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Tilting modules over Auslander algebras of Nakayama algebras with radical cube zero

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Let A' be the Auslander algebra of a finite-dimensional basic connected Nakayama algebra A with radical cube zero and n simple modules. Then the cardinality $\#\mathrm{tilt}\,A'$ of the set consisting of isomorphism classes of basic tilting A'-modules is

$$\#\mathrm{tilt}\,A' = \begin{cases} \dfrac{(1+\sqrt{2})^{2n-2} - (1-\sqrt{2})^{2n-2}}{2\sqrt{2}}, & \text{if } A \text{ is non-self-injective with } n \geq 4; \\ \sqrt{[(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}]^2 + 4}, & \text{if } A \text{ is self-injective with } n \geq 2. \end{cases}$$

Keywords: Basic tilting modules; support τ-tilting modules; Nakayama algebras; Auslander algebras; Pell numbers; Pell–Lucas numbers.

Mathematics Subject Classification 2020: 18G10, 05E10

1. Introduction

Tilting theory is important in representation theory of Artin algebras and homological algebra. There are many related works which made the theory fruitful, see [15, 4, 9] and references therein. In this theory, tilting modules play a central role. So it is fundamental and important to classify tilting modules for a given algebra. An effective method to construct tilting modules is given by mutation [18, 20]. However, the mutation of tilting modules is not always possible. To improve the behavior of

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mutation of tilting modules, Adachi *et al.* [3] introduced support τ-tilting modules as a generalization of tilting modules. They showed that the mutation of support τ-tilting modules is always possible; in particular, τ-tilting modules share many nice properties of tilting modules.

It is shown by Auslander that there is a bijection between classes of representation-finite algebras and Auslander algebras [5]. There are many works on Auslander algebras. Brüstle *et al.* [7] classified tilting modules over the Auslander algebra of $K[x]/\langle x^n\rangle$ and showed that the number of tilting modules is $n!$. Iyama and Zhang [17] classified τ-tilting modules over the Auslander algebra of $K[x]/\langle x^n\rangle$. Recently, Zhang [21] gave a classification of tilting modules over Auslander algebras of Nakayama algebras with radical square zero. On the other hand, algebras with radical cube zero have gained a lot of attention. Hoshino [14] proved the Tachikawa version of the Nakayama conjecture for algebras with radical cube zero. Erdmann and Solberg [8] classified all the possible quivers of finite-dimensional self-injective algebras with radical cube zero and finite complexity. Adachi and Aoki [2] calculated the number of two-term tilting complexes over symmetric algebras with radical cube zero.

In the literature, especially in mathematics and physics, there are a lot of integer numbers, which are used in almost every field of modern sciences. Admittedly, Pell numbers (sequence A000129 in OEIS) and Pell–Lucas numbers (sequence A002203 in OEIS) are very essential in the fields of combinatorics and number theory. The Pell sequence $\{\mathrm{P_n}\}$ are defined by recurrence $\mathrm{P_n} = 2\mathrm{P_{n-1}} + \mathrm{P_{n-2}}$ for any $n \geq 2$ with $\mathrm{P_0} = 0$ and $\mathrm{P_1} = 1$ and the Pell–Lucas sequence $\{\mathrm{Q_n}\}$ by the same recurrence but with initial conditions $\mathrm{Q_0} = \mathrm{Q_1} = 2$. Explicit Binet forms for $\{\mathrm{P_n}\}$ and $\{\mathrm{Q_n}\}$ are $\mathrm{P_n} = \frac{\alpha^{\mathrm{n}}-\beta^{\mathrm{n}}}{\alpha-\beta}$ and $\mathrm{Q_n} = \alpha^{\mathrm{n}} + \beta^{\mathrm{n}}$, where α and β are the roots of the characteristic equation $x^2 - 2x - 1 = 0$. Then one gets $8\mathrm{P_n^2} = \mathrm{Q_n^2} - 4(-1)^{\mathrm{n}}$. Further details about Pell and Pell–Lucas sequences can be found in [6, 10–13].

In this paper, by virtual of Pell and Pell–Lucas sequences, we will determine the number of isomorphism classes of basic tilting modules over Auslander algebras of Nakayama algebras with radical cube zero. Let A be a finite-dimensional algebra over an algebraically closed field. We use $\mathrm{tilt}\,A$ to denote the set consisting of isomorphism classes of basic tilting modules. For a set X, and use $\#X$ to denote the cardinality of X. The following is our main result.

Theorem 1.1 (Theorem 3.8). *Let A be a Nakayama algebra with radical cube zero and n simple modules, and let A' be the Auslander algebra of A.*

(1) *If A is non-self-injective with $n \geq 4$, then*

$$\#\mathrm{tilt}\,A' = \frac{(1+\sqrt{2})^{2n-2} - (1-\sqrt{2})^{2n-2}}{2\sqrt{2}}.$$

(2) *If A is self-injective with $n \geq 2$, then*

$$\#\mathrm{tilt}\,A' = \sqrt{[(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}]^2 + 4}.$$

We also give two examples to illustrate this result.

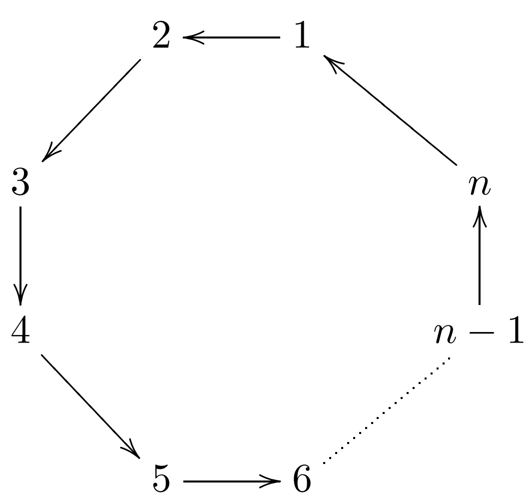
2. Preliminaries

Throughout this paper, A is a finite-dimensional algebra over an algebraically closed field K and τ the Auslander–Reiten translation. We use $\operatorname{mod} A$ to denote the category of finitely generated left A-modules and use $\operatorname{gl.dim} A$ to denote the global dimension of A. For a module $T \in \operatorname{mod} A$, we use $\mathsf{add}\, T$ to denote the subcategory of $\operatorname{mod} A$ consisting of direct summands of finite direct sums of T.

