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Almost split triangles and morphisms determined by objects in extriangulated categories

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ABSTRACT

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an Ext-finite, Krull-Schmidt and k -linear extriangulated category with k a commutative artinian ring. We define an additive subcategory \mathcal{C}_r (respectively, \mathcal{C}_l) of \mathcal{C} in terms of the representable functors from the stable category of \mathcal{C} modulo \mathfrak{s} -injectives (respectively, \mathfrak{s} -projectives) to k -modules, which consists of all \mathfrak{s} -projective (respectively, \mathfrak{s} -injective) objects and objects isomorphic to direct summands of finite direct sums of all third (respectively, first) terms of almost split \mathfrak{s} -triangles. We investigate the subcategories \mathcal{C}_r and \mathcal{C}_l in terms of morphisms determined by objects, and then give equivalent characterizations on the existence of almost split \mathfrak{s} -triangles.

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1. Introduction

In algebra, geometry and topology, exact categories and triangulated categories are two fundamental structures. As expected, exact categories and triangulated categories are not independent of each other. A well-known fact is that triangulated categories which at the same time are abelian must be semisimple [22]. Also, there are a series of ways to produce triangulated categories from abelian ones, such as, taking the stable categories of Frobenius exact categories [11], or taking the homotopy categories or derived categories of complexes over abelian categories [22].

On the other hand, because of the recent development of the cluster theory, it becomes possible to produce abelian categories from triangulated ones, that is, starting from a cluster category and taking a cluster tilting subcategory, one can get a suitable quotient category, which turns out to be abelian [16,23]. In addition, exact categories and triangulated categories share properties in many homological invariants, for example, in the aspect of the approximation theory [1,20,23]. The approximation theory is the main part of relative homological algebra and the representation theory of algebras, and its starting point is to approximate arbitrary objects by a class of suitable subcategories. In this process, the notion of cotorsion pairs [20,21,24] provides a fruitful context, in particular, it is closely related to many important homological structures, such as t -structures, co- t -structures, cluster tilting subcategories, and so on. In general, to transfer the homological properties between exact categories and triangulated categories, one needs to specify to the case of stable categories of Frobenius exact categories, and then lift (or descend) the associated definitions and statements, and finally adapt the proof so that it can be applied to arbitrary exact (or triangulated) categories.

However, it is not easy to do it in general case, especially in the third step above. To overcome the difficulty, Nakaoka and Palu [25] introduced the notion of externally triangulated categories (extriangulated categories for short) by a careful looking what is necessary in the definition of cotorsion pairs in exact and triangulated cases. Under this notion, exact categories with a suitable assumption and extension-closed subcategories of triangulated categories (they may no longer be triangulated categories in general) both are externally triangulated ([25]), and hence, in some levels, it becomes easy to give uniform statements and proofs in the exact and triangulated settings [21,25,28,29].

The Auslander-Reiten theory, initiated in [4,5], plays a crucial role in the representation theory of algebras and related topics, especially in the aspect of understanding the structure of module categories of finite dimensional algebras [3] and that of exact and triangulated categories [8,14,15,18]. As a simultaneous generalization and enhancement of the Auslander-Reiten theory in exact categories and triangulated categories, recently, Iyama, Nakaoka and Palu [13] investigated the Auslander-Reiten theory in extriangulated categories. They gave two different sets of sufficient conditions in the extriangulated category so that the existence of almost split extensions is equivalent to that of an Auslander-Reiten-Serre duality. In this paper, as a continuation of their work,

we will investigate the existence of almost split triangles in extriangulated categories. The paper is organized as follows.

In Section 2, we give some terminologies and some preliminary results.

Let $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ be an Ext-finite, Krull-Schmidt and k -linear extriangulated category with k a commutative artinian ring. In Section 3, we introduce an additive subcategory \mathfrak{C}_r (respectively, \mathfrak{C}_l) of \mathfrak{C} in terms of the representable functors from the stable category of \mathfrak{C} modulo \mathfrak{s} -injectives (respectively, \mathfrak{s} -projectives) to the category of k -modules. For an indecomposable object Y in \mathfrak{C} , if Y is non- \mathfrak{s} -projective (respectively, non- \mathfrak{s} -injective), then $Y \in \mathfrak{C}_r$ (respectively, $Y \in \mathfrak{C}_l$) if and only if there exists an almost split \mathfrak{s} -triangle ending (respectively, starting) at Y (Proposition 3.3). Moreover, we get two quasi-inverse functors τ and τ^- in the stable categories of \mathfrak{C}_r and \mathfrak{C}_l (Theorem 3.7), and the pair (τ^-, τ) forms an adjoint pair (Proposition 3.8).

In Section 4, we mainly characterize the subcategory \mathfrak{C}_r via morphisms determined by objects. We prove that for any $C \in \mathfrak{C}_r$ and $Y \in \mathfrak{C}$, if H is a right $\text{End}_{\mathfrak{C}}(C)$ -submodule of $\mathfrak{C}(C, Y)$ containing the class $\mathcal{P}(C, Y)$ of \mathfrak{s} -projective morphisms, then there exists an \mathfrak{s} -triangle

$$K \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\eta} \gg,$$

such that α is right C -determined, $K \in \text{add}(\tau C)$ and $H = \text{Im } \mathfrak{C}(C, \alpha)$ (Theorem 4.7). Under the so-called WIC condition (see Section 4.2 for the definition), we get that an \mathfrak{s} -deflation is right C -determined for some object C if and only if its intrinsic weak kernel lies in \mathfrak{C}_l (Theorem 4.10). Moreover, we have the following

Theorem 1.1. (Theorem 4.13) *Under the WIC condition, the following statements are equivalent for any non- \mathfrak{s} -projective and indecomposable object C in \mathfrak{C} .*

- (1) $C \in \mathfrak{C}_r$.
- (2) For each object Y and each right $\text{End}_{\mathfrak{C}}(C)$ -submodule H of $\mathfrak{C}(C, Y)$ satisfying $\mathcal{P}(C, Y) \subseteq H$, there exists a right C -determined \mathfrak{s} -deflation $\alpha: X \rightarrow Y$ such that $H = \text{Im } \mathfrak{C}(C, \alpha)$.
- (3) C is an intrinsic weak cokernel of some \mathfrak{s} -inflation $\alpha: X \rightarrow Y$ which is left K -determined for some object K .
- (4) There exists an almost split \mathfrak{s} -triangle ending at C .
- (5) There exists a non-retraction \mathfrak{s} -deflation which is right C -determined.
- (6) There exists an \mathfrak{s} -deflation $\alpha: X \rightarrow Y$ and a morphism $f: C \rightarrow Y$ such that f almost factors through α .

In Section 5, we give some examples to illustrate the subcategories \mathfrak{C}_r and \mathfrak{C}_l .

2. Preliminaries

Throughout \mathfrak{C} is an additive category and $\mathbb{E} : \mathfrak{C}^{\text{op}} \times \mathfrak{C} \rightarrow \mathfrak{Ab}$ is a biadditive functor, where \mathfrak{Ab} is the category of abelian groups.

2.1. \mathbb{E} -extensions

Definition 2.1. ([25, Definitions 2.1 and 2.5]) For any $A, C \in \mathfrak{C}$, there exists a corresponding abelian group $\mathbb{E}(C, A)$.

- (1) An element $\delta \in \mathbb{E}(C, A)$ is called an \mathbb{E} -extension. More formally, an \mathbb{E} -extension is a triple (A, δ, C) .
- (2) The zero element 0 in $\mathbb{E}(C, A)$ is called the *split \mathbb{E} -extension*.

