

Chapter 17

Wakamatsu Tilting Modules, U -Dominant Dimension, and k -Gorenstein Modules

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Abstract Let Λ and Γ be left and right noetherian rings and ${}_{\Lambda}U$ a Wakamatsu tilting module with $\Gamma = \text{End}({}_{\Lambda}T)$. We introduce a new definition of U -dominant dimensions and show that the U -dominant dimensions of ${}_{\Lambda}U$ and U_{Γ} are identical. We characterize k -Gorenstein modules in terms of homological dimensions and the property of double homological functors preserving monomorphisms. We also study a generalization of k -Gorenstein modules, and characterize it in terms of some similar properties of k -Gorenstein modules.

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17.1 Introduction and Main Results

Let Λ be a ring. We use $\text{Mod } \Lambda$ (resp. $\text{Mod } \Lambda^{op}$) to denote the category of left (resp. right) Λ -modules, and use $\text{mod } \Lambda$ (resp. $\text{mod } \Lambda^{op}$) to denote the category of finitely generated left Λ -modules (resp. right Λ -modules).

Definition 17.1.1 ^[7] For a module M in $\text{mod } \Lambda$ (resp. $\text{mod } \Lambda^{op}$) and a positive integer k , M is said to have *dominant dimension* at least k , written as $\text{dom.dim}({}_{\Lambda}M)$ (resp. $\text{dom.dim}(M_{\Lambda}) \geq k$, if each of the first k terms in a minimal injective resolution of M is Λ -flat (resp. Λ^{op} -flat).

For a module T in $\text{Mod } \Lambda$ (resp. $\text{Mod } \Lambda^{op}$), we use $\text{add-lim}_{\Lambda}T$ (resp. $\text{add-lim}T_{\Lambda}$) to denote the subcategory of $\text{Mod } \Lambda$ (resp. $\text{Mod } \Lambda^{op}$) consisting of all modules isomorphic to direct summands of a direct limit of a family of modules in which each is a finite direct sum of copies of ${}_{\Lambda}T$ (resp. T_{Λ}). We now introduce a definition of *U -dominant dimension* as follows.

Definition 17.1.2 Let U be in $\text{Mod } \Lambda$ (resp. $\text{Mod } \Lambda^{op}$) and k a positive integer. For a module M in $\text{Mod } \Lambda$ (resp. $\text{Mod } \Lambda^{op}$), M is said to have U -dominant dimension at least k , written as $U\text{-dom.dim}({}_\Lambda M)$ (resp. $U\text{-dom.dim}(M_\Lambda)) \geq k$, if each of the first k terms in a minimal injective resolution of M can be embedded into a direct limit of a family of modules in which each is a finite direct sum of copies of ${}_\Lambda U$ (resp. U_Λ), that is, each of these terms is in $\text{add-lim}_\Lambda U$ (resp. $\text{add-lim} U_\Lambda$).

Remark 17.1.3 Notice that a module (not necessarily finitely generated) is flat if and only if it is a direct limit of a family of finitely generated free modules (see [15]). So, if putting ${}_\Lambda U = {}_\Lambda \Lambda$ (resp. $U_\Lambda = \Lambda_\Lambda$), then the above definition of U -dominant dimension coincides with that of the usual dominant dimension for any ring Λ .

Tachikawa in [19] showed that if Λ is a left and right artinian ring then the dominant dimensions of ${}_\Lambda \Lambda$ and Λ_Λ are identical. Hoshino in [7] further showed that this result also holds for left and right noetherian rings. Colby and Fuller in [5] gave some equivalent conditions of $\text{dom.dim}({}_\Lambda \Lambda) \geq 1$ (or 2) in terms of the properties of double dual functors (with respect to ${}_\Lambda \Lambda_\Lambda$). These results motivate our interests in establishing the identity of U -dominant dimensions of ${}_\Lambda U$ and U_Γ (where $\Gamma = \text{End}({}_\Lambda U)$) and characterizing the properties of modules with a given U -dominant dimension.

Let T be a module in $\text{mod } \Lambda$. For a module $A \in \text{mod } \Lambda$ and a non-negative integer n , we say that the grade of A with respect to ${}_\Lambda T$, written as $\text{grade}_T A$, is at least n if $\text{Ext}_\Lambda^i(A, T) = 0$ for any $0 \leq i < n$. We say that the strong grade of A with respect to ${}_\Lambda T$, written as $s.\text{grade}_T A$, is at least n if $\text{grade}_T B \geq n$ for all submodules B of A . The notion of the (strong) grade of modules with respect to a given module in $\text{mod } \Lambda^{op}$ is defined dually.

The following is one of the main results in this paper.

Theorem 17.1.4 Let Λ and Γ be left and right noetherian rings and ${}_\Lambda U$ a Wakamatsu tilting module with $\Gamma = \text{End}({}_\Lambda U)$. For a positive integer k , the following statements are equivalent.

- (1) $U\text{-dom.dim}({}_\Lambda U) \geq k$.
- (2) $s.\text{grade}_U \text{Ext}_\Lambda^1(M, U) \geq k$ for any $M \in \text{mod } \Lambda$.
- (3) $\text{Hom}_\Lambda(U, E_i)$ is Γ -flat, where E_i is the $(i + 1)$ -st term in a minimal injective resolution of ${}_\Lambda U$, for any $0 \leq i \leq k - 1$.
- (1)^{op} $U\text{-dom.dim}(U_\Gamma) \geq k$.
- (2)^{op} $s.\text{grade}_U \text{Ext}_\Gamma^1(N, U) \geq k$ for any $N \in \text{mod } \Gamma^{op}$.
- (3)^{op} $\text{Hom}_\Gamma(U, E'_i)$ is Λ^{op} -flat, where E'_i is the $(i + 1)$ -st term in a minimal injective resolution of U_Γ , for any $0 \leq i \leq k - 1$.

Kato in [14] gave a definition of U -dominant dimension as follows, which is different from that of 17.1.2. For a module M in $\text{mod } \Lambda$ (resp. $\text{mod } \Lambda^{op}$), M is said to have U -dominant dimension at least k , written as $U\text{-dom.dim}({}_\Lambda M)$ (resp. $U\text{-dom.dim}(M_\Lambda)) \geq k$, if each of the first k terms in a minimal injective resolution of M is cogenerated by ${}_\Lambda U$ (resp. U_Λ), that is, each of these terms can be embedded into a direct product of copies of ${}_\Lambda U$ (resp. U_Λ). If we adopt the definition of U -dominant dimension given by Kato, then in Theorem 17.1.4 the equivalence of (2), (3), (2)^{op} and (3)^{op} and that (1) implies (3) also hold. However, that (3) does not imply (1) in general. For example, consider Wakamatsu tilting module ${}_{\mathbb{Z}}\mathbb{Z}$ and its injective envelope ${}_{\mathbb{Z}}\mathbb{Q}$, where \mathbb{Z} is the ring of integers and \mathbb{Q} is the field of rational numbers. Then the module ${}_{\mathbb{Z}}\mathbb{Q}$ is flat, but it cannot be embedded into any direct product of copies of ${}_{\mathbb{Z}}\mathbb{Z}$ since $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$.

Corollary 17.1.5 *Let Λ and Γ be left and right noetherian rings and ${}_{\Lambda}U$ a Wakamatsu tilting module with $\Gamma = \text{End}({}_{\Lambda}T)$. Then $U\text{-dom.dim}({}_{\Lambda}U) = U\text{-dom.dim}(U_{\Gamma})$.*

Remark 17.1.6 We do not know whether the conclusion in Corollary 16.1.4 holds for Kato's U -dominant dimension. The answer is positive when Λ and Γ are artinian algebras (see [11, Theorem 1.3]).

Putting ${}_{\Lambda}U_{\Gamma} = {}_{\Lambda}\Lambda_{\Lambda}$, we immediately get the following result, which is due to Hoshino (see [7, Theorem]).

Corollary 17.1.7 *For a left and right noetherian ring Λ , $\text{dom.dim}({}_{\Lambda}\Lambda) = \text{dom.dim}(\Lambda_{\Lambda})$.*

Definition 17.1.8 ^[12] For a non-negative integer k , a module $U \in \text{mod } \Lambda$ with $\Gamma = \text{End}({}_{\Lambda}U)$ is called k -Gorenstein if $s.\text{grade}_U \text{Ext}_{\Gamma}^i(N, U) \geq i$ for any $N \in \text{mod } \Gamma^{op}$ and $1 \leq i \leq k$. Dually, we may define the notion of k -Gorenstein modules in $\text{mod } \Gamma^{op}$.

We introduce a new homological dimension of modules as follows.

Definition 17.1.9 Let Λ be a ring and T in $\text{Mod } \Lambda$. For a module A in $\text{Mod } \Lambda$, if there exists an exact sequence $\cdots \rightarrow T_n \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow A \rightarrow 0$ in $\text{Mod } \Lambda$ with each $T_i \in \text{add-lim}_{\Lambda} T$ for any $i \geq 0$, then we define $T\text{-lim.dim}_{\Lambda}(A) = \inf\{n\}$ and there exists an exact sequence $0 \rightarrow T_n \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow A \rightarrow 0$ in $\text{Mod } \Lambda$ with each $T_i \in \text{add-lim}_{\Lambda} T$ for any $0 \leq i \leq n$. We set $T\text{-lim.dim}_{\Lambda}(A)$ infinity if no such an integer exists. For Λ^{op} -modules, we may define such a dimension dually.

Remark 17.1.10 Putting ${}_{\Lambda}T = {}_{\Lambda}\Lambda$ (resp. $T_{\Lambda} = \Lambda_{\Lambda}$), the dimension defined as above is just the flat dimension of modules.

In [21], Wakamatsu showed that the notion of k -Gorenstein modules is left-right symmetric. We give here some other characterizations of k -Gorenstein modules. The following is another main result in this paper.

Theorem 17.1.11 *Let Λ and Γ be left and right noetherian rings and ${}_{\Lambda}U$ a Wakamatsu tilting module with $\Gamma = \text{End}({}_{\Lambda}T)$. Then, for a positive integer k , the following statements are equivalent.*

- (1) ${}_{\Lambda}U$ is k -Gorenstein.
 - (2) $s.\text{grade}_U \text{Ext}_{\Lambda}^i(M, U) \geq i$ for any $M \in \text{mod } \Lambda$ and $1 \leq i \leq k$.
 - (3) $U\text{-lim.dim}_{\Lambda}(E_i) \leq i$, where E_i is the $(i + 1)$ -st term in a minimal injective resolution of ${}_{\Lambda}U$, for any $0 \leq i \leq k - 1$.
 - (4) $lfd_{\Gamma}(\text{Hom}_{\Lambda}(U, E_i)) \leq i$ for any $0 \leq i \leq k - 1$, where lfd denotes the left flat dimension.
 - (5) $\text{Ext}_{\Gamma}^i(\text{Ext}_{\Lambda}^i(, U), U)$ preserves monomorphisms in $\text{mod } \Lambda$ for any $0 \leq i \leq k - 1$.
- (1)^{op} U_{Γ} is k -Gorenstein.
 - (2)^{op} $s.\text{grade}_U \text{Ext}_{\Gamma}^i(N, U) \geq i$ for any $N \in \text{mod } \Gamma^{op}$ and $1 \leq i \leq k$.
 - (3)^{op} $U\text{-lim.dim}_{\Gamma}(E'_i) \leq i$, where E'_i is the $(i + 1)$ -st term in a minimal injective resolution of U_{Γ} , for any $0 \leq i \leq k - 1$.
 - (4)^{op} $rfd_{\Lambda}(\text{Hom}_{\Gamma}(U, E'_i)) \leq i$ for any $0 \leq i \leq k - 1$, where rfd denotes the right flat dimension.