Recall that a module $T \in \operatorname{mod} A$ is called (*classical*) *tilting* if the projective dimension of T is at most one, $\operatorname{Ext}^1_A(T,T) = 0$ and there is an exact sequence $0 \to A \to T_0 \to T_1 \to 0$ in $\operatorname{mod} A$ with T_0 and T_1 in $\mathsf{add}T$. Also recall that A is called a *Nakayama algebra* if it is both right and left serial, that is, every indecomposable projective module and every indecomposable injective module in $\operatorname{mod} A$ are uniserial.

Proposition 2.1 ([4, Chap. V, Theorem 3.2]). *A basic and connected algebra A is a Nakayama algebra if and only if its ordinary quiver Q_A is one of the following two quivers*:

(1) $1 \to 2 \to 3 \to \cdots \to n-1 \to n$;
(2)



(*with $n \geq 1$ vertices*).

We use $|T|$ to denote the number of pairwise non-isomorphic indecomposable direct summands of T.

Definition 2.2 ([3, 19]). Let T be in $\operatorname{mod} A$.

(1) T is called *τ-rigid* if $\operatorname{Hom}_A(T, \tau T) = 0$ and T is called *τ-tilting* if T is τ-rigid and $|T| = |A|$.
(2) T is called *support τ-tilting* if there exists an idempotent e of A such that T is a τ-tilting $A/\langle e\rangle$-module.

We use $\operatorname{proj} A$ to denote the full subcategory of $\operatorname{mod} A$ consisting of projective modules. Sometimes, it is convenient to view support τ-tilting modules and τ-rigid modules as certain pairs of modules in $\operatorname{mod} A$.

Definition 2.3. Let (T, P) be a pair with $T \in \operatorname{mod} A$ and $P \in \operatorname{proj} A$.

(1) (T, P) is called a *τ-rigid pair* if T is τ-rigid and $\operatorname{Hom}_A(P, T) = 0$.
(2) (T, P) is called a *support τ-tilting pair* if (T, P) is τ-rigid and $|T| + |P| = |A|$.

We use $\mathrm{s}\tau\text{-tilt}A$ to denote the set of isomorphism classes of basic support τ-tilting modules in $\operatorname{mod} A$. For a module $M \in \operatorname{mod} A$, we use $\mathsf{Fac}M$ to denote the full subcategory of $\operatorname{mod} A$ consisting of modules isomorphic to factor modules of finite direct sums of copies of M.

Definition 2.4 ([3]). Let $T, U \in \mathrm{s}\tau\text{-tilt}A$. We call T a *mutation* of U if they have the same indecomposable direct summands except one. Precisely speaking, there are three cases:

(1) $T = V \oplus X$ and $U = V \oplus Y$ with $X \ncong Y$ indecomposable;
(2) $T = U \oplus X$ with X indecomposable;
(3) $U = T \oplus X$ with X indecomposable.

Moreover, we call T a *left mutation* (respectively, *right mutation*) of U if $\mathsf{Fac}T \subsetneq \mathsf{Fac}U$ (respectively, $\mathsf{Fac}T \supsetneq \mathsf{Fac}U$) and write $T = \mu_X^-(U)$ (respectively, $T = \mu_X^+(U)$).

The following result [3, Theorem 2.30] gives a method for computing left mutations. For the convenience, we recall the definition of the Bongartz completion. For a τ-rigid A-module U, we have that $T := \mathrm{P}({}^{\perp}(\tau U))$ is a τ-tilting A-module which is called a *Bongartz completion* of U satisfying $U \in \mathsf{add}T$ and ${}^{\perp}(\tau T) = \mathsf{Fac}T$, where $\mathrm{P}({}^{\perp}(\tau U))$ is the direct sum of one copy of each of the indecomposable Ext-projective objects in ${}^{\perp}(\tau U)$ up to isomorphism.

Lemma 2.5. *Let $T = X \oplus U$ be a basic τ-tilting module which is the Bongartz completion of U with X indecomposable. Let*

$$X \xrightarrow{f} U' \xrightarrow{g} Y \to 0$$

be an exact sequence with f the minimal left $\mathsf{add}U$-approximation. Then we have

(1) *If U is not sincere, then $Y = 0$. In this case, $U = \mu_X^-(T)$ holds and it is a basic support τ-tilting A-module that is not τ-tilting.*
(2) *If U is sincere, then Y is a direct sum of finite copies of an indecomposable A-module Y_1 and is not in $\mathsf{add}T$. In this case, $Y_1 \oplus U = \mu_X^-(T)$ holds and it is a basic τ-tilting A-module.*

We use $K^b(\operatorname{proj} A)$ to denote the bounded homotopy category of $\operatorname{proj} A$.

Definition 2.6 ([3]). Let P be a complex in $K^b(\operatorname{proj} A)$.

(1) P is called *presilting* if $\operatorname{Hom}_{K^b(\operatorname{proj} A)}(P, P[n]) = 0$ for any $n \geq 1$.
(2) P is called *silting* if it is presilting and generates $K^b(\operatorname{proj} A)$ by taking direct sums, direct summands, shifts and mapping cones. In addition, it is called *tilting* if it is also satisfies $\operatorname{Hom}_{K^b(\operatorname{proj} A)}(P, P[n]) = 0$ for all nonzero integers n.
(3) P is called *two-term silting* if it isomorphic to a complex concentrated in degree 0 and -1 in $K^b(\operatorname{proj} A)$.

We use 2-silt A to denote the set of isomorphism classes of basic two-term silting complexes in $K^b(\operatorname{proj} A)$.

Lemma 2.7 ([3, Theorem 3.2]). *There exists a bijection*

$$\text{2-silt}\, A \leftrightarrow s\tau\text{-tilt}\, A$$

given by 2-silt$A \ni P \mapsto H^0(P) \in \mathrm{s}\tau$-tilt$A$ *and* sτ-tilt$A \ni (T,P) \mapsto (P_1 \oplus P \xrightarrow{(f\ 0)} P_0) \in$ 2-silt$A$, *where $f : P_1 \to P_0$ is a minimal projective presentation of T.*

3. Main Result

We begin with the following definition.

Definition 3.1 ([1, Definition 3.2]). Let $\Omega = (\Omega, \geq)$ be a poset and N a subposet of Ω.

