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A bijection triangle in extriangulated categories

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A B S T R A C T

Extriangulated categories were introduced by Nakaoka and Palu, which unify exact categories and extension-closed subcategories of triangulated categories, and recently Iyama, Nakaoka and Palu investigated Auslander-Reiten theory in terms of Auslander-Reiten-Serre duality in extriangulated categories. In this paper, we introduce the notion of Serre duality, as a special type of Auslander-Reiten-Serre duality, and then give an equivalent condition for the existence of Serre duality. On the other hand, we study a bijection triangle in extriangulated categories which involves the restricted Auslander bijection and Auslander-Reiten-Serre duality. Let R be a commutative artinian ring. We show that the restricted Auslander bijection holds true in Hom-finite R-linear Krull-Schmidt extriangulated categories having Auslander-Reiten-Serre duality, and especially obtain the Auslander bijection in Hom-finite R-linear Krull-Schmidt extriangulated categories having Serre duality. We also give some applications, and in particular, we show that a conjecture given by Ringel holds true in this setting.

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1. Introduction

The Auslander-Reiten theory, initiated in [4,5], plays a crucial role in the representation theory of algebras and related topics. As a simultaneous generalization and enhancement of Auslander-Reiten theory in exact categories and triangulated categories, Iyama, Nakaoka and Palu [21] investigated Auslander-Reiten theory in extriangulated categories. In their work, one of important research objects is Auslander-Reiten-Serre duality. On the other hand, the theory of morphisms determined by objects was introduced by Auslander [1,2], and this theory is closely related to many aspects of representation theory of algebras, especially Auslander-Reiten theory [6]. What Auslander has achieved is a clear description of the poset structure of the category of modules as well as a blueprint for interrelating individual modules and families of modules. In [33], Ringel outlined the general setting for Auslander's ideas and gave the Auslander bijection using more clear language. The advantage of Auslander bijection is to reduce the study of morphisms to submodules, and the latter has a geometric feature via the Grassmannians of submodules. In [13,24], it was shown that the Auslander bijection holds true in dualizing varieties over a commutative artin ring and in the category of finitely generated modules over an artin algebra respectively. However, Chen [12] showed that the Auslander bijection may fail in abelian categories with Auslander-Reiten duality, and further one has to consider the restricted Auslander bijection which restricts morphisms to epimorphisms. Inspired by this, one may naturally ask how the (restricted) Auslander bijection acts on the following categories:

- exact categories,
- extension-closed subcategories of triangulated categories,
- some categories which are neither exact categories nor extension-closed subcategories of triangulated categories.

There are many important examples in the related research for the second and the third types of categories, and it is meaningful to explore some topics in this setting. For example, let A be an artin algebra and $K^{[-1,0]}(A\text{-proj})$ the category of complexes of finitely generated projective A-modules concentrated in degrees -1 and 0, with morphisms considered up to homotopy. Then $K^{[-1,0]}(A\text{-proj})$ is an extension-closed subcategory of the bounded homotopy category $K^b(A\text{-proj})$ which is not triangulated (e.g. see [21, Example 6.2]). Let $\mathfrak{C}$ be a triangulated category. Beligiannis introduced the notion of proper classes of triangles in [10], see also [19] for a more general theory in the relative case. As a particular case, let $\mathfrak{C}$ be a compactly generated triangulated category. Beligiannis [10] and independently, Krause [23] studied the class ξ of pure triangles; in this case, $(\mathfrak{C}, \mathbb{E}_\xi, \mathfrak{s}_\xi)$ is neither exact nor triangulated, where

$$\mathbb{E}_\xi(C,A) := \{\delta \in \mathrm{Hom}_{\mathfrak{C}}(C,A[1]) \mid \text{there is a pure triangle} \quad A \longrightarrow B \longrightarrow C \stackrel{\delta}{\longrightarrow} A[1] \ \text{ in } \mathfrak{C}\}$$

for any $A, C \in \mathfrak{C}$, and

$$\mathfrak{s}_\xi(\delta) := [\ A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C\ ]$$

for any $\delta \in \mathbb{E}_\xi(C,A)$ with a pure triangle $A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \stackrel{\delta}{\longrightarrow} A[1]$, see [20, Remark 3.3].

The notion of extriangulated categories was introduced by Nakaoka and Palu in [29], which is a simultaneous generalization of exact categories and extension-closed subcategories of triangulated categories. After that, the study of extriangulated categories has become an active topic, and many results on exact categories and triangulated categories have gotten realization in the setting of extriangulated categories, see [11,21], [26–31], [36–38], and so on. Recently, as one of important implications, Nakaoka and Palu [30] showed that the homotopy category of any exact quasi-category can be equipped with a natural extriangulated structure. As examples, $K^{[-1,0]}(A\text{-proj})$ and $(\mathfrak{C}, \mathbb{E}_\xi, \mathfrak{s}_\xi)$ above are extriangulated categories. For more examples of extriangulated categories, see [21, Section 6], [29, Example 2.13], [34, Example 2.8] and [36, Corollary 4.12 and Remark 4.13]. Following the philosophy of Chen [12], we expect to give the (restricted) Auslander bijection in extriangulated categories.

The paper is organized as follows.

In Section 2, we recall some terminologies and some preliminary results needed in this paper. In particular, we give the definition of the restricted Auslander bijection in extriangulated categories. In Section 3, we introduce the notions of Serre duality and right deflation-classified objects in extriangulated categories. Then we give a necessary and sufficient condition for ensuring the existence of Serre duality in terms of right (left) deflation-classified objects (Theorem 3.5). As a consequence, we show that Serre duality is a special type of Auslander-Reiten-Serre duality (Corollary 3.6). In Section 4, in an extriangulated category $\mathfrak{C}$, we establish a bijection which relates the poset of right equivalence classes of morphisms ending at an object Y with a certain condition on X to the poset of finitely generated $\mathrm{End}_{\mathfrak{C}}(X)$-submodules of $\mathbb{E}(Y,X)$ (Theorems 4.1 and 4.3), which is a key step for obtaining the (restricted) Auslander bijection. In Section 5, we exploit a bijection triangle in a Hom-finite R-linear Krull-Schmidt extriangulated category having Auslander-Reiten-Serre duality (where R is a commutative artin ring), which shows that the restricted Auslander bijection holds true (Theorem 5.4). In Section 6, we give a realization in terms of elements of stable Hom-set for a map relating the restricted Auslander bijection and Auslander-Reiten-Serre duality, and as a corollary we show that a conjecture given in [33, Section 10] holds true in this setting. In particular, we show that if $\mathfrak{C}$ is a Hom-finite R-linear Krull-Schmidt extriangulated category having Serre duality, then the Auslander bijection holds true. Some examples are discussed.

2. Preliminaries

Throughout this paper, R is a commutative artinian ring, and unless otherwise stated we always assume that $\mathfrak{C}$ is an additive category which is skeletally small.

2.1. Extriangulated categories

In [29], Nakaoka and Palu introduced the notion of extriangulated categories by a careful looking what is necessary in the definition of cotorsion pairs in exact and triangulated cases. Under this notion, both exact categories with a suitable assumption and extension-closed subcategories of triangulated categories are extriangulated ([29]), and hence, in some levels, it gives a natural framework in the study of some topics of exact categories and extension-closed subcategories of triangulated categories. Now we briefly recall some notions and some needed properties of extriangulated categories from [29].

In this subsection, $\mathfrak{C}$ is an additive category and $\mathbb{E} : \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathfrak{Ab}$ is a biadditive functor, where $\mathfrak{Ab}$ is the category of abelian groups.

Let $A, C \in \mathfrak{C}$. An element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. Two sequences of morphisms

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A \xrightarrow{x'} B' \xrightarrow{y'} C$$

are said to be *equivalent* if there exists an isomorphism $b \in \mathrm{Hom}_{\mathfrak{C}}(B, B')$ such that $x' = bx$ and $y = y'b$. We denote by $[\ A \xrightarrow{x} B \xrightarrow{y} C\ ]$ the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$. In particular, we write $0 := [\ A \xrightarrow{\binom{\mathrm{Id}_A}{0}} A \oplus C \xrightarrow{(0\ \ \mathrm{Id}_C)} C\ ]$.

For an $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$, we briefly write

$$a_\star \delta := \mathbb{E}(C, a)(\delta) \text{ and } c^\star \delta := \mathbb{E}(c, A)(\delta).$$

For two $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$, a *morphism* from δ to δ' is a pair (a, c) of morphisms with $a \in \mathrm{Hom}_{\mathfrak{C}}(A, A')$ and $c \in \mathrm{Hom}_{\mathfrak{C}}(C, C')$ such that $a_\star \delta = c^\star \delta'$.

Definition 2.1. ([29, Definition 2.9]) Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [\ A \xrightarrow{x} B \xrightarrow{y} C\ ]$ to each $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. Such $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ provided that it satisfies the following condition.

(R) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [\ A \xrightarrow{x} B \xrightarrow{y} C\ ] \text{ and } \mathfrak{s}(\delta') = [\ A' \xrightarrow{x'} B' \xrightarrow{y'} C'\ ].$$

Then for any morphism $(a, c) : \delta \to \delta'$, there exists $b \in \mathrm{Hom}_{\mathfrak{C}}(B, B')$ such that the following diagram

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \end{tikzcd}$$

commutes.

Let $\mathfrak{s}$ be a realization of $\mathbb{E}$. If $\mathfrak{s}(\delta) = [\ A \xrightarrow{x} B \xrightarrow{y} C\ ]$ for some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$, then we say that the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ *realizes* δ; and in the condition (R), we say that the triple (a, b, c) *realizes* the morphism (a, c).

For any two equivalence classes $[\ A \xrightarrow{x} B \xrightarrow{y} C\ ]$ and $[\ A' \xrightarrow{x'} B' \xrightarrow{y'} C'\ ]$, we define

$$[\ A \xrightarrow{x} B \xrightarrow{y} C\ ] \oplus [\ A' \xrightarrow{x'} B' \xrightarrow{y'} C'\ ] := [\ A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'\ ].$$

Definition 2.2. ([29, Definition 2.10]) A realization $\mathfrak{s}$ of $\mathbb{E}$ is called *additive* if it satisfies the following conditions.

(1) For any $A, C \in \mathfrak{C}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C, A)$ satisfies $\mathfrak{s}(0) = 0$.
(2) For any pair of $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$, we have $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.

Definition 2.3. ([29, Definition 2.12]) The triple $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ is called an *externally triangulated* (or *extriangulated* for short) category if it satisfies the following conditions.

(ET1) $\mathbb{E} : \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathfrak{Ab}$ is a biadditive functor.
(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.
(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [\ A \xrightarrow{x} B \xrightarrow{y} C\ ] \text{ and } \mathfrak{s}(\delta') = [\ A' \xrightarrow{x'} B' \xrightarrow{y'} C'\ ].$$

For any commutative diagram

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \end{tikzcd}$$

in $\mathfrak{C}$, there exists a morphism $(a, c) : \delta \to \delta'$ which is realized by the triple (a, b, c).
(ET3)$^{\mathrm{op}}$ Dual of (ET3).

(ET4) Let $\delta \in \mathbb{E}(C,A)$ and $\rho \in \mathbb{E}(F,B)$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [\ A \xrightarrow{x} B \xrightarrow{y} C\ ] \text{ and } \mathfrak{s}(\rho) = [\ B \xrightarrow{u} D \xrightarrow{v} F\ ].$$

Then there exist an object $E \in \mathfrak{C}$, an $\mathbb{E}$-extension ξ with $\mathfrak{s}(\xi) = [\ A \xrightarrow{z} D \xrightarrow{w} E\ ]$, and a commutative diagram

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, equal] & B \arrow[r, "y"] \arrow[d, "u"] & C \arrow[d, "s"] \\ A \arrow[r, "z"] & D \arrow[r, "w"] \arrow[d, "v"] & E \arrow[d, "t"] \\ & F \arrow[r, equal] & F \end{tikzcd}$$

in $\mathfrak{C}$, which satisfy the following compatibilities.

(i) $\mathfrak{s}(y_\star\rho) = [\ C \xrightarrow{s} E \xrightarrow{t} F\ ]$.
(ii) $s^\star\xi = \delta$.
(iii) $x_\star\xi = t^\star\rho$.

(ET4)$^{\mathrm{op}}$ Dual of (ET4).

Let $\mathfrak{C}$ be an extriangulated category. For any $X, Y \in \mathfrak{C}$, it is easy to see that $\mathbb{E}(Y,X)$ is an $\mathrm{End}_{\mathfrak{C}}(X)$-module with the module structure $f \cdot \delta := f_\star\delta$. Any $\mathbb{E}$-extension $\delta \in \mathbb{E}(Y,Z)$ induces a natural transformation (see [29, Definition 3.1])

$$\delta^\sharp : \mathrm{Hom}_{\mathfrak{C}}(Z,-) \longrightarrow \mathbb{E}(Y,-),$$

such that for any $X \in \mathfrak{C}$, there is a map

$$\begin{aligned} \delta^\sharp_X : \mathrm{Hom}_{\mathfrak{C}}(Z,X) &\longrightarrow \mathbb{E}(Y,X), \\ f &\mapsto f_\star\delta \end{aligned}$$

which is a morphism of $\mathrm{End}_{\mathfrak{C}}(X)$-modules. Thus $\mathrm{Im}\,\delta^\sharp_X$ is an $\mathrm{End}_{\mathfrak{C}}(X)$-submodule of $\mathbb{E}(Y,X)$.

Dually, one has the definition of $\delta_\sharp : \mathrm{Hom}_{\mathfrak{C}}(-,Y) \longrightarrow \mathbb{E}(-,Z)$.

