

Minimal right determiners of irreducible morphisms in tree string algebras

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Let Λ be a finite dimensional string algebra over a field with the quiver Q such that the underlying graph of Q is a tree, and let $|\text{Det}(\Lambda)|$ be the number of the minimal right determiners of all irreducible morphisms between indecomposable left Λ -modules. Then we have

$$|\text{Det}(\Lambda)| = 2n - p - q - 1,$$

where n is the number of vertices in Q , $p = |\{i \mid i \text{ is a source in } Q \text{ with two neighbors}\}|$ and q is the number of vertices such that the associated vertex ideals are not zero.

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1. Introduction

In the seminal Philadelphia notes [2], Auslander introduced the notion of morphisms determined by objects, which generalized that of almost split morphisms. However, because “the basic definition may look quite technical and unattractive, at least at first sight” [12, p. 409] such that it is not easy to grasp it, this useful notion and related results in [2] gained the deserved attention until recently, see [7, 9–13].

The (minimal) right determiners of morphisms were introduced in [4]. Krause [9], Chen and Le [7] established a close relation between right determiners of morphisms

and dualizing varieties as well as the Serre duality. Let Λ be an artin algebra and $\text{mod } \Lambda$ the category of finitely generated left Λ -modules. We have the following facts:

- (1) Obviously, an indecomposable right determiner is the minimal one if it exists.
- (2) The minimal right determiner of a morphism in $\text{mod } \Lambda$ is a direct summand of any of its right determiners [4, Proposition XI.2.4].
- (3) A morphism in $\text{mod } \Lambda$ is right C -determined, then f is right $(C \oplus B)$ -determined for any $B \in \text{mod } \Lambda$ [2, Proposition 2.6].

Thus, among all right determiners of a morphism, the minimal one is the most essential one. Moreover, notice that any morphism in $\text{mod } \Lambda$ admits a right determiner [2, Theorem I.3.17] and [11, Theorem 1], so one expects concrete computations leading to a better understanding of the above notions and a classification of the minimal right determiners of a certain class of (irreducible) morphisms. It is very difficult in general and few related results have been known.

Ringel corrected in [11, Theorem 1] a formula in [3, Theorem 2.6] for calculating a right determiner of a morphism in $\text{mod } \Lambda$; and then he reproved in [12, Theorem 3.4] a formula originally in [4] for calculating the minimal right determiner of a morphism in $\text{mod } \Lambda$. Based on these formulas, we determined in [13, Theorems 3.13 and 3.15] the minimal right determiners of all irreducible morphisms between indecomposable modules over a finite dimensional algebra of type \mathbb{A}_n . We use $\text{Det}(\Lambda)$ to denote the set of the minimal right determiners of all irreducible morphisms between indecomposable modules in $\text{mod } \Lambda$, and use $|\text{Det}(\Lambda)|$ to denote the cardinality of $\text{Det}(\Lambda)$.

In this paper, we continue the previous work mentioned above. For a finite dimensional string algebra Λ over a field K with the quiver Q such that the underlying graph of Q is a tree, we will determine the set $\text{Det}(\Lambda)$ completely. The key point is to introduce the so-called vertex ideals. Roughly speaking, the definition of a vertex ideal depends on the restriction of the admissible ideal of KQ to certain full subquiver around that vertex, see Definition 3.3 for details. In particular, if Λ is of type \mathbb{A}_n , then vertex ideals are exactly sink ideals, which played a crucial role in [13]. Our main result is the following.

Theorem 1.1. *Let Λ be a finite dimensional string algebra over a field with the quiver Q such that the underlying graph of Q is a tree. Then we have*

$$|\text{Det}(\Lambda)| = 2n - p - q - 1,$$

where n is the number of vertices in Q , $p = |\{i \mid i \text{ is a source in } Q \text{ with two neighbors}\}|$ and q is the number of vertices such that the associated vertex ideals are not zero.

We prove it in Sec. 3. Note that the proof of Theorem 1.1 is constructive, from which we can determine the set $\text{Det}(\Lambda)$. In fact, it provides explicit computations

and a complete classification of minimal right determiners of all irreducible morphisms between indecomposable modules in our setting. In Sec. 4, we apply Theorem 1.1 to the case of algebras whose quiver is Dynkin type; and in particular, we obtain a unified version of [13, Theorems 3.13 and 3.15]. Finally, we give in Sec. 5 an example of non-Dynkin type to illustrate this theorem.

2. Preliminaries

Throughout this paper, Λ is a finite dimensional algebra over a field K with the quiver Q , $\text{mod } \Lambda$ is the category of finitely generated left Λ -modules and τ is the Auslander–Reiten translation. For an arrow α in Q , $s(\alpha)$ and $e(\alpha)$ are the starting and end points of α , respectively. We use $P(i)$ and $S(i)$ to denote the indecomposable projective and simple modules corresponding to the vertex i , respectively. For a module M in $\text{mod } \Lambda$, we use $\text{Soc}(M)$ and $\text{add}_\Lambda M$ to denote the socle of M and the full subcategory of $\text{mod } \Lambda$ consisting of direct summands of finite direct sums of copies of M , respectively. For a set S , we use $|S|$ to denote the cardinality of S .

The original definition of morphisms determined by objects in [2] is based on the notion of subfunctors determined by objects. However, in the relevant papers, one prefers the following definition since it is easier to understand.

Definition 2.1 ([11, 12]). For a module $C \in \text{mod } \Lambda$, a morphism $f \in \text{Hom}_\Lambda(X, Y)$ is said to be **right determined** by C (simply **C -right determined**) if the following condition is satisfied: given for any $f' \in \text{Hom}_\Lambda(X', Y)$ such that $f'\phi$ factors through f for all $\phi \in \text{Hom}_\Lambda(C, X')$, then f' factors through f ; that is, in the following diagram, if there exists $\phi' \in \text{Hom}_\Lambda(C, X)$ such that $f'\phi = f\phi'$, then there exists $h \in \text{Hom}_\Lambda(X', X)$ such that $f' = fh$.

$$\begin{array}{ccccc}
 C & \xrightarrow{\phi} & X' & \xrightarrow{f'} & Y \\
 \parallel & & \downarrow h & & \parallel \\
 C & \xrightarrow{\phi'} & X & \xrightarrow{f} & Y
 \end{array}$$

In this case, C is called a **right determiner** of f .

Definition 2.2 ([11, p. 984]). Given a morphism $f \in \text{Hom}_\Lambda(B, C)$ with $B = B_1 \oplus B_2$ such that $B_1 \subseteq \text{Ker } f$ and $f|_{B_2}$ is right minimal, then we call $\text{Ker } f|_{B_2}$ the **intrinsic kernel** of f .