(1) We define a new poset $\Omega^{\mathrm{N}} = (\Omega^{\mathrm{N}}, \geq_{\mathrm{N}})$ as follows, where $\mathrm{N}^+ := \{n^+ \mid n \in \mathrm{N}\}$ is a copy of N, and $\omega_1, \omega_2 \in \Omega\backslash\mathrm{N}$ and $n_1, n_2 \in \mathrm{N}$ are arbitrary elements:

$$\Omega^{\mathrm{N}} := \Omega \coprod \mathrm{N}^+,$$

$$\omega_1 \geq_{\mathrm{N}} \omega_2 :\Leftrightarrow \omega_1 \geq \omega_2, n_1 \geq_{\mathrm{N}} n_2 :\Leftrightarrow n_1 \geq n_2,$$

$$\omega_1 \geq_{\mathrm{N}} n_1 :\Leftrightarrow \omega_1 \geq n_1, n_1 \geq_{\mathrm{N}} \omega_1 :\Leftrightarrow n_1 \geq \omega_1,$$

$$n_1^+ \geq_{\mathrm{N}} \omega_1 :\Leftrightarrow n_1 \geq \omega_1, n_1^+ \geq_{\mathrm{N}} n_2 :\Leftrightarrow n_1 \geq n_2,$$

$$\omega_1 \geq_{\mathrm{N}} n_1^+ :\Leftrightarrow \omega_1 \geq n_1, n_1^+ \geq_{\mathrm{N}} n_2^+ :\Leftrightarrow n_1 \geq n_2.$$

In particular, $n_1 \geq_{\mathrm{N}} n_2^+$ never holds. It is easily to check that $(\Omega^{\mathrm{N}}, \geq_{\mathrm{N}})$ forms a poset.

(2) Let $\mathrm{H}(\Omega) := (\Omega, \mathrm{H_a})$ be the Hasse quiver of Ω. We define a new quiver $\mathrm{H}(\Omega)^{\mathrm{N}} := (\Omega^{\mathrm{N}}, \mathrm{H_a^N})$ as follows, where ω_1, ω_2 are arbitrary elements in $\Omega\backslash\mathrm{N}$ and n_1, n_2 are arbitrary elements in N:

$$\mathrm{H_a^N} = \{\omega_1 \to \omega_2 \mid \omega_1 \to \omega_2 \text{ in } \mathrm{H_a}\} \coprod \{\mathrm{n}_2 \to \omega_2 \mid \mathrm{n}_2 \to \omega_2 \text{ in } \mathrm{H_a}\}$$

$$\coprod \{n_1 \to n_2, n_1^+ \to n_2^+ \mid n_1 \to n_2 \text{ in } \mathrm{H_a}\}$$

$$\coprod \{\omega_1 \to n_1^+ \mid \omega_1 \to n_1 \text{ in } \mathrm{H_a}\} \coprod \{\mathrm{n}_1^+ \to \mathrm{n}_1 \mid \mathrm{n}_1 \in \Omega\}.$$

It is easy to check that $\mathrm{H}(\Omega^{\mathrm{N}}) = \mathrm{H}(\Omega)^{\mathrm{N}}$ holds.

Assume that A has an indecomposable projective-injective summand L as an A-module. Moreover, let $S := \operatorname{soc} L$ and $\overline{A} := A/S$. Consider the functor

$$\overline{(-)} := - \otimes_A \overline{A} : \operatorname{mod} A \to \operatorname{mod} \overline{A}.$$

Then $\overline{L} = L/S$. Note that, for every indecomposable A-module $M \not\simeq L$, so we have an isomorphism $\overline{M} \simeq M$ as $\overline{A}$-modules.

Now let $\mathcal{N} := \{N \in \mathrm{s}\tau\text{-tilt}\,\overline{A} \mid \overline{L} \in \mathsf{add}\,N \text{ and } \operatorname{Hom}_A(N, L) = 0\}$. Applying Definition 3.1, we have a poset $(\mathrm{s}\tau\text{-tilt}\,\overline{A})^{\mathcal{N}}$. For any A-module M, we denote by $\alpha(M)$ a basic A-module satisfying $\mathsf{add}\,\alpha(M) = \mathsf{add}\,\overline{M}$.

Lemma 3.2 ([1, Theorem 3.3(1)]). *Let L be an indecomposable projective-injective summand of A as an A-module. Then there is an isomorphism of posets*

$$\mathrm{s}\tau\text{-tilt}\,A \to (\mathrm{s}\tau\text{-tilt}\,\overline{A})^{\mathcal{N}}$$

given by $M \mapsto \alpha(M)$. In particular, we have an isomorphism of Hasse quivers

$$\mathrm{H}(A) \simeq \mathrm{H}(\overline{A})^{\mathcal{N}}.$$

By the definition of $(\mathrm{s}\tau\text{-tilt}\,\overline{A})^{\mathcal{N}}$, we have

$$\#(\mathrm{s}\tau\text{-tilt}\,\overline{A})^{\mathcal{N}} = \#\mathrm{s}\tau\text{-tilt}\,\overline{A} + \#\mathcal{N}.$$

It follows from Lemma 3.2 that

$$\#\mathrm{s}\tau\text{-tilt}\,A = \#\mathrm{s}\tau\text{-tilt}\,\overline{A} + \#\mathcal{N}.$$

This equality will be crucial in proving our main result.

For an algebra A, assume that

$$0 \to A \to I^0(A) \to I^1(A) \to \cdots \to I^i(A) \to \cdots$$

is the minimal injective resolution of ${}_AA$.

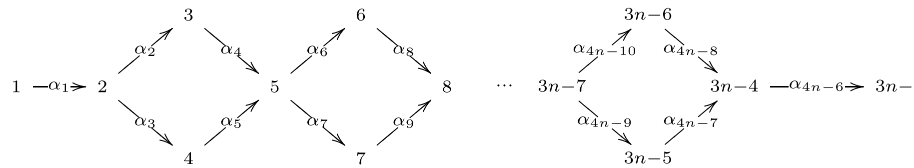
Lemma 3.3 ([16, Theorem 4.5]). *Let $I^0(A)$ be projective and e an idempotent of A such that $\mathsf{add}\,eA = \mathsf{add}\,I^0(A)$. Then the tensor functor $- \otimes_A A/\langle e\rangle$ induces a bijection from* $\operatorname{tilt} A$ *to* $\mathrm{s}\tau\text{-tilt}\,A/\langle e\rangle$.

Recall that A is called an *Auslander algebra* if $\operatorname{gl.dim} A \leq 2$ and both $I^0(A)$ and $I^1(A)$ are projective. Let A be representation-finite with M an additive generator for $\operatorname{mod} A$. Then $A' := \operatorname{End}_A(M)$ is an Auslander algebra [5]. In this case, A' is called the *Auslander algebra* of A.