In [31], Ogawa introduced the notion of defects over extriangulated categories, that is, given an $\mathfrak{s}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, there is an exact sequence

$$\mathrm{Hom}_{\mathfrak{C}}(-,A) \xrightarrow{\mathrm{Hom}_{\mathfrak{C}}(-,x)} \mathrm{Hom}_{\mathfrak{C}}(-,B) \xrightarrow{\mathrm{Hom}_{\mathfrak{C}}(-,y)} \mathrm{Hom}_{\mathfrak{C}}(-,C) \longrightarrow \widetilde{\delta} \longrightarrow 0.$$

The functor $\widetilde{\delta}$ is called a *defect* of δ (see [31, Definition 2.4]). In fact, by [31, Lemma 1.17], $\widetilde{\delta} \cong \operatorname{Im} \delta_\sharp$. Similarly, one can define $\widehat{\delta} := \operatorname{Coker} \operatorname{Hom}_{\mathfrak{C}}(x, -)$ and then $\widehat{\delta} \cong \operatorname{Im} \delta^\sharp$. See [7,15] for more study on defects.

Definition 2.4. ([21, Definition 1.16]) Let $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1) and (ET2).

(1) If a sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes an $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$, then the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ is called an $\mathfrak{s}$-*triangle*, and write it in the following way

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} .$$

In this case, x is called an $\mathfrak{s}$-*inflation*, and y is called an $\mathfrak{s}$-*deflation*.

(2) Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathfrak{s}$-triangles. If a triple (a, b, c) realizes $(a, c) : \delta \to \delta'$ as in the condition (R), then we write it as

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd},$$

and call the triple (a, b, c) a *morphism of* $\mathfrak{s}$-*triangles*.

The following lemma is used frequently in this paper.

Lemma 2.5. ([29, Corollary 3.5]) *Assume that* $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ *satisfies* (ET1), (ET2), (ET3) *and* (ET3)$^{\mathrm{op}}$. *Let*

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[r, dashed, "\delta"] \arrow[d, "c"] & {} \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd},$$

be any morphism of $\mathfrak{s}$-*triangles. Then the following statements are equivalent.*

(1) *a factors through x.*
(2) $a_\star \delta = c^\star \delta' = 0$.
(3) *c factors through y'.*

In particular, in the case $\delta = \delta'$ *and* $(a, b, c) = (\mathrm{Id}_A, \mathrm{Id}_B, \mathrm{Id}_C)$, *we have*

$$x \text{ is a section} \Leftrightarrow \delta \text{ is split} \Leftrightarrow y \text{ is a retraction}.$$

Definition 2.6. Assume that the pair $(\mathfrak{C}, \mathbb{E})$ satisfies (ET1).

(1) Let $f \in \mathrm{Hom}_{\mathfrak{C}}(C', C)$ be a morphism. We call f a *projective morphism* if $\mathbb{E}(f, A) = 0$, and an *injective morphism* if $\mathbb{E}(A, f) = 0$ for any $A \in \mathfrak{C}$.
(2) Let $C \in \mathfrak{C}$. We call C a *projective object* if the identity morphism Id_C is projective, and an *injective object* if the identity morphism Id_C is injective.

We denote by $\mathcal{P}$ (respectively, $\mathcal{I}$) the ideal of $\mathfrak{C}$ consisting of all projective (respectively, injective) morphisms. Define

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathfrak{C}}(C, Y) &:= \mathrm{Hom}_{\mathfrak{C}}(C, Y)/\mathcal{P}(C, Y) \text{ (respectively, } \overline{\mathrm{Hom}}_{\mathfrak{C}}(C, Y) \\ &:= \mathrm{Hom}_{\mathfrak{C}}(C, Y)/\mathcal{I}(C, Y)),\end{aligned}$$

and

$$\underline{\mathfrak{C}} := \mathfrak{C}/\mathcal{P} \text{ (respectively, } \overline{\mathfrak{C}} := \mathfrak{C}/\mathcal{I}).$$

In [21], Iyama, Nakaoka and Palu introduced almost split $\mathbb{E}$-extensions and $\mathfrak{s}$-triangles as follows.

Definition 2.7. ([21, Definition 2.1]) Assume that the pair $(\mathfrak{C}, \mathbb{E})$ satisfies (ET1). A non-split (i.e. non-zero) $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$ is said to be *almost split* if it satisfies the following conditions.

(AS1) $a_\star\delta = 0$ for any non-section $a \in \mathrm{Hom}_{\mathfrak{C}}(A, A')$.
(AS2) $c^\star\delta = 0$ for any non-retraction $c \in \mathrm{Hom}_{\mathfrak{C}}(C', C)$.

Definition 2.8. ([21, Definition 2.7]) Assume that the triple $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1) and (ET2). An $\mathfrak{s}$-triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

in $\mathfrak{C}$ is called *almost split* if δ is an almost split $\mathbb{E}$-extension.

2.2. Auslander-Reiten-Serre duality

In this subsection, $\mathfrak{C}$ is a Hom-finite R-linear Krull-Schmidt extriangulated category. We denote by $D = \mathrm{Hom}_R(-, E)$, where E is the minimal injective cogenerator for R.

Recently, Iyama, Nakaoka and Palu [21] introduced the notion of Auslander-Reiten-Serre duality in order to study the existence of almost split extensions. More precisely, the category $\mathfrak{C}$ is said to *have Auslander-Reiten-Serre duality* provided that there exists an R-linear equivalence $\tau : \underline{\mathfrak{C}} \to \overline{\mathfrak{C}}$ with an R-linear natural isomorphism

$$\Phi_{X,Y} : D\mathbb{E}(X,Y) \longrightarrow \overline{\mathrm{Hom}}_{\mathfrak{C}}(Y, \tau X)$$

for any $X, Y \in \mathfrak{C}$. The equivalence τ is called the *Auslander-Reiten translation* of $\mathfrak{C}$.

Example 2.9. Let A be a finite-dimensional algebra over a field k and A-mod the category of finitely generated left A-modules. We use A-proj to denote the subcategory of A-mod consisting of projective modules, and use A-Gproj to denote the subcategory of A-mod consisting of Gorenstein projective modules (e.g. see [14,16] for the definition).

(1) It is well known that A-mod has Auslander-Reiten-Serre duality. Moreover, if A is self-injective, then the stable category A-$\underline{\mathrm{mod}}$ has Auslander-Reiten-Serre duality ([18]).
(2) The category of finitely presented functors on a dualizing k-variety has Auslander-Reiten-Serre duality ([3, Proposition 3.2]).
(3) If $\mathcal{X}$ is an extension-closed functorially finite subcategory of the bounded homotopy category $K^b(A\text{-proj})$, then $\mathcal{X}$ has Auslander-Reiten-Serre duality ([21, Proposition 6.1]).
(4) If the global dimension of A is finite, then the bounded derived category $D^b(A)$ has Auslander-Reiten-Serre duality ([18]).
(5) If A is Gorenstein (that is, the left and right self-injective dimensions of A are finite), then the stable category A-$\underline{\mathrm{Gproj}}$ has Auslander-Reiten-Serre duality. In fact, since A-Gproj is an extension-closed functorially finite subcategory of A-mod, A-Gproj has almost split sequences, and they induce almost split triangles in A-$\underline{\mathrm{Gproj}}$. Moreover, if A is Gorenstein and A-Gproj is of finite type, then the bounded Gorenstein derived category $D^b_{gp}(A)$ has Auslander-Reiten-Serre duality ([16]).
(6) Let Q be a connected locally finite interval-finite quiver. Then the category $\mathrm{rep}\, Q$ of finitely presented representations of Q has Auslander-Reiten-Serre duality if and only if either Q has neither left infinite paths nor right infinite paths, or Q itself is a left infinite path ([9, Theorem 3.7] and [22, Corollary 4.5]).
(7) Let $\mathcal{A}$ be a Hom-finite k-linear abelian category. Then $\mathcal{A}$ has Auslander-Reiten-Serre duality if and only if it has almost split sequences ([25, Theorem 1.1]).
(8) Let $\mathfrak{C}$ be an Ext-finite k-linear Krull-Schmidt extriangulated category. Then $\mathfrak{C}$ has Auslander-Reiten-Serre duality if and only if it has almost split extensions ([21, Theorem 3.4]).

For the equivalence τ, we denote by τ^- a quasi-inverse of τ. Then it is well known that the pair (τ^-, τ) is an adjoint pair. We denote by the counit $\theta : \tau^-\tau \to \mathrm{Id}_{\underline{\mathfrak{C}}}$ and the unit $\epsilon : \mathrm{Id}_{\overline{\mathfrak{C}}} \to \tau\tau^-$. There is an isomorphism

$$\begin{array}{rcl} \vartheta_{X,Y} : \ \overline{\mathrm{Hom}}_{\mathfrak{C}}(Y, \tau X) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- Y, X). \\ f & \mapsto & \theta_X \tau^-(f) \end{array}$$

for any $X, Y \in \mathfrak{C}$. Following this isomorphism, there is also a natural isomorphism

$$\begin{array}{rccl} \Psi_{X,Y}: & D\mathbb{E}(X,Y) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^{-}Y, X). \\ & f & \mapsto & \vartheta_{\tau^{-}X,Y}(\Phi_{\tau^{-}X,Y}(f)) \end{array}$$

for any $X, Y \in \mathfrak{C}$.

2.3. Morphisms determined by objects

We now recall the concept of morphisms determined by objects, which was introduced by Auslander in [1,2] and closely related to the Auslander-Reiten theory [6] and the Auslander bijections [33].

Definition 2.10. ([1]) Let $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ and $C \in \mathfrak{C}$. Then f is called *right C-determined* (or *right determined by C*) if the following condition is satisfied: given any morphism $f' \in \mathrm{Hom}_{\mathfrak{C}}(X', Y)$ such that $f'\phi$ factors through f for each $\phi \in \mathrm{Hom}_{\mathfrak{C}}(C, X')$, then f' itself factors through f. This can be described by the following commutative diagrams:

$$\begin{tikzcd} C \arrow[r, "\forall\phi"] \arrow[d, dotted, "\exists\phi'"'] & X' \arrow[d, "f'"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & X' \arrow[d, "f'"] \arrow[dl, dotted, "\exists g"'] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

For an object $C \in \mathfrak{C}$, we denote by $\mathrm{add}\, C$ the subcategory consisting of direct summands of finite direct sums of C. If $\mathfrak{C}$ is a Hom-finite R-linear additive category, then morphisms determined by objects have some functorial characterization as follows, see [24,13] for more details.

Lemma 2.11. ([24, Proposition 5.2] and [13, Lemma 2.3]) *Assume that $\mathfrak{C}$ is a Hom-finite R-linear additive category. Let $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$. Define $F^{\alpha} = \mathrm{Coker}\,\mathrm{Hom}_{\mathfrak{C}}(-, \alpha)$. Then α is right C-determined if and only if there is a monomorphism $F^{\alpha} \to D\,\mathrm{Hom}_{\mathfrak{C}}(C', -)$ for some $C' \in \mathrm{add}\, C$.*

Let $Y \in \mathfrak{C}$. For any $C \in \mathfrak{C}$ and any $\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}$-submodule H of $\mathrm{Hom}_{\mathfrak{C}}(C, Y)$, since $D\,\mathrm{Hom}_{\mathfrak{C}}(C, C)$ is an injective cogenerator, there is an embedding

$$\mathrm{Hom}_{\mathfrak{C}}(C, Y)/H \hookrightarrow D\,\mathrm{Hom}_{\mathfrak{C}}(C', C)$$

with $C' \in \mathrm{add}\, C$. This gives a morphism $\varpi : \mathrm{Hom}_{\mathfrak{C}}(-, Y) \to D\,\mathrm{Hom}_{\mathfrak{C}}(C', -)$. Define $F^{(C,H)} = \mathrm{Im}\varpi$.

Lemma 2.12. ([13, Lemma 2.4]) *Assume that $\mathfrak{C}$ is a Hom-finite R-linear additive category. Let H be an* $\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}$*-submodule of* $\mathrm{Hom}_{\mathfrak{C}}(C,Y)$*. Then $\alpha : X \to Y$ is right C-determined and* $\mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C,\alpha) = H$ *if and only if the functor $F^{(C,H)}$ is finitely presented.*

2.4. The Auslander bijection

Definition 2.13. ([33]) Let $\alpha_1 \in \mathrm{Hom}_{\mathfrak{C}}(X_1,Y)$ and $\alpha_2 \in \mathrm{Hom}_{\mathfrak{C}}(X_2,Y)$. Then α_1 and α_2 are called *right equivalent* if α_1 factors through α_2, and α_2 factors through α_1.

Remark 2.14.

(1) This relation is an equivalence relation on the set of all morphisms ending in some object $Y \in \mathfrak{C}$.
(2) If α_1 and α_2 are right equivalent, then α_1 is right C-determined if and only if α_2 is right C-determined.
(3) If α_1 and α_2 are right equivalent, then $\mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C,\alpha_1) = \mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C,\alpha_2)$.
(4) If α_1 and α_2 are right C-determined, then α_1 and α_2 are right equivalent if and only if $\mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C,\alpha_1) = \mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C,\alpha_2)$.

We denote by $[\alpha\rangle$ the right equivalence class of a morphism $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(X,Y)$.

Definition 2.15. ([33]) Let $\alpha_1 \in \mathrm{Hom}_{\mathfrak{C}}(X_1,Y)$ and $\alpha_2 \in \mathrm{Hom}_{\mathfrak{C}}(X_2,Y)$. Define

$$[\alpha_1\rangle \leq [\alpha_2\rangle \text{ provided that } \alpha_1 \text{ factors through } \alpha_2.$$

We denote by $[\to Y\rangle$ the set of right equivalence classes of morphisms ending in Y. Then $\leq$ induces a poset relation on $[\to Y\rangle$.

By Remark 2.14(2), we say that $[\alpha\rangle$ is *right C-determined* if a representative element α is right C-determined. We denote by ${}^{C}[\to Y\rangle$ the subset of $[\to Y\rangle$ consisting of all right equivalence classes that are right C-determined.