Definition 2.3 ([12, p. 418]). An indecomposable projective module $P \in \text{mod } \Lambda$ is said to **almost factor through** $f \in \text{Hom}_\Lambda(M, N)$ provided that there exists a commutative diagram of the following form

$$\begin{array}{ccc}
 \text{rad } P & \xrightarrow{i} & P \\
 \downarrow & & \downarrow h \\
 M & \xrightarrow{f} & N,
 \end{array}$$

where i is the inclusion map and $\text{rad } P$ is the radical of P , such that $\text{Im } h$ is not contained in $\text{Im } f$.

The following is the determiner formula.

Theorem 2.4. *Let f be a morphism in $\text{mod } \Lambda$. Let $C(f)$ be the direct sum of the indecomposable modules of the form $\tau^{-1}K$, where K is an indecomposable direct summand of the intrinsic kernel of f and of the indecomposable projective modules which almost factor through f , one from each isomorphism class. Then we have*

- (1) [12, Theorem 3.4; 11, Theorem 2] and [4, Corollary XI.2.3] f is right C -determined if and only if $C(f) \in \text{add}_\Lambda C$.
- (2) [13, Theorem 2.4(2)] If f is irreducible, then $C(f) = \tau^{-1} \text{Ker } f \oplus (\oplus P_i)$, where all P_i are pairwise nonisomorphic indecomposable projective modules almost factoring through f .

The first assertion in this theorem suggests to call $C(f)$ the **minimal right determiner** of f [11, 12]. We use $\text{Det}(\Lambda)$ to denote the set of (representative of the isomorphism classes of) the minimal right determiners of all irreducible morphisms between indecomposable modules in $\text{mod } \Lambda$.

We use $Q_0 := \{1, \dots, n\}$ and Q_1 to denote the set of vertices and the set of arrows in Q , respectively.

Definition 2.5. [5, p. 534] and [6, p. 157] Let $\Lambda = KQ/I$ with I an ideal of KQ . Then Λ is called a **special biserial algebra** provided the following conditions are satisfied.

- (1) For each $i \in Q_0$, we have $|\{\alpha \in Q_1 \mid s(\alpha) = i\}| \leq 2$ and $|\{\alpha \in Q_1 \mid e(\alpha) = i\}| \leq 2$.
- (2) For $\alpha, \beta, \gamma \in Q_1$ with $e(\alpha) = e(\beta) = s(\gamma)$ and $\alpha \neq \beta$, we have $\gamma\alpha \in I$ or $\gamma\beta \in I$.
- (3) For $\alpha, \beta, \gamma \in Q_1$ with $s(\alpha) = s(\beta) = e(\gamma)$ and $\alpha \neq \beta$, we have $\alpha\gamma \in I$ or $\beta\gamma \in I$.

A special biserial algebra is called a **string algebra** if the following condition is satisfied.

- (4) The ideal I can be generated by zero relations.

3. Tree String Algebras

In this section, Λ is a string algebra with the quiver Q such that the underlying graph of Q is a tree. Then either $\Lambda = KQ$ or $\Lambda = KQ/I$ with I an admissible ideal of KQ . By the definition of string algebras, the former case occurs only if Λ is of type \mathbb{A}_n . In either case, Λ is of finite representation type [6, p. 161, Theorem] or [8, Theorem 1.2(2)], and we may assume that there are n vertices and $n - 1$ arrows in Q . All morphisms considered are irreducible morphisms between indecomposable modules in $\text{mod } \Lambda$.

By [6, p. 147], there are only two types of almost split sequences in mod Λ , that is, the middle term in an almost split sequence is indecomposable or is a direct sum of two indecomposable modules:

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \tag{3.1}$$

and

$$0 \rightarrow L \rightarrow M_1 \oplus M_2 \rightarrow N \rightarrow 0. \tag{3.2}$$

Proposition 3.1 ([6, p. 174, Corollary]). (1) *The only almost split sequences in mod Λ of type (3.1) are those of the form*

$$0 \rightarrow U(\beta) \rightarrow N(\beta) \rightarrow V(\beta) \rightarrow 0$$

with β an arrow in Q .

(2) *The number of almost split sequences of type (3.1) in mod Λ is $n - 1$.*

In the following, we describe the modules in Eq. (3.1) briefly. Let β be an arrow of Q . We denote by β^{-1} a **formal inverse** of β with $s(\beta^{-1}) = e(\beta)$ and $e(\beta^{-1}) = s(\beta)$, and write $(\beta^{-1})^{-1} = \beta$. We form ‘paths’ $c_1 \cdots c_n$ of length $n \geq 1$, where all the c_i are of the form β or β^{-1} and $s(c_i) = e(c_{i+1})$. Define $(c_1 \cdots c_n)^{-1} = c_n^{-1} \cdots c_1^{-1}$ and $s(c_1 \cdots c_n) = s(c_n)$, $e(c_1 \cdots c_n) = e(c_1)$. A path $c_1 \cdots c_n$ of length $n \geq 1$ is called a **string** if $c_{i+1} \neq c_i^{-1}$ for any $1 \leq i \leq n - 1$, and neither subpath $c_i c_{i+1} \cdots c_{i+t}$ nor its inverse belong to the ideal I . Also, the two strings of length 0 is defined just as the trivial path ϵ_i at each vertex i and its inverse.

Let \mathcal{S} be the set of all strings. We say that a string ω **starts (respectively ends) on a peak** if there exists no arrow α such that $\omega\alpha \in \mathcal{S}$ (respectively $\alpha^{-1}\omega \in \mathcal{S}$); similarly, a string **starts (respectively ends) in a deep** if there exists no arrow β such that $\omega\beta^{-1} \in \mathcal{S}$ (respectively $\beta\omega \in \mathcal{S}$). A string $\omega = \alpha_1 \cdots \alpha_n$ is called **direct** if all α_i are arrows, and called **inverse** if its inverse is direct.

For every arrow α in Q , let $N_\alpha = U_\alpha \alpha V_\alpha$ be the unique string with U_α and V_α both inverse and N_α starts in a deep and ends on a peak. By [5, Remarks 3.2(1)], there exists an almost split sequence with an indecomposable middle term:

$$0 \longrightarrow M(U_\alpha) \longrightarrow M(N_\alpha) \longrightarrow M(V_\alpha) \longrightarrow 0$$

for every arrow α , and all almost split sequences of this type are constructed in this way. Here, $M(\omega)$ denotes the indecomposable module corresponding to the string ω . Note that $M(\omega)$ and $M(\omega^{-1})$ are always isomorphic (c.f. [5, 6]).