In the rest of this section, A is a basic connected Nakayama algebra with radical cube zero and n simple modules, A' is the Auslander algebra of A and $\overline{A'} := A'/\langle e\rangle$

where $\mathsf{add}\, eA' = \mathsf{add}\, I^0(A')$ with e an idempotent of A'. The following result gives the structure of A', which is induced from Proposition 2.1 directly.

Proposition 3.4. (1) *If A is non-self-injective with $n \geq 4$, then A' is given by the following quiver Q':*



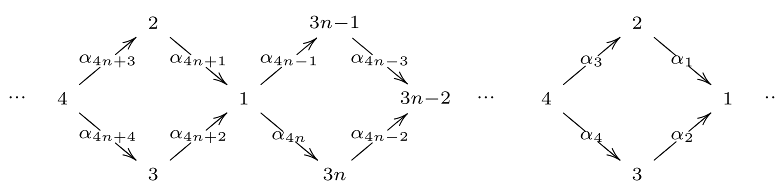
with relations

$$\alpha_{4i+2}\alpha_{4i+4} = \alpha_{4i+3}\alpha_{4i+5}, \quad \alpha_{4i+1}\alpha_{4i+3} = 0$$

for any $0 \leq i \leq n-3$ and

$$\alpha_{4n-7}\alpha_{4n-6} = 0.$$

(2) *If A is self-injective with $n \geq 2$, then A' is given by the following quiver Q':*



with relations

$$\alpha_{4i+3}\alpha_{4i+1} = \alpha_{4i+4}\alpha_{4i+2}, \alpha_{4i+5}\alpha_{4i+3} = 0$$

for any $i \geq 0$.

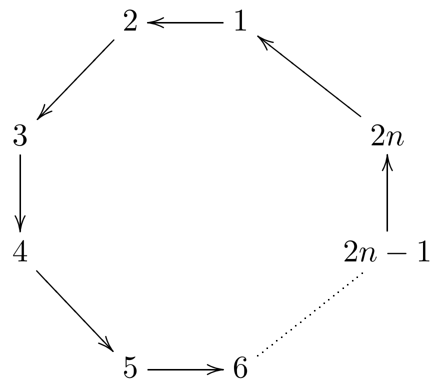
The following proposition is quite essential for the main result.

Proposition 3.5. (1) *If A is non-self-injective with $n \geq 4$, then $\overline{A'}$ is given by the following quiver Q'' :*

$$1 \to 2 \to 3 \to \cdots \to 2n-4 \to 2n-3$$

with $\operatorname{rad}^2 KQ'' = 0$.

(2) *If A is self-injective with $n \geq 2$, then $\overline{A'}$ is given by the following quiver Q'' :*



with $\operatorname{rad}^2 KQ'' = 0$.

The following proposition gives some properties of indecomposable direct summands of tilting A'-modules.

Proposition 3.6. *Let T be a tilting module in $\operatorname{mod} A'$. Then we have*

(1) *The number of indecomposable projective-injective direct summands of T is n.*
(2) *The simple direct summand of T is either projective or a simple socle of an indecomposable projective A'-module.*
(3) *For any indecomposable non-projective-injective direct summand M of T, the Loewy length of M' which is the mutation of T on M is at most three.*

Proof. (1) By Proposition 3.4, we can easily get the number of indecomposable projective-injective direct summands of T. Since T is faithful, we have an epimorphism $T^n \twoheadrightarrow \mathbb{D}A'$, where $\mathbb{D} = \operatorname{Hom}_K(-, K)$ is the ordinary dual. If P is an indecomposable projective-injective module, then P is a direct summand of T.

If A is non-self-injective with $n(\geq 4)$ simple modules, then A' has $3n - 3$ simple modules and the indecomposable projective-injective modules are $P(1), P(2), P(3), P(6), \ldots, P(3n-6)$. If A is self-injective with $n(\geq 2)$ simple modules, then A' has $3n$ simple modules and the indecomposable projective-injective modules are $P(3), P(6), \ldots, P(3n)$.

(2) If A is non-self-injective with $n(\geq 4)$ simple modules, then for any indecomposable projective module $P \in \operatorname{mod} A'$, $\operatorname{soc} P$ is either $S(3n-4)$ or $S(3i-3)$ with $2 \leq i \leq n$. Then by Lemma 2.5, we can verify directly that the simple direct summand of T is either projective or a simple socle of an indecomposable projective A'-module.

If A is self-injective with $n(\geq 2)$ simple modules, then for any indecomposable projective module $P \in \operatorname{mod} A'$, $\operatorname{soc} P = S(3i)$ with $1 \leq i \leq n$. Then by Lemma 2.5, we can verify directly that the simple direct summand of T is a simple socle of an indecomposable projective A'-module.

(3) If A is non-self-injective, then the quiver Q' of A' is as in Proposition 3.4(1). The indecomposable projective modules in $\operatorname{mod} A'$ are as follows:

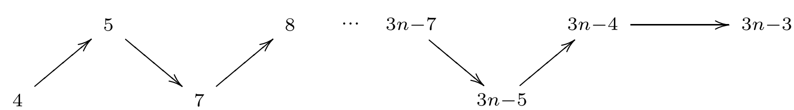
$$P(1) = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}, \quad P(2) = \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix}, \quad P(3) = \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \\ & 9 & \end{smallmatrix}, \quad P(4) = \begin{smallmatrix} 4 \\ 5 \\ 6 \end{smallmatrix}, \quad P(5) = \begin{smallmatrix} & 5 & \\ 6 & & 7 \\ & 8 & \\ & 9 & \end{smallmatrix},$$

$$P(6) = \begin{smallmatrix} & 6 & \\ & 8 & \\ 9 & & 10 \\ & 11 & \\ & 12 & \end{smallmatrix}, \quad P(7) = \begin{smallmatrix} 7 \\ 8 \\ 9 \end{smallmatrix}, \quad \ldots,$$

$$P(3n-7) = \begin{smallmatrix} & 3n-7 & \\ 3n-6 & & 3n-5 \\ & 3n-4 & \end{smallmatrix}, \quad P(3n-6) = \begin{smallmatrix} 3n-6 \\ 3n-4 \\ 3n-3 \end{smallmatrix}, \quad P(3n-5) = \begin{smallmatrix} 3n-5 \\ 3n-4 \end{smallmatrix},$$

$$P(3n-4) = \begin{smallmatrix} 3n-4 \\ 3n-3 \end{smallmatrix}, \quad P(3n-3) = \begin{smallmatrix} 3n-3 \end{smallmatrix}.$$