We denote by $\mathrm{Sub}_{\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}}\mathrm{Hom}_{\mathfrak{C}}(C,Y)$ the poset formed by $\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}$-submodules of $\mathrm{Hom}_{\mathfrak{C}}(C,Y)$. Then by Remark 2.14(3), we have a well-defined map

$$\begin{array}{rccl} \eta_{C,Y}: & [\to Y\rangle & \longrightarrow & \mathrm{Sub}_{\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}}\mathrm{Hom}_{\mathfrak{C}}(C,Y). \\ & [\alpha\rangle & \mapsto & \mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C,\alpha) \end{array}$$

Remark 2.16. The restriction of $\eta_{C,Y}$ on ${}^{C}[\to Y\rangle$ is injective and reflects the orders, that is, for $[\alpha_1\rangle, [\alpha_2\rangle \in {}^{C}[\to Y\rangle$, we have that $[\alpha_1\rangle \leq [\alpha_2\rangle$ if and only if $\eta_{C,Y}([\alpha_1\rangle) \subseteq \eta_{C,Y}([\alpha_2\rangle)$.

If the map $\eta_{C,Y} : {}^{C}[\to Y\rangle \longrightarrow \mathrm{Sub}_{\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}}\mathrm{Hom}_{\mathfrak{C}}(C,Y)$ above is surjective, then we say that *the Auslander bijection at Y relative to C* holds, see [12,33] for more details.

2.5. The restricted Auslander bijection

Let $\mathfrak{C}$ be an extriangulated category.

Since each $\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}$-submodule of $\underline{\mathrm{Hom}}_{\mathfrak{C}}(C,Y)$ corresponds to a unique $\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}$-submodule of $\mathrm{Hom}_{\mathfrak{C}}(C,Y)$ containing $\mathcal{P}(C,Y)$, the poset $\mathrm{Sub}_{\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}}\underline{\mathrm{Hom}}_{\mathfrak{C}}(C,Y)$ is viewed as a subset of $\mathrm{Sub}_{\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}}\mathrm{Hom}_{\mathfrak{C}}(C,Y)$.

In the rest of this paper, we assume that $\mathfrak{C}$ satisfies the following *weak idempotent completeness* (WIC for short) given originally in [29, Condition 5.8].

WIC Condition:

(1) Let $f\in\mathrm{Hom}_{\mathfrak{C}}(A,B)$, $g\in\mathrm{Hom}_{\mathfrak{C}}(B,C)$ be any composable pair of morphisms. If gf is an $\mathfrak{s}$-inflation, then so is f.
(2) Let $f\in\mathrm{Hom}_{\mathfrak{C}}(A,B)$, $g\in\mathrm{Hom}_{\mathfrak{C}}(B,C)$ be any composable pair of morphisms. If gf is an $\mathfrak{s}$-deflation, then so is g.

Under this condition, we know that if $\alpha_1\in\mathrm{Hom}_{\mathfrak{C}}(X_1,Y)$ and $\alpha_2\in\mathrm{Hom}_{\mathfrak{C}}(X_2,Y)$ are right equivalent, then α_1 is an $\mathfrak{s}$-deflation if and only if so is α_2, and we can define

$$[\to Y\rangle_{\mathrm{def}} := \{[\alpha\rangle\in[\to Y\rangle \mid \alpha \text{ is an } \mathfrak{s}\text{-deflation}\}.$$

Clearly, $\mathcal{P}(C,Y)\subseteq \mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C,\alpha)$ for any $[\alpha\rangle\in[\to Y\rangle_{\mathrm{def}}$. This implies the following map

$$\begin{array}{rccl}\eta_{C,Y}: & [\to Y\rangle_{\mathrm{def}} & \longrightarrow & \mathrm{Sub}_{\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}}\underline{\mathrm{Hom}}_{\mathfrak{C}}(C,Y).\\ & [\alpha\rangle & \mapsto & \mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C,\alpha)/\mathcal{P}(C,Y)\end{array}$$

Set ${}^{C}[\to Y\rangle_{\mathrm{def}} := [\to Y\rangle_{\mathrm{def}}\cap {}^{C}[\to Y\rangle$. Then we have a map

$$\begin{array}{rccl}\eta_{C,Y}: & {}^{C}[\to Y\rangle_{\mathrm{def}} & \longrightarrow & \mathrm{Sub}_{\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}}\underline{\mathrm{Hom}}_{\mathfrak{C}}(C,Y),\\ & [\alpha\rangle & \mapsto & \mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C,\alpha)/\mathcal{P}(C,Y)\end{array}$$

which is injective by Remark 2.16. Inspired by [12], we introduce the following

Definition 2.17. If the map $\eta_{C,Y}: {}^{C}[\to Y\rangle_{\mathrm{def}}\longrightarrow \mathrm{Sub}_{\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}}\underline{\mathrm{Hom}}_{\mathfrak{C}}(C,Y)$ above is surjective, then we say that *the restricted Auslander bijection at* Y *relative to* C holds.

3. Extriangulated categories having Serre duality

In this section, $\mathfrak{C}$ is a Hom-finite R-linear Krull-Schmidt extriangulated category.

Definition 3.1. We say that $\mathfrak{C}$ has *Serre duality* if there exists an R-linear auto-equivalence $\tau:\mathfrak{C}\to\mathfrak{C}$ with a natural isomorphism

$$\varphi_{X,Y} : D\mathbb{E}(X,Y) \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(Y, \tau X) \tag{3.1}$$

for any $X, Y \in \mathfrak{C}$.

If $\mathfrak{C}$ has Serre duality as above, then for any projective object P, we have

$$\mathrm{Hom}_{\mathfrak{C}}(\tau P, \tau P) \cong D\mathbb{E}(P, \tau P) = 0,$$

which implies $\tau P = 0$. Thus τ induces a functor $\tau : \underline{\mathfrak{C}} \to \mathfrak{C}$. Similarly, τ^{-} induces a functor $\tau^{-} : \overline{\mathfrak{C}} \to \mathfrak{C}$.

In [13], Chen and Le introduced the notion of right epimorphism-classified objects in abelian categories in order to describe the existence of Serre duality. Inspired by this, we give the following definition. We will show that $\mathfrak{s}$-deflations satisfying the following assumptions are essential in this process.

Definition 3.2. An object $Y \in \mathfrak{C}$ is called *right deflation-classified* if the following conditions are satisfied.

(RDC1) Each $\mathfrak{s}$-deflation $\alpha : X \to Y$ is right C-determined for some $C \in \mathfrak{C}$.
(RDC2) For any $C \in \mathfrak{C}$ and $H \in \mathrm{Sub}_{\mathrm{End}_{\mathfrak{C}}(C)^{\mathrm{op}}} \mathrm{Hom}_{\mathfrak{C}}(C, Y)$, there exists an $\mathfrak{s}$-deflation $\alpha : X \to Y$ such that α is right C-determined and $\mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C, \alpha) = H$.

The category $\mathfrak{C}$ is said to *have right determined deflations* if each object in $\mathfrak{C}$ is right deflation-classified. Dually, *left inflation-classified objects* and *the category having left determined inflations* are defined.

Let $\mathcal{T}$ be a triangulated category with $\mathbb{E} := \mathrm{Hom}_{\mathcal{T}}(-, -[1])$. Then $\mathcal{T}$ is an extriangulated category in the sense of [29, Proposition 3.22]. In this case, each morphism in $\mathcal{T}$ is an $\mathfrak{s}$-deflation, thus each morphism $\alpha : X \to Y$ in $\mathcal{T}$ with Y right deflation-classified is right C-determined by some $C \in \mathcal{T}$.

The following result shows that if $\mathfrak{C}$ has right determined deflations, then all morphisms determined by some objects are exactly all $\mathfrak{s}$-deflations.

Proposition 3.3. *Assume that $Y \in \mathfrak{C}$ is right deflation-classified. Then for any $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$, α is right C-determined for some $C \in \mathfrak{C}$ if and only if α is an $\mathfrak{s}$-deflation.*

Proof. The sufficiency follows from (RDC1). For the necessity, set $H := \mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C, \alpha)$. By (RDC2), there exists an $\mathfrak{s}$-deflation $\alpha' : X' \to Y$ such that α' is right C-determined and $\mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C, \alpha') = H$. By Remark 2.16(4), α and α' are right equivalent, which shows that α is an $\mathfrak{s}$-deflation. $\square$

In [13, Proposition 3.3], it was proved that each indecomposable non-projective right epimorphism-classified object can be viewed as the ending term for some almost split sequence in abelian categories. Now we can show the existence of almost split $\mathfrak{s}$-triangles ending at right deflation-classified objects in the setting of extriangulated categories.

Proposition 3.4. *Assume that $Y \in \mathfrak{C}$ is right deflation-classified. If Y is indecomposable and non-projective, then there exists an almost split $\mathfrak{s}$-triangle*

$$K \xrightarrow{\iota} X \xrightarrow{\alpha} Y \dashrightarrow .$$

Proof. Set $H := \operatorname{rad}\operatorname{End}_{\mathfrak{C}}(Y)$. By (RDC2), there exists an $\mathfrak{s}$-deflation $\alpha : X \to Y$ such that α is right Y-determined and $\operatorname{Im}\operatorname{Hom}_{\mathfrak{C}}(Y, \alpha) = \operatorname{rad}\operatorname{End}_{\mathfrak{C}}(Y)$. Without loss of generality, we may assume that α is right minimal. Take an $\mathfrak{s}$-triangle

$$K \xrightarrow{\iota} X \xrightarrow{\alpha} Y \dashrightarrow .$$

We claim that it is almost split. Indeed, we have

- α is minimal right almost split: For any non-retraction $f \in \operatorname{Hom}_{\mathfrak{C}}(X', Y)$, clearly fg is non-retraction for any $g \in \operatorname{Hom}_{\mathfrak{C}}(Y, X')$, that is, $fg \in \operatorname{rad}\operatorname{End}_{\mathfrak{C}}(Y)$ since Y is indecomposable. But $\operatorname{Im}\operatorname{Hom}_{\mathfrak{C}}(Y, \alpha) = \operatorname{rad}\operatorname{End}_{\mathfrak{C}}(Y)$, so fg factors through α. Moreover, since α is right Y-determined, we have that f factors through α.
- K is non-injective: Since $\operatorname{Id}_Y \notin \operatorname{rad}\operatorname{End}_{\mathfrak{C}}(Y)$.
- K is indecomposable: Suppose $K = \bigoplus_{i=1}^{n} K_i$ with all K_i indecomposable. Since ι is not a section, there exists some K_i with $1 \le i \le n$ such that the projection $K \to K_i$ does not factor through ι. Consider the following morphism of $\mathfrak{s}$-triangles

$$\begin{tikzcd} K \arrow[r, "\iota"] \arrow[d, "\iota_i"] & X \arrow[r, "\alpha"] \arrow[d, dotted, "t"] & Y \arrow[r, dashed] \arrow[d, equal] & {} \\ K_i \arrow[r] & Z \arrow[r, "\beta"] & Y \arrow[r, dashed] & {} \end{tikzcd}$$

  Then β is not a retraction by Lemma 2.5. Now for any non-retraction $s \in \operatorname{Hom}_{\mathfrak{C}}(L, Y)$, s factors through α, and thus s factors through β, which shows that β is right almost split. Moreover, K_i is indecomposable implies that β is right minimal. Thus β is also minimal right almost split. It follows that t is an isomorphism, and hence ι_i is an isomorphism, that is, K is indecomposable.

Therefore, the $\mathfrak{s}$-triangle

$$K \xrightarrow{\iota} X \xrightarrow{\alpha} Y \dashrightarrow$$

is almost split by [21, Proposition 2.5]. $\square$

Now assume that $Y \in \mathfrak{C}$ is right deflation-classified. For any $Z \in \mathfrak{C}$, setting $H = 0$, then by (RDC2) there exists an $\mathfrak{s}$-deflation $\alpha : X \to Y$ such that α is right Z-determined and $\operatorname{Im}\operatorname{Hom}_{\mathfrak{C}}(Z,\alpha) = 0$. Moreover, for any projective morphism $f : Z \to Y$, f factors through α, that is, $f \in \operatorname{Im}\operatorname{Hom}_{\mathfrak{C}}(Z,\alpha)$, which implies $f = 0$. Thus, if $\mathfrak{C}$ has right determined deflations, then $\mathcal{P} = \{0\}$ and $\mathfrak{C} = \underline{\mathfrak{C}}$. Similarly, if $\mathfrak{C}$ has left determined inflations, then $\mathcal{I} = \{0\}$ and $\mathfrak{C} = \overline{\mathfrak{C}}$.

In the following result, we give an equivalent characterization for extriangulated categories having Serre duality, which is an extriangulated version of [13, Theorem 3.4].

Theorem 3.5. *The category $\mathfrak{C}$ has Serre duality if and only if $\mathfrak{C}$ has right determined deflations and left determined inflations.*

Proof. Assume that $\mathfrak{C}$ has Serre duality as in (3.1). Let $Y \in \mathfrak{C}$. For any $\mathfrak{s}$-deflation $\alpha \in \operatorname{Hom}_{\mathfrak{C}}(X,Y)$, take an $\mathfrak{s}$-triangle

$$K \longrightarrow X \xrightarrow{\alpha} Y \dashrightarrow .$$

By [29, Proposition 3.3], there is an exact sequence

$$\operatorname{Hom}_{\mathfrak{C}}(-,X) \xrightarrow{\operatorname{Hom}_{\mathfrak{C}}(-,\alpha)} \operatorname{Hom}_{\mathfrak{C}}(-,Y) \longrightarrow \mathbb{E}(-,K) .$$

By assumption, $\mathbb{E}(-,K) \cong D\operatorname{Hom}_{\mathfrak{C}}(\tau^{-}K,-)$. Thus there is a monomorphism

$$\operatorname{Coker}\operatorname{Hom}_{\mathfrak{C}}(-,\alpha) \to D\operatorname{Hom}_{\mathfrak{C}}(\tau^{-}K,-).$$

By Lemma 2.11, α is right $\tau^{-}K$-determined and (RDC1) holds.