Let $a \in \mathfrak{C}(A, A')$ and $c \in \mathfrak{C}(C', C)$. Then we have the following commutative diagram

$$\begin{array}{ccc}
 \mathbb{E}(C, A) & \xrightarrow{\mathbb{E}(C, a)} & \mathbb{E}(C, A') \\
 \downarrow \mathbb{E}(c, A) & \searrow \mathbb{E}(c, a) & \downarrow \mathbb{E}(c, A') \\
 \mathbb{E}(C', A) & \xrightarrow{\mathbb{E}(C', a)} & \mathbb{E}(C', A')
 \end{array}$$

in \mathfrak{Ab} . For an \mathbb{E} -extension (A, δ, C) , we briefly write $a_*\delta := \mathbb{E}(C, a)(\delta)$ and $c^*\delta := \mathbb{E}(c, A)(\delta)$. Then

$$\mathbb{E}(c, a)(\delta) = c^*a_*\delta = a_*c^*\delta.$$

Definition 2.2. ([25, Definition 2.3]) Given two \mathbb{E} -extensions (A, δ, C) and (A', δ', C') . A *morphism* from δ to δ' is a pair (a, c) of morphisms, where $a \in \mathfrak{C}(A, A')$ and $c \in \mathfrak{C}(C, C')$, such that $a_*\delta = c^*\delta'$. In this case, we write $(a, c) : \delta \rightarrow \delta'$.

Now let $A, C \in \mathfrak{C}$. Two sequences of morphisms

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A \xrightarrow{x'} B' \xrightarrow{y'} C$$

are said to be *equivalent* if there exists an isomorphism $b \in \mathfrak{C}(B, B')$ such that the following diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{x} & B & \xrightarrow{y} & C \\
 \parallel & & \downarrow \cong b & & \parallel \\
 A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C
 \end{array}$$

commutes. We denote by $[A \xrightarrow{x} B \xrightarrow{y} C]$ the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$.

In particular, we write $0 := [A \xrightarrow{\binom{1}{0}} A \oplus C \xrightarrow{\binom{0}{1}} C]$.

Note that, for any pair $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$, since \mathbb{E} is biadditive, there exists a natural isomorphism

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A').$$

We define the symbol $\delta \oplus \delta'$ to be the element in $\mathbb{E}(C \oplus C', A \oplus A')$ corresponding to the element $(\delta, 0, 0, \delta')$ in $\mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A')$ through the above isomorphism.

Definition 2.3. ([25, Definition 2.9]) Let \mathfrak{s} be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to each \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$. The \mathfrak{s} is called a *realization* of \mathbb{E} provided that it satisfies the following condition.

(R) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of \mathbb{E} -extensions with

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C] \text{ and } \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

Then for any morphism $(a, c) : \delta \rightarrow \delta'$, there exists $b \in \mathfrak{C}(B, B')$ such that the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow a & & \downarrow b & & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

commutes.

Let \mathfrak{s} be a realization of \mathbb{E} . If $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ for some \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$, then we say that the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes δ ; and in the condition (R), we say that the triple (a, b, c) realizes the morphism (a, c) .

For any two equivalence classes $[A \xrightarrow{x} B \xrightarrow{y} C]$ and $[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, we define

$$[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] := [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'].$$

Definition 2.4. ([25, Definition 2.10]) A realization \mathfrak{s} of \mathbb{E} is called *additive* if it satisfies the following conditions.

- (1) For any $A, C \in \mathfrak{C}$, the split \mathbb{E} -extension $0 \in \mathbb{E}(C, A)$ satisfies $\mathfrak{s}(0) = 0$.
- (2) For any pair of \mathbb{E} -extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$, we have $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.

Let \mathfrak{s} be an additive realization of \mathbb{E} . By [25, Remark 2.11], if the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes 0 in $\mathbb{E}(C, A)$, then x is a section and y is a retraction.

2.2. Externally triangulated categories

Definition 2.5. ([25, Definition 2.12]) Let \mathfrak{C} be an additive category. The triple $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ is called an *externally triangulated* (or *extriangulated* for short) category if it satisfies the following conditions.

- (ET1) $\mathbb{E} : \mathfrak{C}^{\text{op}} \times \mathfrak{C} \rightarrow \mathfrak{Ab}$ is a biadditive functor.
- (ET2) \mathfrak{s} is an additive realization of \mathbb{E} .
- (ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of \mathbb{E} -extensions with

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C] \text{ and } \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

For any commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow a & & \downarrow b & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

in \mathfrak{C} , there exists a morphism $(a, c) : \delta \rightarrow \delta'$ which is realized by the triple (a, b, c) .

(ET3)^{op} Dual of (ET3).

(ET4) Let $\delta \in \mathbb{E}(C, A)$ and $\rho \in \mathbb{E}(F, B)$ be any pair of \mathbb{E} -extensions with

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C] \text{ and } \mathfrak{s}(\rho) = [B \xrightarrow{u} D \xrightarrow{v} F].$$

Then there exist an object $E \in \mathfrak{C}$, an \mathbb{E} -extension ξ with $\mathfrak{s}(\xi) = [A \xrightarrow{z} D \xrightarrow{w} E]$, and a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \parallel & & \downarrow u & & \downarrow s \\ A & \xrightarrow{z} & D & \xrightarrow{w} & E \\ & & \downarrow v & & \downarrow t \\ & & F & \xlongequal{\quad} & F \end{array}$$

in \mathfrak{C} , which satisfy the following compatibilities.

(i) $\mathfrak{s}(y_*\rho) = [C \xrightarrow{s} E \xrightarrow{t} F]$.

(ii) $s^*\xi = \delta$.

(iii) $x_*\xi = t^*\rho$.

(ET4)^{op} Dual of (ET4).

For examples of extriangulated categories, see [12, Remark 3.3], [25, Example 2.13], [28, Example 2.8] and [29, Corollary 4.12 and Remark 4.13].

Definition 2.6. ([13, Definition 1.16]) Let $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1) and (ET2).

- (1) If a sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes an \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$, then the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ is called an \mathfrak{s} -triangle, and write it in the following way

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \triangleright .$$

In this case, x is called an \mathfrak{s} -inflation, and y is called an \mathfrak{s} -deflation.

- (2) Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \triangleright$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'} \triangleright$ be any pair of \mathfrak{s} -triangles. If a triple (a, b, c) realizes $(a, c) : \delta \rightarrow \delta'$ as in the condition (R), then we write it as

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \triangleright \\ \downarrow a & & \downarrow b & & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{\delta'} \triangleright \end{array} ,$$

and call the triple (a, b, c) a *morphism of \mathfrak{s} -triangles*.

Remark 2.7. Let $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1) and (ET2), and let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \triangleright$ be an \mathfrak{s} -triangle.

- (1) For any $a \in \mathfrak{C}(A, A')$, there exists a morphism of \mathfrak{s} -triangles

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \triangleright \\ \downarrow a & & \downarrow & & \parallel \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \xrightarrow{a_*\delta} \triangleright \end{array} .$$

(2) For any $c \in \mathfrak{C}(C', C)$, there exists a morphism of \mathfrak{s} -triangles

$$\begin{array}{ccccc} A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \xrightarrow{c^* \delta} & \gg \\ \parallel & & \downarrow & & \downarrow c & & \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \gg \end{array}$$

The following lemma is used frequently in this paper.

Lemma 2.8. ([25, Corollary 3.5]) *Assume that $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1), (ET2), (ET3), $(ET3)^{op}$. Let*

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \gg \\ \downarrow a & & \downarrow b & & \downarrow c & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \xrightarrow{\delta'} & \gg \end{array}$$

be any morphism of \mathfrak{s} -triangles. Then the following statements are equivalent.

- (1) a factors through x .
- (2) $a_* \delta = c^* \delta' = 0$.
- (3) c factors through y' .