(5)^{op} $\text{Ext}_{\Lambda}^i(\text{Ext}_{\Gamma}^i(-, U), U)$ preserves monomorphisms in $\text{mod } \Gamma^{\text{op}}$ for any $0 \leq i \leq k - 1$.

Let Λ and Γ be left and right noetherian rings and ${}_{\Lambda}U$ a Wakamatsu tilting module with $\Gamma = \text{End}({}_{\Lambda}U)$. By Theorems 17.1.4 and 17.1.11, if U has U -dominant dimension at least k , then it is k -Gorenstein.

Recall that a left and right noetherian ring Λ is called k -Gorenstein if the flat dimension of the i -th term in a minimal injective resolution of ${}_{\Lambda}\Lambda$ is at most $i - 1$ for any $1 \leq i \leq k$. Auslander showed in [6, Theorem 3.7] that the notion of k -Gorenstein rings is left-right symmetric. Following Definition 17.1.8 and [6, Theorem 3.7], a left and right noetherian ring Λ is k -Gorenstein if it is k -Gorenstein as a Λ -module. So, by Theorem 17.1.11, we have the following corollary, which develops this Auslander’s result.

Corollary 17.1.12 *Let Λ and Γ be left and right noetherian rings. Then, for a positive integer k , the following statements are equivalent.*

- (1) Λ is k -Gorenstein.
- (2) $s.\text{grade}_{\Lambda}\text{Ext}_{\Lambda}^i(M, \Lambda) \geq i$ for any $M \in \text{mod } \Lambda$ and $1 \leq i \leq k$.
- (3) The flat dimension of the i -th term in a minimal injective resolution of ${}_{\Lambda}\Lambda$ is at most $i - 1$ for any $1 \leq i \leq k$.
- (4) $\text{Ext}_{\Lambda}^i(\text{Ext}_{\Lambda}^i(-, \Lambda), \Lambda)$ preserves monomorphisms in $\text{mod } \Lambda$ for any $0 \leq i \leq k - 1$.
- (2)^{op} $s.\text{grade}_{\Lambda}\text{Ext}_{\Lambda}^i(N, \Lambda) \geq i$ for any $N \in \text{mod } \Lambda^{\text{op}}$ and $1 \leq i \leq k$.
- (3)^{op} The flat dimension of the i -th term in a minimal injective resolution of Λ_{Λ} is at most $i - 1$ for any $1 \leq i \leq k$.
- (4)^{op} $\text{Ext}_{\Lambda}^i(\text{Ext}_{\Lambda}^i(-, \Lambda), \Lambda)$ preserves monomorphisms in $\text{mod } \Lambda^{\text{op}}$ for any $0 \leq i \leq k - 1$.

The paper is organized as follows. In Section 17.2, we give some properties of Wakamatsu tilting modules. For example, let Λ and Γ be left and right noetherian rings and ${}_{\Lambda}U$ a Wakamatsu tilting module with $\Gamma = \text{End}({}_{\Lambda}U)$. If ${}_{\Lambda}U$ is k -Gorenstein for all k , then the left and right injective dimensions of ${}_{\Lambda}U_{\Gamma}$ are identical provided that both of them are finite. We shall prove our main results in Section 17.3. As applications of the results obtained in Section 17.3, we characterize in Section 17.4 U -dominant dimension of U at least one and two in terms of the properties of $\text{Hom}(\text{Hom}(-, U), U)$ preserving monomorphisms and being left exact, respectively. Motivated by the work of Auslander and Reiten in [3], we study in Section 17.5 a generalization of k -Gorenstein modules, which is however not left-right symmetric. We characterize this generalization in terms of some properties similar to that of k -Gorenstein modules. At the end of this section, we generalize the result of Wakamatsu on the symmetry of k -Gorenstein modules.

17.2 Wakamatsu Tilting Modules

In this section, we give some properties of Wakamatsu tilting modules with finite homological dimensions.

Definition 17.2.1 Let Λ be a ring. A module ${}_{\Lambda}U$ in $\text{mod } \Lambda$ is called a *Wakamatsu tilting module* if ${}_{\Lambda}U$ is self-orthogonal (that is, $\text{Ext}_{\Lambda}^i({}_{\Lambda}U, {}_{\Lambda}U) = 0$ for any $i \geq 1$), and possessing an exact sequence:

$$0 \rightarrow {}_{\Lambda}\Lambda \rightarrow U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_i \rightarrow \cdots$$

such that: (1) all term U_i are direct summands of finite direct sums of copies of ${}_{\Lambda}U$, that is, $U_i \in \text{add}_{\Lambda}U$, and (2) after applying the functor $\text{Hom}_{\Lambda}(\ , U)$ the sequence is still exact. The definition of Wakamatsu tilting modules in $\text{mod } \Lambda^{op}$ is given dually (see [20] or [21]).

Let Λ and Γ be rings. Recall that a bimodule ${}_{\Lambda}U_{\Gamma}$ is called a faithfully balanced self-orthogonal bimodule if it satisfies the following conditions:

- (1) ${}_{\Lambda}U \in \text{mod } \Lambda$ and $U_{\Gamma} \in \text{mod } \Gamma^{op}$.
- (2) The natural maps $\Lambda \rightarrow \text{End}(U_{\Gamma})$ and $\Gamma \rightarrow \text{End}({}_{\Lambda}U)^{op}$ are isomorphisms.
- (3) $\text{Ext}_{\Lambda}^i({}_{\Lambda}U, {}_{\Lambda}U) = 0$ and $\text{Ext}_{\Gamma}^i(U_{\Gamma}, U_{\Gamma}) = 0$ for any $i \geq 1$.

The following result is [21, Corollary 3.2].

Proposition 17.2.2 *Let Λ be a left noetherian ring and Γ a right noetherian ring. For a bimodule ${}_{\Lambda}U_{\Gamma}$, the following statements are equivalent.*

- (1) ${}_{\Lambda}U$ is a Wakamatsu tilting module with $\Gamma = \text{End}({}_{\Lambda}U)$.
- (2) U_{Γ} is a Wakamatsu tilting module with $\Lambda = \text{End}(U_{\Gamma})$.
- (3) ${}_{\Lambda}U_{\Gamma}$ is a faithfully balanced self-orthogonal bimodule.

In the rest of this paper, we shall freely use the properties of Wakamatsu tilting modules in Proposition 17.2.2 without pointing it out explicitly.

Recall from [16] that a module U in $\text{mod } \Lambda$ is called a tilting module of projective dimension $\leq r$ if it satisfies the following conditions:

- (1) The projective dimension of ${}_{\Lambda}U$ is at most r .
- (2) ${}_{\Lambda}U$ is self-orthogonal.
- (3) There exists an exact sequence in $\text{mod } \Lambda$:

$$0 \rightarrow \Lambda \rightarrow U_0 \rightarrow U_1 \rightarrow \dots \rightarrow U_r \rightarrow 0$$

such that each $U_i \in \text{add}_{\Lambda}U$ for any $0 \leq i \leq r$.

By Proposition 17.2.2 and [16, Theorem 1.5], we have the following result.

Corollary 17.2.3 *Let Λ be a left noetherian ring, Γ a right noetherian ring and ${}_{\Lambda}U$ a Wakamatsu tilting module with $\Gamma = \text{End}({}_{\Lambda}U)$. If the projective dimensions of ${}_{\Lambda}U$ and U_{Γ} are finite, then ${}_{\Lambda}U_{\Gamma}$ is a tilting bimodule (that is, both ${}_{\Lambda}U$ and U_{Γ} are tilting) with the left and right projective dimensions identical.*

For a module A in $\text{Mod } \Lambda$ (resp. $\text{Mod } \Lambda^{op}$), we use $l.\text{id}_{\Lambda}(A)$ (resp. $r.\text{id}_{\Lambda}(A)$) to denote the left (resp. right) injective dimension of A .

Lemma 17.2.4 *Let Λ and Γ be rings and ${}_{\Lambda}U_{\Gamma}$ a bimodule.*

- (1) *If Γ is a right noetherian ring, then $r.\text{id}_{\Gamma}(U) = \sup\{l.f.d_{\Gamma}(\text{Hom}_{\Lambda}(U, E)) \mid E \text{ is injective}\}$. Moreover, $r.\text{id}_{\Gamma}(U) = l.f.d_{\Gamma}(\text{Hom}_{\Lambda}(U, Q))$ for any injective cogenerator ${}_{\Lambda}Q$ for $\text{Mod } \Lambda$.*
- (2) *If Λ is a left noetherian ring, then $l.\text{id}_{\Lambda}(U) = \sup\{r.f.d_{\Lambda}(\text{Hom}_{\Gamma}(U, E')) \mid E' \text{ is injective}\}$. Moreover, $l.\text{id}_{\Lambda}(U) = r.f.d_{\Lambda}(\text{Hom}_{\Gamma}(U, Q'))$ for any injective cogenerator Q'_{Γ} for $\text{Mod } \Gamma^{op}$.*

Proof (1) By [4, Chapter VI, Proposition 5.3], for any $i \geq 1$, we have the following isomorphism:

$$\mathrm{Tor}_i^\Gamma(B, \mathrm{Hom}_\Lambda(U, E)) \cong \mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^i(B, U), E) \tag{1}$$

for any $B \in \mathrm{mod} \Gamma^{op}$ and ${}_\Lambda E$ injective.

If $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, E)) \leq n (< \infty)$ for any injective module ${}_\Lambda E$, then the isomorphism (1) induces $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^{n+1}(B, U), E) \cong \mathrm{Tor}_{n+1}^\Gamma(B, \mathrm{Hom}_\Lambda(U, E)) = 0$. Now taking ${}_\Lambda E$ as an injective cogenerator for $\mathrm{mod} \Lambda$, we see that $\mathrm{Ext}_\Gamma^{n+1}(B, U) = 0$ and $r.\mathrm{id}_\Gamma(U) \leq n$.

Conversely, if $r.\mathrm{id}_\Gamma(U) = n (< \infty)$, then $\mathrm{Ext}_\Gamma^{n+1}(B, U) = 0$ for any $B \in \mathrm{mod} \Gamma^{op}$ and $\mathrm{Tor}_{n+1}^\Gamma(B, \mathrm{Hom}_\Lambda(U, E)) = 0$ for any injective module ${}_\Lambda E$ by the isomorphism (1).

Let Y be any module in $\mathrm{Mod} \Gamma^{op}$. Then $Y = \varinjlim Y_\alpha$ (where Y_α ranges over all finitely generated submodules of Y). It is well known that the functor Tor_i commutes with \varinjlim for any $i \geq 0$, so $\mathrm{Tor}_{n+1}^\Gamma(Y, \mathrm{Hom}_\Lambda(U, E)) \cong \varinjlim \mathrm{Tor}_{n+1}^\Gamma(Y_\alpha, \mathrm{Hom}_\Lambda(U, E)) = 0$ by the above argument. This implies that $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, E)) \leq n$. Consequently, we conclude that the first equality holds.

The above argument in fact proves the second equality.

(2) It is similar to the proof of (1). □

Let ${}_\Lambda U_\Gamma$ be a bimodule. For a module A in $\mathrm{Mod} \Lambda$ (resp. $\mathrm{Mod} \Gamma^{op}$), we call $\mathrm{Hom}_\Lambda(\Lambda A, {}_\Lambda U_\Gamma)$ (resp. $\mathrm{Hom}_\Gamma(A_\Gamma, {}_\Lambda U_\Gamma)$) the dual module of A with respect to ${}_\Lambda U_\Gamma$, and denote either of these modules by A^* . For a homomorphism f between Λ -modules (resp. Γ^{op} -modules), we put $f^* = \mathrm{Hom}(f, {}_\Lambda U_\Gamma)$. We use $\sigma_A : A \rightarrow A^{**}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$ to denote the canonical evaluation homomorphism. A is called U -torsionless (resp. U -reflexive) if σ_A is a monomorphism (resp. an isomorphism).