We give the following useful remark.

Remark 3.2. In [13], the algebra Λ is assumed to be of type \mathbb{A}_n . We point out that all results from 3.1 to 3.8 in [13] hold true in the setting of this paper even without changing the proofs there. To avoid repeating, we will not list these results in details here, but cite them directly when needed.

Note that an irreducible morphism is either a proper monomorphism or a proper epimorphism. By [13, Corollary 3.7], we have

$$\begin{aligned} & \{C(f) \mid f \text{ is an epic irreducible morphism in mod } \Lambda\} \\ &= \{\text{the last terms in almost split sequences of type (3.1)} \\ & \qquad \qquad \qquad \text{as in Proposition 3.1}\}, \end{aligned}$$

and its cardinality is $n - 1$. So, in the following, we only need to determine the minimal right determiners of all irreducible monomorphisms.

For $\alpha \in Q_1$, recall from [1, p. 43] that $s(\alpha)$ and $e(\alpha)$ are called the **neighbors** of $e(\alpha)$ and $s(\alpha)$, respectively. By the definition of string algebras, we can give a complete classification of the vertices in Q as follows.

(v1) The vertex i_1 with a unique neighbor:

$$i_1 \longrightarrow \cdots \tag{v1.1}$$

and

$$i_1 \longleftarrow \cdots \tag{v1.2}$$

(v2) The vertex i_2 with two neighbors:

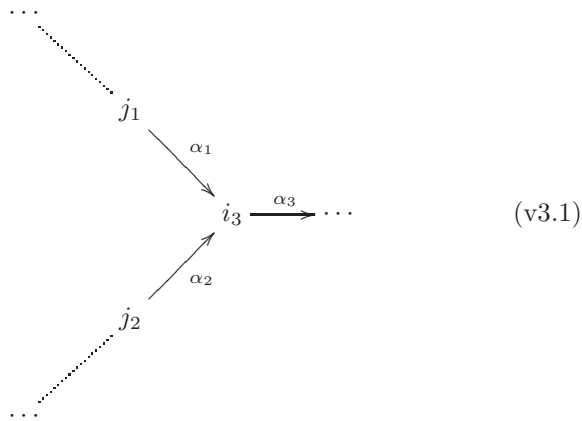
$$\cdots \longleftarrow i_2 \longrightarrow \cdots, \tag{v2.1}$$

$$\cdots \longrightarrow i_2 \longleftarrow \cdots \tag{v2.2}$$

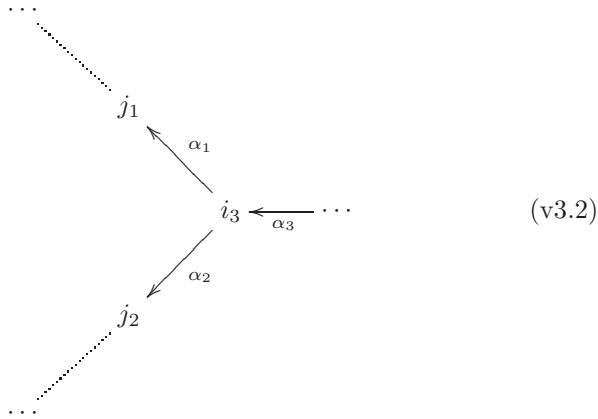
and

$$\cdots \longrightarrow i_2 \longrightarrow \cdots \tag{v2.3}$$

(v3) The vertex i_3 with three neighbors:

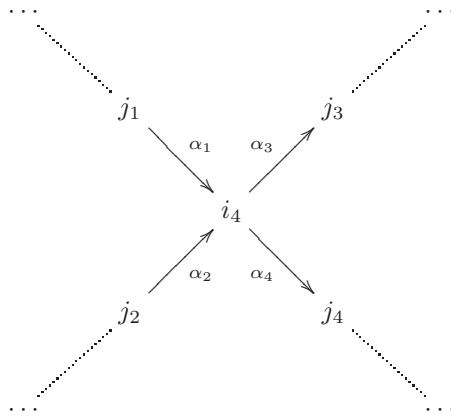


such that at least one in $\{\alpha_3\alpha_1, \alpha_3\alpha_2\}$ is in I ; and



such that at least one in $\{\alpha_1\alpha_3, \alpha_2\alpha_3\}$ is in I .

(v4) The vertex i_4 with four neighbors:



such that at least one of the two sets $\{\alpha_3\alpha_1, \alpha_4\alpha_2\}$ and $\{\alpha_4\alpha_1, \alpha_3\alpha_2\}$ is in I .

For convenience sake, we denote the subquivers in (v3) with 4 vertices including the vertex i_3 and its 3 neighbors by X_{i_3} , and denote the subquivers in (v4) with 5 vertices including the vertex i_4 and its 4 neighbors by X_{i_4} . A full subquiver of Q between two vertices i and j is denoted by $\langle i, j \rangle$. For a subquiver Q' of Q , we write $I|_{Q'} := I \cap KQ'$. The following definition is crucial in the sequel.

Definition 3.3. For the vertex i in Q , we define the **vertex ideal** J_i of Λ according to the above classification of vertices as follows:

(1) For the sink i_1 of type (v1.2), define

$$J_{i_1} = \begin{cases} 0, & \text{if there exists } j \in Q_0 \text{ such that } |\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2, \\ & \langle j, i_1 \rangle \text{ is linear and } I|_{\langle j, i_1 \rangle} = 0; \\ \Lambda, & \text{if } \Lambda \text{ is a path algebra with a unique sink } i_1; \\ I, & \text{otherwise.} \end{cases}$$

(2) For the sink i_2 of type (v2.2), define

$$J_{i_2} = \begin{cases} 0, & \text{if there exists } j \in Q_0 \text{ such that } |\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2, \\ & \langle j, i_2 \rangle \text{ is linear and } I|_{\langle j, i_2 \rangle} = 0; \\ \Lambda, & \text{if } \Lambda \text{ is a path algebra with a unique sink } i_2; \\ I, & \text{otherwise.} \end{cases}$$

(3) For the vertex i_3 of type (v3), define

$$J_{i_3} = \begin{cases} 0, & \begin{cases} \text{(a) if } i_3 \text{ is of type (v3.1); or} \\ \text{(b) if } i_3 \text{ is of type (v3.2) and there exists } j \in Q_0 \text{ such that} \\ \quad |\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2, \langle j, i_3 \rangle \text{ is linear, } I|_{\langle j, i_3 \rangle} = 0, \\ \quad I|_{\langle j, j_1 \rangle} \neq 0 \text{ and } I|_{\langle j, j_2 \rangle} \neq 0; \end{cases} \\ I|_{X_{i_3}}, & \text{otherwise.} \end{cases}$$

(4) For the vertex i_4 of type (v4), define

$$J_{i_4} = \begin{cases} 0, & \text{if there exists } j \in Q_0 \text{ such that } |\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2, \\ & \langle j, i_4 \rangle \text{ is linear, } I|_{\langle j, i_4 \rangle} = 0, I|_{\langle j, j_3 \rangle} \neq 0 \text{ and } I|_{\langle j, j_4 \rangle} \neq 0; \\ I|_{X_{i_4}}, & \text{otherwise.} \end{cases}$$

By [13, Corollary 3.2], we have that the minimal right determiner of any irreducible monomorphism is indecomposable projective. The following lemma gives some criteria for determining when an indecomposable projective module is the minimal right determiner of an irreducible monomorphism.