Now let $C \in \mathfrak{C}$ and H an $\operatorname{End}_{\mathfrak{C}}(C)^{\mathrm{op}}$-submodule of $\operatorname{Hom}_{\mathfrak{C}}(C,Y)$. Consider the morphism $\varpi : \operatorname{Hom}_{\mathfrak{C}}(-,Y) \to D\operatorname{Hom}_{\mathfrak{C}}(C',-)$ defined just before Lemma 2.12, where $C' \in \operatorname{add} C$ and $F^{(C,H)} = \operatorname{Im}\varpi$. By assumption, $D\operatorname{Hom}_{\mathfrak{C}}(C',-) \cong \mathbb{E}(-,\tau C')$, and by combining ϖ we have a morphism $\varpi' : \operatorname{Hom}_{\mathfrak{C}}(-,Y) \to \mathbb{E}(-,\tau C')$ with $\operatorname{Im}\varpi' \cong \operatorname{Im}\varpi = F^{(C,H)}$. Let $\varpi'_Y(\mathrm{Id}_Y) = \delta \in \mathbb{E}(Y,\tau C')$ with an $\mathfrak{s}$-triangle $\tau C' \longrightarrow X \xrightarrow{\alpha} Y \dashrightarrow$. By [29, Proposition 3.3], there is an exact sequence

$$\operatorname{Hom}_{\mathfrak{C}}(-,X) \xrightarrow{\operatorname{Hom}_{\mathfrak{C}}(-,\alpha)} \operatorname{Hom}_{\mathfrak{C}}(-,Y) \xrightarrow{\delta_\sharp} \mathbb{E}(-,\tau C') .$$

We have $(\delta_\sharp)_Y(\mathrm{Id}_Y) = \delta = \varpi'_Y(\mathrm{Id}_Y)$. Thus by the Yoneda lemma, we have $\delta_\sharp = \varpi'$, and hence $\operatorname{Im}\delta_\sharp = \operatorname{Im}\varpi' \cong F^{(C,H)}$, which shows that $F^{(C,H)}$ is finitely presented. By Lemma 2.12, (RDC2) holds. This shows that Y is right deflation-classified. By the arbitrariness of Y, $\mathfrak{C}$ has right determined deflations. Dually, we have that $\mathfrak{C}$ has left determined inflations.

Conversely, assume that $\mathfrak{C}$ has right determined deflations and left determined inflations. Then we have $\mathfrak{C} = \overline{\mathfrak{C}} = \underline{\mathfrak{C}}$. For any indecomposable non-projective object Y, there exists an almost split $\mathfrak{s}$-triangle ending at Y by Proposition 3.4. Dually, for any indecomposable non-injective object X, there exists an almost split $\mathfrak{s}$-triangle starting from X. It follows that $\mathfrak{C}$ has almost split $\mathfrak{s}$-triangles. By [21, Theorem 3.4], $\mathfrak{C}$ has Auslander-Reiten-Serre duality (τ, φ). In particular, since $\mathfrak{C} = \overline{\mathfrak{C}} = \underline{\mathfrak{C}}$, we have that (τ, φ) is a Serre duality. □

By the proof of Theorem 3.5, we know that if $\mathfrak{C}$ has Serre duality, then $\mathfrak{C}$ has Auslander-Reiten-Serre duality. In fact, we can get the following corollary.

Corollary 3.6. *$\mathfrak{C}$ has Serre duality if and only if $\mathfrak{C}$ has Auslander-Reiten-Serre duality and $\mathcal{P} = \{0\} = \mathcal{I}$.*

Example 3.7.

(1) Let $\mathrm{coh}(\mathbf{C})$ be the category of coherent sheaves for a weighted projective line $\mathbf{C}$. Then $\mathrm{coh}(\mathbf{C})$ has Serre duality ([17, 2.2]).
(2) Let A be a finite-dimensional algebra over a field k. Then A-mod has no Serre duality.
(3) The category $\mathrm{rep}\, Q$ appeared in Example 2.9(6) has no Serre duality.
(4) For any triangulated category, it has Auslander-Reiten-Serre duality if and only if it has Serre duality. Thus all the categories A-$\underline{\mathrm{mod}}$, $D^b(A)$, A-$\underline{\mathrm{Gproj}}$ and $D^b_{gp}(A)$ appeared in Example 2.9 have Serre duality under the corresponding assumptions.

4. A map from ${}^{\tau^{-1}X}[\to Y\rangle_{\mathrm{def}}$ to $\mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(X)}\, \mathbb{E}(Y, X)$

In this section, $\mathfrak{C}$ is an extriangulated category. Let $X, Y \in \mathfrak{C}$ and

$$Z \longrightarrow W \xrightarrow{\alpha} Y \overset{\delta_\alpha}{\dashrightarrow}$$

be an $\mathfrak{s}$-triangle. From the argument below Definition 2.3, we know that $\mathrm{Im}\, {\delta_\alpha}^\sharp_X$ is an $\mathrm{End}_{\mathfrak{C}}(X)$-submodule of $\mathbb{E}(Y, X)$. Following this, we define

$$\begin{aligned} \xi_{X,Y}: \ [\to Y\rangle_{\mathrm{def}} &\longrightarrow \mathrm{Sub}_{\mathrm{End}_{\mathfrak{C}}(X)}\, \mathbb{E}(Y, X). \\ [\alpha\rangle &\mapsto \mathrm{Im}\, {\delta_\alpha}^\sharp_X \end{aligned}$$

Claim 1. $\xi_{X,Y}$ is well defined.

Indeed, if $[\alpha_1\rangle = [\alpha_2\rangle$, then

$$\mathrm{Im}\, {\delta_1}^\sharp_X = \mathrm{Coker}\, \mathrm{Hom}_{\mathfrak{C}}(\alpha_1, X) = \mathrm{Coker}\, \mathrm{Hom}_{\mathfrak{C}}(\alpha_2, X) = \mathrm{Im}\, {\delta_2}^\sharp_X.$$

It is well known that there is an equivalence

$$\mathrm{Hom}_{\mathfrak{C}}(-,X) : \mathrm{add}\, X \longrightarrow \mathrm{End}_{\mathfrak{C}}(X)\text{-proj},$$

where $\mathrm{End}_{\mathfrak{C}}(X)$-proj is the subcategory of finitely generated projective $\mathrm{End}_{\mathfrak{C}}(X)$-modules.

By the Yoneda lemma, there are natural isomorphisms

$$\begin{array}{rcl} \mathbb{E}(Y,Z) & \longrightarrow & \mathrm{Hom}_{\mathrm{End}_{\mathfrak{C}}(X)}(\mathrm{Hom}_{\mathfrak{C}}(Z,X),\mathbb{E}(Y,X)) \\ \delta & \mapsto & \delta^{\sharp}_{X} \end{array} \tag{4.1}$$

and

$$\begin{array}{rcl} \mathrm{Hom}_{\mathfrak{C}}(Y,Z) & \longrightarrow & \mathrm{Hom}_{\mathrm{End}_{\mathfrak{C}}(X)}(\mathrm{Hom}_{\mathfrak{C}}(Z,X),\mathrm{Hom}_{\mathfrak{C}}(Y,X)) \\ f & \mapsto & \mathrm{Hom}_{\mathfrak{C}}(f,X) \end{array} \tag{4.2}$$

for any $Z \in \mathrm{add}\, X$.

We denote by ${}_X[\to Y\rangle_{\mathrm{def}}$ the subset of $[\to Y\rangle_{\mathrm{def}}$ consisting of those classes $[\alpha\rangle$ that have a representative element α such that there exists an $\mathfrak{s}$-triangle

$$X_1 \longrightarrow W \xrightarrow{\alpha} Y \overset{\delta_\alpha}{\dashrightarrow}$$

with $X_1 \in \mathrm{add}\, X$. In this case, $\mathrm{Hom}_{\mathfrak{C}}(X_1,X)$ is a finitely generated projective $\mathrm{End}_{\mathfrak{C}}(X)$-module, and hence $\xi_{X,Y}([\alpha\rangle) = \mathrm{Im}\, {\delta_\alpha}^{\sharp}_{X}$ is a finitely generated $\mathrm{End}_{\mathfrak{C}}(X)$-module.

We denote by $\mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(X)}\, \mathbb{E}(Y,X)$ the subset of $\mathrm{Sub}_{\mathrm{End}_{\mathfrak{C}}(X)}\, \mathbb{E}(Y,X)$ consisting of finitely generated $\mathrm{End}_{\mathfrak{C}}(X)$-modules. The above $\xi_{X,Y}$ induces a well-defined map

$$\begin{array}{rrcl} \xi_{X,Y} : & {}_X[\to Y\rangle_{\mathrm{def}} & \longrightarrow & \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(X)}\, \mathbb{E}(Y,X). \\ & [\alpha\rangle & \mapsto & \mathrm{Im}\, {\delta_\alpha}^{\sharp}_{X} \end{array}$$

Claim 2. $\xi_{X,Y}$ is bijective.

Proof. Let

$$X_1 \longrightarrow W_1 \xrightarrow{\alpha_1} Y \overset{\delta_1}{\dashrightarrow} \quad \text{and} \quad X_2 \longrightarrow W_2 \xrightarrow{\alpha_2} Y \overset{\delta_2}{\dashrightarrow}$$

be two $\mathfrak{s}$-triangles satisfying $\mathrm{Im}\, {\delta_1}^{\sharp}_{X} = \mathrm{Im}\, {\delta_2}^{\sharp}_{X}$. Suppose $X_2 \in \mathrm{add}\, X$. Consider the following diagram of exact rows

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{C}}(X_1, X) \arrow[r, "{\delta_1}^{\sharp}_X"] & \mathrm{Im}\,{\delta_1}^{\sharp}_X \arrow[r] & 0 \\
\mathrm{Hom}_{\mathfrak{C}}(X_2, X) \arrow[r, "{\delta_2}^{\sharp}_X"] & \mathrm{Im}\,{\delta_2}^{\sharp}_X \arrow[r] \arrow[u, tail] & 0.
\end{tikzcd}$$

Since $X_2 \in \mathrm{add}\, X$, $\mathrm{Hom}_{\mathfrak{C}}(X_2, X) \in \mathrm{End}_{\mathfrak{C}}(X)$-proj, which induces the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{C}}(X_1, X) \arrow[r, "{\delta_1}^{\sharp}_X"] & \mathrm{Im}\,{\delta_1}^{\sharp}_X \arrow[r] & 0 \\
\mathrm{Hom}_{\mathfrak{C}}(X_2, X) \arrow[r, "{\delta_2}^{\sharp}_X"] \arrow[u, dotted, "w"'] & \mathrm{Im}\,{\delta_2}^{\sharp}_X \arrow[r] \arrow[u, tail] & 0.
\end{tikzcd}$$

By (4.2), there exists $u \in \mathrm{Hom}_{\mathfrak{C}}(X_1, X_2)$ such that $\mathrm{Hom}_{\mathfrak{C}}(u, X) = w$, and hence

$${\delta_2}^{\sharp}_X = {\delta_1}^{\sharp}_X \,\mathrm{Hom}_{\mathfrak{C}}(u, X).$$

Thus for any $f \in \mathrm{Hom}_{\mathfrak{C}}(X_2, X)$, we have

$$f_\star \delta_2 = {\delta_2}^{\sharp}_X(f) = ({\delta_1}^{\sharp}_X \,\mathrm{Hom}_{\mathfrak{C}}(u, X))(f) = {\delta_1}^{\sharp}_X(fu) = (fu)_\star \delta_1 = f_\star u_\star \delta_1.$$

Furthermore, since $X_2 \in \mathrm{add}\, X$, we may assume that there exists $\iota : X_2 \to X$ and $p : X \to X_2$ such that $\mathrm{Id}_{X_2} = p\iota$. Thus we have

$$\begin{aligned}
\delta_2 =& {\mathrm{Id}_{X_2}}_\star \delta_2 = (p\iota)_\star \delta_2 = p_\star(\iota_\star \delta_2) = p_\star(\iota_\star u_\star \delta_1) \\
&= (p\iota)_\star (u_\star \delta_1) = {\mathrm{Id}_{X_2}}_\star u_\star \delta_1 = u_\star \delta_1,
\end{aligned}$$

which means that (u, Id_Y) is a morphism from δ_1 to δ_2. By (ET2), there exists $v \in \mathrm{Hom}_{\mathfrak{C}}(W_1, W_2)$ such that the following diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "u"] & W_1 \arrow[r, "\alpha_1"] \arrow[d, dotted, "v"] & Y \arrow[r, dashed, "\delta_1"] \arrow[d, equal] & {} \\
X_2 \arrow[r] & W_2 \arrow[r, "\alpha_2"] & Y \arrow[r, dashed, "\delta_2"] & {}
\end{tikzcd}$$

commutes. In particular, α_1 factors through α_2. Similarly, if $X_1 \in \mathrm{add}\, X$, then α_2 factors through α_1. Consequently, we conclude that $[\alpha_1\rangle = [\alpha_2\rangle$ if $\mathrm{Im}\,{\delta_1}^{\sharp}_X = \mathrm{Im}\,{\delta_2}^{\sharp}_X$ and $X_1, X_2 \in \mathrm{add}\, X$. Thus $\xi_{X,Y}$ is injective.

Now, let F be any finitely generated $\mathrm{End}_{\mathfrak{C}}(X)$-submodule of $\mathbb{E}(Y, X)$. Then there is a morphism $f : \mathrm{Hom}_{\mathfrak{C}}(X_1, X) \to \mathbb{E}(Y, X)$ with $X_1 \in \mathrm{add}\, X$ and $\mathrm{Im}\, f = F$. By (4.1), there exists an $\mathbb{E}$-extension $\delta \in \mathbb{E}(Y, X_1)$ such that $\mathrm{Im}\,\delta^{\sharp}_X = f$. Thus $\xi_{X,Y}$ is surjective. $\square$

By Claims 1 and 2, we have the following theorem, which extends [12, Proposition 2.4].

Theorem 4.1. *The map*

$$\begin{array}{rccl} \xi_{X,Y}: & {}_X[\to Y\rangle_{\mathrm{def}} & \longrightarrow & \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(X)}\,\mathbb{E}(Y,X). \\ & [\alpha\rangle & \mapsto & \mathrm{Im}\,\delta_{\alpha}{}^{\sharp}_{X} \end{array}$$

is an anti-isomorphism of posets.