Lemma 17.2.5 *Let Λ be a left noetherian ring, Γ any ring and ${}_\Lambda U_\Gamma$ a bimodule. If $\Lambda = \mathrm{End}(U_\Gamma)$, U_Γ is self-orthogonal and $r.\mathrm{id}_\Gamma(U) \leq n$, then $\bigoplus_{i=0}^n V_i$ is an injective cogenerator for $\mathrm{Mod} \Lambda$, where V_i is the $(i + 1)$ -st term in an injective resolution of ${}_\Lambda U$ for any $0 \leq i \leq n$.*

Proof Let A be any module in $\mathrm{mod} \Lambda$. Since $r.\mathrm{id}_\Gamma(U) \leq n$, $\mathrm{Ext}_\Gamma^i(X, U) = 0$ for any $X \in \mathrm{mod} \Gamma^{op}$ and $i \geq n + 1$. Then, by the assumption and [13, Theorem 2.2], it is easy to see that A is U -reflexive provided that $\mathrm{Ext}_\Lambda^t(A, U) = 0$ for any $1 \leq i \leq n$.

Let S be any simple Λ -module. Then $\mathrm{Ext}_\Lambda^t(S, U) \neq 0$ for some t with $0 \leq t \leq n$ (Otherwise, S is U -reflexive by the above argument and hence $S \cong S^{**} = 0$, a contradiction.)

Let

$$0 \rightarrow {}_\Lambda U \rightarrow V_0 \rightarrow V_1 \rightarrow \dots \rightarrow V_i \rightarrow \dots$$

be an injective resolution of ${}_\Lambda U$. Set $W_t = \mathrm{Im}(V_{t-1} \rightarrow V_t)$. We then get the following exact sequences:

$$\mathrm{Hom}_\Lambda(S, W_t) \rightarrow \mathrm{Ext}_\Lambda^t(S, U) \rightarrow 0,$$

$$0 \rightarrow \mathrm{Hom}_\Lambda(S, W_t) \rightarrow \mathrm{Hom}_\Lambda(S, V_t).$$

Because $\mathrm{Ext}_\Lambda^t(S, U) \neq 0$, $\mathrm{Hom}_\Lambda(S, W_t) \neq 0$ and $\mathrm{Hom}_\Lambda(S, V_t) \neq 0$. So $\mathrm{Hom}_\Lambda(S, \bigoplus_{i=0}^n V_i) \neq 0$ and hence $\bigoplus_{i=0}^n V_i$ is an injective cogenerator for $\mathrm{Mod} \Lambda$ by [1, Proposition 18.15]. □

As an application to Theorem 17.1.11, we have the following result.

Proposition 17.2.6 *Let Λ and Γ be left and right noetherian rings and ${}_\Lambda U$ a Wakamatsu tilting module with $\Gamma = \mathrm{End}({}_\Lambda U)$. If ${}_\Lambda U$ is k -Gorenstein for all k and both $l.\mathrm{id}_\Lambda(U)$ and $r.\mathrm{id}_\Gamma(U)$ are finite, then $l.\mathrm{id}_\Lambda(U) = r.\mathrm{id}_\Gamma(U)$.*

Proof Assume that $l.\mathrm{id}_\Lambda(U) = m < \infty$ and $r.\mathrm{id}_\Gamma(U) = n < \infty$. Since ${}_\Lambda U$ is k -Gorenstein for all k , by Theorem 17.1.11, we have that $l.\mathrm{fd}_\Gamma(\mathrm{Hom}_\Lambda(U, \bigoplus_{i=0}^m E_i)) \leq m$, where E_i is the $(i + 1)$ -st term in a minimal injective resolution of ${}_\Lambda U$ for any $i \geq 0$.

By Proposition 17.2.2, ${}_{\Lambda}U_{\Gamma}$ is a faithfully balanced self-orthogonal bimodule. If $m < n$, then, by Lemmas 17.2.5 and 17.2.4, we have that $\bigoplus_{i=0}^n E_i \cong \bigoplus_{i=0}^m E_i$ is an injective cogenerator for $\text{Mod } \Lambda$ and $l.\text{fd}_{\Gamma}(\text{Hom}_{\Lambda}(U, \bigoplus_{i=0}^m E_i)) = n$, which is a contradiction. So we have that $m \geq n$. According to the symmetry of k -Gorenstein modules, we can prove $n \geq m$ similarly. \square

Proposition 17.2.7 *Let Λ be a left and right artinian ring and ${}_{\Lambda}U$ a Wakamatsu tilting module with $\Lambda = \text{End}({}_{\Lambda}U)$. If ${}_{\Lambda}U$ is k -Gorenstein for all k , then $l.\text{id}_{\Lambda}(U) = r.\text{id}_{\Lambda}(U)$.*

Proof By Theorem 17.1.11, for any $i \geq 1$ and $M \in \text{mod } \Lambda$ or $\text{mod } \Lambda^{op}$, we have that $\text{s.grade}_U \text{Ext}_{\Lambda}^i(M, U) \geq i$. By Proposition 17.2.2, ${}_{\Lambda}U_{\Lambda}$ is a faithfully balanced self-orthogonal bimodule. It then follows from [9, Theorem] and its dual statement that $l.\text{id}_{\Lambda}(U)$ is finite if and only if $r.\text{id}_{\Lambda}(U)$ is finite. Now our conclusion follows from Proposition 17.2.6. \square

Putting ${}_{\Lambda}U = {}_{\Lambda}\Lambda$, we immediately have the following result, which generalizes [2, Corollary 5.5(b)].

Corollary 17.2.8 *Let Λ be a left and right artinian ring. If Λ is k -Gorenstein for all k , then $l.\text{id}_{\Lambda}(\Lambda) = r.\text{id}_{\Lambda}(\Lambda)$.*

17.3 The Proof of Main Results

In this section, we prove Theorems 17.1.4 and 17.1.11.

From now on, Λ and Γ are left and right noetherian rings and ${}_{\Lambda}U$ is a Wakamatsu tilting module with $\Gamma = \text{End}({}_{\Lambda}U)$. We always assume that

$$0 \rightarrow {}_{\Lambda}U \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_i \rightarrow \cdots$$

is a minimal injective resolution of ${}_{\Lambda}U$, and

$$0 \rightarrow U_{\Gamma} \rightarrow E'_0 \rightarrow E'_1 \rightarrow \cdots \rightarrow E'_i \rightarrow \cdots$$

is a minimal injective resolution of U_{Γ} and k is a positive integer.

Lemma 17.3.1 *Let ${}_{\Lambda}E$ be injective. Then $l.\text{fd}_{\Gamma}(\text{Hom}_{\Lambda}(U, E)) = U\text{-lim.dim}_{\Lambda}(E)$.*

Proof We first prove that $U\text{-lim.dim}_{\Lambda}(E) \leq l.\text{fd}_{\Gamma}(\text{Hom}_{\Lambda}(U, E))$. Without loss of generality, assume that $l.\text{fd}_{\Gamma}(\text{Hom}_{\Lambda}(U, E)) = n < \infty$. Then there exists an exact sequence:

$$0 \rightarrow Q_n \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow \text{Hom}_{\Lambda}(U, E) \rightarrow 0$$

in $\text{Mod } \Gamma$ with each Q_i Γ -flat for any $0 \leq i \leq n$. By [4, Chapter VI, Proposition 5.3], we have that

$$\text{Tor}_j^{\Gamma}(U, \text{Hom}_{\Lambda}(U, E)) \cong \text{Hom}_{\Lambda}(\text{Ext}_{\Gamma}^j(U, U), E) = 0$$

for any $j \geq 1$. Then we easily get an exact sequence:

$$0 \rightarrow U \otimes_{\Gamma} Q_n \rightarrow \cdots \rightarrow U \otimes_{\Gamma} Q_1 \rightarrow U \otimes_{\Gamma} Q_0 \rightarrow U \otimes_{\Gamma} \text{Hom}_{\Lambda}(U, E) \rightarrow 0.$$

Because each Q_i is a direct limit of finitely generated free Γ -modules, $U \otimes_{\Gamma} Q_i \in \text{add-lim}_{\Lambda} U$ for any $0 \leq i \leq n$. On the other hand, $U \otimes_{\Gamma} \text{Hom}_{\Lambda}(U, E) \cong \text{Hom}_{\Lambda}(\text{Hom}_{\Gamma}(U, U), E) \cong E$ by [18, p.47]. So we conclude that $U\text{-lim.dim}_{\Lambda}(E) \leq n$.

We next prove that $lfd_\Gamma(\text{Hom}_\Lambda(U, E)) \leq U\text{-lim.dim}_\Lambda(E)$. Assume that $U\text{-lim.dim}_\Lambda(E) = n < \infty$. Then there exists an exact sequence:

$$0 \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow E \rightarrow 0 \tag{2}$$

in $\text{Mod } \Lambda$ with each X_i in $\text{add-lim}_\Lambda U$ for any $0 \leq i \leq n$. Since ${}_\Lambda U$ is finitely generated, by [17, Theorem 3.2], for any direct system $\{M_\alpha\}_{\alpha \in I}$ and $j \geq 0$, we have that $\text{Ext}_\Lambda^j(U, \varinjlim M_\alpha) \cong \varinjlim \text{Ext}_\Lambda^j(U, M_\alpha)$. From this fact we know that $\text{Ext}_\Lambda^j(U, X_i) = 0$ and $\text{Hom}_\Lambda(U, X_i)$ is in $\text{add-lim}_\Gamma \Gamma$ for any $j \geq 1$ and $0 \leq i \leq n$. So each $\text{Hom}_\Lambda(U, X_i)$ is Γ -flat for any $0 \leq i \leq n$ and by applying the functor $\text{Hom}_\Lambda(U, _)$ to the exact sequence (2) we obtain the following exact sequence:

$$0 \rightarrow \text{Hom}_\Lambda(U, X_n) \rightarrow \cdots \rightarrow \text{Hom}_\Lambda(U, X_1) \rightarrow \text{Hom}_\Lambda(U, X_0) \rightarrow \text{Hom}_\Lambda(U, E) \rightarrow 0.$$

Hence $lfd_\Gamma(\text{Hom}_\Lambda(U, E)) \leq n$. The proof is finished. □

Lemma 17.3.2 *Let m be an integer with $m \geq -k$. Then the following statements are equivalent.*

- (1) $U\text{-lim.dim}_\Lambda(\bigoplus_{i=0}^{k-1} E_i) \leq k + m$.
- (2) $s.\text{grade}_U \text{Ext}_\Gamma^{k+m+1}(N, U) \geq k$ for any $N \in \text{mod } \Gamma^{op}$.
- (3) $lfd_\Gamma(\text{Hom}_\Lambda(U, E_i)) \leq k + m$ for any $0 \leq i \leq k - 1$.

Proof By Lemma 17.3.1, we have (1) \Leftrightarrow (3).