Lemma 3.4. *For an irreducible monomorphism f in $\text{mod } \Lambda$, the following statements are equivalent.*

- (1) $P(i) = C(f)$.
- (2) *There exists an irreducible monomorphism $f_1 : X \rightarrow P(j)$ with X indecomposable such that $C(f_1) = P(i) = C(f)$.*
- (3) $P(i)$ almost factors through f .
- (4) $S(i) = \text{Soc}(\text{Coker } f)$.

If one of the above equivalent conditions is satisfied and j is as in (2), then the subquiver $\langle j, i \rangle$ is linear and $I|_{\langle j, i \rangle} = 0$.

Proof. By [13, Theorem 3.5(1)], we have (1) \Leftrightarrow (2). By Theorem 2.4(2) and [13, Remark 3.3], we have (1) \Leftrightarrow (3). By [13, Corollary 3.4(1)], we have (1) \Leftrightarrow (4).

If one of the above equivalent conditions is satisfied and j is as in (2), then $P(i)$ almost factors through f_1 and $\text{Hom}_\Lambda(P(i), P(j)) \neq 0$, which indicates that there exists a nonzero path from j to i in Q , that is, $\langle j, i \rangle$ is linear and $I|_{\langle j, i \rangle} = 0$. \square

We need some further preparation.

Lemma 3.5. For a vertex $i \in Q_0$, we have

- (1) $|\{\alpha \in Q_1 \mid s(\alpha) = i\}| = 1$ if and only if $\text{rad } P(i)$ is indecomposable. In this case, for any irreducible monomorphism $X \rightarrow P(i)$ with X indecomposable, we have $X \cong \text{rad } P(i)$.
- (2) $|\{\alpha \in Q_1 \mid s(\alpha) = i\}| = 2$ if and only if $\text{rad } P(i) = M_i \oplus N_i$ with M_i and N_i indecomposable. In this case, for any irreducible monomorphism $X \rightarrow P(i)$ with X indecomposable, there exists an indecomposable module $Y \in \text{mod } \Lambda$, such that $\text{rad } P(i) \cong X \oplus Y$.

Proof. The first assertions in (1) and (2) are well known. If $X \rightarrow P(i)$ is an irreducible monomorphism, then X is isomorphic to a submodule of the unique maximal submodule $\text{rad } P(i)$ of $P(i)$, and hence isomorphic to a direct summand of $\text{rad } P(i)$ by [4, Lemma V.5.1(b)]. \square

The following three lemmas are useful.

Lemma 3.6. Let $i \in Q_0$ with $|\{\alpha \in Q_1 \mid s(\alpha) = i\}| = 1$. Then $P(i) = C(\text{rad } P(i) \hookrightarrow P(i))$.

Proof. It follows from Lemmas 3.5(1) and 3.4. \square

Lemma 3.7. Let $i \in Q_0$ with $|\{\alpha \in Q_1 \mid s(\alpha) = i\}| \neq 1$ and $P(i) \in \text{Det}(\Lambda)$.

- (1) If j is as in Lemma 3.4(2), then $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2$.
- (2) If the vertex i is a sink i_1 of type (v1.2) (respectively a sink i_2 of type (v 2.2)), then $J_{i_1} = 0$ (respectively $J_{i_2} = 0$).

Proof. (1) Let $i \in Q_0$ with $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| \neq 1$ and $P(i) \in \text{Det}(\Lambda)$. If j is as in Lemma 3.4(2), then the subquiver $\langle j, i \rangle$ is linear and $I|_{\langle j, i \rangle} = 0$ by Lemma 3.4. If $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 0$ (that is, j is a sink), then $\text{rad } P(j) = 0$, and hence $f_1 = 0$ by Lemma 3.5(1), a contradiction. If $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 1$, then $C(f_1) = P(j)$ by Lemmas 3.5(1) and 3.6. It is clear that $i \neq j$, so we have $C(f_1) \neq P(i)$, also a contradiction. Consequently we conclude that $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2$.

(2) If the vertex i is a sink i_1 of type (v1.2) (respectively a sink i_2 of type (v2.2)), then $J_{i_1} = 0$ (respectively $J_{i_2} = 0$) by (1) and the definition of vertex ideals. \square

Lemma 3.8. *Let $i \in Q_0$ with $|\{\alpha \in Q_1 \mid s(\alpha) = i\}| = 2$. Then $P(i) \neq C(f)$ for any irreducible monomorphism $f : X \rightarrow P(i)$.*

Proof. Let $i \in Q_0$ with $|\{\alpha \in Q_1 \mid s(\alpha) = i\}| = 2$. Suppose $P(i) = C(f)$ for some irreducible monomorphism $f : X \rightarrow P(i)$. It is clear that f can be assumed to be an inclusion. By Lemma 3.5(2), we have that $\text{rad } P(i) = M_i \oplus N_i$ with M_i and N_i indecomposable and that either $X = M_i$ or $X = N_i$. Consider the following diagram:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & 0 & \xleftarrow{K} & \overset{i}{K} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & K & \xleftarrow{1_K} & K & \longrightarrow & 0 & \longrightarrow & \cdots
 \end{array}$$

or

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & 0 & \xleftarrow{K} & \overset{i}{K} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & 0 & \xleftarrow{K} & K & \xrightarrow{1_K} & K & \longrightarrow & \cdots
 \end{array}$$

In either diagram, the above is part of the representation of $S(i)$ around i and the below is part of the representation of $\text{Coker } f$ around i . Notice that neither the left square in the first diagram nor the right square in the second diagram is commutative, so $S(i)$ is not a submodule of $\text{Coker } f$. It implies $S(i) \neq \text{Soc}(\text{Coker } f)$, which contradicts Lemma 3.4. The assertion follows. \square

The following is a key step toward proving the main result.