In particular, in the case $\delta = \delta'$ and $(a, b, c) = (\text{Id}_A, \text{Id}_B, \text{Id}_C)$, we have

$$x \text{ is a section} \Leftrightarrow \delta \text{ is split} \Leftrightarrow y \text{ is a retraction.}$$

Definition 2.9. Let $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1) and (ET2).

- (1) Let $f \in \mathfrak{C}(C', C)$ be a morphism. We call f an \mathfrak{s} -projective morphism if $\mathbb{E}(f, A) = 0$, and an \mathfrak{s} -injective morphism if $\mathbb{E}(A, f) = 0$ for any $A \in \mathfrak{C}$.
- (2) Let $C \in \mathfrak{C}$. We call C an \mathfrak{s} -projective object if the identity morphism Id_C is \mathfrak{s} -projective, and an \mathfrak{s} -injective object if the identity morphism Id_C is \mathfrak{s} -injective.

We call an \mathfrak{s} -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$ split if δ is a split \mathbb{E} -extension.

Lemma 2.10. Let $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1), (ET2), (ET3) and $(ET3)^{op}$, and let $f \in \mathfrak{C}(C', C)$ be a morphism. Then the following statements are equivalent.

- (1) f is \mathfrak{s} -projective.
- (2) f factors through any \mathfrak{s} -deflation $y: B \rightarrow C$.

(3) For any \mathfrak{s} -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$, if there exists a morphism of \mathfrak{s} -triangles

$$\begin{array}{ccccc}
 A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{f^*\delta} \gg \\
 \parallel & & \downarrow g & & \downarrow f \\
 A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \gg,
 \end{array} \tag{2.1}$$

then the top \mathfrak{s} -triangle is split.

Proof. (1) \Leftrightarrow (3) It follows from the definition of \mathfrak{s} -projective morphisms.

(3) \Rightarrow (2) Since y' is a retraction by Lemma 2.8, it follows that there exists a morphism $y'' : C' \rightarrow B'$ such that $y' \circ y'' = \text{Id}_{C'}$. Then $y \circ (g \circ y'') = f \circ y' \circ y'' = f$, and hence (2) holds.

(2) \Rightarrow (3) For any \mathfrak{s} -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$, consider the diagram (2.1). By assumption, f factors through y , and so x' is a section by Lemma 2.8. Thus $f^*\delta = 0$, that is, the top \mathfrak{s} -triangle is split. \square

We denote by \mathcal{P} (respectively, \mathcal{I}) the ideal of \mathfrak{C} consisting of all \mathfrak{s} -projective (respectively, \mathfrak{s} -injective) morphisms. The *stable category* (respectively, *costable category*) of \mathfrak{C} is defined as the ideal quotient

$$\underline{\mathfrak{C}} := \mathfrak{C}/\mathcal{P} \text{ (respectively, } \overline{\mathfrak{C}} := \mathfrak{C}/\mathcal{I}).$$

Lemma 2.11. Let $\alpha : X \rightarrow Y$ be an \mathfrak{s} -deflation. Then a morphism $f : T \rightarrow Y$ factors through α in \mathfrak{C} if and only if \underline{f} factors through $\underline{\alpha}$ in $\underline{\mathfrak{C}}$.

Proof. It suffices to show the sufficiency. Assume that \underline{f} factors through $\underline{\alpha}$ in $\underline{\mathfrak{C}}$. Then there exists some morphism $g : T \rightarrow X$ in \mathfrak{C} such that $\underline{f} = \underline{\alpha} \circ \underline{g}$ in $\underline{\mathfrak{C}}$. We have that $f - \alpha \circ g$ is \mathfrak{s} -projective in \mathfrak{C} . Since α is an \mathfrak{s} -deflation, there exists some morphism $h : T \rightarrow X$ such that $f - \alpha \circ g = \alpha \circ h$ in \mathfrak{C} . It follows that $f = \alpha \circ (g + h)$, factoring through α in \mathfrak{C} . \square

From now on, unless otherwise stated, we assume that the extriangulated category $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ is an Ext-finite, Krull-Schmidt and k -linear extriangulated category with k a commutative artinian ring. Here, an extriangulated category $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ is k -linear if $\mathfrak{C}(A, B)$ and $\mathbb{E}(A, B)$ are k -modules such that the following compositions

$$\begin{aligned}
 \mathfrak{C}(A, B) \times \mathfrak{C}(B, C) &\rightarrow \mathfrak{C}(A, C), \\
 \mathfrak{C}(A, B) \times \mathbb{E}(B, C) \times \mathfrak{C}(C, D) &\rightarrow \mathbb{E}(A, D),
 \end{aligned}$$

are k -linear for any $A, B, C, D \in \mathfrak{C}$; and is *Ext-finite* if $\mathbb{E}(A, B)$ is a finitely generated k -module for any $A, B \in \mathfrak{C}$.

2.3. Almost split extensions

In [13], Iyama, Nakaoka and Palu introduced the notion of almost split \mathbb{E} -extensions.

Definition 2.12. ([13, Definition 2.1]) A non-split (i.e. non-zero) \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$ is said to be *almost split* if it satisfies the following conditions.

- (AS1) $a_*\delta = 0$ for any non-section $a \in \mathfrak{C}(A, A')$.
- (AS2) $c^*\delta = 0$ for any non-retraction $c \in \mathfrak{C}(C', C)$.

Definition 2.13. ([13, Definition 2.7]) Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$ be an \mathfrak{s} -triangle in \mathfrak{C} . It is called *almost split* if δ is an almost split \mathbb{E} -extension.

The following class of morphisms is basic to understand almost split \mathfrak{s} -triangles.

Definition 2.14. ([13, Definition 2.8]) Let \mathfrak{C} be an additive category and A an object in \mathfrak{C} . A morphism $a: A \rightarrow B$ which is not a section is called *left almost split* if

- any morphism $A \rightarrow B'$ which is not a section factors through a .

Dually, a morphism $a: B \rightarrow A$ which is not a retraction is called *right almost split* if

- any morphism $B' \rightarrow A$ which is not a retraction factors through a .

It was showed that for an \mathfrak{s} -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$, it is almost split if and only if x is left almost split and y is right almost split ([13]).

Recall that a morphism $f: X \rightarrow Y$ is called *right minimal*, if each $g \in \text{End}_{\mathfrak{C}}(X)$ with $f \circ g = f$ is an automorphism. Dually, a morphism $f: X \rightarrow Y$ is called *left minimal*, if each $g \in \text{End}_{\mathfrak{C}}(Y)$ with $g \circ f = f$ is an automorphism.

By [13, Propositions 2.5 and 2.10], for an \mathfrak{s} -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$, we have

- if x is left almost split, then A is indecomposable and y is right minimal;
- if y is right almost split, then C is indecomposable and x is left minimal.

So if the \mathfrak{s} -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$ is almost split, then x is left minimal, y is right minimal and A, C are indecomposable.

3. Two subcategories and two functors

3.1. Two subcategories \mathfrak{C}_r and \mathfrak{C}_l

Let k be a commutative artinian ring and \check{k} the minimal injective cogenerator for the category $k\text{-mod}$ of finitely generated k -modules. We write $D := \text{Hom}_k(-, \check{k})$.

The following two lemmas are essentially contained in [13, Proposition 3.1] and its proof.

Lemma 3.1. *Let \mathfrak{C} be a k -linear extriangulated category and*

$$X \longrightarrow E \longrightarrow Y \xrightarrow{\eta} \triangleright$$

an almost split \mathfrak{s} -triangle, and let $\gamma \in DE(Y, X)$ such that $\gamma(\eta) \neq 0$.