(2) \Rightarrow (3) We proceed by using induction on i . Suppose that $s.\text{grade}_U \text{Ext}_\Gamma^{k+m+1}(N, U) \geq k$ for any $N \in \text{mod } \Gamma^{op}$. We first prove $lfd_\Gamma(\text{Hom}_\Lambda(U, E_0)) \leq k + m$. By assumption, we have $\text{Hom}_\Lambda(\text{Ext}_\Gamma^{k+m+1}(N, U), U) = 0$. We now claim that $\text{Hom}_\Lambda(\text{Ext}_\Gamma^{k+m+1}(N, U), E_0) = 0$. For if otherwise, then there exists $0 \neq f : \text{Ext}_\Gamma^{k+m+1}(N, U) \rightarrow E_0$ and $\text{Im } f \cap U \neq 0$ (since U is essential in E_0). Hence, there exists a submodule $X (= f^{-1}(\text{Im } f \cap U))$ of $\text{Ext}_\Gamma^{k+m+1}(N, U)$ such that $\text{Hom}_\Lambda(X, U) \neq 0$, which contradicts $s.\text{grade}_U \text{Ext}_\Gamma^{k+m+1}(N, U) \geq k$. It follows easily from [4, Chapter VI, Proposition 5.3] that $lfd_\Gamma(\text{Hom}_\Lambda({}_\Lambda U_\Gamma, E_0)) \leq k + m$.

Now suppose $i \geq 1$. Consider the exact sequence:

$$0 \rightarrow K_{i-1} \rightarrow E_{i-1} \rightarrow K_i \rightarrow 0$$

where $K_{i-1} = \text{Ker}(E_{i-1} \rightarrow E_i)$ and $K_i = \text{Im}(E_{i-1} \rightarrow E_i)$. Then for any $X \subset \text{Ext}_\Gamma^{k+m+1}(N, U)$, we have an exact sequence:

$$\text{Hom}_\Lambda(X, E_{i-1}) \rightarrow \text{Hom}_\Lambda(X, K_i) \rightarrow \text{Ext}_\Lambda^1(X, K_{i-1}) \rightarrow 0 \tag{3}$$

Since $s.\text{grade}_U \text{Ext}_\Gamma^{k+m+1}(N, U) \geq k$ and $1 \leq i \leq k - 1$, $\text{Ext}_\Lambda^1(X, K_{i-1}) \cong \text{Ext}_\Lambda^1(X, U) = 0$. By induction assumption, $lfd_\Gamma(\text{Hom}_\Lambda(U, E_{i-1})) \leq k + m$. It follows from [4, Chapter VI, Proposition 5.3] that $\text{Hom}_\Lambda(\text{Ext}_\Gamma^{k+m+1}(N, U), E_{i-1}) \cong \text{Tor}_{k+m+1}^\Gamma(N, \text{Hom}_\Lambda(U, E_{i-1})) = 0$. Since E_{i-1} is injective, $\text{Hom}_\Lambda(X, E_{i-1}) = 0$. It follows from the exactness of the sequence (3) that $\text{Hom}_\Lambda(X, K_i) = 0$. Observe that E_i is the injective envelope of K_i ; by using a similar argument to the case $i = 0$, we can show that $\text{Hom}_\Lambda(\text{Ext}_\Gamma^{k+m+1}(M, U), E_i) = 0$. Hence, we have that $lfd_\Gamma(\text{Hom}_\Lambda({}_\Lambda U_\Gamma, E_i)) \leq k + m$.

(3) \Rightarrow (2) Suppose that $lfd_\Gamma(\text{Hom}_\Lambda(U, \bigoplus_{i=0}^{k-1} E_i)) \leq k + m$. Then, by [4, Chapter VI, Proposition 5.3], we have that $\text{Hom}_\Lambda(\text{Ext}_\Gamma^{k+m+1}(N, U), \bigoplus_{i=0}^{k-1} E_i) = 0$ for any $N \in \text{mod } \Gamma^{op}$. Let X be any submodule of $\text{Ext}_\Gamma^{k+m+1}(N, U)$. Then $\text{Hom}_\Lambda(X, \bigoplus_{i=0}^{k-1} E_i) = 0$. Put $K_0 = U$ and $K_i = \text{Im}(E_{i-1} \rightarrow E_i)$ for any $1 \leq i \leq k - 1$. Then $\text{Hom}_\Lambda(X, K_i) = 0$ for any $0 \leq i \leq k - 1$. It is not difficult to prove that $\text{Ext}_\Lambda^{i+1}(X, K_0) \cong \text{Ext}_\Lambda^1(X, K_i)$ and $\text{Ext}_\Lambda^1(X, K_i) \cong \text{Hom}_\Lambda(X, K_{i+1})$ for any $0 \leq i \leq k - 2$. Hence we conclude that $\text{Hom}_\Lambda(X, U) = 0 = \text{Ext}_\Lambda^i(X, U)$ for any $1 \leq i \leq k - 1$. This completes the proof. □

Putting $m = -1$, then by Lemma 17.3.2, we have the following

Corollary 17.3.3 (1) $U\text{-lim.dim}_\Lambda(\bigoplus_{i=0}^{k-1} E_i) \leq k - 1$ if and only if $s.\text{grade}_U \text{Ext}_\Gamma^k(N, U) \geq k$ for any $N \in \text{mod } \Gamma^{op}$ if and only if $l.\text{fd}_\Gamma(\text{Hom}_\Lambda(U, \bigoplus_{i=0}^{k-1} E_i)) \leq k - 1$.

(2) $U\text{-lim.dim}_\Lambda(E_i) \leq i$ for any $0 \leq i \leq k - 1$ if and only if $s.\text{grade}_U \text{Ext}_\Gamma^i(N, U) \geq i$ for any $N \in \text{mod } \Gamma^{op}$ and $1 \leq i \leq k$ if and only if $l.\text{fd}_\Gamma(\text{Hom}_\Lambda(U, E_i)) \leq i$ for any $0 \leq i \leq k - 1$.

Let M be in $\text{mod } \Lambda$ (resp. $\text{mod } \Gamma^{op}$) and $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$ be a projective presentation of M in $\text{mod } \Lambda$ (resp. $\text{mod } \Gamma^{op}$). Then we have an exact sequence:

$$0 \rightarrow M^* \rightarrow P_0^* \xrightarrow{f^*} P_1^* \rightarrow \text{Coker } f^* \rightarrow 0.$$

We call $\text{Coker } f^*$ the transpose (with respect to ${}_\Lambda U_\Gamma$) of M , and denote it by $\text{Tr}_U M$.

For a positive integer k , recall from [10] that M is called U - k -torsionfree if $\text{Ext}_\Gamma^i(\text{Tr}_U M, U)$ (resp. $\text{Ext}_\Lambda^i(\text{Tr}_U M, U) = 0$ for any $1 \leq i \leq k$. We call M U - k -syzygy if there exists an exact sequence $0 \rightarrow M \rightarrow X_0 \rightarrow X_1 \rightarrow \dots \xrightarrow{f} X_{k-1}$ with all X_i in $\text{add}_\Lambda U$ (resp. $\text{add}_U \Gamma$), and denote M by $\Omega_U^k(\text{Coker } f)$. Putting ${}_\Lambda U_\Gamma = {}_\Lambda \Lambda_\Lambda$, then, in this case, the notions of U - k -torsionfree modules and U - k -syzygy modules are just that of k -torsionfree modules and k -syzygy modules respectively (see [3] for the definitions of k -torsionfree modules and k -syzygy modules). We use $\mathcal{T}_U^k(\text{mod } \Lambda)$ (resp. $\mathcal{T}_U^k(\text{mod } \Gamma^{op})$) and $\Omega_U^k(\text{mod } \Lambda)$ (resp. $\Omega_U^k(\text{mod } \Gamma^{op})$) to denote the full subcategory of $\text{mod } \Lambda$ (resp. $\text{mod } \Gamma^{op}$) consisting of U - k -torsionfree modules and U - k -syzygy modules, respectively. It is not difficult to verify that $\mathcal{T}_U^k(\text{mod } \Lambda) \subseteq \Omega_U^k(\text{mod } \Lambda)$ and $\mathcal{T}_U^k(\text{mod } \Gamma^{op}) \subseteq \Omega_U^k(\text{mod } \Gamma^{op})$.

The following result generalizes [3, Proposition 1.6(a)].

Lemma 17.3.4 *The following statements are equivalent.*

- (1) $\text{grade}_U \text{Ext}_\Gamma^{i+1}(M, U) \geq i$ for any $M \in \text{mod } \Lambda$ and $1 \leq i \leq k - 1$.
- (1)^{op} $\text{grade}_U \text{Ext}_\Gamma^{i+1}(N, U) \geq i$ for any $N \in \text{mod } \Gamma^{op}$ and $1 \leq i \leq k - 1$.

If one of the above equivalent conditions holds, then $\mathcal{T}_U^i(\text{mod } \Lambda) = \Omega_U^i(\text{mod } \Lambda)$ and $\mathcal{T}_U^i(\text{mod } \Gamma^{op}) = \Omega_U^i(\text{mod } \Gamma^{op})$ for any $1 \leq i \leq k$.

Proof The equivalence of (1) and (1)^{op} was proved in [12, Lemma 3.3]. The latter assertion follows from [10, Theorem 3.1]. □

Putting $m = 0$, then by Lemma 17.3.2, we have the following result, in which the second assertion is just [3, Proposition 2.2] when ${}_\Lambda U_\Gamma = {}_\Lambda \Lambda_\Lambda$.

Corollary 17.3.5 (1) $U\text{-lim.dim}_\Lambda(\bigoplus_{i=0}^{k-1} E_i) \leq k$ if and only if $s.\text{grade}_U \text{Ext}_\Gamma^{k+1}(N, U) \geq k$ for any $N \in \text{mod } \Gamma^{op}$ if and only if $l.\text{fd}_\Gamma(\text{Hom}_\Lambda(U, \bigoplus_{i=0}^{k-1} E_i)) \leq k$.

(2) $U\text{-lim.dim}_\Lambda(E_i) \leq i + 1$ for any $0 \leq i \leq k - 1$ if and only if $s.\text{grade}_U \text{Ext}_\Gamma^{i+1}(N, U) \geq i$ for any $N \in \text{mod } \Gamma^{op}$ and $1 \leq i \leq k$ if and only if $l.\text{fd}_\Gamma(\text{Hom}_\Lambda(U, E_i)) \leq i + 1$ for any $0 \leq i \leq k - 1$. In this case, $\mathcal{T}_U^i(\text{mod } \Gamma^{op}) = \Omega_U^i(\text{mod } \Gamma^{op})$ for any $1 \leq i \leq k$.

Proof Our assertions follow from Lemmas 16.3.2 and 16.3.4. □

Putting $m = -k$, then by Lemma 17.3.2, we have the following

Corollary 17.3.6 *The following statements are equivalent.*

- (1) $U\text{-dom.dim}(\wedge U) \geq k$.
- (2) $s.\text{grade}_U \text{Ext}_\Gamma^1(N, U) \geq k$ for any $N \in \text{mod } \Gamma^{op}$.
- (3) $\text{Hom}_\wedge(U, E_i)$ is Γ -flat for any $0 \leq i \leq k - 1$.

Dually, we have the following

Corollary 17.3.7 *The following statements are equivalent.*

- (1) $U\text{-dom.dim}(U_\Gamma) \geq k$.
- (2) $s.\text{grade}_U \text{Ext}_\Lambda^1(M, U) \geq k$ for any $M \in \text{mod } \Lambda$.
- (3) $\text{Hom}_\Gamma(U, E'_i)$ is Λ^{op} -flat for any $0 \leq i \leq k - 1$.

The following two results are cited from [11].