In what follows, let $\mathfrak{C}$ be a Hom-finite R-linear Krull-Schmidt extriangulated category having Auslander-Reiten-Serre duality. The following proposition is an extriangulated version of [12, Proposition 4.5].

Proposition 4.2. *Let*

$$X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y \overset{\delta}{\dashrightarrow}$$

be an $\mathfrak{s}$-triangle. Then

(1) *β is right $\tau^{-}X$-determined.*
(2) *If β is right minimal, then β is right C-determined for some $C \in \mathfrak{C}$ if and only if $\tau^{-}X \in \mathrm{add}\,C$.*

Consequently, ${}_X[\to Y\rangle_{\mathrm{def}} = {}^{\tau^{-}X}[\to Y\rangle_{\mathrm{def}}$.

Proof. (1) It follows from [35, Lemma 4.6].

(2) The sufficiency follows from (1). It suffices to prove the necessity. We will prove that each indecomposable direct summand X' of X satisfies $\tau^{-}X' \in \mathrm{add}\,C$. First of all, the composition of $\mathfrak{s}$-inflations $X' \xrightarrow{i} X \xrightarrow{\alpha} Z$ is not a section, where i is the inclusion. Otherwise, assume that αi is a section and let $X \cong X' \oplus X''$. Then

$$X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y \overset{\delta}{\dashrightarrow}$$

is isomorphic to

$$X' \oplus X'' \xrightarrow{\left(\begin{smallmatrix} \mathrm{Id}_{X'} & * \end{smallmatrix}\right)} X' \oplus Z' \xrightarrow{\beta} Y \overset{\delta}{\dashrightarrow}.$$

Thus $\beta(X') = 0$, which contradicts with the assumption that β is right minimal. Hence X' is not an injective object. Then by [21, Theorem 3.4], there is an almost split $\mathfrak{s}$-triangle

$$X' \xrightarrow{\alpha'} W \xrightarrow{\beta'} \tau^- X' \overset{\sigma}{\dashrightarrow} .$$

We have the following commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\alpha'"] \arrow[d, "i"] & W \arrow[r, "\beta'"] \arrow[d, dotted, "s"] & \tau^- X' \arrow[r, dashed, "\sigma"] \arrow[d, dotted, "t"] & {} \\ X \arrow[r, "\alpha"] & Z \arrow[r, "\beta"] & Y \arrow[r, dashed, "\delta"] & {} . \end{tikzcd}$$

In particular, $i_\star \sigma = t^\star \delta$.

Suppose $\tau^- X' \notin \operatorname{add} C$. Then any $f \in \operatorname{Hom}_{\mathfrak{C}}(C, \tau^- X')$ is not a retraction, and hence factors through β', that is, $f = \beta' g$ for some $g \in \operatorname{Hom}_{\mathfrak{C}}(C, W)$. Thus $tf = t(\beta' g) = \beta(sg)$. Moreover, since β is right C-determined, there exists $h \in \operatorname{Hom}_{\mathfrak{C}}(\tau^- X', Z)$ such that $t = \beta h$. Consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "w"] \arrow[d, equal] & W' \arrow[r] \arrow[d, dotted] & \tau^- X' \arrow[r, dashed, "t^\star \delta"] \arrow[d, "t"] & {} \\ X \arrow[r, "\alpha"] & Z \arrow[r, "\beta"] & Y \arrow[r, dashed, "\delta"] & {} . \end{tikzcd}$$

By Lemma 2.5, we have that Id_X factors through w and w is a section, and moreover $i_\star \sigma = t^\star \delta = 0$. Consider the following commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\alpha'"] \arrow[d, "i"] & W \arrow[r] \arrow[d, dotted] & \tau^- X' \arrow[r, dashed, "\sigma"] \arrow[d, equal] & {} \\ X \arrow[r] & W'' \arrow[r, "\beta"] & \tau^- X' \arrow[r, dashed, "i_\star \sigma"] & {} . \end{tikzcd}$$

By Lemma 2.5 again, the condition $i_\star \sigma = 0$ implies that there exists $w' \in \operatorname{Hom}_{\mathfrak{C}}(W, X)$ such that $i = w' \alpha'$. Finally, since i is a section, α' is also a section, which is a contradiction. Thus we have $\tau^- X' \in \operatorname{add} C$. $\square$

By Theorem 4.1 and Proposition 4.2, we get the following

Theorem 4.3. *The map*

$$\begin{aligned} \xi_{X,Y} : {}^{\tau^- X}[\to Y\rangle_{\mathrm{def}} \quad &\longrightarrow \quad \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(X)} \mathbb{E}(Y, X). \\ [\alpha\rangle \quad &\mapsto \quad \operatorname{Im} \delta_{\alpha}{}^{\sharp}_{X} \end{aligned}$$

is an anti-isomorphism of posets.

5. The restricted Auslander bijection induced by ARS-duality

In this section, $\mathfrak{C}$ is a Hom-finite R-linear Krull-Schmidt extriangulated category having Auslander-Reiten-Serre duality. In this case, any $\mathrm{End}_{\mathfrak{C}}(X)$-submodule of $\mathbb{E}(Y,X)$ is finitely generated. In [12, Lemma 4.2], it was shown that there is a bijection between $\mathrm{sub}_{\mathrm{End}_{\mathcal{A}}(X)}\mathrm{Ext}^1_{\mathcal{A}}(Y,X)$ and $\mathrm{sub}_{\mathrm{End}_{\mathcal{A}}(X)^{\mathrm{op}}}\underline{\mathrm{Hom}}_{\mathcal{A}}(\tau^- X,Y)$ over an abelian category $\mathcal{A}$ admitting Auslander-Reiten duality. We now show that the Auslander-Reiten-Serre duality in the extriangulated category $\mathfrak{C}$ still induces the following bijection of two posets.

Lemma 5.1. *There is a bijection*

$$\Upsilon_{X,Y}:\mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(X)}\mathbb{E}(Y,X)\longrightarrow \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(X)^{\mathrm{op}}}\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X,Y)$$

such that for any $\mathrm{End}_{\mathfrak{C}}(X)$*-submodule* F *of* $\mathbb{E}(Y,X)$*,* $\Upsilon_{X,Y}(F)=H$ *is defined by an exact sequence*

$$0\longrightarrow H\longrightarrow \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X,Y)\xrightarrow{D(i)\Psi^{-1}_{Y,X}} DF\longrightarrow 0$$

where $i:F\to\mathbb{E}(Y,X)$ *is the inclusion.*

The bijection $\Upsilon_{X,Y}$ *is an anti-isomorphism of posets.*

Proof. Let $0\to F\xrightarrow{i}\mathbb{E}(Y,X)$ be the inclusion. Then $D\mathbb{E}(Y,X)\xrightarrow{D(i)}DF\to 0$ is exact. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r] & \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X,Y) \arrow[r, "D(i)\Psi^{-1}_{Y,X}"] \arrow[d, "\Psi^{-1}_{Y,X}", "\cong"'] & DF \arrow[r] \arrow[d, equal] & 0 \\
 & & D\mathbb{E}(Y,X) \arrow[r, "D(i)"] & DF \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The bijection follows from the well-known fact that for a finitely generated module M over an artin algebra Λ, there is a bijection between $\mathrm{sub}_{\Lambda}(M)$ and $\mathrm{sub}_{\Lambda^{\mathrm{op}}}(DM)$, sending $L\in\mathrm{sub}_{\Lambda}(M)$ to the kernel of the projection $DM\to DL$. Note that the previous process is just given by this map. □

Given an $\mathfrak{s}$-triangle

$$X'\xrightarrow{\alpha}Z\xrightarrow{\beta}Y\overset{\delta}{\dashrightarrow}.$$

Let $f\in\mathrm{Hom}_{\mathfrak{C}}(X',X)$ be an injective morphism. Then we have a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "\alpha"] \arrow[d, "f"] & Z \arrow[r, "\beta"] \arrow[d, dotted] & Y \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
X \arrow[r, "\alpha'"] & Z' \arrow[r, "\beta'"] & Y \arrow[r, dashed, "f_\star\delta"] & {} .
\end{tikzcd}$$

Moreover, $\delta_X^\sharp(f) = f_\star\delta = 0$ by Definition 2.6. This shows that the map

$$\begin{array}{rrcl} \delta_X^\sharp: & \mathrm{Hom}_{\mathfrak{C}}(X', X) & \longrightarrow & \mathbb{E}(Y, X) \\ & f & \mapsto & f_\star\delta \end{array}$$

vanishes on $\mathcal{I}(X', X)$. Therefore, we have a map

$$\begin{array}{rrcl} \delta_X^\sharp: & \overline{\mathrm{Hom}}_{\mathfrak{C}}(X', X) & \longrightarrow & \mathbb{E}(Y, X). \\ & f & \mapsto & f_\star\delta \end{array}$$

Dually, the map

$$\begin{array}{rrcl} (\delta_\sharp)_X: & \mathrm{Hom}_{\mathfrak{C}}(X, Y) & \longrightarrow & \mathbb{E}(X, X') \\ & f & \mapsto & f^\star\delta \end{array}$$

vanishes on $\mathcal{P}(X, Y)$, and hence induces a map

$$\begin{array}{rrcl} (\delta_\sharp)_X: & \underline{\mathrm{Hom}}_{\mathfrak{C}}(X, Y) & \longrightarrow & \mathbb{E}(X, X'). \\ & f & \mapsto & f^\star\delta \end{array}$$

For any $X \in \mathfrak{C}$, there are natural isomorphisms

$$\Phi_{X,-}^{-1}: \overline{\mathrm{Hom}}_{\mathfrak{C}}(-, \tau X) \longrightarrow D\mathbb{E}(X, -)$$

and

$$\Psi_{-,X}^{-1}: \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, -) \longrightarrow D\mathbb{E}(-, X).$$

Set

$$\lambda_X := \Phi_{X,\tau X}^{-1}(\overline{\mathrm{Id}_{\tau X}}) \in D\mathbb{E}(X, \tau X), \quad \underline{\mu_X} := \Psi_{X,\tau X}(\lambda_X) \in \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^-\tau X, X),$$
$$\kappa_X := \Psi_{\tau^- X,X}^{-1}(\underline{\mathrm{Id}_{\tau^- X}}) \in D\mathbb{E}(\tau^- X, X), \quad \overline{\nu_X} := \Phi_{\tau^- X,X}(\kappa_X) \in \overline{\mathrm{Hom}}_{\mathfrak{C}}(X, \tau\tau^- X).$$

Then we have the following commutative diagrams

$$\begin{tikzcd}
\mathbb{E}(X,\tau X) \arrow[r, "{\mathbb{E}(\mu_X,\tau X)}"] \arrow[dr, "\lambda_X"'] & \mathbb{E}(\tau^-\tau X,\tau X) \arrow[d, "\kappa_{\tau X}"] \\
& E
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathbb{E}(\tau^- X,X) \arrow[r, "{\mathbb{E}(\tau^- X,\nu_X)}"] \arrow[dr, "\kappa_X"'] & \mathbb{E}(\tau^- X,\tau\tau^- X) \arrow[d, "\lambda_{\tau^- X}"] \\
& E
\end{tikzcd},$$

that is,

$$\begin{aligned}
\lambda_X &= \kappa_{\tau X}\mathbb{E}(\mu_X,\tau X),\\
\kappa_X &= \lambda_{\tau^- X}\mathbb{E}(\tau^- X,\nu_X).
\end{aligned} \tag{5.1}$$

Indeed, by definition $\lambda_X = \Psi^{-1}_{X,\tau X}(\underline{\mu_X})$. Consider the following commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^-\tau X,\tau^-\tau X) \arrow[r, "\Psi^{-1}_{\tau^-\tau X,\tau X}"] \arrow[d, "{\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^-\tau X,\underline{\mu_X})}"'] & D\mathbb{E}(\tau^-\tau X,\tau X) \arrow[d, "{D\mathbb{E}(\mu_X,\tau X)}"] \\
\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^-\tau X,X) \arrow[r, "\Psi^{-1}_{X,\tau X}"] & D\mathbb{E}(X,\tau X).
\end{tikzcd}$$

Then we have

$$\begin{aligned}
\lambda_X &= \Psi^{-1}_{X,\tau X}(\underline{\mu_X})\\
&= \Psi^{-1}_{X,\tau X}\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^-\tau X,\underline{\mu_X})(\underline{\mathrm{Id}_{\tau^-\tau X}})\\
&= D\mathbb{E}(\mu_X,\tau X)\Psi^{-1}_{\tau^-\tau X,\tau X}(\underline{\mathrm{Id}_{\tau^-\tau X}})\\
&= D\mathbb{E}(\mu_X,\tau X)(\kappa_{\tau X})\\
&= \kappa_{\tau X}\mathbb{E}(\mu_X,\tau X).
\end{aligned}$$

Similarly, we have $\kappa_X = \lambda_{\tau^- X}\mathbb{E}(\tau^- X,\nu_X)$.