(1) *For each M , we have a non-degenerated k -bilinear map*

$$\langle -, - \rangle_M : \overline{\mathfrak{C}}(M, X) \times \mathbb{E}(Y, M) \longrightarrow \check{k}, \quad (\overline{f}, \mu) \mapsto \gamma(f_*\mu).$$

If moreover $\overline{\mathfrak{C}}(M, X) \in k\text{-mod}$ for each M , then the induced map

$$\phi_{Y,M} : \overline{\mathfrak{C}}(M, X) \longrightarrow DE(Y, M), \quad \overline{f} \mapsto \langle \overline{f}, - \rangle_M,$$

is an isomorphism and natural in M with $\gamma = \phi_{Y,X}(\overline{\text{Id}}_X)$.

(2) *For each M , we have a non-degenerated k -bilinear map*

$$\langle -, - \rangle_M : \mathbb{E}(M, X) \times \underline{\mathfrak{C}}(Y, M) \longrightarrow \check{k}, \quad (\mu, \underline{g}) \mapsto \gamma(g^*\mu).$$

If moreover $\underline{\mathfrak{C}}(Y, M) \in k\text{-mod}$ for each M , then the induced map

$$\psi_{X,M} : \underline{\mathfrak{C}}(Y, M) \longrightarrow DE(M, X), \quad \underline{g} \mapsto \langle -, \underline{g} \rangle_M,$$

is an isomorphism and natural in M with $\gamma = \psi_{X,Y}(\underline{\text{Id}}_Y)$.

Lemma 3.2. *Let \mathfrak{C} be a k -linear extriangulated category and $Y \in \mathfrak{C}$ an indecomposable object.*

- (1) *Assume $DE(Y, Z) \in k\text{-mod}$ for each $Z \in \mathfrak{C}$. If the functor $DE(Y, -)$ is isomorphic to $\overline{\mathfrak{C}}(-, Y')$ for some Y' , which has a non- \mathfrak{s} -injective indecomposable direct summand, then there exists an almost split \mathfrak{s} -triangle ending at Y .*
- (2) *Assume $DE(Z, Y) \in k\text{-mod}$ for each $Z \in \mathfrak{C}$. If the functor $DE(-, Y)$ is isomorphic to $\underline{\mathfrak{C}}(Y', -)$ for some Y' , which has a non- \mathfrak{s} -projective indecomposable direct summand, then there exists an almost split \mathfrak{s} -triangle starting at Y .*

We introduce two full subcategories of \mathfrak{C} as follows.

$$\begin{aligned} \mathfrak{C}_r &= \{X \in \mathfrak{C} \mid \text{the functor } D\mathbb{E}(X, -): \overline{\mathfrak{C}} \rightarrow k\text{-mod is representable}\}, \\ \mathfrak{C}_l &= \{X \in \mathfrak{C} \mid \text{the functor } D\mathbb{E}(-, X): \underline{\mathfrak{C}} \rightarrow k\text{-mod is representable}\}. \end{aligned}$$

Here, both \mathfrak{C}_r and \mathfrak{C}_l are additive subcategories which are closed under direct summands. As a consequence of Lemmas 3.1 and 3.2, we have the following description of indecomposable objects in \mathfrak{C}_r and \mathfrak{C}_l .

Proposition 3.3. *Let Y be an indecomposable object in \mathfrak{C} .*

- (1) *If Y is non- \mathfrak{s} -projective, then $Y \in \mathfrak{C}_r$ if and only if there exists an almost split \mathfrak{s} -triangle ending at Y .*
- (2) *If Y is non- \mathfrak{s} -injective, then $Y \in \mathfrak{C}_l$ if and only if there exists an almost split \mathfrak{s} -triangle starting at Y .*

We also have the following easy observation.

Proposition 3.4. *Let X and Y be two objects in \mathfrak{C} .*

- (1) *If $X \in \mathfrak{C}_r$ and $X \simeq Y$ in $\underline{\mathfrak{C}}$, then $Y \in \mathfrak{C}_r$.*
- (2) *If $X \in \mathfrak{C}_l$ and $X \simeq Y$ in $\overline{\mathfrak{C}}$, then $Y \in \mathfrak{C}_l$.*

Proof. (1) The assertion follows from the fact that $D\mathbb{E}(X, -) \simeq D\mathbb{E}(Y, -)$ as functors.
 (2) It is dual to (1). \square

3.2. Two functors τ and τ^-

For each Y in \mathfrak{C}_r , we define τY to be an object in \mathfrak{C} such that there exists an isomorphism of functors

$$\phi_Y: \overline{\mathfrak{C}}(-, \tau Y) \longrightarrow D\mathbb{E}(Y, -).$$

Then τ gives a map from the objects of \mathfrak{C}_r to that of \mathfrak{C} . Dually, for each X in \mathfrak{C}_l , we define $\tau^- X$ to be an object in \mathfrak{C} such that there exists an isomorphism of functors

$$\psi_X: \underline{\mathfrak{C}}(\tau^- X, -) \longrightarrow D\mathbb{E}(-, X).$$

Then τ^- gives a map from the objects of \mathfrak{C}_l to that of \mathfrak{C} .

Lemma 3.5. *Let X and Y be two objects in \mathfrak{C} .*

- (1) *If $X, Y \in \mathfrak{C}_r$ and $X \simeq Y$ in $\underline{\mathfrak{C}}$, then $\tau X \simeq \tau Y$ in $\overline{\mathfrak{C}}$.*

(2) If $X, Y \in \mathfrak{C}_l$ and $X \simeq Y$ in $\overline{\mathfrak{C}}$, then $\tau^-X \simeq \tau^-Y$ in $\underline{\mathfrak{C}}$.

Proof. (1) We observe that $\overline{\mathfrak{C}}(-, \tau X) \simeq \overline{\mathfrak{C}}(-, \tau Y)$, since they are both isomorphic to $D\mathbb{E}(X, -) \simeq D\mathbb{E}(Y, -)$. Then the assertion follows from the Yoneda’s lemma.

(2) It is dual to (1). \square

We denote by $\underline{\mathfrak{C}}_r$ the image of \mathfrak{C}_r under the canonical functor $\mathfrak{C} \rightarrow \underline{\mathfrak{C}}$, and by $\overline{\mathfrak{C}}_l$ the image of \mathfrak{C}_l under the canonical functor $\mathfrak{C} \rightarrow \overline{\mathfrak{C}}$. Then we have

Proposition 3.6.

- (1) τ induces a functor from $\underline{\mathfrak{C}}_r$ to $\overline{\mathfrak{C}}_l$.
- (2) τ^- induces a functor from $\overline{\mathfrak{C}}_l$ to $\underline{\mathfrak{C}}_r$.

Moreover, we have

- (3) If $Y \in \mathfrak{C}_r$, then $Y \simeq \tau^- \tau Y$ in $\underline{\mathfrak{C}}_r$.
- (4) If $Y \in \mathfrak{C}_l$, then $Y \simeq \tau \tau^- Y$ in $\overline{\mathfrak{C}}_l$.

Proof. (1) Let $Y \in \mathfrak{C}_r$. We may assume that Y is indecomposable and non- \mathfrak{s} -projective. By Lemma 3.3(1), there exists an almost split \mathfrak{s} -triangle

$$X \longrightarrow E \longrightarrow Y \xrightarrow{\eta} \dots$$

Then we have $\overline{\mathfrak{C}}(-, X) \simeq D\mathbb{E}(Y, -)$ and $\underline{\mathfrak{C}}(Y, -) \simeq D\mathbb{E}(-, X)$ by Lemma 3.1. We then obtain $X \in \mathfrak{C}_l$. It follows from the Yoneda’s lemma that $\tau Y \simeq X$ in $\overline{\mathfrak{C}}$, and $\tau^-X \simeq Y$ in $\underline{\mathfrak{C}}$. So $\tau Y \in \mathfrak{C}_l$ by Proposition 3.4(2), and hence $\tau^- \tau Y \simeq \tau^-X \simeq Y$ in $\underline{\mathfrak{C}}$. Here, the first isomorphism follows from Lemma 3.5(2).