Lemma 17.3.8 ([11, Corollary 2.5]) *$\text{Hom}_\wedge(U, E_0)$ is Γ -flat if and only if $\text{Hom}_\Gamma(U, E'_0)$ is Λ^{op} -flat.*

Lemma 17.3.9 ([11, Lemma 2.6]) *Let X be in $\text{mod } \Lambda$ (resp. $\text{mod } \Gamma^{op}$) and n a non-negative integer. If $\text{grade}_U X \geq n$ and $\text{grade}_U \text{Ext}_\Lambda^n(X, U)$ (resp. $\text{grade}_U \text{Ext}_\Gamma^n(X, U)$) $\geq n + 1$, then $\text{grade}_U X \geq n + 1$.*

Lemma 17.3.10 *If $U\text{-dom.dim}(U_\Gamma) \geq k$, then $U\text{-dom.dim}(\wedge U) \geq k$.*

Proof When $k = 1$, by Corollary 16.3.7, $\text{Hom}_\wedge(U, E'_0)$ is Λ^{op} -flat. Then, by Lemma 17.3.8, $\text{Hom}_\wedge(U, E_i)$ is Γ -flat. So $U\text{-dom.dim}(\wedge U) \geq 1$ by Corollary 16.3.6.

Now suppose $k \geq 2$. By induction assumption, $U\text{-dom.dim}(\wedge U) \geq k - 1$. So, by Corollary 16.3.6, we have that $s.\text{grade}_U \text{Ext}_\Gamma^1(N, U) \geq k - 1$ for any $N \in \text{mod } \Gamma^{op}$.

Let X be any submodule of $\text{Ext}_\Gamma^1(N, U)$. Then $\text{grade}_U X \geq k - 1$. By assumption and Corollary 16.3.7, $\text{grade}_U \text{Ext}_\Gamma^i(X, U) \geq k$ for any $i \geq 1$. It follows from Lemma 17.3.9 that $\text{grade}_U X \geq k$. So $s.\text{grade}_U \text{Ext}_\Gamma^1(N, U) \geq k$ and hence $U\text{-dom.dim}(\wedge U) \geq k$ by Corollary 16.3.6. □

Proof of Theorem 17.1.4. By Corollary 16.3.6 we have that (1) \Leftrightarrow (2)^{op} \Leftrightarrow (3), and by Lemma 17.3.10 we have that (1) \Rightarrow (1)^{op}. The other implications follow from the symmetry. □

We now begin to prove Theorem 17.1.11.

Lemma 17.3.11 ([12, Lemma 3.2]) *If $s.\text{grade}_U \text{Ext}_\Gamma^{i+1}(X, U) \geq i$ for any $X \in \text{mod } \Lambda$ (resp. $\text{mod } \Gamma^{op}$) and $1 \leq i \leq k - 1$, then each k -syzygy module in $\text{mod } \Lambda$ (resp. $\text{mod } \Gamma^{op}$) is in $\Omega_U^k(\text{mod } \Lambda)$ (resp. $\Omega_U^k(\text{mod } \Gamma^{op})$).*

Theorem 17.3.12 *The following statements are equivalent.*

- (1) $s.\text{grade}_U \text{Ext}_\Lambda^i(M, U) \geq i$ for any $M \in \text{mod } \Lambda$ and $1 \leq i \leq k$.
- (2) $\text{Ext}_\Gamma^i(\text{Ext}_\Lambda^i(\cdot, U), U)$ preserves monomorphisms in $\text{mod } \Lambda$ for any $0 \leq i \leq k - 1$.

Proof We proceed by using induction on k .

(1) \Rightarrow (2) Let

$$0 \rightarrow X \xrightarrow{f} Y \rightarrow Z \rightarrow 0 \tag{4}$$

be an exact sequence in $\text{mod } \Lambda$.

Suppose $k = 1$. By assumption, $\text{s.grade}_U \text{Ext}_\Lambda^1(Z, U) \geq 1$. Since $\text{Coker } f^*$ is a submodule of $\text{Ext}_\Lambda^1(Z, U)$, $(\text{Coker } f^*)^* = 0$ and $0 \rightarrow X^{**} \xrightarrow{f^{**}} Y^{**}$ is exact.

Now suppose $k \geq 2$. From the exact sequence (4), we get an exact sequence:

$$\text{Ext}_\Lambda^{k-1}(Z, U) \xrightarrow{\alpha} \text{Ext}_\Lambda^{k-1}(Y, U) \xrightarrow{\beta} \text{Ext}_\Lambda^{k-1}(X, U) \xrightarrow{\gamma} \text{Ext}_\Lambda^k(Z, U).$$

Set $A = \text{Im } \alpha$, $B = \text{Im } \beta$ and $C = \text{Im } \gamma$. By (1), we have that $\text{grade}_U A \geq k - 1$, $\text{grade}_U B \geq k - 1$ and $\text{grade}_U C \geq k$. Then we get the following exact sequences:

$$0 \rightarrow \text{Ext}_\Gamma^{k-1}(B, U) \rightarrow \text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^{k-1}(Y, U), U),$$

$$0 \rightarrow \text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^{k-1}(X, U), U) \rightarrow \text{Ext}_\Gamma^{k-1}(B, U).$$

Thus we get a composition of monomorphisms:

$$\text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^{k-1}(X, U), U) \hookrightarrow \text{Ext}_\Gamma^{k-1}(B, U) \hookrightarrow \text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^{k-1}(Y, U), U),$$

which is also a monomorphism.

(2) \Rightarrow (1) Suppose $k = 1$. Let M be in $\text{mod } \Lambda$ and X a submodule of $\text{Ext}_\Lambda^1(M, U)$. Because $\text{Ext}_\Lambda^1(M, U)$ is in $\text{mod } \Gamma^{op}$, X is also in $\text{mod } \Gamma^{op}$. So there exist a positive integer t and an exact sequence:

$$0 \rightarrow U^t \xrightarrow{f} L \rightarrow M \rightarrow 0$$

such that the induced exact sequence:

$$L^* \xrightarrow{f^*} (U^t)^* \rightarrow \text{Ext}_\Lambda^1(M, U)$$

has the property that $X \cong \text{Coker } f^*$. By assumption, f^{**} is monic, so $X^* \cong \text{Ker } f^{**} = 0$. Hence we conclude that $\text{s.grade}_U \text{Ext}_\Lambda^1(M, U) \geq 1$.

Now suppose $k \geq 2$. By induction assumption, for any $M \in \text{mod } \Lambda$, we have that $\text{s.grade}_U \text{Ext}_\Lambda^i(M, U) \geq i$ for any $1 \leq i \leq k - 1$ and $\text{s.grade}_U \text{Ext}_\Lambda^k(M, U) \geq k - 1$. By [10, Theorem 3.1], $\Omega_U^i(\text{mod } \Lambda) = \mathcal{T}_U^i(\text{mod } \Lambda)$ for any $1 \leq i \leq k$.

Let

$$\dots \xrightarrow{g_{i+1}} P_i \xrightarrow{g_i} \dots \xrightarrow{g_2} P_1 \xrightarrow{g_1} P_0 \rightarrow M \rightarrow 0$$

be a projective resolution of M in $\text{mod } \Lambda$. Notice that $\text{Coker } g_k$ is a $(k - 1)$ -syzygy module in $\text{mod } \Lambda$, so it is in $\Omega_U^{k-1}(\text{mod } \Lambda)$ by Lemma 17.3.11 and hence in $\mathcal{T}_U^{k-1}(\text{mod } \Lambda)$. Thus $\text{Ext}_\Gamma^i(\text{Coker } g_k^*, U) = 0$ for any $1 \leq i \leq k - 1$.

Let X be a submodule of $\text{Ext}_\Lambda^k(M, U)$. Then $\text{grade}_U X \geq k - 1$. By [9, Lemma 2], there exists an embedding $0 \rightarrow X \rightarrow \text{Coker } g_k^*$. By assumption, we then have an exact sequence:

$$0 \rightarrow \text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^{k-1}(X, U), U) \rightarrow \text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^{k-1}(\text{Coker } g_k^*, U), U) = 0,$$

which implies that $\text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^{k-1}(X, U), U) = 0$.

On the other hand, $\text{s.grade}_U \text{Ext}_\Gamma^{k-1}(X, U) \geq k - 1$ by [21, Theorem 7.5]. So $\text{s.grade}_U \text{Ext}_\Gamma^{k-1}(X, U) \geq k$. It follows from Lemma 17.3.9 that $\text{grade}_U X \geq k$ and $\text{s.grade}_U \text{Ext}_\Gamma^k(M, U) \geq k$. We are done. \square

Proof of Theorem 17.1.11. By definition, we have (1) \Leftrightarrow (2). By Corollary 17.3.5(2), we have that (3) \Leftrightarrow (2)^{op} \Leftrightarrow (4). By Theorem 16.3.12 and [21, Theorem 7.5], we have that (5) \Leftrightarrow (2) \Leftrightarrow (2)^{op}. The other implications follow from the symmetry. \square

17.4 Exactness of the Double Dual

As applications to the results in Section 17.3, we give in this section some characterizations of $()^{**}$ preserving monomorphisms and being left exact, respectively.

As an immediate consequence of Theorem 17.1.11, we have the following result, which generalizes [5, Theorem 1] and [7, Proposition 3.1].

Proposition 17.4.1 *The following statements are equivalent.*

- (1) $U\text{-dom.dim}(\wedge U) \geq 1$.
 - (2) $s.\text{grade}_U \text{Ext}_\Lambda^1(M, U) \geq 1$ for any $M \in \text{mod } \Lambda$.
 - (3) $E_0 \in \text{add-lim}_\Lambda U$.
 - (4) $()^{**}$ preserves monomorphisms in $\text{mod } \Lambda$.
- (1)^{op} $U\text{-dom.dim}(U_\Gamma) \geq 1$.
 - (2)^{op} $s.\text{grade}_U \text{Ext}_\Gamma^1(N, U) \geq 1$ for any $N \in \text{mod } \Gamma^{\text{op}}$.
 - (3)^{op} $E'_0 \in \text{add-lim} U_\Gamma$.
 - (4)^{op} $()^{**}$ preserves monomorphisms in $\text{mod } \Gamma^{\text{op}}$.

Lemma 17.4.2 *Assume that $U\text{-dom.dim}(\wedge U) \geq k$. Then, for a module M in $\text{mod } \Lambda$, $\text{grade}_U M \geq k$ if $M^* = 0$.*

Proof For any $M \in \text{mod } \Lambda$ and $i \geq 1$, we have an exact sequence

$$\text{Hom}_\Lambda(M, E_{i-1}) \rightarrow \text{Hom}_\Lambda(M, K_i) \rightarrow \text{Ext}_\Lambda^i(M, U) \rightarrow 0 \tag{5}$$

where $K_i = \text{Im}(E_{i-1} \rightarrow E_i)$.

Suppose $U\text{-dom.dim}(\wedge U) \geq k$. Then each E_i is in $\text{add-lim}_\Lambda U$ for any $0 \leq i \leq k - 1$. So, for a given $M \in \text{mod } \Lambda$ with $M^* = 0$, we have that $\text{Hom}_\Lambda(M, E_i) = 0$ by [17, Theorem 3.2] and $\text{Hom}_\Lambda(M, K_i) = 0$ for any $0 \leq i \leq k - 1$. Then by the exactness of the sequence (5), $\text{Ext}_\Lambda^i(M, U) = 0$ for any $1 \leq i \leq k - 1$, and so $\text{grade}_U M \geq k$. \square

Lemma 17.4.3 *If $[\text{Ext}_\Lambda^1(M, U)]^* = 0$ for any $M \in \text{mod } \Lambda$, then N^* is U -reflexive for any $N \in \text{mod } \Gamma^{\text{op}}$.*

Proof By the dual statements of [10, Proposition 4.2 and Corollary 4.2]. \square

We now characterize U -dominant dimension of U at least two. The following result generalizes [5, Theorem 2] and [8, Proposition E].

