type or the square of the radical of $R$ is zero, then any finitely generated left $R$-module has an ultimately closed projective resolution ([14, p.341]). Following [21], a module $N \in \text{Mod} S$ is called $C$-adstatic if $\mu_N$ is an isomorphism.

**Proposition 4.8.** Let $N \in \text{Mod} S$ and $n \geq 1$. If $\text{Tor}^S_{1 \leq i \leq n}(C, N) = 0$, then $N$ is $C$-adstatic provided that one of the following conditions is satisfied.

1. $\text{pd}_R C \leq n$.
2. $RC$ has an ultimately closed projective resolution at $n$.

**Proof.** (1) It follows directly from Lemma 4.7(1).

(2) Let

$$
\cdots \xrightarrow{g_{n+1}} P_n \xrightarrow{g_n} \cdots \xrightarrow{g_2} P_1 \xrightarrow{g_1} P_0 \xrightarrow{g_0} C \to 0
$$

be a projective resolution of $C$ in $\text{Mod} R$ ultimately closed at $n$. Then $\text{Im} g_n \cong \oplus_j W_j$ such that each $W_j$ is isomorphic to a direct summand of some $\text{Im} g_{i_j}$ with $i_j < n$. Let $N \in \text{Mod} S$ with $\text{Tor}^S_{1 \leq i \leq n}(C, N) = 0$. By Lemma 4.7(2), we have

$$
\text{Ext}^1_R(\text{Im} g_{i_j}, \text{Ker}(1_C \otimes f_{n+1})) \cong \text{Ext}^1_R(\text{Ker}(1_C \otimes f_{n+1})) = 0.
$$

Because $W_j$ is isomorphic to a direct summand of some $\text{Im} g_{i_j}$, we have $\text{Ext}^1_R(W_j, \text{Ker}(1_C \otimes f_{n+1})) = 0$ for any $j$, which implies

$$
\text{Ext}^{n+1}_R(C, \text{Ker}(1_C \otimes f_{n+1})) \\
\cong \text{Ext}^1_R(\text{Im} g_n, \text{Ker}(1_C \otimes f_{n+1})) \\
\cong \text{Ext}^1_R(\oplus_j W_j, \text{Ker}(1_C \otimes f_{n+1})) \\
\cong \sum_j \text{Ext}^1_R(W_j, \text{Ker}(1_C \otimes f_{n+1})) \\
= 0.
$$

Then by Lemma 4.7(2), we conclude that $\text{Ext}^1_{R}^{0 \leq i \leq n+1}(C, \text{Ker}(1_C \otimes f_{n+1})) = 0$. Similar to the above argument we get $\text{Ext}^{n+2}_R(C, \text{Ker}(1_C \otimes f_{n+1})) = 0$. It follows from Lemma 4.7(1) that $\mu_N$ is an isomorphism and $N$ is $C$-adstatic.

**Corollary 4.9.** For any $n \geq 1$, a module $N \in \text{Mod} S$ satisfying $\text{Tor}^S_{0 \leq i \leq n}(C, N) = 0$ implies $N = 0$ provided that one of the following conditions is satisfied.

1. $\text{pd}_R C \leq n$. 


(2) \( RC \) has an ultimately closed projective resolution at \( n \).

Proof. Let \( N \in \text{Mod} \ S \) with \( \text{Tor}^S_{0 \leq i \leq n}(C, N) = 0 \). By Proposition 4.8, we have that \( N \) is \( C \)-adstatic and \( N \cong (C \otimes S N)_* = 0 \). \( \square \)

We now are in a position to give the following

**Theorem 4.10.** If \( RC \) has an ultimately closed projective resolution, then

\[
\mathcal{A}_C(S) = C_S^\top = \mathcal{I}_C(S).
\]

Proof. By the definition of \( \mathcal{A}_C(S) \) and Lemma 4.3, we have \( \mathcal{A}_C(S) \subseteq C_S^\top \subseteq \mathcal{I}_C(S) \).