By Proposition 3.5(1), the quiver Q'' of $\overline{A'}$ is as follows:



with the relation $\operatorname{rad}^2 KQ'' = 0$. The indecomposable projective modules in $\operatorname{mod} \overline{A'}$ are as follows:

$$P'(4) = \begin{smallmatrix} 4 \\ 5 \end{smallmatrix}, \quad P'(5) = \begin{smallmatrix} 5 \\ 7 \end{smallmatrix}, \quad P'(7) = \begin{smallmatrix} 7 \\ 8 \end{smallmatrix}, \quad \ldots, \quad P'(3n-7) = \begin{smallmatrix} 3n-7 \\ 3n-5 \end{smallmatrix},$$

$$P'(3n-5) = \begin{smallmatrix} 3n-5 \\ 3n-4 \end{smallmatrix}, \quad P'(3n-4) = \begin{smallmatrix} 3n-4 \\ 3n-3 \end{smallmatrix}, \quad P'(3n-3) = \begin{smallmatrix} 3n-3 \end{smallmatrix}.$$

The maximal tilting A'-module is

$$T = P(1) \oplus P(2) \oplus P(3) \oplus \cdots \oplus P(3n-3).$$

By (1), the indecomposable projective-injective direct summands of T are

$$P(1), P(2), P(3), P(6), \ldots, P(3n-6).$$

The maximal support τ-tilting $\overline{A'}$-module is

$$T' = P'(4) \oplus P'(5) \oplus P'(7) \oplus \cdots \oplus P'(3n-3).$$

For any $i \in \{3j-2, 3j-1, 3n-3 \mid 2 \leq j \leq n-1\}$, we have a correspondence between $P(i)$ and $P'(i)$ by Lemma 3.3. Let L be an indecomposable direct summand of T'. Then there exists a module L' which is the mutation of T' on L by Lemma 2.5. We have that the Lowey length of L is at most two and the Lowey length of L' is at most one. Thus, if M is an indecomposable non-projective-injective direct summand of T. Then there exists a module M' which is the mutation of T on M. We have that the Lowey length of M is at most four and the Lowey length of M' is at most three.

If A is self-injective, then the quiver Q' of A' is as in Proposition 3.4(2). The indecomposable projective modules in $\operatorname{mod} A'$ are as follows:

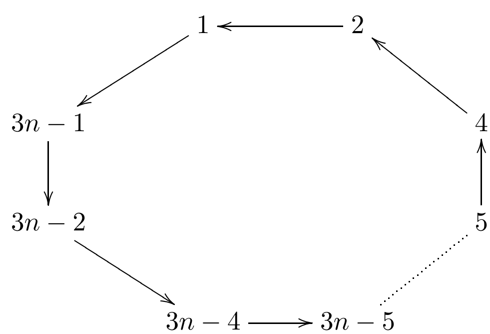
$$P(1) = \begin{smallmatrix} & 1 & \\ 3n-1 & & 3n \\ & 3n-2 & \\ & 3n-3 & \end{smallmatrix}, \quad P(2) = \begin{smallmatrix} 2 \\ 1 \\ 3n \end{smallmatrix}, \quad P(3) = \begin{smallmatrix} & 3 & \\ & 1 & \\ 3n-1 & & 3n \\ & 3n-2 & \\ & 3n-3 & \end{smallmatrix}, \quad P(4) = \begin{smallmatrix} & 4 & \\ 2 & & 3 \\ & 1 & \\ & 3n & \end{smallmatrix},$$

$$P(5) = \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix}, \quad P(6) = \begin{smallmatrix} & 6 & \\ & 4 & \\ 2 & & 3 \\ & 1 & \\ & 3n & \end{smallmatrix}, \quad \ldots,$$

$$P(3n-3) = \begin{smallmatrix} & 3n-3 & \\ & 3n-5 & \\ 3n-7 & & 3n-6 \\ & 3n-8 & \\ & 3n-9 & \end{smallmatrix}, \quad P(3n-2) = \begin{smallmatrix} & 3n-2 & \\ 3n-4 & & 3n-3 \\ & 3n-5 & \\ & 3n-6 & \end{smallmatrix}, \quad P(3n-1) = \begin{smallmatrix} 3n-1 \\ 3n-2 \\ 3n-3 \end{smallmatrix},$$

$$P(3n) = \begin{smallmatrix} & 3n & \\ & 3n-2 & \\ 3n-4 & & 3n-3. \\ & 3n-5 & \\ & 3n-6 & \end{smallmatrix}$$

By Proposition 3.5(2), the quiver Q'' of $\overline{A'}$ is as follows:



with the relation $\operatorname{rad}^2 KQ'' = 0$. The indecomposable projective modules in $\operatorname{mod} \overline{A'}$ are as follows:

$$P'(1) = \begin{smallmatrix} 1 \\ 3n-1 \end{smallmatrix}, \quad P'(2) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \quad P'(4) = \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}, \quad \ldots, \quad P'(3n-2) = \begin{smallmatrix} 3n-2 \\ 3n-4 \end{smallmatrix},$$

$$P'(3n-1) = \begin{smallmatrix} 3n-1 \\ 3n-2 \end{smallmatrix}.$$

The maximal tilting A'-module is

$$T = P(1) \oplus P(2) \oplus P(3) \oplus \cdots \oplus P(3n).$$

By (1), the indecomposable projective-injective direct summands of T are as follows:

$$P(3), P(6), \ldots, P(3n).$$

The maximal support τ-tilting $\overline{A'}$-module is

$$T' = P'(1) \oplus P'(2) \oplus P'(4) \oplus \cdots \oplus P'(3n-1).$$

For any $i \in \{3j-2, 3j-1 \mid 1 \leq j \leq n\}$, we have a correspondence between $P(i)$ and $P'(i)$ by Lemma 3.3. Let L be an indecomposable direct summand of T'. Then there exists a module L' which is the mutation of T' on L by Lemma 2.5. We have that the Lowey length of L is two and the Lowey length of L' is at most one. Thus, if M is an indecomposable non-projective-injective direct summand of T and M' is the module which is the mutation of T on M, then the Lowey length of M is at most four and the Lowey length of M' is at most three. $\square$

The following proposition calculates the number of support τ-tilting modules in $\mathrm{mod}\,\overline{A'}$.