Proposition 17.4.4 *The following statements are equivalent.*

- (1) $U\text{-dom.dim}(\wedge U) \geq 2$.
- (2) $(\)^{**} : \text{mod } \Lambda \rightarrow \text{mod } \Lambda$ is left exact.
- (3) $(\)^{**} : \text{mod } \Lambda \rightarrow \text{mod } \Lambda$ preserves monomorphisms and $\text{Ext}_\Gamma^1(\text{Ext}_\Lambda^1(X, U), U) = 0$ for any $X \in \text{mod } \Lambda$.
- (1)^{op} $U\text{-dom.dim}(U_\Gamma) \geq 2$.
- (2)^{op} $(\)^{**} : \text{mod } \Gamma^{op} \rightarrow \text{mod } \Gamma^{op}$ is left exact.
- (3)^{op} $(\)^{**} : \text{mod } \Gamma^{op} \rightarrow \text{mod } \Gamma^{op}$ preserves monomorphisms and $\text{Ext}_\Lambda^1(\text{Ext}_\Gamma^1(Y, U), U) = 0$ for any $Y \in \text{mod } \Gamma^{op}$.

Proof By Theorem 17.1.4, we have (1) \Leftrightarrow (1)^{op}. By symmetry, we only need to prove that (1) \Rightarrow (2) and (2)^{op} \Rightarrow (3) \Rightarrow (1)^{op}.

(1) \Rightarrow (2) Assume that $U\text{-dom.dim}(\wedge U) \geq 2$ and $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ is an exact sequence in $\text{mod } \Lambda$. By Proposition 17.4.1, α^{**} is monic. By Theorem 17.1.4 and [4, Chapter VI, Proposition 5.3], we have that $\text{Hom}_\Gamma(U, E'_0)$ is Λ^{op} -flat and $\text{Hom}_\Gamma(\text{Ext}_\Lambda^1(C, U), E'_0) = 0$. Since $\text{Coker } \alpha^*$ is isomorphic to a submodule of $\text{Ext}_\Lambda^1(C, U)$, $\text{Hom}_\Gamma(\text{Coker } \alpha^*, E'_0) = 0$ and $(\text{Coker } \alpha^*)^* = 0$. Then by Lemma 17.4.2, we have that $\text{grade}_U \text{Coker } \alpha^* \geq 2$ and $\text{Ext}_\Gamma^1(\text{Coker } \alpha^*, U) = 0$. It follows easily that $0 \rightarrow A^{**} \xrightarrow{\alpha^{**}} B^{**} \xrightarrow{\beta^{**}} C^{**}$ is exact.

(2)^{op} \Rightarrow (3) By Proposition 17.4.1, $(\)^{**} : \text{mod } \Gamma^{op} \rightarrow \text{mod } \Gamma^{op}$ preserves monomorphisms and $U\text{-dom.dim}(\wedge U) = U\text{-dom.dim}(U_\Gamma) \geq 1$. By Theorem 17.1.4, for any $X \in \text{mod } \Lambda$, we have that $s.\text{grade}_U \text{Ext}_\Lambda^1(X, U) \geq 1$ and $[\text{Ext}_\Lambda^1(X, U)]^* = 0$.

Let

$$0 \rightarrow K \xrightarrow{f} Q \xrightarrow{g} \text{Ext}_\Lambda^1(X, U) \rightarrow 0$$

be an exact sequence in $\text{mod } \Gamma^{op}$ with Q projective. Then, by (2)^{op}, f^{**} is a monomorphism and hence an isomorphism. So f^{***} is also an isomorphism. On the other hand, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q^* & \xrightarrow{f^*} & K^* & \longrightarrow & \text{Ext}_\Gamma^1(\text{Ext}_\Lambda^1(X, U), U) \longrightarrow 0 \\ & & \downarrow \sigma_{Q^*} & & \downarrow \sigma_{K^*} & & \\ & & Q^{***} & \xrightarrow{f^{***}} & K^{***} & & \end{array}$$

It follows from Lemma 16.4.3 that Q^* and K^* are U -reflexive. So σ_{Q^*} and σ_{K^*} are isomorphisms and hence f^* is an isomorphism. Consequently we have that $\text{Ext}_\Gamma^1(\text{Ext}_\Lambda^1(X, U), U) = 0$.

(3) \Rightarrow (1)^{op} Suppose that (3) holds. Then $U\text{-dom.dim}(U_\Gamma) \geq 1$ by Proposition 17.4.1.

Let A be in $\text{mod } \Lambda$ and B any submodule of $\text{Ext}_\Lambda^1(A, U)$ in $\text{mod } \Gamma^{op}$. Since $U\text{-dom.dim}(U_\Gamma) \geq 1$, by Theorem 17.1.4 and [4, Chapter VI, Proposition 5.3], we have that $\text{Hom}_\Gamma(U, E'_0)$ is Λ^{op} -flat and $\text{Hom}_\Gamma(\text{Ext}_\Lambda^1(A, U), E'_0) = 0$. So $\text{Hom}_\Gamma(B, E'_0) = 0$ and hence $\text{Hom}_\Gamma(B, E'_0/U_\Gamma) \cong \text{Ext}_\Gamma^1(B, U_\Gamma)$. On the other hand, $\text{Hom}_\Gamma(B, E'_0) = 0$ implies $B^* = 0$. Then by [13, Lemma 2.1], we have that $B \cong \text{Ext}_\Lambda^1(\text{Tr}_U B, U)$ with $\text{Tr}_U B$ in $\text{mod } \Lambda$. By (3), $\text{Hom}_\Gamma(B, E'_0/U) \cong \text{Ext}_\Gamma^1(B, U) \cong \text{Ext}_\Gamma^1(\text{Ext}_\Lambda^1(\text{Tr}_U B, U), U) = 0$. Then by using a similar argument to the proof of (2) \Rightarrow (3) in Lemma 17.3.2, we have that $\text{Hom}_\Gamma(\text{Ext}_\Lambda^1(A, U), E'_1) = 0$ (note: E'_1 is the injective envelope of E'_0/U). It follows from [4, Chapter VI, Proposition 5.3] that $\text{Hom}_\Gamma(U, E'_1)$ is Λ^{op} -flat and thus $U\text{-dom.dim}(U_\Gamma) \geq 2$ by Theorem 17.1.4. \square

17.5 A Generalization of k -Gorenstein Modules

In this section, we study a generalization of k -Gorenstein modules, which is however not left-right symmetric. We characterize this generalization in terms of some properties similar to that of k -Gorenstein modules. The results obtained here develop the main result of Auslander and Reiten in [3].

We begin with the following equivalent characterizations of $U\text{-lim.dim}_\Lambda(E_0) \leq 1$ as follows, which generalizes [8, Proposition D].

Proposition 17.5.1 *The following statements are equivalent.*

- (1) $U\text{-lim.dim}_\Lambda(E_0) \leq 1$.
- (2) σ_X is an essential monomorphism for any U -torsionless module X in $\text{mod } \Lambda$.
- (3) f^{**} is a monomorphism for any monomorphism $f : X \rightarrow Y$ in $\text{mod } \Lambda$ with Y U -torsionless.
- (4) f^{**} is a monomorphism for any monomorphism $f : X \rightarrow Y$ in $\text{mod } \Lambda$ with X and Y U -torsionless.
- (5) $\text{grade}_U \text{Ext}_\Lambda^1(X, U) \geq 1$ for any $X \in \text{mod } \Lambda$.
- (6) $s.\text{grade}_U \text{Ext}_\Gamma^2(N, U) \geq 1$ for any $N \in \text{mod } \Gamma^{op}$.

Proof (1) \Leftrightarrow (6) follows from Corollary 16.3.5(2) and (3) \Rightarrow (4) is trivial.

(1) \Rightarrow (2) Suppose $U\text{-lim.dim}_\Lambda(E_0) \leq 1$. Then by Lemma 17.3.1, we have that $l.\text{fd}_\Gamma(\text{Hom}_\Lambda(U, E_0)) \leq 1$.

Assume that X is U -torsionless in $\text{mod } \Lambda$. Then $\text{Coker}\sigma_X \cong \text{Ext}_\Gamma^2(\text{Tr}_U X, U)$ by [13, Lemma 2.1]. By [4, Chapter VI, Proposition 5.3], we have that $\text{Hom}_\Lambda(\text{Coker}\sigma_X, E_0) \cong \text{Hom}_\Lambda(\text{Ext}_\Gamma^2(\text{Tr}_U X, U), E_0) \cong \text{Tor}_2^\Gamma(\text{Tr}_U X, \text{Hom}_\Lambda(U, E_0)) = 0$. Then $A^* = 0$ for any submodule A of $\text{Coker}\sigma_X$, which implies that any non-zero submodule of $\text{Coker}\sigma_X$ is not U -torsionless.

Let B be a submodule of X^{**} with $X \cap B = 0$. Then $B \cong B/(X \cap B) \cong (X + B)/X$ is isomorphic to a submodule of $\text{Coker}\sigma_X$. On the other hand, B is clearly U -torsionless. So $B = 0$ and hence σ_X is essential.

(2) \Rightarrow (3) Let $f : X \rightarrow Y$ be monic in $\text{mod } \Lambda$ with Y U -torsionless. Then $f^{**}\sigma_X = \sigma_Y f$ is monic. By (2), σ_X is an essential monomorphism, so f^{**} is monic.

(4) \Rightarrow (5) Let X be in $\text{mod } \Lambda$ and $0 \rightarrow Y \xrightarrow{g} P \rightarrow X \rightarrow 0$ an exact sequence in $\text{mod } \Lambda$ with P projective. It is easy to see that $[\text{Ext}_\Lambda^1(X, U)]^* \cong \text{Ker}g^{**}$. On the other hand, since ${}_\Lambda U_\Gamma$ is a faithfully balanced bimodule, P is U -reflexive and Y is U -torsionless. So g^{**} is monic by (4) and hence $\text{Ker}g^{**} = 0$ and $[\text{Ext}_\Lambda^1(X, U)]^* = 0$.

(5) \Rightarrow (1) Let M be in $\text{mod } \Gamma^{op}$ and $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ a projective resolution of M in $\text{mod } \Gamma^{op}$. Put $N = \text{Coker}(P_2 \rightarrow P_1)$. By [13, Lemma 2.1], $\text{Ext}_\Gamma^2(M, U) \cong \text{Ext}_\Gamma^1(N, U) \cong \text{Ker}\sigma_{\text{Tr}_U N}$. On the other hand, since N is U -torsionless, $\text{Ext}_\Lambda^1(\text{Tr}_U N, U) \cong \text{Ker}\sigma_N = 0$.