For each morphism $f: Y \rightarrow Y'$ in \mathfrak{C}_r , we define the morphism $\tau(f): \tau Y \rightarrow \tau Y'$ in $\overline{\mathfrak{C}}_l$ such that the following diagram commutes

$$\begin{CD} \overline{\mathfrak{C}}(-, \tau Y) @>\phi_Y>> D\mathbb{E}(Y, -) \\ @V\overline{\mathfrak{C}}(-, \tau(f))VV @VV D\mathbb{E}(f, -)V \\ \overline{\mathfrak{C}}(-, \tau Y') @>\phi_{Y'}>> D\mathbb{E}(Y', -). \end{CD}$$

Here, the existence and uniqueness of $\tau(f)$ are guaranteed by the Yoneda’s lemma. Then it follows that τ is a functor from $\underline{\mathfrak{C}}_r$ to $\overline{\mathfrak{C}}_l$. Moreover, if f is \mathfrak{s} -projective, then $D\mathbb{E}(f, -) = 0$ and thus $\tau(f) = 0$ in $\overline{\mathfrak{C}}_l$. Thus τ induces a functor from $\underline{\mathfrak{C}}_r$ to $\overline{\mathfrak{C}}_l$ which we still denote by τ .

(2) Similarly, we have a functor $\tau^-: \overline{\mathfrak{C}}_l \rightarrow \underline{\mathfrak{C}}_r$. For each $\overline{g}: X \rightarrow X'$ in $\overline{\mathfrak{C}}_l$, the morphism $\tau^-(\overline{g}): \tau^-X \rightarrow \tau^-X'$ is given by the following commutative diagram

$$\begin{array}{ccc}
 \underline{\mathfrak{C}}(\tau^- X', -) & \xrightarrow{\psi_{X'}} & D\mathbb{E}(-, X') \\
 \underline{\mathfrak{C}}(\tau^-(\bar{g}), -) \downarrow & & \downarrow D\mathbb{E}(-, g) \\
 \underline{\mathfrak{C}}(\tau^- X, -) & \xrightarrow{\psi_X} & D\mathbb{E}(-, X).
 \end{array}$$

(3) Since $\tau Y \simeq X$ in $\bar{\mathfrak{C}}$, we have $\tau^- \tau Y \simeq \tau^- X \simeq Y$ in $\underline{\mathfrak{C}}$ by Lemma 3.5(2). Of course, $\tau^- \tau Y \simeq Y$ in $\underline{\mathfrak{C}}_\tau$.

(4) It is similar to (3). \square

For each $Y \in \underline{\mathfrak{C}}_\tau$, set

$$\underline{\theta}_Y := \psi_{\tau Y, Y}^{-1}(\phi_{Y, \tau Y}(\overline{\text{Id}_{\tau Y}})): \tau^- \tau Y \longrightarrow Y$$

in $\underline{\mathfrak{C}}_\tau$. Dually, for each $X \in \bar{\mathfrak{C}}_l$, set

$$\bar{\xi}_X := \phi_{\tau^- X, X}^{-1}(\psi_{X, \tau^- X}(\underline{\text{Id}_{\tau^- X}})): X \longrightarrow \tau \tau^- X$$

in $\bar{\mathfrak{C}}_l$. The following result shows that the functors τ and τ^- are mutually quasi-inverse equivalences between $\underline{\mathfrak{C}}_\tau$ and $\bar{\mathfrak{C}}_l$.

Theorem 3.7. *Under the above definitions,*

$$\underline{\theta} : \tau^- \tau \rightarrow \text{Id}_{\underline{\mathfrak{C}}_\tau} \text{ and } \bar{\xi} : \text{Id}_{\bar{\mathfrak{C}}_l} \rightarrow \tau \tau^-$$

are both natural isomorphisms. This implies that the functors τ and τ^- are quasi-inverse to each other.

Proof. We only prove that $\underline{\theta} : \tau^- \tau \rightarrow \text{Id}_{\underline{\mathfrak{C}}_\tau}$ is a natural isomorphism, the other is similar.

We first prove that $\underline{\theta} : \tau^- \tau \rightarrow \text{Id}_{\underline{\mathfrak{C}}_\tau}$ is a natural transformation. Indeed, for each $\underline{f} : Y \rightarrow Y'$ in $\underline{\mathfrak{C}}_\tau$, consider the following diagram

$$\begin{array}{ccccc}
 \bar{\mathfrak{C}}(\tau Y, \tau Y) & \xrightarrow{\phi_{Y, \tau Y}} & D\mathbb{E}(Y, \tau Y) & \xleftarrow{\psi_{\tau Y, Y}} & \underline{\mathfrak{C}}(\tau^- \tau Y, Y) \\
 \bar{\mathfrak{C}}(\tau Y, \tau(\underline{f})) \downarrow & & D\mathbb{E}(f, \tau Y) \downarrow & & \underline{\mathfrak{C}}(\tau^- \tau Y, \underline{f}) \downarrow \\
 \bar{\mathfrak{C}}(\tau Y, \tau Y') & \xrightarrow{\phi_{Y', \tau Y}} & D\mathbb{E}(Y', \tau Y) & \xleftarrow{\psi_{\tau Y, Y'}} & \underline{\mathfrak{C}}(\tau^- \tau Y, Y').
 \end{array}$$

The left square commutes by the definition of $\tau(\underline{f})$, and the right square commutes since the isomorphism $\psi_{\tau Y}$ is natural. By a diagram chasing, we have

$$\tau(\underline{f}) = \phi_{Y', \tau Y}^{-1}(\psi_{\tau Y, Y'}(\underline{f} \circ \underline{\theta}_Y)).$$

We have the following commutative diagram

$$\begin{array}{ccccc}
 \bar{\mathfrak{C}}(\tau Y', \tau Y') & \xrightarrow{\phi_{Y', \tau Y'}} & DE(Y', \tau Y') & \xleftarrow{\psi_{\tau Y', Y'}} & \underline{\mathfrak{C}}(\tau^- \tau Y', Y') \\
 \downarrow \bar{\mathfrak{C}}(\tau \underline{f}, \tau Y') & & \downarrow DE(Y', \tau \underline{f}) & & \downarrow \underline{\mathfrak{C}}(\tau^- \tau \underline{f}, Y') \\
 \bar{\mathfrak{C}}(\tau Y, \tau Y') & \xrightarrow{\phi_{Y', \tau Y}} & DE(Y', \tau Y) & \xleftarrow{\psi_{\tau Y, Y'}} & \underline{\mathfrak{C}}(\tau^- \tau Y, Y').
 \end{array}$$

The right square commutes by the definition of $\tau^- \tau \underline{f}$. By a diagram chasing, we have

$$\tau(\underline{f}) = \phi_{Y', \tau Y}^{-1}(\psi_{\tau Y, Y'}(\underline{\theta}_{Y'} \circ \tau^- \tau(\underline{f}))).$$

We then obtain $\underline{f} \circ \theta_Y = \underline{\theta}_{Y'} \circ \tau^- \tau(\underline{f})$. It follows that $\underline{\theta}$ is a natural transformation.