Proposition 3.7. (1) *If A is non-self-injective with $n \geq 4$, then*

$$\#\mathrm{s}\tau\text{-tilt}\overline{A'} = \frac{(1+\sqrt{2})^{2n-2} - (1-\sqrt{2})^{2n-2}}{2\sqrt{2}}.$$

(2) *If A is self-injective with $n \geq 2$, then*

$$\#\mathrm{s}\tau\text{-tilt}\,\overline{A'} = \sqrt{[(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}]^2 + 4}.$$

Proof. We only need to prove the case of radical square zero Nakayama algebra A by Lemma 3.3 and Proposition 3.5. Set $\mathrm{P_n} := \#\mathrm{s}\tau\text{-tilt}\,A$.

(1) If A is non-self-injective, then the quiver Q of A is

$$1 \to 2 \to 3 \to \cdots \to m-1 \to m$$

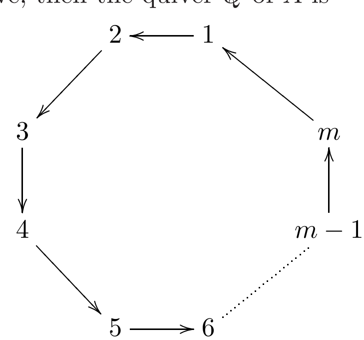
with the relation $\mathrm{rad}^2 KQ = 0$. Let $L = \frac{1}{2}$ be an indecomposable projective-injective summand of A. Then $\mathrm{soc}\ L = 2, \overline{L} = 1$ and $\overline{A} = A/\,\mathrm{soc}\ L$ is given by the following quiver:

$$1, 2 \to 3 \to \cdots \to m-1 \to m.$$

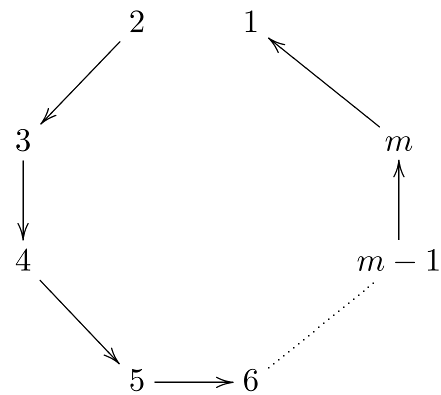
Thus, $\#\mathrm{s}\tau\text{-tilt}\overline{A} = 2\mathrm{P_{m-1}}$.

By calculating $\mathcal{N} := \{N \in \mathrm{s}\tau\text{-tilt}\overline{A} \mid \overline{L} \in \mathsf{add}\,N \text{ and } \mathrm{Hom}_A(N, L) = 0\}$, we get that the set $\mathcal{N}$ contains the module 1 but does not contain modules 2, $\frac{1}{2}$ and $\frac{2}{3}$. So we have $\#\mathcal{N} = \mathrm{P_{m-2}}$ and hence $\mathrm{P_m} = 2\mathrm{P_{m-1}} + \mathrm{P_{m-2}}$ by Lemma 3.2. It is a Pell-sequence (sequence A000129 in OEIS) and $\mathrm{P_m} = \frac{(1+\sqrt{2})^{m+1} - (1-\sqrt{2})^{m+1}}{2\sqrt{2}}$. By letting $m = 2n-3$, we get the desired assertion.

(2) If A is self-injective, then the quiver Q of A is



with the relation $\mathrm{rad}^2 KQ = 0$. Let $L = \frac{1}{2}$ be an indecomposable projective-injective summand of A. Then $\mathrm{soc}\,L = 2$, $\overline{L} = 1$ and $\overline{A} = A/\mathrm{soc}\ L$ is given by the following quiver:



Thus, $\#\mathrm{s}\tau\text{-tilt}\,\overline{A} = \mathrm{P_m}$.

Similar to (1), we have $\#\mathcal{N} = \mathrm{P_{m-2}}$ and hence $\mathrm{Q_m} = \mathrm{P_m} + \mathrm{P_{m-2}}$. Applying $\mathrm{P_m} = 2\mathrm{P_{m-1}} + \mathrm{P_{m-2}}$ from (1), we get $\mathrm{Q_m} = 2\mathrm{Q_{m-1}} + \mathrm{Q_{m-2}}$. It is a Pell–Lucas sequence (sequence A002203 in OEIS) and

$$\mathrm{Q_m} = \sqrt{[(1+\sqrt{2})^m - (1-\sqrt{2})^m]^2 + 4(-1)^m}.$$

By letting $m = 2n$, we get the desired assertion. □

We now are in a position to give the main result.

Theorem 3.8. (1) *If A is non-self-injective with $n \geq 4$, then*

$$\#\mathrm{tilt}\,A' = \frac{(1+\sqrt{2})^{2n-2} - (1-\sqrt{2})^{2n-2}}{2\sqrt{2}}.$$

(2) *If A is self-injective with $n \geq 2$, then*

$$\#\mathrm{tilt}\,A' = \sqrt{[(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}]^2 + 4}.$$

Proof. Using the correspondence in Lemma 3.3, we can see that the number of tilting modules in $\mathrm{mod}\,A'$ is equal to the number of support τ-tilting modules in $\mathrm{mod}\,\overline{A'}$ which we have proved in Proposition 3.7. □

As a consequence, we have the following corollary.

Corollary 3.9. (1) *If A is non-self-injective with $n \geq 4$, then*

$$\#2\text{-silt}\overline{A'} = \frac{(1+\sqrt{2})^{2n-2} - (1-\sqrt{2})^{2n-2}}{2\sqrt{2}}.$$

(2) *If A is self-injective with $n \geq 2$, then*

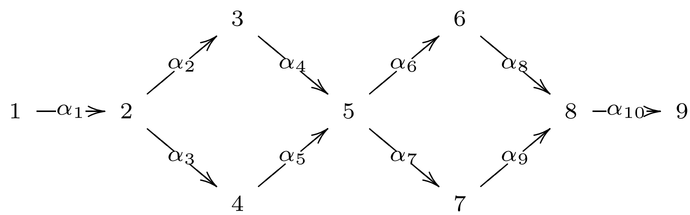
$$\#2\text{-silt}\overline{A'} = \sqrt{[(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}]^2 + 4}.$$

Proof. This follows from Lemma 2.7 and Proposition 3.7. □

4. Examples

In this section, we give two examples to illustrate the theorem in Sec. 3.