Let X be any finitely generated submodule of $\text{Ext}_\Gamma^2(M, U)$ and $f_1 : X \rightarrow \text{Ext}_\Gamma^2(M, U) (\cong \text{Ker}\sigma_{\text{Tr}_U N})$ the inclusion, and let f be the composition: $X \xrightarrow{f_1} \text{Ext}_\Gamma^2(M, U) \xrightarrow{g} \text{Tr}_U N$, where g is a monomorphism. Then $\sigma_{\text{Tr}_U N} f = 0$ and $f^* \sigma_{\text{Tr}_U N}^* = (\sigma_{\text{Tr}_U N} f)^* = 0$. But $\sigma_{\text{Tr}_U N}^*$ is epic by [1, Proposition 20.14], so $f^* = 0$. Hence, by applying the functor $\text{Hom}_\Lambda(_, U)$ to the exact sequence $0 \rightarrow X \xrightarrow{f} \text{Tr}_U N \rightarrow \text{Coker}f \rightarrow 0$, we have that $X^* \cong \text{Ext}_\Lambda^1(\text{Coker}f, U)$ and then $X^{**} \cong [\text{Ext}_\Lambda^1(\text{Coker}f, U)]^* = 0$ by (5), which implies that $X^* = 0$ since X^* is a direct summand of X^{**} ($= 0$). By using a similar argument to the proof of (2) \Rightarrow (3) in Lemma 17.3.2, we can prove that $l.\text{fd}_\Gamma(\text{Hom}_\Lambda(U, E_0)) \leq 1$. Therefore $U\text{-lim.dim}_\Lambda(E_0) \leq 1$ by Lemma 17.3.1. \square

By Proposition 17.4.1, we have that $E_0 \in \text{add-lim}_\Lambda U$ if and only if $E'_0 \in \text{add-lim} U_\Gamma$, that is, $U\text{-lim.dim}_\Lambda(E_0) = 0$ if and only if $U\text{-lim.dim}_\Gamma(E'_0) = 0$. However, in general, we don't have the fact that $U\text{-lim.dim}_\Lambda(E_0) \leq 1$ if and only if $U\text{-lim.dim}_\Gamma(E'_0) \leq 1$ even when ${}_\Lambda U_\Gamma = {}_\Lambda \Lambda_\Lambda$.

Example 17.5.2 We use I_0 and I'_0 to denote the injective envelope of ${}_\Lambda \Lambda$ and Λ_Λ , respectively. Consider the following example. Let K be a field and Δ the quiver:

$$1 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 2 \xrightarrow{\gamma} 3$$

(1) If $\Lambda = K\Delta/(\alpha\beta\alpha)$. Then $l\text{fd}_\Lambda(I_0) = 1$ and $r\text{fd}_\Lambda(I'_0) \geq 2$. (2) If $\Lambda = K\Delta/(\gamma\alpha, \beta\alpha)$. Then $l\text{fd}_\Lambda(I_0) = 2$ and $r\text{fd}_\Lambda(I'_0) = 1$.

Compare the following result with Theorem 17.3.12.

Theorem 17.5.3 *The following statements are equivalent.*

- (1) $\text{grade}_U \text{Ext}_\Lambda^i(M, U) \geq i$ for any $M \in \text{mod } \Lambda$ and $1 \leq i \leq k$.
- (2) $\text{Ext}_\Gamma^i(\text{Ext}_\Lambda^i(_, U), U)$ preserves monomorphisms $X \rightarrow Y$ with both X and Y torsionless in $\text{mod } \Lambda$ for any $0 \leq i \leq k - 1$.

Proof We proceed by using induction on k . The case $k = 1$ follows from Proposition 17.5.1. Now suppose $k \geq 2$.

(1) \Rightarrow (2) Let A be a torsionless module in $\text{mod } \Lambda$. Then there exists an exact sequence in $\text{mod } \Lambda$ with P projective:

$$0 \rightarrow A \rightarrow P \rightarrow B \rightarrow 0.$$

By (1), for any $1 \leq i \leq k - 1$, we have that $\text{grade}_U \text{Ext}_\Lambda^i(A, U) = \text{grade}_U \text{Ext}_\Lambda^{i+1}(B, U) \geq i + 1$, which implies that $\text{Ext}_\Gamma^i(\text{Ext}_\Lambda^i(A, U), U) = 0$. The desired conclusion follows trivially.

(2) \Rightarrow (1) By induction assumption, for any $M \in \text{mod } \Lambda$, we have that $\text{grade}_U \text{Ext}_\Lambda^i(M, U) \geq i$ for any $1 \leq i \leq k - 1$ and $\text{grade}_U \text{Ext}_\Lambda^k(M, U) \geq k - 1$. So it suffices to prove that $\text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^k(M, U), U) = 0$.

Let

$$0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$$

be an exact sequence in $\text{mod } \Lambda$ with P projective. Then by (2), we have the following exact sequence:

$$0 \rightarrow \text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^{k-1}(K, U), U) \rightarrow \text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^{k-1}(P, U), U).$$

But the last term in this sequence is always zero, so $\text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^k(M, U), U) \cong \text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^{k-1}(K, U), U) = 0$. □

Compare the following result with [21, Theorem 7.5].

Theorem 17.5.4 *The following statements are equivalent.*

- (1) $s.\text{grade}_U \text{Ext}_\Gamma^{i+1}(N, U) \geq i$ for any $N \in \text{mod } \Gamma^{op}$ and $1 \leq i \leq k$.
- (2) $\text{grade}_U \text{Ext}_\Lambda^i(M, U) \geq i$ for any $M \in \text{mod } \Lambda$ and $1 \leq i \leq k$.

Proof We proceed by using induction on k . The case $k = 1$ follows from Proposition 17.5.1. Now suppose $k \geq 2$.

(1) \Rightarrow (2) By induction assumption, for any $M \in \text{mod } \Lambda$, we have that $\text{grade}_U \text{Ext}_\Lambda^i(M, U) \geq i$ for any $1 \leq i \leq k - 1$ and $\text{grade}_U \text{Ext}_\Lambda^k(M, U) \geq k - 1$. Then $\mathcal{T}_U^i(\text{mod } \Lambda) = \Omega_U^i(\text{mod } \Lambda)$ for any $1 \leq i \leq k$ by Lemma 17.3.4.

Let

$$\dots \rightarrow P_i \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

be an exact sequence in $\text{mod } \Lambda$ with each P_i projective for any $i \geq 0$. By [9, Lemma 2], we have the following exact sequence:

$$0 \rightarrow \text{Ext}_\Lambda^k(M, U) \rightarrow \text{Tr}_U \Omega_\Lambda^{k-1}(M) \rightarrow P_{k+1}^* \rightarrow \text{Tr}_U \Omega_\Lambda^k(M) \rightarrow 0. \tag{6}$$

Notice that $\Omega_\Lambda^{k-1}(M)$ is $(k-1)$ -syzygy and $\Omega_\Lambda^k(M)$ is k -syzygy, so, by Lemma 16.3.11, $\Omega_\Lambda^{k-1}(M)$ (resp. $\Omega_\Lambda^k(M)$) is in $\Omega_U^{k-1}(\text{mod } \Lambda)$ (resp. $\Omega_U^k(\text{mod } \Lambda)$) and hence is in $\mathcal{T}_U^{k-1}(\text{mod } \Lambda)$ (resp. $\mathcal{T}_U^k(\text{mod } \Lambda)$). It follows that $\text{Ext}_\Gamma^i(\text{Tr}_U \Omega_\Lambda^{k-1}(M), U) = 0$ for any $1 \leq i \leq k-1$ and $\text{Ext}_\Gamma^i(\text{Tr}_U \Omega_\Lambda^k(M), U) = 0$ for any $1 \leq i \leq k$. In addition, $P_{k+1}^* \in \text{add } U_\Gamma$, so $\text{Ext}_\Gamma^i(P_{k+1}^*, U) = 0$ for any $i \geq 1$. Thus from the exact sequence (6) we get an embedding:

$$0 \rightarrow \text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^k(M, U), U) \rightarrow \text{Ext}_\Gamma^{k+1}(\text{Tr}_U \Omega_\Lambda^k(M), U).$$

Then, by (1), we have that $\text{grade}_U \text{Ext}_\Gamma^{k-1}(\text{Ext}_\Lambda^k(M, U), U) \geq k$. Consequently, $\text{grade}_U \text{Ext}_\Lambda^k(M, U) \geq k$ by Lemma 17.3.9.

(2) \Rightarrow (1) By induction assumption, for any $N \in \text{mod } \Gamma^{op}$, we have that $\text{s.grade}_U \text{Ext}_\Gamma^{i+1}(N, U) \geq i$ for any $1 \leq i \leq k-1$ and $\text{s.grade}_U \text{Ext}_\Gamma^{k+1}(N, U) \geq k-1$. Then $\mathcal{T}_U^i(\text{mod } \Gamma^{op}) = \Omega_U^i(\text{mod } \Gamma^{op})$ for any $1 \leq i \leq k$ by Lemma 17.3.4.

Let X be a submodule of $\text{Ext}_\Gamma^{k+1}(N, U)$. Then $\text{grade}_U X \geq k-1$. By [9, Lemma 2], there exists an exact sequence:

$$0 \rightarrow X \xrightarrow{f} \text{Tr}_U \Omega_\Gamma^k(N) \rightarrow \text{Coker } f \rightarrow 0. \tag{7}$$

Notice that $\Omega_\Gamma^k(N)$ is k -syzygy, so, by Lemma 17.3.11, it is in $\Omega_U^k(\text{mod } \Gamma^{op})$ and hence is in $\mathcal{T}_U^k(\text{mod } \Gamma^{op})$. It follows that $\text{Ext}_\Lambda^i(\text{Tr}_U \Omega_\Gamma^k(N), U) = 0$ for any $1 \leq i \leq k$. So from the exact sequence (7) we get that $\text{Ext}_\Lambda^{k-1}(X, U) \cong \text{Ext}_\Lambda^k(\text{Coker } f, U)$. By (2), $\text{grade}_U \text{Ext}_\Lambda^{k-1}(X, U) = \text{grade}_U \text{Ext}_\Lambda^k(\text{Coker } f, U) \geq k$. It follows from Lemma 17.3.9 that $\text{grade}_U X \geq k$ and $\text{s.grade}_U \text{Ext}_\Gamma^{k+1}(N, U) \geq k$. \square

Recall that a full subcategory \mathcal{X} of $\text{mod } \Lambda$ (resp. $\text{mod } \Gamma^{op}$) is said to be closed under extensions if the middle term B of any short sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is in \mathcal{X} provided that the end terms A and C are in \mathcal{X} .

The following is the main result in this section.

Theorem 17.5.5 *The following statements are equivalent.*

- (1) $\text{s.grade}_U \text{Ext}_\Gamma^{i+1}(N, U) \geq i$ for any $N \in \text{mod } \Gamma^{op}$ and $1 \leq i \leq k$.
- (2) $U\text{-lim.dim}_\Lambda(E_i) \leq i + 1$ for any $0 \leq i \leq k-1$.
- (3) $l.f.d_\Gamma(\text{Hom}_\Lambda(U, E_i)) \leq i + 1$ for any $0 \leq i \leq k-1$.
- (4) $\text{grade}_U \text{Ext}_\Lambda^i(M, U) \geq i$ for any $M \in \text{mod } \Lambda$ and $1 \leq i \leq k$.
- (5) $\text{Ext}_\Gamma^i(\text{Ext}_\Lambda^i(_, U), U)$ preserves monomorphisms $X \rightarrow Y$ with both X and Y torsionless in $\text{mod } \Lambda$ for any $0 \leq i \leq k-1$.

If one of the above equivalent conditions holds, then $\Omega_U^i(\text{mod } \Gamma^{op}) (= \mathcal{T}_U^i(\text{mod } \Gamma^{op}))$ is closed under extensions for any $1 \leq i \leq k$.

Proof By Corollary 17.3.5(2), we have that (1) \Leftrightarrow (2) \Leftrightarrow (3). It follows from Theorems 17.5.4 and 17.5.3 that (1) \Leftrightarrow (4) \Leftrightarrow (5). The last assertion follows from [10, Theorem 3.3]. \square

We use I_i (resp. I'_i) to denote the $(i + 1)$ -st term in a minimal injective resolution of ${}_{\Lambda}\Lambda$ (resp. Λ_{Λ}) for any $i \geq 0$. The following corollary generalizes [3]. In [3, Theorem 4.7], the assumption of Λ being a noetherian algebra is necessary for proving (5) \Rightarrow (3). But here the assumption of Λ being a left and right noetherian ring is enough for all of the implications.