Next we prove that $\underline{\theta}_Y$ is an isomorphism for each $Y \in \mathfrak{C}_r$. We may assume that Y is indecomposable and non- \mathfrak{s} -projective in \mathfrak{C} . Let $\alpha = \psi_{\tau Y, \tau^- \tau Y}(\underline{\text{Id}}_{\tau^- \tau Y})$ in $DE(\tau^- \tau Y, \tau Y)$ and let $\beta = \phi_{Y, \tau Y}(\underline{\text{Id}}_{\tau Y})$ in $DE(Y, \tau Y)$. By the definition of $\underline{\theta}_Y$, we have $\beta = \psi_{\tau Y, Y}(\underline{\theta}_Y)$. Consider the following commutative diagram

$$\begin{array}{ccc}
 \underline{\mathfrak{C}}(\tau^- \tau Y, \tau^- \tau Y) & \xrightarrow{\psi_{\tau Y, \tau^- \tau Y}} & DE(\tau^- \tau Y, \tau Y) \\
 \downarrow \underline{\mathfrak{C}}(\tau^- \tau Y, \underline{\theta}_Y) & & \downarrow DE(\theta_Y, \tau Y) \\
 \underline{\mathfrak{C}}(\tau^- \tau Y, Y) & \xrightarrow{\psi_{\tau Y, Y}} & DE(Y, \tau Y).
 \end{array}$$

Since $\psi_{\tau Y, \tau^- \tau Y}(\underline{\text{Id}}_{\tau^- \tau Y}) = \alpha$ and $\underline{\mathfrak{C}}(\tau^- \tau Y, \underline{\theta}_Y)(\underline{\text{Id}}_{\tau^- \tau Y}) = \underline{\theta}_Y$, we have

$$\beta = DE(\theta_Y, \tau Y)(\alpha) = \alpha \circ \mathbb{E}(\theta_Y, \tau Y).$$

By Lemma 3.3(1), there exists an almost split \mathfrak{s} -triangle

$$X \longrightarrow E \longrightarrow Y \xrightarrow{\eta} \cdot$$

By Lemma 3.1(1), we have a natural isomorphism $\phi': \bar{\mathfrak{C}}(-, X) \rightarrow DE(Y, -)$ such that $\phi'_X(\overline{\text{Id}}_X)(\eta) \neq 0$. Setting $\beta' := \phi'_X(\overline{\text{Id}}_X)$, we have $\beta'(\eta) \neq 0$. By the Yoneda's lemma, there exists some $s: X \rightarrow \tau Y$ such that $\bar{\mathfrak{C}}(-, s) = \phi_Y^{-1} \circ \phi'$. We obtain

$$\beta' = \phi'_X(\overline{\text{Id}}_X) = (\phi_{Y, X} \circ \bar{\mathfrak{C}}(X, s))(\overline{\text{Id}}_X) = \phi_{Y, X}(\bar{s}).$$

Consider the following commutative diagram

$$\begin{array}{ccc}
 \overline{\mathfrak{C}}(\tau Y, \tau Y) & \xrightarrow{\phi_{Y, \tau Y}} & D\mathbb{E}(Y, \tau Y) \\
 \overline{\mathfrak{C}}(s, \tau Y) \downarrow & & \downarrow D\mathbb{E}(Y, s) \\
 \overline{\mathfrak{C}}(X, \tau Y) & \xrightarrow{\phi_{Y, X}} & D\mathbb{E}(Y, X).
 \end{array}$$

Since $\phi_{Y, \tau Y}(\overline{\text{Id}}_{\tau Y}) = \beta$ and $\overline{\mathfrak{C}}(s, \tau Y)(\overline{\text{Id}}_{\tau Y}) = \overline{s}$, we have

$$\beta' = DE(Y, s)(\beta) = \beta \circ \mathbb{E}(Y, s) = \alpha \circ \mathbb{E}(\theta_Y, \tau Y) \circ \mathbb{E}(Y, s).$$

Thus we have

$$0 \neq \beta'(\eta) = \alpha(\theta_Y^*(s_*\eta)) = \alpha(s_*(\theta_Y^*\eta)),$$

which implies that the \mathfrak{s} -triangle

$$X \longrightarrow E' \longrightarrow \tau^- \tau Y \xrightarrow{\theta_Y^* \eta} \gg$$

is non-split. We claim that $\theta_Y : \tau^- \tau Y \rightarrow Y$ is a retraction in \mathfrak{C} . Otherwise, suppose that $\theta_Y : \tau^- \tau Y \rightarrow Y$ is not a retraction in \mathfrak{C} . Since η is almost split, we have the following commutative diagram

$$\begin{array}{ccccccc}
 X & \longrightarrow & E' & \longrightarrow & \tau^- \tau Y & \xrightarrow{\theta_Y^* \eta} & \gg \\
 \parallel & & \downarrow & \swarrow \text{dotted} & \downarrow & & \\
 X & \longrightarrow & E & \longrightarrow & Y & \xrightarrow{\eta} & \gg
 \end{array}$$

By Lemma 2.8, the top \mathfrak{s} -triangle is split, which is a contradiction. Thus θ_Y is an isomorphism in $\underline{\mathfrak{C}}$ since we have already known $\tau^- \tau Y \simeq Y$ in $\underline{\mathfrak{C}}$. \square

The following proposition shows that the pair of functors (τ^-, τ) forms an adjoint pair with unit ξ and counit θ .

Proposition 3.8. *We have*

- (1) $\tau(\theta_Y) \circ \overline{\xi_{\tau Y}} = \overline{\text{Id}}_{\tau Y}$ for each $Y \in \underline{\mathfrak{C}}_r$.
- (2) $\theta_{\tau^- X} \circ \tau^-(\overline{\xi_X}) = \overline{\text{Id}}_{\tau^- X}$ for each $X \in \overline{\mathfrak{C}}_l$.

Proof. We only prove (1). Consider the following diagram

$$\begin{array}{ccccc}
 \overline{\mathfrak{C}}(\tau Y, \tau \tau^{-\tau} Y) & \xrightarrow{\phi_{\tau^{-\tau} Y, \tau Y}} & D\mathbb{E}(\tau^{-\tau} Y, \tau Y) & \xleftarrow{\psi_{\tau Y, \tau^{-\tau} Y}} & \underline{\mathfrak{C}}(\tau^{-\tau} Y, \tau^{-\tau} Y) \\
 \downarrow \overline{\mathfrak{C}}(\tau Y, \tau(\theta_Y)) & & \downarrow D\mathbb{E}(\theta_Y, \tau Y) & & \downarrow \underline{\mathfrak{C}}(\tau^{-\tau} Y, \theta_Y) \\
 \overline{\mathfrak{C}}(\tau Y, \tau Y) & \xrightarrow{\phi_{Y, \tau Y}} & D\mathbb{E}(Y, \tau Y) & \xleftarrow{\psi_{\tau Y, Y}} & \underline{\mathfrak{C}}(\tau^{-\tau} Y, Y).
 \end{array}$$

The left square commutes by the definition of $\tau(\theta_Y)$, and the right square commutes since $\psi_{\tau Y}$ is natural. By the definitions of θ_Y and $\overline{\xi_{\tau Y}}$, we have

$$\begin{aligned}
 \phi_{Y, \tau Y}^{-1}(\psi_{\tau Y, Y}(\theta_Y)) &= \overline{\text{Id}_{\tau Y}}, \\
 \phi_{\tau^{-\tau} Y, \tau Y}^{-1}(\psi_{\tau Y, \tau^{-\tau} Y}(\text{Id}_{\tau^{-\tau} Y})) &= \overline{\xi_{\tau Y}}.
 \end{aligned}$$

By the above commutative diagram, we have

$$\overline{\text{Id}_{\tau Y}} = \overline{\mathfrak{C}}(\tau Y, \tau(\theta_Y))(\overline{\xi_{\tau Y}}) = \tau(\theta_Y) \circ \overline{\xi_{\tau Y}}. \quad \square$$

4. Characterizing the subcategory \mathfrak{C}_r via morphisms determined by objects

4.1. \mathfrak{s} -deflations determined by objects

We first recall the concept of morphisms being determined by objects, which was introduced by Auslander in [2] and closely related to the Auslander-Reiten theory [3] and Auslander bijections [27].