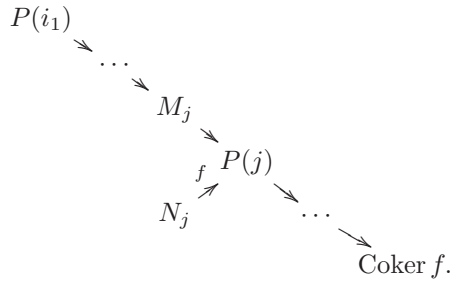
Theorem 3.9. *For a vertex i_k in Q , $P(i_k) \in \text{Det}(\Lambda)$ if and only if i_k is one of the following types.*

- (1) (1.1) a source i_1 of type (v1.1);
 - (1.2) a sink i_1 of type (v1.2) and $J_{i_1} = 0$.
- (2) (2.1) a sink i_2 of type (v2.2) and $J_{i_2} = 0$;
 - (2.2) i_2 of type (v2.3).
- (3) (3.1) i_3 of type (v3.1);
 - (3.2) i_3 of type (v3.2) and $J_{i_3} = 0$.
- (4) i_4 of type (v4) and $J_{i_4} = 0$.

Proof. (1) If i_1 is a source of type (v1.1), then $P(i_1) = C(\text{rad } P(i_1) \hookrightarrow P(i_1))$ by Lemma 3.6.

Let i_1 be a sink of type (v1.2). If $J_{i_1} = 0$, then there exists $j \in Q_0$ such that $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2$, $\langle j, i_1 \rangle$ is linear and $I|_{\langle j, i_1 \rangle} = 0$. By Lemma 3.5(2), we have

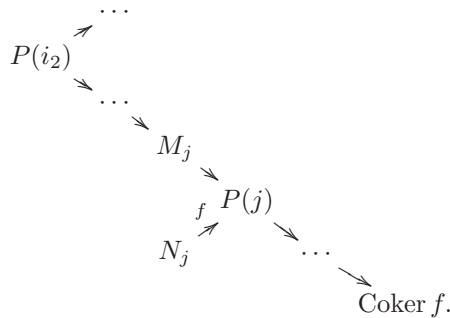
$\text{rad } P(j) = M_j \oplus N_j$ with M_j and N_j indecomposable. So there exists a subquiver of the Auslander–Reiten quiver of $\text{mod } \Lambda$ as follows:



It is straightforward to calculate that $\text{Soc}(\text{Coker } f) = S(i_1)$. So $P(i_1) = C(f)$ by Lemma 3.4. Conversely, if $P(i_1) \in \text{Det}(\Lambda)$, then $J_{i_1} = 0$ by Lemma 3.7(2).

(2) Let i_2 be a source of type (v2.1). If $P(i_2) \in \text{Det}(\Lambda)$, then by Lemmas 3.4 and 3.7(1), there exists an irreducible monomorphism $f_1 : X \rightarrow P(j)$ with X indecomposable such that $P(i_2) = C(f_1)$, $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2$, the subquiver $\langle j, i_2 \rangle$ is linear and $I|_{\langle j, i_2 \rangle} = 0$, and hence $j = i_2$. It contradicts Lemma 3.8. Thus, we have $P(i_2) \notin \text{Det}(\Lambda)$.

Let i_2 be a sink of type (v2.2). If $J_{i_2} = 0$, then there exists $j \in Q_0$ such that $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2$, $\langle j, i_2 \rangle$ is linear and $I|_{\langle j, i_2 \rangle} = 0$. By Lemma 3.5(2), we have $\text{rad } P(j) = M_j \oplus N_j$ with M_j and N_j indecomposable. So there exists a subquiver of the Auslander–Reiten quiver of $\text{mod } \Lambda$ as follows:



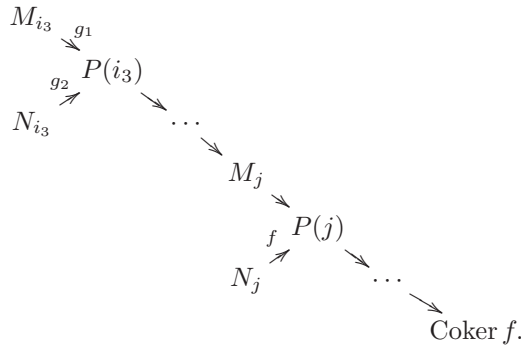
It is straightforward to calculate that $\text{Soc}(\text{Coker } f) = S(i_2)$. So $P(i_2) = C(f)$ by Lemma 3.4. Conversely, if $P(i_2) \in \text{Det}(\Lambda)$, then $J_{i_2} = 0$ by Lemma 3.7(2).

If i_2 is of type (v2.3), then $P(i_2) = C(\text{rad } P(i_2) \hookrightarrow P(i_2))$ by Lemma 3.6.

(3) If i_3 is of type (v3.1), then $P(i_3) = C(\text{rad } P(i_3) \hookrightarrow P(i_3))$ by Lemma 3.6 again.

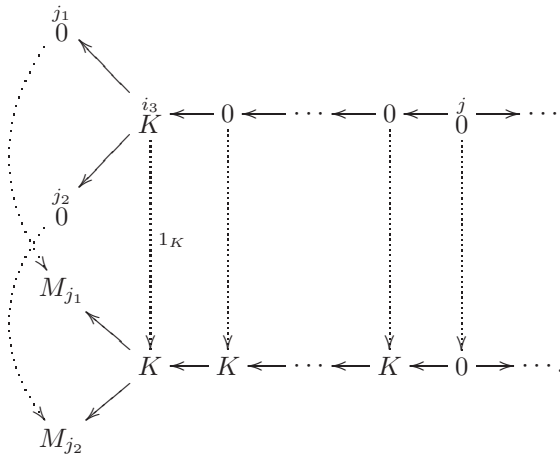
Let i_3 be of type (v3.2). If $J_{i_3} = 0$, then there exists $j \in Q_0$ such that $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2$, $\langle j, i_3 \rangle$ is linear, $I|_{\langle j, i_3 \rangle} = 0$, $I|_{\langle j, j_3 \rangle} \neq 0$ and $I|_{\langle j, k_3 \rangle} \neq 0$.

By Lemma 3.5(2), we have $\text{rad} P(j) = M_j \oplus N_j$ with M_j and N_j indecomposable. So there exists a subquiver of the Auslander–Reiten quiver of $\text{mod } \Lambda$ as follows.