Corollary 17.5.6 *The following statements are equivalent.*

- (1) $s.\text{grade}_{\Lambda} \text{Ext}_{\Lambda}^{i+1}(N, \Lambda) \geq i$ for any $N \in \text{mod } \Lambda^{op}$ and $1 \leq i \leq k$.
- (2) $\text{lfid}_{\Lambda}(I_i) \leq i + 1$ for any $0 \leq i \leq k - 1$.
- (3) $\text{grade}_{\Lambda} \text{Ext}_{\Lambda}^i(M, \Lambda) \geq i$ for any $M \in \text{mod } \Lambda$ and $1 \leq i \leq k$.
- (4) $\text{Ext}_{\Lambda}^i(\text{Ext}_{\Lambda}^i(-, \Lambda), \Lambda)$ preserves monomorphisms $X \rightarrow Y$ with both X and Y torsionless in $\text{mod } \Lambda$ for any $0 \leq i \leq k - 1$.
- (5) $\Omega_{\Lambda}^i(\text{mod } \Lambda^{op})$ is closed under extensions for any $1 \leq i \leq k$.
- (6) $\text{add}\Omega_{\Lambda}^i(\text{mod } \Lambda^{op})$ (the subcategory of $\text{mod } \Lambda^{op}$ whose objects are those modules which are direct summands of i -th syzygies) is closed under extensions for any $1 \leq i \leq k$.

Proof By Theorem 17.5.5, we have that (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4). The equivalence of (1), (5) and (6) follows from the dual statements of [3, Theorem 1.7]. \square

At the end of this section, we generalize the result of Wakamatsu on the symmetry of k -Gorenstein modules.

Proposition 17.5.7 *Assume that m is a non-negative integer and $U\text{-lim.dim}_{\Lambda}(E_i) \leq i + 1$ for any $0 \leq i \leq m - 1$.*

- (1) *If $U\text{-lim.dim}_{\Gamma}(\bigoplus_{i=0}^m E'_i) \leq m$, then $U\text{-lim.dim}_{\Lambda}(E_m) \leq m$; Especially, if $\text{lid}_{\Lambda}(U) \leq m$, then $U\text{-lim.dim}_{\Lambda}(E_m) \leq m$.*
- (2) *For a positive integer k , if $U\text{-lim.dim}_{\Gamma}(\bigoplus_{i=0}^m E'_i) \leq m$ and $U\text{-lim.dim}_{\Gamma}(E'_{m+j}) \leq m + j$ for any $1 \leq j \leq k - 1$, then $U\text{-lim.dim}_{\Lambda}(E_{m+j}) \leq m + j$ for any $0 \leq j \leq k - 1$.*

Proof The case $m = 0$ follows from Theorem 16.1.8. Now suppose $m \geq 1$.

(1) By Corollaries 17.3.5 and 17.3.3, it suffices to prove that if $s.\text{grade}_U \text{Ext}_{\Gamma}^{i+1}(N, U) \geq i$ for any $N \in \text{mod } \Gamma^{op}$ and $1 \leq i \leq m$ and $s.\text{grade}_U \text{Ext}_{\Lambda}^{m+1}(M, U) \geq m + 1$ for any $M \in \text{mod } \Lambda$, then $s.\text{grade}_U \text{Ext}_{\Gamma}^{m+1}(N, U) \geq m + 1$ for any $N \in \text{mod } \Gamma^{op}$.

Suppose that

$$\cdots \rightarrow Q_i \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow N \rightarrow 0 \tag{8}$$

is a projective resolution of N in $\text{mod } \Gamma^{op}$.

By Lemma 17.3.4, we have that $\mathcal{T}_U^i(\text{mod } \Gamma^{op}) = \Omega_U^i(\text{mod } \Gamma^{op})$ for any $1 \leq i \leq m + 1$. Notice that $\text{Coker}(Q_{m+1} \rightarrow Q_m)$ is m -syzygy, so, by Lemma 17.3.11, it is in $\Omega_U^m(\text{mod } \Gamma^{op})$ and hence is in $\mathcal{T}_U^m(\text{mod } \Gamma^{op})$, which implies that $\text{Ext}_{\Lambda}^i(\text{Tr}_U \Omega_{\Gamma}^m(N), U) = 0$ for any $1 \leq i \leq m$.

Let X be a submodule of $\text{Ext}_{\Gamma}^{m+1}(N, U)$. Then $\text{grade}_U X \geq m$. By [9, Lemma 2], we have an exact sequence:

$$0 \rightarrow X \xrightarrow{f} \text{Tr}_U \Omega_{\Gamma}^m(N) \rightarrow \text{Coker } f \rightarrow 0.$$

We then get an embedding $0 \rightarrow \text{Ext}_{\Lambda}^m(X, U) \rightarrow \text{Ext}_{\Lambda}^{m+1}(\text{Coker } f, U)$. By assumption, $s.\text{grade}_U \text{Ext}_{\Lambda}^{m+1}(\text{Coker } f, U) \geq m + 1$. So $\text{grade}_U \text{Ext}_{\Lambda}^m(X, U) \geq m + 1$. It follows from Lemma 17.3.9 that $\text{grade}_U X \geq m + 1$ and $s.\text{grade}_U \text{Ext}_{\Gamma}^{m+1}(N, U) \geq m + 1$.

By Lemma 17.2.4(2) and the dual statement of Lemma 17.3.1, we have that $U\text{-lim.dim}_{\Gamma}(\bigoplus_{i=0}^k E'_i) \leq \text{lid}_{\Lambda}(U)$. So the latter assertion follows from the former one.

(2) We proceed by using induction on k . The case $k = 1$ is just (1).

Now suppose $k \geq 2$. By induction assumption, we have that $U\text{-lim.dim}_\Lambda(E_i) \leq i + 1$ for any $0 \leq i \leq m - 1$ and $U\text{-lim.dim}_\Lambda(E_{m+j}) \leq m + j$ for any $0 \leq j \leq k - 2$. By Corollaries 17.3.5 and 17.3.3, for any $N \in \text{mod } \Gamma^{op}$, we have that $\text{s.grade}_U \text{Ext}_\Gamma^{i+1}(N, U) \geq i$ for any $1 \leq i \leq m$ and $\text{s.grade}_U \text{Ext}_\Gamma^{m+j}(N, U) \geq m + j$ for any $1 \leq j \leq k - 1$. By Corollary 17.3.3, it suffices to prove that $\text{s.grade}_U \text{Ext}_\Gamma^{m+k}(N, U) \geq m + k$.

Suppose that N has a projective resolution as (8). By Lemma 17.3.4, we have that $T_U^i(\text{mod } \Gamma^{op}) = \Omega_U^i(\text{mod } \Gamma^{op})$ for any $1 \leq i \leq m + k$. Notice that $\text{Coker}(Q_{m+k} \rightarrow Q_{m+k-1})$ is $(m + k - 1)$ -syzygy, so, by Lemma 17.3.11, it is in $\Omega_U^{m+k-1}(\text{mod } \Gamma^{op})$ and hence is in $T_U^{m+k-1}(\text{mod } \Gamma^{op})$, which implies that $\text{Ext}_\Lambda^i(\text{Tr}_U \Omega_\Gamma^{m+k-1}(N), U) = 0$ for any $1 \leq i \leq m + k - 1$.

By assumption, $U\text{-lim.dim}_\Gamma(\bigoplus_{i=0}^m E'_i) \leq m$ and $U\text{-lim.dim}_\Gamma(E'_{m+j}) \leq m + j$ for any $1 \leq j \leq k - 1$. Then, by Corollary 17.3.3, we have that $\text{s.grade}_U \text{Ext}_\Lambda^{m+k}(M, U) \geq m + k$ for any $M \in \text{mod } \Lambda$.

Let X be a submodule of $\text{Ext}_\Gamma^{m+k}(N, U)$. Then $\text{grade}_U X \geq m + k - 1$. By [9, Lemma 2], we have an exact sequence:

$$0 \rightarrow X \xrightarrow{f} \text{Tr}_U \Omega_\Gamma^{m+k-1}(N) \rightarrow \text{Coker } f \rightarrow 0.$$

We then get an embedding $0 \rightarrow \text{Ext}_\Lambda^{m+k-1}(X, U) \rightarrow \text{Ext}_\Lambda^{m+k}(\text{Coker } f, U)$. Since $\text{s.grade}_U \text{Ext}_\Lambda^{m+k}(\text{Coker } f, U) \geq m + k$, $\text{grade}_U \text{Ext}_\Lambda^{m+k-1}(X, U) \geq m + k$. It follows from Lemma 16.3.9 that $\text{grade}_U X \geq m + k$ and $\text{s.grade}_U \text{Ext}_\Gamma^{m+k}(N, U) \geq m + k$. \square

Putting $m = 0$, by Proposition 17.5.7(2), $U\text{-lim.dim}_\Lambda(E_i) \leq i$ for any $0 \leq i \leq k - 1$ if $U\text{-lim.dim}_\Gamma(E'_i) \leq i$ for any $0 \leq i \leq k - 1$. Combining this result with Corollary 17.3.3(2) and their dual statements, we then get the symmetry of k -Gorenstein modules (see [21, Theorem 7.5]).

Putting ${}_\Lambda U_\Gamma = {}_\Lambda \Lambda_\Lambda$, the following corollary is an immediate consequence of Proposition 17.5.7, which is a generalization of the result of Auslander on the symmetry of k -Gorenstein rings.

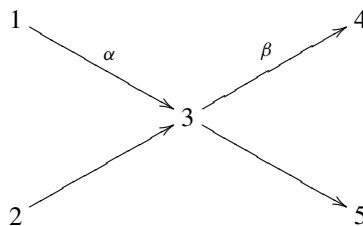
Corollary 17.5.8 *Assume that m is a non-negative integer and $\text{lfd}_\Lambda(I_i) \leq i + 1$ for any $0 \leq i \leq m - 1$.*

(1) *If $\text{rfd}_\Lambda(\bigoplus_{i=0}^m I'_i) \leq m$, then $\text{lfd}_\Lambda(I_m) \leq m$; Especially, if $\text{l.id}_\Lambda(\Lambda) \leq m$, then $\text{lfd}_\Lambda(I_m) \leq m$.*

(2) *For a positive integer k , if $\text{rfd}_\Lambda(\bigoplus_{i=0}^m I'_i) \leq m$ and $\text{rfd}_\Lambda(I'_{m+j}) \leq m + j$ for any $1 \leq j \leq k - 1$, then $\text{lfd}_\Lambda(I_{m+j}) \leq m + j$ for any $0 \leq j \leq k - 1$.*

When $m = 0$, the result in Corollary 17.5.8(2) is equivalent to the symmetry of k -Gorenstein rings (see [6, Theorem 3.7]). In the following, we give an example satisfying the conditions in Corollary 17.5.8 for the case $m = 1$ and $k = 2$ as follows.

Example 17.5.9 Let K be a field and Λ a finite dimensional K -algebra which is given by the quiver:



modulo the ideal $\beta\alpha$. Then $\text{lfd}_\Lambda(I_0) = \text{lfd}_\Lambda(I_1) = \text{rfd}_\Lambda(I'_0) = \text{rfd}_\Lambda(I'_1) = 1$, $\text{lfd}_\Lambda(I_2) = \text{rfd}_\Lambda(I'_2) = 2$ and $\text{l.id}_\Lambda(\Lambda) = \text{r.id}_\Lambda(\Lambda) = 2$.

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