Example 4.1. Let A be an algebra given by the quiver Q: $1 \to 2 \to 3 \to 4$ with $\operatorname{rad}^3 KQ = 0$. The corresponding Auslander algebra A' is given by the quiver Q':



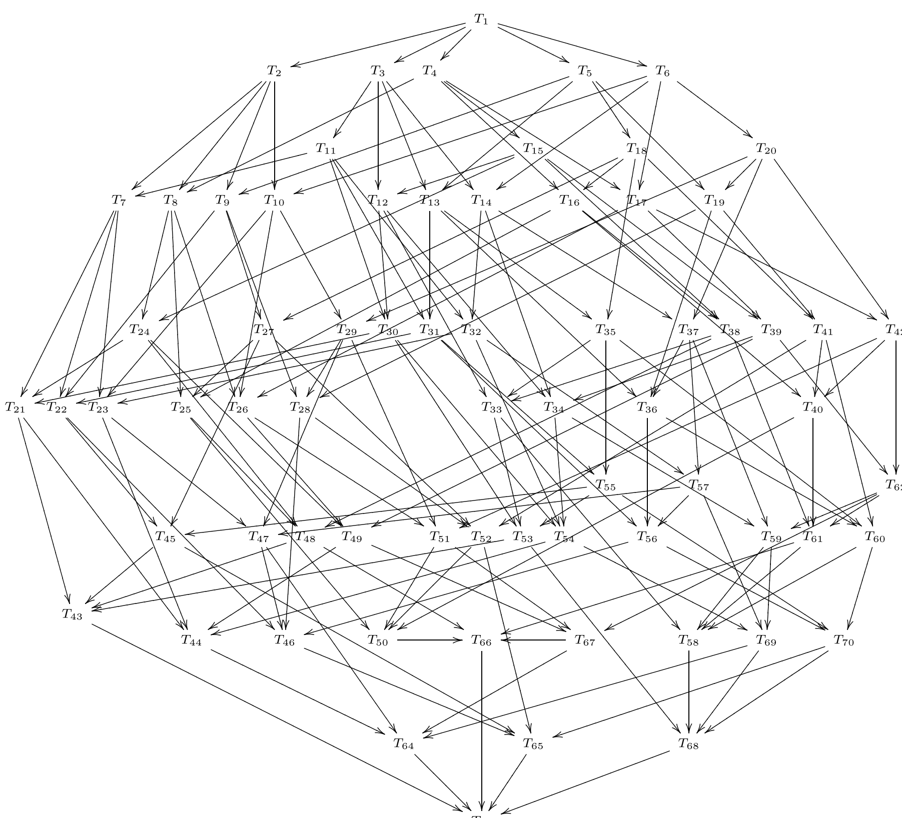
with relations

$$\alpha_{4i+2}\alpha_{4i+4} = \alpha_{4i+3}\alpha_{4i+5}, \quad \alpha_{4i+1}\alpha_{4i+3} = 0$$

for $i = 0, 1$ and

$$\alpha_9\alpha_{10} = 0.$$

Putting $n = 4$ in Theorem 3.8(1), we get $\#\text{tilt}\, A' = 70$. The basic tilting A'-modules are presented by the following quiver Q'':



where

$$T_1 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_2 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_3 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_4 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_5 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9,$$

$$T_6 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_7 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_8 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_9 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9,$$

$$T_{10} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{11} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_{12} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_{13} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9,$$

$$T_{14} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{15} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_{16} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus 6 \oplus 9,$$

$$T_{17} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{18} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9,$$

$$T_{19} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{20} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix} \oplus 8,$$

$$T_{21} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_{22} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9,$$

$$T_{23} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{24} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_{25} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus 6 \oplus 9,$$

$$T_{26} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{27} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9,$$

$$T_{28} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{29} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix} \oplus 8,$$

$$T_{30} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 9,$$

$$T_{31} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9,$$

$$T_{32} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{33} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus 6 \oplus 9,$$

$$T_{34} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{35} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9,$$

$$T_{36} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{37} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix} \oplus 8,$$

$$T_{38} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus 6 \oplus 9,$$

$$T_{39} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{40} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{41} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{42} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix} \oplus 8,$$

$$T_{43} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus 6 \oplus 9,$$

$$T_{44} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{45} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus 9,$$

$$T_{46} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{47} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix} \oplus 8,$$

$$T_{48} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus 6 \oplus 9,$$

$$T_{49} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}8\\9\end{smallmatrix} \oplus 8,$$

$$T_{50} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{51} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix} \oplus 8,$$

$$T_{52} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&5&\\6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}6&&7\\&8&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\end{smallmatrix},$$

$$T_{53} = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\&5&\\6&&7\\&8&\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}3&&4\\&5&\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}6\\8\\9\end{smallmatrix} \oplus \begin{smallmatrix}3\\5\\6\end{smallmatrix} \oplus 6 \oplus 9,$$

$$T_{54} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 8 \\ 9 \end{smallmatrix} \oplus 8,$$

$$T_{55} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix} 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus 9,$$

$$T_{56} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 7 \\ 8 \end{smallmatrix} \oplus \begin{smallmatrix} 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix},$$

$$T_{57} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 7 \\ 8 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix} \oplus 8,$$

$$T_{58} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix},$$

$$T_{59} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix} \oplus 8,$$

$$T_{60} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix} 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix},$$

$$T_{61} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix},$$

$$T_{62} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix} \oplus 8,$$

$$T_{63} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix},$$

$$T_{64} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix} \oplus 8,$$

$$T_{65} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix} 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix},$$

$$T_{66} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix},$$

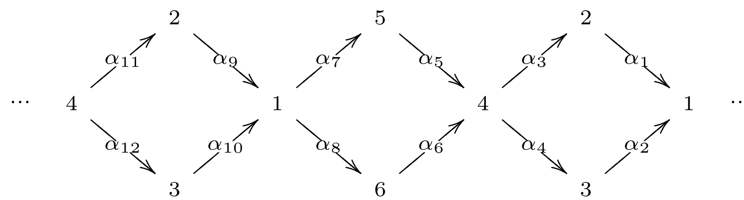
$$T_{67} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix} \oplus 8,$$

$$T_{68} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix},$$

$$T_{69} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix} \oplus 8,$$

$$T_{70} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 5 & \\ 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 3 & & 4 \\ & 5 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \\ 9 \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix} 6 & & 7 \\ & 8 & \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 8 \end{smallmatrix}.$$