It is straightforward to calculate that $\text{Soc}(\text{Coker } f) = S(i_3)$. So $P(i_3) = C(f)$ by Lemma 3.4.

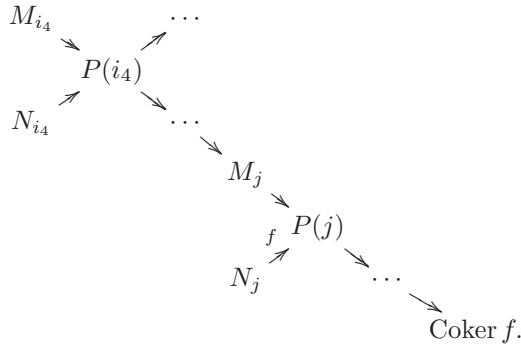
Conversely, if $P(i_3) \in \text{Det}(\Lambda)$, then by Lemma 3.4, there exists an irreducible monomorphism $f_1 : X \rightarrow P(j)$ with X indecomposable such that $C(f_1) = P(i_3)$, the subquiver $\langle j, i_3 \rangle$ is linear and $I|_{\langle j, i_3 \rangle} = 0$. By Lemmas 3.8 and 3.7(1), we have $j \neq i_3$ and $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2$. Parts of the representations of $S(i_3)$ and $\text{Coker } f$ around i_3 are shown as below.



Because $S(i_3) = \text{Soc}(\text{Coker } f)$ by Lemma 3.4, the above diagram is commutative. It implies $M_{j_1} = 0 = M_{j_2}$. So $I|_{\langle j, j_1 \rangle} \neq 0$ and $I|_{\langle j, j_2 \rangle} \neq 0$. Thus, we have $J_{i_3} = 0$ by the definition of vertex ideals.

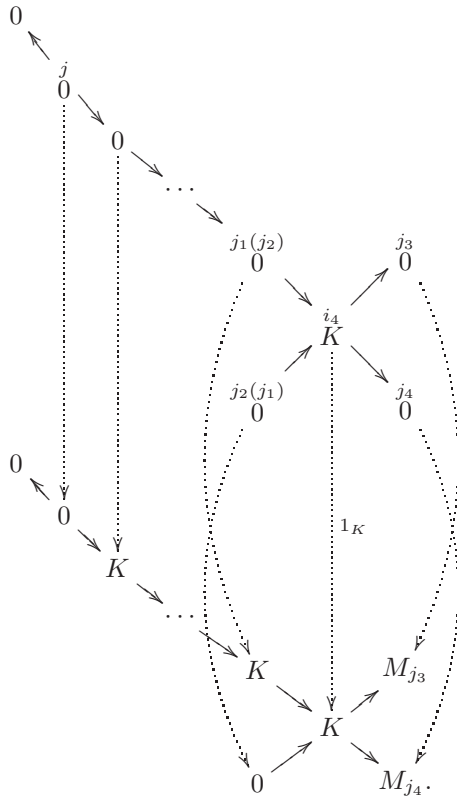
(4) Let i_4 be of type (v4). If $J_{i_4} = 0$, then there exists $j \in Q_0$ such that $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2$, $\langle j, i_4 \rangle$ is linear, $I|_{\langle j, i_4 \rangle} = 0$, $I|_{\langle j, j_3 \rangle} \neq 0$ and $I|_{\langle j, j_4 \rangle} \neq 0$. By Lemma 3.5(2), we have $\text{rad} P(j) = M_j \oplus N_j$ with M_j and N_j indecomposable.

So there exists a subquiver of the Auslander–Reiten quiver of mod Λ as follows.



It is straightforward to calculate that $\text{Soc}(\text{Coker } f) = S(i_4)$. So $P(i_4) = C(f)$ by Lemma 3.4.

Conversely, if $P(i_4) \in \text{Det}(\Lambda)$, then by Lemma 3.4, there exists an irreducible monomorphism $f_1 : X \rightarrow P(j)$ with X indecomposable such that $C(f_1) = P(i_4)$, the subquiver $\langle j, i_4 \rangle$ is linear and $I|_{\langle j, i_4 \rangle} = 0$. By Lemmas 3.8 and 3.7(1), we have $j \neq i_4$ and $|\{\alpha \in Q_1 \mid s(\alpha) = j\}| = 2$. Parts of the representations of $S(i_4)$ and $\text{Coker } f$ around i_4 are shown as below.



Because $S(i_4) = \text{Soc}(\text{Coker } f)$ by Lemma 3.4, the above diagram is commutative. It implies $M_{j_3} = 0 = M_{j_4}$. So $I|_{\langle j, j_3 \rangle} \neq 0$ and $I|_{\langle j, j_4 \rangle} \neq 0$. Thus, we have $J_{i_4} = 0$ by the definition of vertex ideals. □

We are now in a position to give the main result in this paper.

Theorem 3.10. *Set*

$$p := |\{i \mid i \text{ is a source of type (v2.1)}\}|,$$

$$q := |\{J_{i_j} \neq 0 \mid 1 \leq j \leq 4\}|.$$

Then we have

$$|\text{Det}(\Lambda)| = 2n - p - q - 1.$$

Proof. By [13, Corollary 3.7], we have that the number of the (nonprojective) minimal right determiners of all irreducible epimorphisms is $n - 1$. By Theorem 3.9, we have that the number of the (projective) minimal right determiners of all irreducible monomorphisms is $n - p - q$. So we have

$$|\text{Det}(\Lambda)| = (n - 1) + (n - p - q) = 2n - p - q - 1. \quad \square$$

The following two results show that the distribution of the projective minimal right determiners can determine the orientation of a quiver in some cases. The first one is a generalization of [13, Corollary 3.12].

Proposition 3.11. *Assume that there are no vertices of type (v4) in Q and $j \in Q_0$ is a sink of type (v1.2). Then the following statements are equivalent.*

- (1) *The projective minimal right determiners are $\{P(i) \mid 1 \leq i \leq n \text{ but } i \neq j\}$.*
- (2) *j is the unique sink in Q .*

Proof. (2) \Rightarrow (1) Assume that j is the unique sink in Q . Let $i \in Q_0$ with $i \neq j$. Then i is one of the following types: (v1.1), (v2.3), (v3.1). Now the assertion follows from Theorem 3.9.