Now given an $\mathfrak{s}$-triangle

$$X' \longrightarrow Z \longrightarrow Y \overset{\delta}{\dashrightarrow}.$$

Then for any $X\in\mathfrak{C}$, there are the following commutative diagrams

$$\begin{tikzcd}
D\mathbb{E}(X,X') \arrow[r, "D(\delta_\sharp)_X"] & D\underline{\mathrm{Hom}}_{\mathfrak{C}}(X,Y) \\
\overline{\mathrm{Hom}}_{\mathfrak{C}}(X',\tau X) \arrow[u, "\Phi^{-1}_{X,X'}"] \arrow[r, "\delta^\sharp_{\tau X}"] & \mathbb{E}(Y,\tau X) \arrow[u, "{D(\Psi^{-1}_{Y,\tau X}\underline{\mathrm{Hom}}_{\mathfrak{C}}(\underline{\mu_X},Y))}"']
\end{tikzcd} \tag{5.2}$$

and

$$\begin{tikzcd} D\mathbb{E}(Y,X) \arrow[r, "D\delta^\sharp_X"] & D\overline{\mathrm{Hom}}_{\mathfrak{C}}(X',X) \\ \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X,Y) \arrow[u, "\Psi^{-1}_{Y,X}"] \arrow[r, "(\delta_\sharp)_{\tau^- X}"] & \mathbb{E}(\tau^- X,X'). \arrow[u, "D(\Phi^{-1}_{\tau^- X,X'}\overline{\mathrm{Hom}}_{\mathfrak{C}}(X',\overline{\nu_X}))"'] \end{tikzcd} \tag{5.3}$$

Indeed, let $\Omega := \Psi^{-1}_{Y,\tau X}\underline{\mathrm{Hom}}_{\mathfrak{C}}(\underline{\mu_X},Y) : \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^-\tau X,Y) \to D\mathbb{E}(Y,\tau X)$. For any $\overline{g} \in \overline{\mathrm{Hom}}_{\mathfrak{C}}(X',\tau X)$ and $\underline{h} \in \underline{\mathrm{Hom}}_{\mathfrak{C}}(X,Y)$, we have

$$\begin{aligned} ((D\Omega)\delta^\sharp_{\tau X})(\overline{g})(\underline{h}) &= (\delta^\sharp_{\tau X}(\overline{g})\Omega)(\underline{h}) \\ &= (g_\star\delta)(\Omega(\underline{h})) \\ &= (\Omega(\underline{h}))(g_\star\delta) \\ &= \Psi^{-1}_{Y,\tau X}(\underline{h\mu_X})(g_\star\delta), \end{aligned}$$

where the third equality follows from the canonical isomorphism $\mathbb{E}(Y,\tau X) \cong DD\mathbb{E}(Y,\tau X)$. By the naturality of Ψ^{-1}, we have

$$\Psi^{-1}_{Y,\tau X}(\underline{h\mu_X})(g_\star\delta) = \kappa_{\tau Y}(\mu_X^\star h^\star g_\star\delta).$$

On the other hand,

$$\begin{aligned} (D(\delta_\sharp)_X\Phi^{-1}_{X,X'})(\overline{g})(\underline{h}) &= (\Phi^{-1}_{X,X'}(\overline{g}(\delta_\sharp)_X)(\underline{h}) \\ &= \Phi^{-1}_{X,X'}(\overline{g}(h^\star\delta). \end{aligned}$$

By the naturality of Φ^{-1}, we have

$$\Phi^{-1}_{X,X'}(\overline{g}(h^\star\delta) = \lambda_X(h^\star g_\star\delta).$$

By (5.1), $\lambda_X = \kappa_{\tau X}\mathbb{E}(\mu_X,\tau X)$, and hence

$$\begin{aligned} \lambda_X(h^\star g_\star\delta) &= (\kappa_{\tau X}\mathbb{E}(\mu_X,\tau X))(h^\star g_\star\delta) \\ &= \kappa_{\tau X}(\mu_X^\star h^\star g_\star\delta). \end{aligned}$$

Thus $((D\Omega)\delta^\sharp_{\tau X})(\overline{g})(\underline{h}) = (D(\delta_\sharp)_X\Phi^{-1}_{X,X'})(\overline{g})(\underline{h})$, and therefore $(D\Omega)\delta^\sharp_{\tau X} = D(\delta_\sharp)_X\Phi^{-1}_{X,X'}$, that is, the diagram (5.2) is commutative. Similarly, we can prove that the diagram (5.3) is commutative.

Remark 5.2. According to the commutative diagrams (5.2) and (5.3), it is easy to see that there are exact sequences

$$0 \longrightarrow \mathrm{Ker}(\delta^\sharp)_{\tau X} \longrightarrow \overline{\mathrm{Hom}}_{\mathfrak{C}}(X',\tau X) \xrightarrow{D(i_1)\Phi^{-1}_{X,X'}} D\,\mathrm{Im}(\delta_\sharp)_X \longrightarrow 0, \tag{5.4}$$

and

$$0 \longrightarrow \mathrm{Ker}(\delta_\sharp)_{\tau^- X} \longrightarrow \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y) \xrightarrow{D(i_2)\Psi^{-1}_{Y,X}} D\,\mathrm{Im}\,\delta^\sharp_X \longrightarrow 0, \tag{5.5}$$

where $i_1 : \mathrm{Im}(\delta_\sharp)_X \to \mathbb{E}(X, X')$ and $i_2 : \mathrm{Im}\,\delta^\sharp_X \to \mathbb{E}(Y, X)$ are the corresponding inclusions.

Given an $\mathfrak{s}$-triangle

$$X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y \overset{\delta}{\dashrightarrow} .$$

For any $C \in \mathfrak{C}$, by [29, Proposition 3.3] there are the following exact sequences

$$\mathrm{Hom}_{\mathfrak{C}}(C, X) \xrightarrow{\mathrm{Hom}_{\mathfrak{C}}(C,\alpha)} \mathrm{Hom}_{\mathfrak{C}}(C, Z) \xrightarrow{\mathrm{Hom}_{\mathfrak{C}}(C,\beta)} \mathrm{Hom}_{\mathfrak{C}}(C, Y) \xrightarrow{(\delta_\sharp)_C} \mathbb{E}(C, X) \xrightarrow{\mathbb{E}(C,\alpha)} \mathbb{E}(C, Z)$$

and

$$\underline{\mathrm{Hom}}_{\mathfrak{C}}(C, Z) \xrightarrow{\underline{\mathrm{Hom}}_{\mathfrak{C}}(C,\beta)} \underline{\mathrm{Hom}}_{\mathfrak{C}}(C, Y) \xrightarrow{(\delta_\sharp)_C} \mathbb{E}(C, X) \xrightarrow{\mathbb{E}(C,\alpha)} \mathbb{E}(C, Z).$$

We claim that $\mathrm{Im}\,\underline{\mathrm{Hom}}_{\mathfrak{C}}(C, \beta) = \mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C, \beta)/\mathcal{P}(C, Y)$. First of all, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{C}}(C, X) \arrow[r, "\mathrm{Hom}_{\mathfrak{C}}(C,\alpha)"] & \mathrm{Hom}_{\mathfrak{C}}(C, Z) \arrow[r] \arrow[d] & \mathrm{Im}\,\mathrm{Hom}_{\mathfrak{C}}(C, \beta) \arrow[r] \arrow[d, "p"] & 0 \\
 & \underline{\mathrm{Hom}}_{\mathfrak{C}}(C, Z) \arrow[r] \arrow[d] & \mathrm{Im}\,\underline{\mathrm{Hom}}_{\mathfrak{C}}(C, \beta) \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 &
\end{tikzcd}$$

Moreover, we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{P}(C,Y) \arrow[r, equal] \arrow[d] & \mathcal{P}(C,Y) \arrow[d] & & \\
0 \arrow[r] & \operatorname{Im}\operatorname{Hom}_{\mathfrak{C}}(C,\beta) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathfrak{C}}(C,Y) \arrow[r] \arrow[d] & \mathbb{E}(C,X) \arrow[r, "\mathbb{E}(C,\alpha)"] \arrow[d, equal] & \mathbb{E}(C,Z) \arrow[d, equal] \\
0 \arrow[r] & \operatorname{Im}\underline{\operatorname{Hom}}_{\mathfrak{C}}(C,\beta) \arrow[r] \arrow[d] & \underline{\operatorname{Hom}}_{\mathfrak{C}}(C,Y) \arrow[r] \arrow[d] & \mathbb{E}(C,X) \arrow[r, "\mathbb{E}(C,\alpha)"] & \mathbb{E}(C,Z). \\
 & 0 & 0 & &
\end{tikzcd}$$

In particular, we have $\operatorname{Im}\underline{\operatorname{Hom}}_{\mathfrak{C}}(C,\beta) = \operatorname{Im}\operatorname{Hom}_{\mathfrak{C}}(C,\beta)/\mathcal{P}(C,Y)$.

Now we have a well-defined map

$$\begin{aligned} \eta_{C,Y}: \ [\to Y\rangle_{\mathrm{def}} &\longrightarrow \operatorname{sub}_{\operatorname{End}_{\mathfrak{C}}(C)^{\mathrm{op}}} \underline{\operatorname{Hom}}_{\mathfrak{C}}(C,Y). \\ [\beta\rangle &\mapsto \operatorname{Im}\underline{\operatorname{Hom}}_{\mathfrak{C}}(C,\beta) \end{aligned}$$

For any $X \in \mathfrak{C}$, since τ^- is an equivalence, we can identity via τ^- the $\operatorname{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}$-module structure on $\underline{\operatorname{Hom}}_{\mathfrak{C}}(\tau^- X, Y)$ with the corresponding $\operatorname{End}_{\mathfrak{C}}(X)^{\mathrm{op}}$-module structure. Thus we can identity the poset $\operatorname{sub}_{\operatorname{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}} \underline{\operatorname{Hom}}_{\mathfrak{C}}(\tau^- X, Y)$ with $\operatorname{sub}_{\operatorname{End}_{\mathfrak{C}}(X)^{\mathrm{op}}} \underline{\operatorname{Hom}}_{\mathfrak{C}}(\tau^- X, Y)$. Under this identification, we have the bijection

$$\Upsilon_{X,Y}: \operatorname{sub}_{\operatorname{End}_{\mathfrak{C}}(X)} \mathbb{E}(Y,X) \longrightarrow \operatorname{sub}_{\operatorname{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}} \underline{\operatorname{Hom}}_{\mathfrak{C}}(\tau^- X, Y).$$

Now we can obtain the following commutative triangle, which extends [12, Proposition 4.4].

Proposition 5.3. *For any $X, Y \in \mathfrak{C}$, we have the following commutative triangle*

$$\begin{tikzcd}
 & \operatorname{sub}_{\operatorname{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}} \underline{\operatorname{Hom}}_{\mathfrak{C}}(\tau^- X, Y) & \\
{[\to Y\rangle_{\mathrm{def}}} \arrow[ur, "\eta_{\tau^- X, Y}"] \arrow[rr, "\xi_{X,Y}"] & & \operatorname{sub}_{\operatorname{End}_{\mathfrak{C}}(X)} \mathbb{E}(Y,X). \arrow[ul, "\Upsilon_{X,Y}"']
\end{tikzcd}$$

Proof. For any $[\beta\rangle \in [\to Y\rangle_{\mathrm{def}}$, there is an $\mathfrak{s}$-triangle

$$X' \xrightarrow{\alpha} Z \xrightarrow{\beta} Y \overset{\delta}{\dashrightarrow}.$$

We have an exact sequence

$$\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^{-}X,Z)\xrightarrow{\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^{-}X,\beta)}\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^{-}X,Y)\xrightarrow{(\delta_{\sharp})_{\tau^{-}X}}\mathbb{E}(\tau^{-}X,X').$$

By definition, we have

$$\eta_{\tau^{-}X,Y}([\beta\rangle)=\mathrm{Im}\,\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^{-}X,\beta)=\mathrm{Ker}\,(\delta_{\sharp})_{\tau^{-}X},$$
$$\xi_{X,Y}([\beta\rangle)=\mathrm{Im}\,\delta^{\sharp}_{X}.$$

It follows from Lemma 5.1 and Remark 5.2 that $\Upsilon_{X,Y}(\mathrm{Im}\,\delta^{\sharp}_{X})=\mathrm{Ker}\,(\delta_{\sharp})_{\tau^{-}X}$. Thus $\eta_{\tau^{-}X,Y}=\Upsilon_{X,Y}\xi_{X,Y}$. □

We are now in a position to give the following commutative bijection triangle, which shows that the restricted Auslander bijection holds true under the assumption that $\mathfrak{C}$ has Auslander-Reiten-Serre duality. It is an extriangulated version of [12, Theorem 4.6].

Theorem 5.4. *For any $X,Y\in\mathfrak{C}$, the following bijection triangle*

$$\begin{tikzcd}[column sep=small] & \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^{-}X)^{\mathrm{op}}}\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^{-}X,Y) & \\ {}_{X}[\to Y\rangle_{\mathrm{def}}={}^{\tau^{-}X}[\to Y\rangle_{\mathrm{def}} \arrow[ur, "\eta_{\tau^{-}X,Y}"] \arrow[rr, "\xi_{X,Y}"] & & \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(X)}\mathbb{E}(Y,X) \arrow[ul, "\Upsilon_{X,Y}"'] \end{tikzcd}$$

is commutative. In particular, we get the restricted Auslander bijection at Y relative to $\tau^{-}X$

$$\eta_{\tau^{-}X,Y}:{}^{\tau^{-}X}[\to Y\rangle_{\mathrm{def}}\longrightarrow\mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^{-}X)^{\mathrm{op}}}\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^{-}X,Y),$$

which is an isomorphism of posets.

Proof. It follows from Theorem 4.3, Lemma 5.1 and Proposition 5.3. □

6. Applications

6.1. A realization from $D\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^{-}X,Y)$ to $\mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^{-}X)^{\mathrm{op}}}\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^{-}X,Y)$

In this subsection, $\mathfrak{C}$ is an extriangulated category. In order to establish a desired map from $D\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^{-}X,Y)$ to $\mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^{-}X)^{\mathrm{op}}}\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^{-}X,Y)$, we first give some equivalent descriptions for $\mathfrak{s}$-deflations being right minimal, and then obtain some conditions for ensuring that any $\mathfrak{s}$-deflation is right equivalent to a right minimal $\mathfrak{s}$-deflation.

Lemma 6.1. *For an $\mathfrak{s}$-triangle*

$$X\longrightarrow Z\xrightarrow{\alpha}Y\overset{\delta}{\dashrightarrow},$$

the following statements are equivalent.