Now let \( N \in \mathcal{I}_C(S) \) and let \( f : C \otimes S N \to B \) be a \( B_C(R) \)-preenvelope of \( C \otimes S N \) in \( \text{Mod} R \) as in Theorem 3.7. Because \( B_C(R) \) is injectively coresolving in \( \text{Mod} R \) by [11, Theorem 6.2], \( f \) is monic. By Proposition 4.8, \( \mu_N \) is an isomorphism. Then by Theorem 3.7(1), we have a monic \( \mathcal{A}_C(S) \)-preenvelope

\[
f^0 : N \to A^0
\]

of \( N \), where \( f^0 = f_\ast \mu_N \) and \( A^0 = B_\ast \). So we have a \( \text{Hom}_S(\_, \mathcal{A}_C(S)) \)-exact sequence

\[
0 \to N \xrightarrow{f^0} A^0 \to N^1 \to 0
\]

in \( \text{Mod} S \), where \( N^1 = \text{Coker} f^0 \). Because \( A^0 \in \mathcal{I}_C(S) \), we have \( N^1 \in \mathcal{I}_C(S) \). Similar to the above argument, we get a \( \text{Hom}_S(\_, \mathcal{A}_C(S)) \)-exact exact sequence

\[
0 \to N^1 \xrightarrow{f^1} A^1 \to N^2 \to 0
\]

in \( \text{Mod} S \) with \( A^1 \in \mathcal{A}_C(S) \) and \( N^2 \in \mathcal{I}_C(S) \). Repeating this procedure, we get a \( \text{Hom}_S(\_, \mathcal{A}_C(S)) \)-exact exact sequence

\[
0 \to N \xrightarrow{f^0} A^0 \xrightarrow{f^1} A^1 \xrightarrow{f^2} \cdots \xrightarrow{f^i} A^i \xrightarrow{f^{i+1}} \cdots
\]

in \( \text{Mod} S \) with all \( A^i \) in \( \mathcal{A}_C(S) \). Because \( \mathcal{I}_C(S) \subseteq \mathcal{A}_C(S) \) by [11, Corollary 6.1], this exact sequence is \( \text{Hom}_S(\_, \mathcal{I}_C(S)) \)-exact. By [17, Theorem 3.11(1)], there exists a \( \text{Hom}_S(\_, \mathcal{A}_C(S)) \)-exact exact sequence

\[
0 \to A^i \to U^i_0 \to U^i_1 \to \cdots \to U^i_j \to \cdots
\]

in \( \text{Mod} S \) with all \( U^i_j \) in \( \mathcal{I}_C(S) \) for any \( i, j \geq 0 \). Then by [12, Corollary 3.9], we get the following \( \text{Hom}_S(\_, \mathcal{A}_C(S)) \)-exact exact sequence

\[
0 \to N \to U^0_0 \to U^0_1 \oplus U^1_0 \to \cdots \to \oplus_{i=0}^n U^n_{n-i} \to \cdots
\]

in \( \text{Mod} S \) with all terms in \( \mathcal{I}_C(S) \). It follows from [17, Theorem 3.11(1)] that \( N \in \mathcal{A}_C(S) \). The proof is finished. \( \square \)

We use \( \text{pd}_{\text{Sop}} C \) and \( \text{fd}_{\text{Sop}} C \) to denote the projective and flat dimensions of \( C_S \) respectively.

**Corollary 4.11.** If \( RC \) has an ultimately closed projective resolution, then the following statements are equivalent for any \( n \geq 0 \).

(1) \( \text{pd}_{\text{Sop}} C \leq n \).
(2) $A_C(S) \text{-pd}_S N \leq n$ for any $N \in \text{Mod } S$.

Proof. Assume that $RC$ has an ultimately closed projective resolution. By Theorem 4.10, we have $A_C(S) = C_S^\top$. Then it is easy to see that $C_S$ is flat (equivalently, projective) if and only if $A_C(S) = \text{Mod } S$, so the assertion for the case $n = 0$ follows. Now let $N \in \text{Mod } S$ and $n \geq 1$.

(2) $\Rightarrow$ (1) By (2) and Theorem 4.4, we have $\Omega^n(N) \in A_C(S) \subseteq C_S^\top$. Then by the dimension shifting, we have $\text{Tor}^S_{2n+1}(C, N) = 0$, and so $\text{pd}_{S^{op}} C = \text{fd}_{S^{op}} C \leq n$.

(1) $\Rightarrow$ (2) If $\text{pd}_{S^{op}} C \leq n$, then $\Omega^n(N) \in C_S^\top$ by the dimension shifting. By Theorem 4.10, we have $\Omega^n(N) \in A_C(S)$ and $A_C(S) \text{-pd}_S N \leq n$. □

Acknowledgements. This research was partially supported by NSFC (Grant No. 11971225). The author thanks the referee for useful suggestions.

References