(1) \Rightarrow (2) Assume that the projective minimal right determiners are $\{P(i) \mid 1 \leq i \leq n \text{ but } i \neq j\}$. Let $i \in Q_0$ with $i \neq j$. Because $P(i) \in \text{Det}(\Lambda)$ by (1), we have that i is not of type (v2.1) by Theorem 3.9. So, to show that i is not a sink, it suffices to show that i is not of any one of the following types: (v1.2), (v2.2), (v3.2). Assume that i is of one of these three types. It follows from Lemmas 3.4 and 3.8 that there exists $k_1 \in Q_0$ with $k_1 \neq i$ such that $|\{\alpha \in Q_1 \mid s(\alpha) = k_1\}| = 2$ and $\langle k_1, i \rangle$ is linear. It is clear that $k_1 \neq j$. So $P(k_1) \in \text{Det}(\Lambda)$ and k_1 is of type either (v2.1) or (v3.2).

If k_1 is of type (v2.1), then by Theorem 3.9, we have $P(k_1) \notin \text{Det}(\Lambda)$, a contradiction. If k_1 is of type (v3.2), then by Lemmas 3.4 and 3.8 again, there exists $k_2 \in Q_0$ with $k_2 \neq k_1$ such that $|\{\alpha \in Q_1 \mid s(\alpha) = k_2\}| = 2$ and $\langle k_1, k_2 \rangle$ is linear. It

is clear that $k_2 \neq j$. So $P(k_2) \in \text{Det}(\Lambda)$ and k_2 is of type either (v2.1) or (v3.2). By the same reason as above, we have that k_2 is of type (v3.2) but not of type (v2.1). Note that the quiver Q is acyclic. So, continuing this process, we have that there are infinitely many vertices of type (v3.2). It contradicts the fact that Q is finite.

Consequently, we conclude that j is the unique sink in Q . □

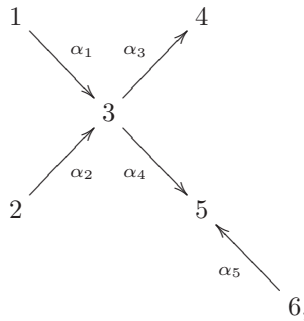
Similarly, we have the following.

Proposition 3.12. *Assume that there are no vertices of type (v4) in Q and $j \in Q_0$ is a sink of type (v2.2). Then the following statements are equivalent.*

- (1) *The projective minimal right determiners are $\{P(i) \mid 1 \leq i \leq n \text{ but } i \neq j\}$.*
- (2) *j is the unique sink in Q .*

The following example illustrates that the assumption “there are no vertices of type (v4) in Q ” is necessary for Propositions 3.11 and 3.12.

Example 3.13. Let Q be the quiver

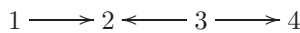


and let $\Lambda = KQ/I$ such that at least one of the two sets $\{\alpha_3\alpha_1, \alpha_4\alpha_2\}$ and $\{\alpha_4\alpha_1, \alpha_3\alpha_2\}$ is in the admissible ideal I of KQ (that is, Λ is a string algebra). Then the projective minimal right determiners in $\text{mod } \Lambda$ are $\{P(1), P(2), P(4), P(5), P(6)\}$ by Theorem 3.9. But the vertex 4 is the unique sink of type (v1.2) and the vertex 5 is the unique sink of type (v2.2) in Q .

The following example illustrates that the assumption “ $j \in Q_0$ is a sink of type (v1.2)” in Proposition 3.11 and the assumption “ $j \in Q_0$ is a sink of type (v2.2)” in Proposition 3.12 are necessary, and that neither of the source counterparts of these two propositions holds true.

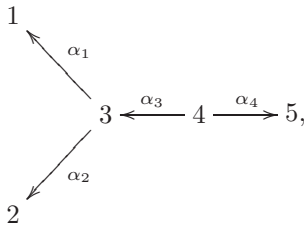
Example 3.14.

- (1) Let Q be the quiver



and $\Lambda = KQ$. Then the projective minimal right determiners in $\text{mod } \Lambda$ are $\{P(1), P(2), P(4)\}$ by Theorem 3.9.

(2) Let Q be the quiver



and $\Lambda = KQ/I$ a bound quiver algebra. If I is generated by $\{\alpha_1\alpha_3, \alpha_2\alpha_3\}$, then by Theorem 3.9, the projective minimal right determiners are $\{P(1), P(2), P(3), P(5)\}$. If I is generated by $\{\alpha_1\alpha_3\}$, then by Theorem 3.9 again, the projective minimal right determiners are $\{P(1), P(2), P(5)\}$.

4. Algebra Whose Quiver is Dynkin Type

In this section, the quiver Q is of Dynkin type and $J_{i_1}, J_{i_2}, J_{i_3}$ are as in Definition 3.3. Note that there are no vertex ideals of type J_{i_4} in this case.

It is trivial that if Λ is of type \mathbb{A}_n , that is, the underlying graph of Q is of the form

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n,$$

then Λ is string. So by Theorem 3.10, we immediately have the following.

Corollary 4.1. *If Λ of type \mathbb{A}_n , then we have*

$$|\text{Det}(\Lambda)| = 2n - p - q - 1,$$

where $p = |\{i \mid i \text{ is a source in } Q \text{ with } 2 \leq i \leq n - 1\}|$ and $q = |\{J_{i_j} \neq 0 \mid j = 1, 2\}|$.

Let Λ be of type \mathbb{A}_n . Then there are no vertex ideals of type J_{i_3} or J_{i_4} .

(1) If there is a unique sink in Q , then we have the following facts.

(1.1) There are no sources i with $2 \leq i \leq n - 1$.

(1.2) Either $J_{i_1} \neq 0$ or $J_{i_2} \neq 0$.

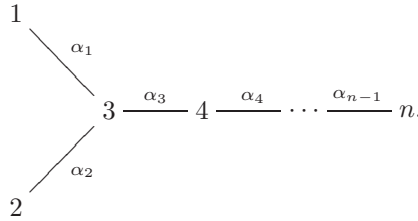
So $p = 0$ and $q = 1$, and hence $|\text{Det}(\Lambda)| = 2n - 2$ by Corollary 4.1.

(2) If Λ is a path algebra with at least two sinks in Q , then $|\{J_{i_1} \neq 0\}| = 0 = |\{J_{i_2} \neq 0\}|$. By Corollary 4.1, we have $|\text{Det}(\Lambda)| = 2n - p - 1$. Thus [13, Theorem 3.13] follows.

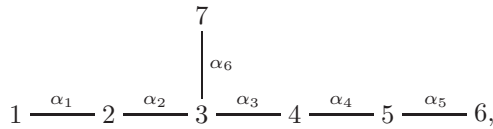
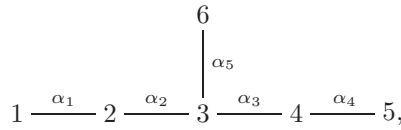
(3) If Λ is a bound quiver algebra with at least two sinks in Q , then it is straightforward to check that the notion of vertex ideals is exactly that of sink ideals in [13, Definition 3.14]. By Corollary 4.1, we have $|\text{Det}(\Lambda)| = 2n - p - q - 1$. Thus [13, Theorem 3.15] follows.