Definition 4.1. ([2,27]) Let $f \in \mathfrak{C}(X, Y)$ and $C \in \mathfrak{C}$. We call f *right C-determined* (or *right determined by C*) and call C a *right determiner* of f , if the following condition is satisfied: each $g \in \mathfrak{C}(T, Y)$ factors through f , provided that for each $h \in \mathfrak{C}(C, T)$ the morphism $g \circ h$ factors through f .

If moreover C is a direct summand of any right determiner of f , we call C a *minimal right determiner* of f .

Lemma 4.2. Consider a morphism of \mathfrak{s} -triangles as follows:

$$\begin{array}{ccccccc}
 X & \xrightarrow{e'} & Y' & \xrightarrow{f'} & Z' & \dashrightarrow & \gg \\
 \parallel & & \downarrow g' & & \downarrow g & & \\
 X & \xrightarrow{e} & Y & \xrightarrow{f} & Z & \dashrightarrow & \gg .
 \end{array}$$

If f is right C -determined for some object C , then f' is also right C -determined.

Proof. Let $h: T \rightarrow Z'$ be a morphism such that for each $u: C \rightarrow T$, there exists some morphism $v: C \rightarrow Y'$ satisfying $h \circ u = f' \circ v$. Then we have $g \circ h \circ u = g \circ f' \circ v = f \circ g' \circ v$. Since f is right C -determined, there exists some morphism $s: T \rightarrow Y$ such that $g \circ h = f \circ s$. We illustrate these by a commutative diagram as follows:

$$\begin{array}{ccccc}
 & & C & & \\
 & & \downarrow \nabla u & & \\
 & & T & & \\
 & & \downarrow \nabla h & & \\
 X & \xrightarrow{e'} & Y' & \xrightarrow{f'} & Z' & \xrightarrow{\delta'} & \gg \\
 \parallel & & \downarrow g' & \swarrow s & \downarrow g & & \\
 X & \xrightarrow{e} & Y & \xrightarrow{f} & Z & \xrightarrow{\delta} & \gg .
 \end{array}$$

Thus, to show that f' is right C -determined, it suffices to find a morphism $t: T \rightarrow Y'$ such that $h = f' \circ t$. Indeed, by [25, Proposition 3.3], we have the following commutative diagram with exact rows

$$\begin{array}{ccccc}
 \mathfrak{C}(T, Y') & \xrightarrow{\mathfrak{C}(T, f')} & \mathfrak{C}(T, Z') & \xrightarrow{\delta'_\#} & \mathbb{E}(T, X) \\
 \mathfrak{C}(T, g') \downarrow & & \mathfrak{C}(T, g) \downarrow & & \parallel \\
 \mathfrak{C}(T, Y) & \xrightarrow{\mathfrak{C}(T, f)} & \mathfrak{C}(T, Z) & \xrightarrow{\delta_\#} & \mathbb{E}(T, X).
 \end{array}$$

Since $\mathfrak{C}(T, g)(h) = g \circ h = f \circ s = \mathfrak{C}(T, f)(s) \in \text{Im } \mathfrak{C}(T, f) = \text{Ker } \delta_\#$, we have

$$\delta'_\#(h) = \delta_\#(\mathfrak{C}(T, g)(h)) = 0,$$

and hence $h \in \text{Ker } \delta'_\# = \text{Im } \mathfrak{C}(T, f')$. Thus there exists a morphism $t: T \rightarrow Y'$ such that $h = f' \circ t$. \square

Given two objects X and Y , we denote by $\text{rad}_{\mathfrak{C}}(X, Y)$ the set of morphisms $f: X \rightarrow Y$, that is, for any object Z and any morphisms $g: Z \rightarrow X$ and $h: Y \rightarrow Z$, the morphism $h \circ f \circ g$ lies in $\text{rad } \text{End}_{\mathfrak{C}}(Z)$. Then $\text{rad}_{\mathfrak{C}}$ forms an ideal of \mathfrak{C} . By [17, Corollary 2.10], we have

$$\text{rad}_{\mathfrak{C}}(X, Y) = \{f: X \rightarrow Y \mid f \circ g \in \text{rad } \text{End}_{\mathfrak{C}}(Y), \text{ for each } g: Y \rightarrow X\}. \tag{4.1}$$

A morphism $g: Z \rightarrow Y$ is said to *almost factor through* $f: X \rightarrow Y$, if g does not factor through f , and for each object T and each morphism $h \in \text{rad}_{\mathfrak{C}}(T, Z)$, the morphism $g \circ h$ factors through f ([26]).

Proposition 4.3. Consider a morphism of \mathfrak{s} -triangles as follows:

$$\begin{array}{ccccccc} X & \xrightarrow{e'} & Y' & \xrightarrow{f'} & Z' & \xrightarrow{\delta'} & \gg \\ \parallel & & \downarrow g' & & \downarrow g & & \\ X & \xrightarrow{e} & Y & \xrightarrow{f} & Z & \xrightarrow{\delta} & \gg . \end{array}$$

If Z' is indecomposable and g almost factors through f , then f' is right almost split.

Proof. Since g does not factor through f , f' is not a retraction. Given an object T , assume that $h: T \rightarrow Z'$ is not a retraction. Since Z' is indecomposable, $h \in \text{rad}_{\mathfrak{C}}(T, Z')$ by (4.1). Thus there exists $s \in \mathfrak{C}(T, Y)$ such that $g \circ h = f \circ s$. By using an argument similar to that in the proof of Lemma 4.2, there exists $t \in \mathfrak{C}(T, Y')$ such that $h = f' \circ t$. It follows that f' is right almost split. \square

Lemma 4.4. Let C be an object and $\alpha: X \rightarrow Y$ an \mathfrak{s} -deflation. Then the following statements are equivalent.

- (1) α is right C -determined in \mathfrak{C} .
- (2) $\underline{\alpha}$ is right C -determined in $\underline{\mathfrak{C}}$.

Proof. (1) \Rightarrow (2) Let $f \in \mathfrak{C}(T, Y)$ such that for each $g \in \mathfrak{C}(C, T)$, the morphism $\underline{f} \circ \underline{g}$ factors through $\underline{\alpha}$ in $\underline{\mathfrak{C}}$. By Lemma 2.11, the morphism $f \circ g$ factors through α in \mathfrak{C} . Since α is right C -determined in \mathfrak{C} by (1), f factors through α in \mathfrak{C} . It follows that \underline{f} factors through $\underline{\alpha}$ in $\underline{\mathfrak{C}}$, and hence $\underline{\alpha}$ is right C -determined in $\underline{\mathfrak{C}}$.

(2) \Rightarrow (1) Let $f \in \mathfrak{C}(T, Y)$ such that for each $g \in \mathfrak{C}(C, T)$, the morphism $f \circ g$ factors through α in \mathfrak{C} . Then $\underline{f} \circ \underline{g}$ factors through $\underline{\alpha}$ in $\underline{\mathfrak{C}}$. Since $\underline{\alpha}$ is right C -determined in $\underline{\mathfrak{C}}$ by (2), we have that \underline{f} factors through $\underline{\alpha}$ in $\underline{\mathfrak{C}}$. By Lemma 2.11, the morphism f factors through α in \mathfrak{C} . It follows that α is right C -determined in \mathfrak{C} . \square

Proposition 4.5. Let $C, C' \in \mathfrak{C}$ such that $C \simeq C'$ in $\underline{\mathfrak{C}}$. Then an \mathfrak{s} -deflation $\alpha: X \rightarrow Y$ is right C -determined if and only if it is right C' -determined.