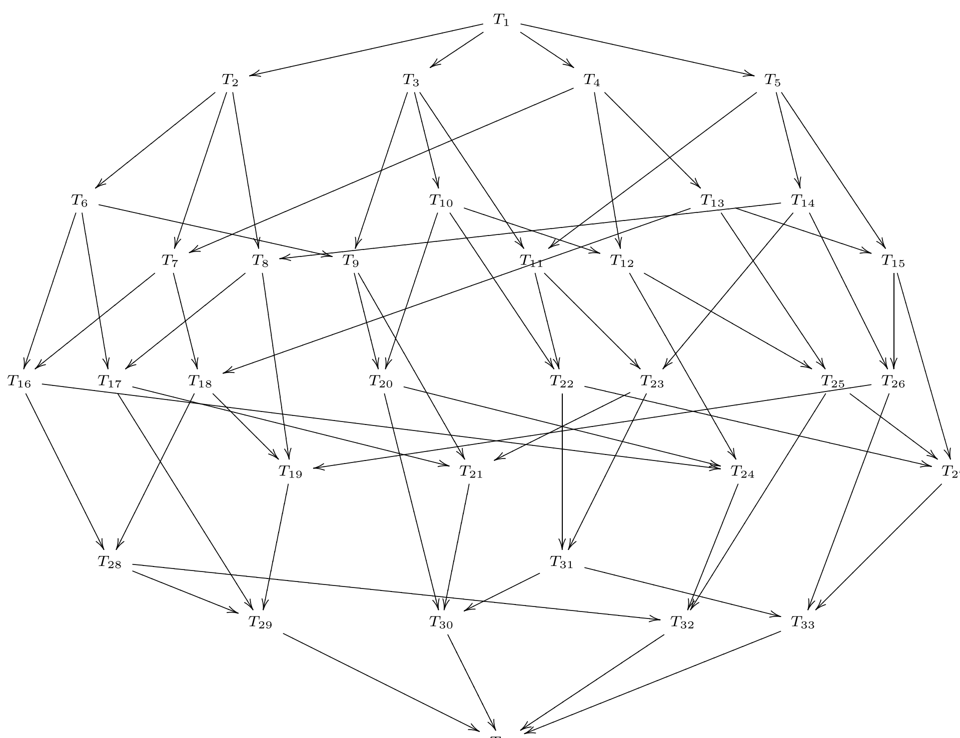
Example 4.2. Let A be an algebra given by the quiver Q: $1 \rightleftarrows 2$ with $\operatorname{rad}^3 KQ = 0$. The corresponding Auslander algebra A' is given by the quiver Q':



with relations

$$\alpha_{4i+3}\alpha_{4i+1} = \alpha_{4i+4}\alpha_{4i+2}, \quad \alpha_{4i+5}\alpha_{4i+3} = 0$$

for any $i \geq 0$. Putting $n = 2$ in Theorem 3.8(2), we get $\#\operatorname{tilt} A' = 34$. The basic tilting A'-modules are presented by the following quiver Q'':



where

$$T_1 = \begin{smallmatrix} & 1 & \\ 5 & & 6 \\ & 4 & \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 1 & \\ 5 & & 6 \\ & 4 & \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} & 4 & \\ 2 & & 3 \\ & 1 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 6 & \\ & 4 & \\ 2 & & 3 \\ & 1 & \\ & 6 & \end{smallmatrix},$$

$$T_2 = \begin{smallmatrix} 2 & & 3 \\ & 1 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 1 & \\ 5 & & 6 \\ & 4 & \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} & 4 & \\ 2 & & 3 \\ & 1 & \\ & 6 & \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 6 & \\ & 4 & \\ 2 & & 3 \\ & 1 & \\ & 6 & \end{smallmatrix},$$

$$T_3 = \begin{smallmatrix} &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_4 = \begin{smallmatrix} &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_5 = \begin{smallmatrix} &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_6 = \begin{smallmatrix} 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_7 = \begin{smallmatrix} 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_8 = \begin{smallmatrix} 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_9 = 3 \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_{10} = \begin{smallmatrix} &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_{11} = \begin{smallmatrix} &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_{12} = \begin{smallmatrix} &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_{13} = \begin{smallmatrix} &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_{14} = \begin{smallmatrix} 3 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_{15} = \begin{smallmatrix} &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix} 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_{16} = \begin{smallmatrix} 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_{17} = \begin{smallmatrix} 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} &3& \\ &1& \\ 5&&6 \\ &4& \\ &3& \end{smallmatrix} \oplus \begin{smallmatrix} &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} &6& \\ &4& \\ 2&&3 \\ &1& \\ &6& \end{smallmatrix},$$

$$T_{18}=\begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}2\\1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{19}=\begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}2\\1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}3\\1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{20}=3\oplus\begin{smallmatrix}4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}6\\4\\3\end{smallmatrix}\oplus\begin{smallmatrix}5\\4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{21}=3\oplus\begin{smallmatrix}4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}3\\1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{22}=\begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}6\\4\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{23}=\begin{smallmatrix}3\\1\\6\end{smallmatrix}\oplus\begin{smallmatrix}4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}&4&\\2&&3\\&1&\\&6&\end{smallmatrix}\oplus\begin{smallmatrix}1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{24}=3\oplus\begin{smallmatrix}6\\4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}5\\4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{25}=\begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}6\\4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{26}=\begin{smallmatrix}3\\1\\6\end{smallmatrix}\oplus\begin{smallmatrix}2\\1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{27}=\begin{smallmatrix}&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}6\\4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{28}=\begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{29}=\begin{smallmatrix}2&&3\\&1&\\&6&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}3\\1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{30}=3\oplus\begin{smallmatrix}4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}6\\4\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{31}=\begin{smallmatrix}3\\1\\6\end{smallmatrix}\oplus\begin{smallmatrix}4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}6\\4\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\6\end{smallmatrix}\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{32}=3\oplus\begin{smallmatrix}6\\4\\3\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\&1&\\5&&6\\&4&\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}5&&6\\&4&\\&3&\end{smallmatrix}\oplus 6\oplus\begin{smallmatrix}&6&\\&4&\\2&&3\\&1&\\&6&\end{smallmatrix},$$

$$T_{33} = \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ & 1 & \\ 5 & & 6 \\ & 4 & \\ & 3 & \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}1\\6\end{smallmatrix} \oplus \begin{smallmatrix} & 6 & \\ & 4 & \\ 2 & & 3 \\ & 1 & \\ & 6 & \end{smallmatrix},$$

$$T_{34} = \begin{smallmatrix}6\\4\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} & 3 & \\ & 1 & \\ 5 & & 6 \\ & 4 & \\ & 3 & \end{smallmatrix} \oplus 6 \oplus \begin{smallmatrix}3\\1\\6\end{smallmatrix} \oplus \begin{smallmatrix} & 6 & \\ & 4 & \\ 2 & & 3 \\ & 1 & \\ & 6 & \end{smallmatrix}.$$

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