In conclusion, Corollary 4.1 is a unified version of [13, Theorems 3.13 and 3.15].

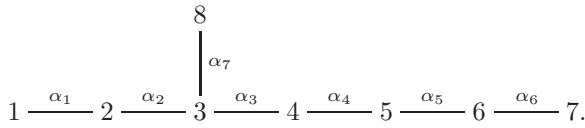
If Λ is of type \mathbb{D}_n , then the underlying graph of Q is of the form



If Λ is of type \mathbb{E}_n with $6 \leq n \leq 8$, then the underlying graph of Q is of the form



or



In the above four cases, we denote the subquivers in Q with 4 vertices including the vertex 3 and its 3 neighbors by X_3 . By the definition of string algebras, we have.

Proposition 4.2. *Let Λ be of type $\mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$ or \mathbb{E}_8 . Then Λ is a string algebra if and only if Λ is a bound quiver algebra and $I|_{X_3} \neq 0$.*

By Proposition 4.2 and Theorem 3.10, we have the following two corollaries.

Corollary 4.3. *Let Λ be a bound quiver algebra of type \mathbb{D}_n with $I|_{X_3} \neq 0$. Then we have*

$$|\text{Det}(\Lambda)| = 2n - p - q - 1,$$

where $p = |\{i \mid i \text{ is a source in } Q \text{ with } 4 \leq i \leq n-1\}|$ and $q = |\{J_{i_j} \neq 0 \mid j = 1, 2, 3\}|$.

Corollary 4.4. *Let Λ be a bound quiver algebra of type \mathbb{E}_n with $6 \leq n \leq 8$ and $I|_{X_3} \neq 0$. Then we have*

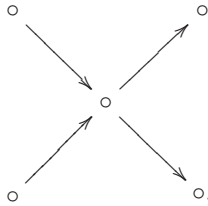
$$|\text{Det}(\Lambda)| = 2n - p - q - 1,$$

where $p = |\{i \mid i \text{ is a source in } Q \text{ with } i \neq 1, 3, n-1, n\}|$ and $q = |\{J_{i_j} \neq 0 \mid j = 1, 2, 3\}|$.

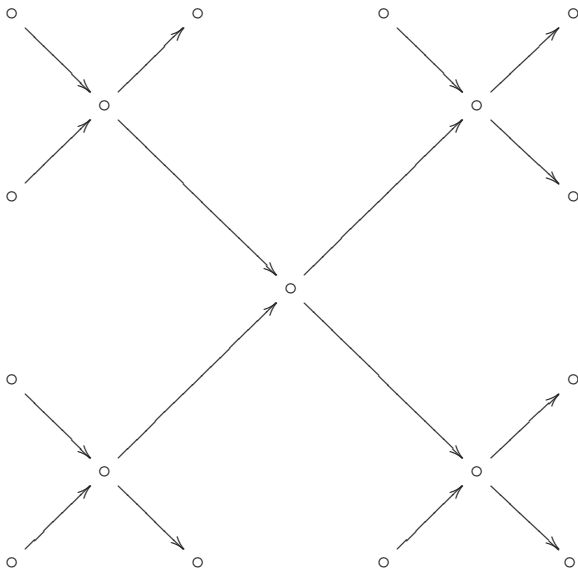
5. An Example of Non-Dynkin Type

In this section, we give an example of non-Dynkin type to illustrate Theorem 3.10.

Example 5.1. Let $Q^{(0)}$ be the quiver with a unique vertex but no arrows: \circ . Let $Q^{(1)}$ be the quiver



and $Q^{(2)}$ the quiver:



We call the following the **first step**: $Q^{(1)}$ is a quiver of type X_{i_4} , which is obtained by adding 4 vertices of type (v1.1) around the unique vertex in $Q^{(0)}$; and call the following the **second step**: $Q^{(2)}$ is obtained by adding 3 vertices of type (v1.1) around each vertex of type (v1.1) in $Q^{(1)}$ such that all the 4 new branches in $Q^{(2)}$ are of type X_{i_4} . Inductively, in the n th step, $Q^{(n)}$ is obtained from $Q^{(n-1)}$ by adding 3 new vertices of type (v1.1) around each of all $4 \times 3^{n-2}$ vertices of type (v1.1) in $Q^{(n-1)}$ such that all the $4 \times 3^{n-2}$ new branches are of type X_{i_4} . In $Q^{(n)}$, the number of vertices is $2 \times 3^n - 1$.

For any $n \geq 1$, let $\Lambda^{(n)} = KQ^{(n)}/I$ with I the admissible ideal of $KQ^{(n)}$ generated by all the paths of length 2. Notice that there are no vertices of type (v2) or (v3), so $|\{J_{i_2} \neq 0\}| = 0 = |\{J_{i_3} \neq 0\}|$. It is easy to see that $|\{J_{i_1} \neq 0\}| = 0$.

If $n \geq 2$, then we have that some $J_{i_4} \neq 0$ if and only if it is the vertex ideal of a vertex of type (v4) in the outermost ring of $Q^{(n)}$. So $|\{J_{i_4} \neq 0\}| = 4 \times 3^{n-2}$, and hence by Theorem 3.10, for any $n \geq 2$, we have

$$|\text{Det}(\Lambda^{(n)})| = 2(2 \times 3^n - 1) - 0 - 4 \times 3^{n-2} - 1 = 32 \times 3^{n-2} - 3,$$

where the number of the projective minimal right determiners is $14 \times 3^{n-2} - 1$ and the number of the nonprojective ones is $2 \times 3^n - 2$. Moreover, by Theorem 3.9, we have that $P(i) \notin \text{Det}(\Lambda^{(n)})$ if and only if i is one of the $4 \times 3^{n-2}$ vertices added in the $(n-1)$ th step.

If $n = 1$, then $|\{J_{i_4} \neq 0\}| = 1$. So by Theorem 3.10, we have

$$|\text{Det}(\Lambda^{(1)})| = 2 \times 5 - 0 - 1 - 1 = 8,$$

where both the number of the projective minimal right determiners and the number of the no-projective ones are 4. By Theorem 3.9, we have that $P(i) \in \text{Det}(\Lambda^{(1)})$ if and only if i is one of the four vertices added in the first step.

Acknowledgment

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