(1) *α is right minimal.*
(2) *Any $u \in \mathrm{Hom}_{\mathfrak{C}}(X,X)$ satisfying $u_\star\delta = \delta$ is an automorphism.*
(3) *The map $\delta_X^\sharp : \mathrm{Hom}_{\mathfrak{C}}(X,X) \to \mathbb{E}(Y,X)$ of $\mathrm{End}_{\mathfrak{C}}(X)$-modules is right minimal.*

Proof. (1) $\Rightarrow$ (2) Assume that $u \in \mathrm{Hom}_{\mathfrak{C}}(X,X)$ satisfies $u_\star\delta = \delta$. Then $u_\star\delta = \mathrm{Id}_Y^\star\,\delta$, and hence we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"] & Z \arrow[r, "\alpha"] \arrow[d, dotted, "v"] & Y \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ X \arrow[r] & Z \arrow[r, "\alpha"] & Y \arrow[r, dashed, "\delta"] & {} \end{tikzcd}.$$

Thus $\alpha = \alpha v$. Since α is right minimal by (1), v is an automorphism. Thus u is an automorphism by [29, Corollary 3.6].

(2) $\Rightarrow$ (1) For any $v \in \mathrm{Hom}_{\mathfrak{C}}(Z,Z)$, if $\alpha v = \alpha$, then we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, dotted, "u"] & Z \arrow[r, "\alpha"] \arrow[d, "v"] & Y \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ X \arrow[r] & Z \arrow[r, "\alpha"] & Y \arrow[r, dashed, "\delta"] & {} \end{tikzcd}.$$

This means that $u_\star\delta = \delta$, hence u is an automorphism by (2). It follows from [29, Corollary 3.6] that v is an automorphism and α is right minimal.

(2) $\Rightarrow$ (3) Assume that there is a commutative diagram

$$\begin{tikzcd} & \mathrm{Hom}_{\mathfrak{C}}(X,X) \arrow[dl, "w"'] \arrow[d, "\delta_X^\sharp"] \\ \mathrm{Hom}_{\mathfrak{C}}(X,X) \arrow[r, "\delta_X^\sharp"] & \mathbb{E}(Y,X). \end{tikzcd}$$

By (4.2), there exists $u \in \mathrm{Hom}_{\mathfrak{C}}(X,X)$ such that $w = \mathrm{Hom}_{\mathfrak{C}}(u,X)$. Thus $\delta_X^\sharp\,\mathrm{Hom}_{\mathfrak{C}}(u,X) = \delta_X^\sharp$, which yields

$$u_\star\delta = \delta_X^\sharp(u) = (\delta_X^\sharp\,\mathrm{Hom}_{\mathfrak{C}}(u,X))(\mathrm{Id}_X) = \delta_X^\sharp(\mathrm{Id}_X) = \mathrm{Id}_{X\star}\delta = \delta.$$

By (2), u is an automorphism, and hence $w = \mathrm{Hom}_{\mathfrak{C}}(u,X)$ is an automorphism, which shows that $\delta_X^\sharp$ is right minimal.

(3) $\Rightarrow$ (2) Assume that $u \in \mathrm{Hom}_{\mathfrak{C}}(X,X)$ satisfies $u_\star\delta = \delta$. Then for any $f \in \mathrm{Hom}_{\mathfrak{C}}(X,X)$, we have

$$(\delta_X^\sharp \operatorname{Hom}_{\mathfrak{C}}(u,X))(f) = \delta_X^\sharp(\operatorname{Hom}_{\mathfrak{C}}(u,X)(f)) = \delta_X^\sharp(fu) = (fu)_\star\delta = f_\star u_\star \delta = f_\star\delta = \delta_X^\sharp(f)$$

and thus $\delta_X^\sharp \operatorname{Hom}_{\mathfrak{C}}(u,X) = \delta_X^\sharp$. Since $\delta_X^\sharp$ is right minimal by (3), $\operatorname{Hom}_{\mathfrak{C}}(u,X)$ is an automorphism, and hence u is also an automorphism. □

Now we give some equivalent condition for $\mathfrak{s}$-deflations being (right equivalent to) a right minimal morphism, which extends [12, Proposition 3.2 and Corollary 3.3].

Proposition 6.2. *For an $\mathfrak{s}$-triangle*

$$X' \longrightarrow Z \xrightarrow{\alpha} Y \overset{\delta}{\dashrightarrow}$$

with $X' \in \operatorname{add} X$, the following statements are equivalent.

(1) *α is right minimal.*
(2) *The map $\delta_X^\sharp : \operatorname{Hom}_{\mathfrak{C}}(X',X) \to \mathbb{E}(Y,X)$ of $\operatorname{End}_{\mathfrak{C}}(X)$-modules is right minimal.*

Proof. (1) ⇒ (2) Assume that there is a commutative diagram

$$\begin{tikzcd} & \operatorname{Hom}_{\mathfrak{C}}(X',X) \arrow[dl, "w"'] \arrow[d, "\delta_X^\sharp"] \\ \operatorname{Hom}_{\mathfrak{C}}(X',X) \arrow[r, "\delta_X^\sharp"] & \mathbb{E}(Y,X). \end{tikzcd}$$

Since $X' \in \operatorname{add} X$, there exists $u \in \operatorname{Hom}_{\mathfrak{C}}(X',X')$ such that $w = \operatorname{Hom}_{\mathfrak{C}}(u,X)$ by (4.2). Thus for any $f \in \operatorname{Hom}_{\mathfrak{C}}(X',X)$, we have

$$f_\star u_\star \delta = (fu)_\star\delta = \delta_X^\sharp(fu) = \delta_X^\sharp \operatorname{Hom}_{\mathfrak{C}}(u,X)(f) = (\delta_X^\sharp w)(f) = \delta_X^\sharp(f) = f_\star\delta.$$

Moreover, since $X' \in \operatorname{add} X$, we may assume that there exists $\iota : X' \to X$ and $p : X \to X'$ such that $\mathrm{Id}_{X'} = p\iota$. Then

$$\begin{aligned} \delta &= \mathrm{Id}_{X'\star}\delta = (p\iota)_\star\delta = p_\star(\iota_\star\delta) = p_\star(\iota_\star u_\star \delta) \\ &= (p\iota)_\star u_\star\delta = \mathrm{Id}_{X'\star} u_\star \delta = u_\star\delta. \end{aligned}$$

Since α is right minimal by (1), u is an automorphism by Lemma 6.1. It follows that w is an automorphism, which implies that $\delta_X^\sharp$ is right minimal.

(2) ⇒ (1) Let $u \in \operatorname{Hom}_{\mathfrak{C}}(X',X')$ satisfying $\delta = u_\star\delta$. Then for any $f \in \operatorname{Hom}_{\mathfrak{C}}(X',X)$, we have

$$(\delta_X^\sharp \operatorname{Hom}_{\mathfrak{C}}(u,X))(f) = \delta_X^\sharp(\operatorname{Hom}_{\mathfrak{C}}(u,X)(f)) = \delta_X^\sharp(fu) = (fu)_\star\delta = f_\star u_\star \delta = f_\star\delta = \delta_X^\sharp(f)$$

and thus $\delta_X^\sharp \operatorname{Hom}_{\mathfrak{C}}(u,X) = \delta_X^\sharp$. Since $\delta_X^\sharp$ is right minimal by (2), $\operatorname{Hom}_{\mathfrak{C}}(u,X)$ is an automorphism, and hence u is also an automorphism. By Lemma 6.1, we have that α is right minimal. □

Corollary 6.3. *For an $\mathfrak{s}$-triangle*

$$X' \longrightarrow Z \xrightarrow{\alpha} Y \overset{\delta}{\dashrightarrow}$$

with $X' \in \operatorname{add} X$, the following statements are equivalent.

(1) *α is right equivalent to a right minimal morphism.*
(2) *The $\operatorname{End}_{\mathfrak{C}}(X)$-module $\operatorname{Im}\delta_X^\sharp$ has a projective cover.*

Proof. (1) ⇒ (2) Assume that α is right equivalent to a right minimal morphism $\alpha' \in \operatorname{Hom}_{\mathfrak{C}}(Z',Y)$. Then α' is an $\mathfrak{s}$-deflation and there is an $\mathfrak{s}$-triangle

$$X'' \longrightarrow Z' \xrightarrow{\alpha'} Y \overset{\delta'}{\dashrightarrow} .$$

By Definition 2.13, there are $v \in \operatorname{Hom}_{\mathfrak{C}}(Z,Z')$ and $v' \in \operatorname{Hom}_{\mathfrak{C}}(Z',Z)$ such that $\alpha = \alpha' v$ and $\alpha' = \alpha v'$. Thus $\alpha' = \alpha'(vv')$. Since α' is right minimal, we have that vv' is an automorphism. Moreover, by (ET3)$^{\mathrm{op}}$ there exist $u \in \operatorname{Hom}_{\mathfrak{C}}(X',X'')$ and $u' \in \operatorname{Hom}_{\mathfrak{C}}(X'',X')$ such that we have the following morphisms of $\mathfrak{s}$-triangles

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d, dotted, "u'"] & Z' \arrow[r, "\alpha'"] \arrow[d, "v'"] & Y \arrow[r, dashed, "\delta'"] \arrow[d, equal] & {} \\
X' \arrow[r] \arrow[d, dotted, "u"] & Z \arrow[r, "\alpha"] \arrow[d, "v"] & Y \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
X'' \arrow[r] & Z' \arrow[r, "\alpha'"] & Y \arrow[r, dashed, "\delta'"] & {} .
\end{tikzcd}$$

By [29, Corollary 3.6], uu' is also an automorphism. This implies that X'' is a direct summand of X' and $X'' \in \operatorname{add} X$, and hence $\operatorname{Hom}_{\mathfrak{C}}(X'',X) \in \operatorname{End}_{\mathfrak{C}}(X)\text{-}\operatorname{proj}$. By Proposition 6.2, ${\delta'}_X^\sharp : \operatorname{Hom}_{\mathfrak{C}}(X'',X) \to \operatorname{Im}{\delta'}_X^\sharp$ is a projective cover of $\operatorname{Im}{\delta'}_X^\sharp$. Since $\operatorname{Im}\delta_X^\sharp = \operatorname{Im}{\delta'}_X^\sharp$ by Claim 1 of Section 4, we get a projective cover ${\delta'}_X^\sharp : \operatorname{Hom}_{\mathfrak{C}}(X'',X) \to \operatorname{Im}\delta_X^\sharp$.

(2) ⇒ (1) By (2), there exists a projective cover $w : \operatorname{Hom}_{\mathfrak{C}}(X''',X) \twoheadrightarrow \operatorname{Im}\delta_X^\sharp$ with $X''' \in \operatorname{add} X$. Then the morphism $w : \operatorname{Hom}_{\mathfrak{C}}(X''',X) \to \mathbb{E}(Y,X)$ is right minimal. By (4.1), there exists an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(Y,X''')$ such that $w = {\delta''}_X^\sharp$ and $\operatorname{Im}\delta_X^\sharp = \operatorname{Im} w = \operatorname{Im}{\delta''}_X^\sharp$. Let

$$X''' \longrightarrow Z'' \xrightarrow{\alpha''} Y \overset{\delta''}{\dashrightarrow}$$

be an $\mathfrak{s}$-triangle. By Proposition 6.2, α'' is right minimal. By using an argument similar to that of Claim 2 in Section 4, we get that α is right equivalent to α'', which completes the proof. $\square$

In what follows, let $\mathfrak{C}$ be a Hom-finite R-linear Krull-Schmidt extriangulated category having Auslander-Reiten-Serre duality. Under this assumption, the ring $\mathrm{End}_{\mathfrak{C}}(X)$ is an artin algebra for any $X \in \mathfrak{C}$, and hence any $\mathrm{End}_{\mathfrak{C}}(X)$-module has a projective cover. Thus for any $[\alpha\rangle \in [\to Y\rangle_{\mathrm{def}}$, there exists a right minimal $\mathfrak{s}$-deflation α' such that $[\alpha\rangle = [\alpha'\rangle$ in $[\to Y\rangle_{\mathrm{def}}$. Following this, we only need to consider right minimal $\mathfrak{s}$-deflations.

Let $X \in \mathfrak{C}$ and $X' \in \operatorname{add} X$. For any $\mathfrak{s}$-triangle

$$X' \longrightarrow Z \xrightarrow{\alpha} Y \overset{\delta}{\dashrightarrow}$$

with α right minimal, by Proposition 4.2 there exists a map

$$\begin{aligned} \Re:\ \mathbb{E}(Y,X') &\longrightarrow {}^{\tau^- X}[\to Y\rangle_{\mathrm{def}}. \\ \delta &\mapsto [\alpha\rangle \end{aligned}$$

Now we define a map

$$\Xi : D\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X', Y) \longrightarrow \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}}\, \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y)$$

to be the following composition

$$\begin{aligned} D\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X', Y) &\xrightarrow{D\Psi_{Y,X'}} \mathbb{E}(Y,X') \xrightarrow{\Re} {}^{\tau^- X}[\to Y\rangle_{\mathrm{def}} \\ &\xrightarrow{\eta_{\tau^- X, Y}} \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}}\, \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y). \end{aligned}$$

Now we give a realization for the image of the functor Ξ, which extends [12, Proposition 5.3].

Proposition 6.4. *For any R-linear map $\theta \in D\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X', Y)$, we have*

$$\Xi(\theta) = \{f \in \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y) \mid \theta(fg) = 0 \text{ for any } g \in \mathrm{Hom}_{\mathfrak{C}}(\tau^- X', \tau^- X)\}.$$

Proof. Let $\delta := D\Psi_{Y,X'}(\theta)$ and let

$$X' \longrightarrow Z \xrightarrow{\alpha} Y \overset{\delta}{\dashrightarrow}$$

be an $\mathfrak{s}$-triangle with α right minimal. Consider the following diagram

$$\begin{tikzcd} & & & \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y) \\ \mathbb{E}(Y, X') \arrow[r, "\Re"] & {}_{\tau^- X}[\to Y\rangle_{\mathrm{def}} \arrow[urr, "\eta_{\tau^- X, Y}"] \arrow[rr, "\xi_{X,Y}"] & & \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(X)} \mathbb{E}(Y, X). \arrow[u, "\Upsilon_{X,Y}"'] \end{tikzcd}$$

We have

$$\Xi(\theta) = \eta_{\tau^- X, Y} \Re(\delta) = \eta_{\tau^- X, Y}([\alpha\rangle) = \Upsilon_{X,Y} \xi_{X,Y}([\alpha\rangle) = \Upsilon_{X,Y}(\mathrm{Im}\, \delta_X^\sharp) = H,$$

where H is defined by the following exact sequence

$$0 \longrightarrow H \longrightarrow \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y) \xrightarrow{D(i)\Psi_{Y,X}^{-1}} D\,\mathrm{Im}\, \delta_X^\sharp \longrightarrow 0$$

with $i : \mathrm{Im}\, \delta_X^\sharp \to \mathbb{E}(Y, X)$ an inclusion.

Consider the following R-module

$$S = \big\{\theta' \in D\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y) \mid \text{ there is } g \in \mathrm{Hom}_{\mathfrak{C}}(\tau^- X', \tau^- X) \text{ with } \theta'(f) = \theta(fg) \\ \text{for each } f \in \mathrm{Hom}_{\mathfrak{C}}(\tau^- X, Y)\big\}$$

and let $i' : S \to D\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y)$ be an inclusion. Note that $\mathrm{Im}\, \delta_X^\sharp = \{u_\star \delta \in \mathbb{E}(Y, X) \mid u \in \mathrm{Hom}_{\mathfrak{C}}(X', X)\}$. We apply the naturality of Ψ on the second variable and the equivalence τ^-. Then $D\Psi_{Y,X}$ identifies S with $\mathrm{Im}\, \delta_X^\sharp$. Thus we have a commutative diagram

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y) \arrow[r, "D(i')"] \arrow[d, "\Psi_{Y,X}^{-1}"'] & DS \arrow[d, "\Psi_{X,Y}^{-1}"] \\ D\mathbb{E}(Y, X) \arrow[r, "D(i)"] & D\,\mathrm{Im}\, \delta_X^\sharp. \end{tikzcd}$$

It follows that there is an exact sequence

$$0 \longrightarrow H \longrightarrow \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y) \xrightarrow{D(i')} DS \longrightarrow 0.$$

Note that

$$\begin{aligned} D(i') : \quad & \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y) \to DS \\ & \qquad\qquad f \quad \mapsto \quad (D(i'))(f) : \quad S \to E \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \theta' \mapsto \theta'(f). \end{aligned}$$

Thus $f \in H$ if and only if $\theta'(f) = 0$ for each $\theta' \in S$, and if and only if $\theta(fg) = 0$ for each $g \in \mathrm{Hom}_{\mathfrak{C}}(\tau^- X', \tau^- X)$. Therefore we have

$$\Xi(\theta) = \{f \in \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y) \mid \theta(fg) = 0 \text{ for any } g \in \mathrm{Hom}_{\mathfrak{C}}(\tau^- X', \tau^- X)\}. \quad \square$$

In [33, Section 10], Ringel conjectured that one attaches to a linear map $\alpha \in D\underline{\mathrm{Hom}}_A(\tau^- X, Y)$ the largest $\mathrm{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}$-submodule of $\underline{\mathrm{Hom}}_A(\tau^- X, Y)$ contained in $\mathrm{Ker}\,\alpha$, where A is an artin algebra.

By Proposition 6.4, we have the following result, which shows that Ringel's conjecture holds true in the setting of extriangulated categories.

Corollary 6.5. *For any R-linear map $\theta \in D\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y)$, we have*

$$\Xi(\theta) = \{f \in \underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y) \mid \theta(fg) = 0 \textit{ for any } g \in \mathrm{End}_{\mathfrak{C}}(\tau^- X)\},$$

which is the largest $\mathrm{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}$-submodule of $\underline{\mathrm{Hom}}_{\mathfrak{C}}(\tau^- X, Y)$ contained in $\mathrm{Ker}\,\theta$.

As some particular cases, we have that Ringel's conjecture holds true in the following categories:

- Hom-finite R-linear Krull-Schmidt exact categories having Auslander-Reiten-Serre duality.
- Hom-finite R-linear Krull-Schmidt triangulated categories having Auslander-Reiten-Serre duality.
- Hom-finite R-linear abelian categories having Auslander-Reiten duality ([12, Corollary 5.4]).

6.2. The Auslander bijection over extriangulated category having Serre duality

By Proposition 3.3 and Theorem 3.5, if $\mathfrak{C}$ has Serre duality, then ${}^{C}[\to Y\rangle_{\mathrm{def}} = {}^{C}[\to Y\rangle$ for any $C \in \mathfrak{C}$. Moreover, in this case, $\mathfrak{C}$ has Auslander-Reiten-Serre duality by Corollary 3.6. It follows from Theorem 5.4 that the Auslander bijection holds true in extriangulated category having Serre duality, that is, we have

Corollary 6.6. *Let $\mathfrak{C}$ be a Hom-finite R-linear Krull-Schmidt extriangulated category having Serre duality. For any $X, Y \in \mathfrak{C}$, the following bijection triangle*

$$\begin{tikzcd}[column sep=small]
& \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}} \mathrm{Hom}_{\mathfrak{C}}(\tau^- X, Y) & \\
{}^{\tau^- X}[\to Y\rangle \arrow[ur, "\eta_{\tau^- X, Y}"] \arrow[rr, "\xi_{X,Y}"] & & \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(X)} \mathbb{E}(Y, X) \arrow[ul, "\Upsilon_{X,Y}"']
\end{tikzcd}$$

is commutative. In particular, we get the Auslander bijection at Y relative to $\tau^- X$

$$\eta_{\tau^- X, Y} : {}^{\tau^- X}[\to Y\rangle \longrightarrow \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}} \mathrm{Hom}_{\mathfrak{C}}(\tau^- X, Y),$$

which is an isomorphism of posets.

At the end of the paper, we see some examples as follows.

1. Let $\mathcal{T}$ be a triangulated category. Then by [29, Proposition 3.22], we have that $\mathcal{T}$ is an extriangulated category with $\mathbb{E} = \mathrm{Hom}_{\mathcal{T}}(-,-[1])$, and for any $\delta \in \mathbb{E}(Y,X) = \mathrm{Hom}_{\mathcal{T}}(Y,X[1])$, take a distinguished triangle

$$X \xrightarrow{x} Z \xrightarrow{y} Y \xrightarrow{\delta} X[1],$$

and define

$$\mathfrak{s}(\delta) = [\ X \xrightarrow{x} Z \xrightarrow{y} Y\ ].$$

In this case, each morphism in $\mathcal{T}$ is an $\mathfrak{s}$-deflation, and hence ${}^{\tau^- X}[\to Y\rangle_{\mathrm{def}} = {}^{\tau^- X}[\to Y\rangle$. Moreover, $\mathcal{P} = \mathcal{I} = \{0\}$ in $\mathcal{T}$. It follows from Corollary 3.6 that $\mathcal{T}$ has Serre duality if and only if it has Auslander-Reiten-Serre duality. By Corollary 6.6, we have the following

Corollary 6.7. *Let $\mathcal{T}$ be a Hom-finite R-linear Krull-Schmidt triangulated category having Auslander-Reiten-Serre duality. Then the Auslander bijection at Y relative to $\tau^- X$*

$$\eta_{\tau^- X, Y} : {}^{\tau^- X}[\to Y\rangle \longrightarrow \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{T}}(\tau^- X, Y)$$

holds, which is an isomorphism of posets.

2. Let $\mathcal{T}$ be a compactly generated triangulated category and ξ the class of pure triangles. Then $(\mathcal{T}, \mathbb{E}_\xi, \mathfrak{s}_\xi)$ is an extriangulated category which is neither exact nor triangulated in general, where

$$\mathbb{E}_\xi(C,A) := \{\delta \in \mathrm{Hom}_{\mathcal{T}}(C, A[1]) \mid \text{there is a pure triangle} \\ A \longrightarrow B \longrightarrow C \xrightarrow{\delta} A[1] \ \text{ in } \mathcal{T}\}$$

for any $A, C \in \mathcal{T}$ and

$$\mathfrak{s}_\xi(\delta) = [\ A \xrightarrow{f} B \xrightarrow{g} C\ ]$$

for any $\delta \in \mathbb{E}_\xi(C,A)$ with a pure triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta} A[1].$$

Note that the morphism δ is called a *phantom map* in [23].

Condition ($\diamond$): For any non-zero object $X \in \mathcal{T}$, there is a non-zero phantom map ending at X, and a non-zero phantom map starting from X.

In fact, if $(\mathcal{T}, \mathbb{E}_\xi, \mathfrak{s}_\xi)$ satisfies Condition ($\diamond$), then $\mathcal{P} = \{0\} = \mathcal{I}$ (see [23, Lemma 1.4]). By Theorem 5.4 and Corollary 6.6, we have the following

Corollary 6.8. *Let $(\mathcal{T}, \mathbb{E}_\xi, \mathfrak{s}_\xi)$ be a Hom-finite R-linear Krull-Schmidt extriangulated category having Auslander-Reiten-Serre duality. Then the restricted Auslander bijection at Y relative to $\tau^- X$*

$$\eta_{\tau^- X,Y} : {}^{\tau^- X}[\to Y\rangle_{\mathrm{def}} \longrightarrow \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}} \underline{\mathrm{Hom}}_{\mathcal{T}}(\tau^- X, Y)$$

holds, which is an isomorphism of posets. Moreover, if $(\mathcal{T}, \mathbb{E}_\xi, \mathfrak{s}_\xi)$ satisfies Condition ($\diamond$), then the Auslander bijection at Y relative to $\tau^- X$

$$\eta_{\tau^- X,Y} : {}^{\tau^- X}[\to Y\rangle \longrightarrow \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^- X)^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{T}}(\tau^- X, Y)$$

holds, which is an isomorphism of posets.

3. Let A be an artin algebra and $C^{[-1,0]}(A\text{-proj})$ the category of complexes of finitely generated projective A-modules concentrated in degrees -1 and 0. Let $K^{[-1,0]}(A\text{-proj})$ be the category whose objects are the same of $C^{[-1,0]}(A\text{-proj})$, but morphisms are considered up to homotopy. Then $C^{[-1,0]}(A\text{-proj})$ is an Ext-finite R-linear Krull-Schmidt exact sequence which is not abelian, and $K^{[-1,0]}(A\text{-proj})$ is an Ext-finite R-linear Krull-Schmidt extriangulated category which is not a triangulated category. It follows from [8, Proposition 5.4] and [21, Proposition 6.1] that both $C^{[-1,0]}(A\text{-proj})$ and $K^{[-1,0]}(A\text{-proj})$ have Auslander-Reiten-Serre duality. By Corollary 3.6, $C^{[-1,0]}(A\text{-proj})$ has no Serre duality. Moreover, it is easy to check that $(0 \to A)$ is a non-zero projective object in $K^{[-1,0]}(A\text{-proj})$ (e.g. see [32, Proposition 4.39]). It follows from Corollary 3.6 that $K^{[-1,0]}(A\text{-proj})$ also has no Serre duality. By Theorem 5.4, we have the following

Corollary 6.9. *The restricted Auslander bijection at Y relative to $\tau^- X$ in $C^{[-1,0]}(A\text{-proj})$ or $K^{[-1,0]}(A\text{-proj})$ holds.*

However, we do not know whether the Auslander bijection in $C^{[-1,0]}(A\text{-proj})$ or $K^{[-1,0]}(A\text{-proj})$ holds true.

Now we take $Q = (Q_{-1} \xrightarrow{q} Q_0) \in K^{[-1,0]}(A\text{-proj})$ with q epic. For any $P = (P_{-1} \xrightarrow{p} P_0) \in K^{[-1,0]}(A\text{-proj})$ and $f : P \to Q[1]$, we have the following commutative diagram

$$\begin{tikzcd} & P_{-1} \arrow[r, "p"] \arrow[dl, dotted, "s"'] \arrow[d, "f_{-1}"] & P_0 \arrow[dl, dotted, "0"] \\ Q_{-1} \arrow[r, "q"] & Q_0, & \end{tikzcd}$$

which shows that $f = 0$. Thus $\mathrm{Hom}_{K^{[-1,0]}(A\text{-proj})}(P, Q[1]) = 0$. Using the Auslander bijection triangle in this case, we have $\eta_{\tau^- Q, P} = 0$, which implies

$$^{\tau^- Q}[\to P\rangle_{\mathrm{def}} = \mathrm{sub}_{\mathrm{End}_{\mathfrak{C}}(\tau^- Q)^{\mathrm{op}}} \underline{\mathrm{Hom}}_{K^{[-1,0]}(A\text{-proj})}(\tau^- Q, P) = \{0\}.$$

For example, let A be the path algebra given by the quiver

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} c \xrightarrow{\gamma} d$$

modulo the relation $\gamma\beta\alpha = 0$. Then the Auslander-Reiten quiver of the extriangulated category $K^{[-1,0]}(A\text{-proj})$ is as follows (see [32, Example 4.61]):

$$\begin{tikzcd}[column sep=small]
 & & & & (0 \to P_d) \arrow[ddr] & & {[P_b \to 0]} \arrow[ddr] & & \\
 & & (0 \to P_c) \arrow[dr] & & {[P_a \to 0]} \arrow[dr] & & & & \\
 & (0 \to P_b) \arrow[ur] \arrow[dr] & & (P_a \to P_c) \arrow[uur] \arrow[ur] \arrow[dr] & & {[P_b \to P_d]} \arrow[uur] \arrow[dr] & & {[P_c \to 0]} \arrow[dr] & \\
(0 \to P_a) \arrow[ur] & & (P_a \to P_b) \arrow[ur] & & {[P_b \to P_c]} \arrow[ur] & & {[P_c \to P_d]} \arrow[ur] & & {[P_d \to 0]}.
\end{tikzcd}$$

Here the symbol $[\cdot \xrightarrow{q} \cdot]$ denotes the morphism q to be epic.

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