Proof. Note that $\underline{\alpha}$ is right C -determined in $\underline{\mathfrak{C}}$ if and only if $\underline{\alpha}$ is right C' -determined in $\underline{\mathfrak{C}}$. Then the assertion follows by applying Lemma 4.4 twice. \square

The following lemma generalizes [27, Corollary 3.5].

Lemma 4.6. Let

$$K \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\eta} \gg$$

be an \mathfrak{s} -triangle with $K \in \mathfrak{C}_l$. Then α is right τ^-K -determined.

Proof. Let $f \in \mathfrak{C}(T, Y)$ such that for each $g \in \mathfrak{C}(\tau^-K, T)$, the morphism $f \circ g$ factors through α , that is, there exists the following commutative diagram

$$\begin{array}{ccccc}
 & & \tau^-K & & \\
 & & \downarrow \vee g & & \\
 & & T & & \\
 & \circlearrowleft & \downarrow \vee f & \eta & \\
 K \longrightarrow X & \xrightarrow{\alpha} & Y & \dashrightarrow & .
 \end{array}$$

By Lemma 2.8, $(f \circ g)^*\eta = 0$. Since $K \in \mathfrak{C}_i$, there exists a natural isomorphism

$$\phi: \underline{\mathfrak{C}}(\tau^-K, -) \longrightarrow DE(-, K).$$

Set $\gamma := \phi_{\tau^-K}(\underline{\text{Id}}_{\tau^-K})$. By the naturality of ϕ , we have the following commutative diagram

$$\begin{array}{ccc}
 \underline{\mathfrak{C}}(\tau^-K, \tau^-K) & \xrightarrow{\phi_{\tau^-K}} & DE(\tau^-K, K) \\
 \underline{\mathfrak{C}}(\tau^-K, \underline{g}) \downarrow & & \downarrow DE(g, K) \\
 \underline{\mathfrak{C}}(\tau^-K, T) & \xrightarrow{\phi_T} & DE(T, K).
 \end{array}$$

Thus

$$\phi_T(\underline{g}) = DE(g, K)(\gamma) = \gamma \circ E(g, K),$$

and hence

$$\phi_T(\underline{g})(f^*\eta) = \gamma(g^*f^*\eta) = \gamma((f \circ g)^*\eta) = 0.$$

Note that $\phi_T(\underline{g})$ runs over all maps in $DE(T, K)$, when \underline{g} runs over all morphisms in $\underline{\mathfrak{C}}(\tau^-K, T)$. It follows that $f^*\eta = 0$ and hence, by Lemma 2.8, the morphism f factors through α , that is, there exists the following commutative diagram

$$\begin{array}{ccccc}
 & & \tau^-K & & \\
 & & \downarrow \vee g & & \\
 & & T & & \\
 & \circlearrowleft & \downarrow \vee f & \eta & \\
 K \longrightarrow X & \xrightarrow{\alpha} & Y & \dashrightarrow & .
 \end{array}$$

Thus α is right τ^-K -determined. \square

Note that $\mathcal{P}(C, Y)$ is the subset of $\mathfrak{C}(C, Y)$ consisting of all \mathfrak{s} -projective morphisms. The following existence theorem generalizes [3, Corollary XI.3.4] to extriangulated categories.

Theorem 4.7. *Let $C \in \mathfrak{C}_r$ and $Y \in \mathfrak{C}$, and let H be a right $\text{End}_{\mathfrak{C}}(C)$ -submodule of $\mathfrak{C}(C, Y)$ satisfying $\mathcal{P}(C, Y) \subseteq H$. Then there exists an \mathfrak{s} -triangle*

$$K \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\eta} \gg,$$

such that α is right C -determined, $K \in \text{add}(\tau C)$ and $H = \text{Im } \mathfrak{C}(C, \alpha)$.

Proof. By assumption, $\tau C \in \mathfrak{C}_l$ and $\tau^- \tau C \simeq C$ in $\underline{\mathfrak{C}}$. Then there is a natural isomorphism

$$\phi: \underline{\mathfrak{C}}(C, -) \longrightarrow D\mathbb{E}(-, \tau C).$$

Set $\gamma := \phi_C(\text{Id}_C)$. By the naturality of ϕ , for each object Z and each $f \in \mathfrak{C}(C, Z)$, we have the following commutative diagram

$$\begin{array}{ccc} \underline{\mathfrak{C}}(C, C) & \xrightarrow{\phi_C} & D\mathbb{E}(C, \tau C) \\ \mathfrak{C}(C, f) \downarrow & & \downarrow D\mathbb{E}(f, \tau C) \\ \underline{\mathfrak{C}}(C, Z) & \xrightarrow{\phi_Z} & D\mathbb{E}(Z, \tau C). \end{array}$$

So

$$\phi_Z(\underline{f}) = D\mathbb{E}(f, \tau C)(\gamma) = \gamma \circ \mathbb{E}(f, \tau C).$$

Then for each $\mu \in \mathbb{E}(Z, \tau C)$, we have

$$\phi_Z(\underline{f})(\mu) = \gamma(\mathbb{E}(f, \tau C)(\mu)) = \gamma(f^* \mu).$$

Set $\underline{H} := H/\mathcal{P}(C, Y)$ and

$$\underline{H}^\perp := \{ \mu \in \mathbb{E}(Y, \tau C) \mid \phi_Y(\underline{h})(\mu) = 0 \text{ for each } \underline{h} \in \underline{H} \}.$$

We observe that \underline{H}^\perp is a left $\text{End}_{\mathfrak{C}}(C)$ -submodule of $\mathbb{E}(Y, \tau C)$. Here, for any $f \in \text{End}_{\mathfrak{C}}(C)$ and $\mu \in \mathbb{E}(Y, \tau C)$, the action of f on μ is given by $\tau(\underline{f})_* \mu$. Then there exists finitely many $\eta_1, \eta_2, \dots, \eta_n$ in $\mathbb{E}(Y, \tau C)$ such that $\underline{H}^\perp = \sum_{i=1}^n \eta_i \text{End}_{\mathfrak{C}}(C)$. Assume that we have an \mathfrak{s} -triangle

$$\tau C \longrightarrow X_i \xrightarrow{\alpha_i} Y \xrightarrow{\eta_i} \gg$$

for each $i = 1, 2, \dots, n$. Then α_i is right $\tau^- \tau C$ -determined by Lemma 4.6, and thus α_i is right C -determined by Proposition 4.5. Note that $\bigoplus_{i=1}^n \alpha_i$ is an \mathfrak{s} -deflation. It is easy to verify that $\bigoplus_{i=1}^n \alpha_i$ is right C -determined.

Consider the following commutative diagram

$$\begin{array}{ccccccc}
 \bigoplus_{i=1}^n \tau C & \longrightarrow & X & \xrightarrow{\alpha} & Y & \xrightarrow{\Delta^*(\bigoplus \eta_i)} & \gg \\
 \parallel & & \downarrow & & \downarrow \Delta & & \\
 \bigoplus_{i=1}^n \tau C & \longrightarrow & \bigoplus_{i=1}^n X_i & \xrightarrow{\bigoplus_{i=1}^n \alpha_i} & \bigoplus_{i=1}^n Y_i & \xrightarrow{\bigoplus \eta_i} & \gg,
 \end{array}$$

where $\Delta = (\text{Id}_Y, \text{Id}_Y, \dots, \text{Id}_Y)^{\text{tr}}$. We have that α is an \mathfrak{s} -deflation and $\bigoplus_{i=1}^n \tau C \in \text{add}(\tau C)$. By Lemma 4.2, α is right C -determined. By a direct verification, we have

$$\text{Im } \mathfrak{C}(C, \alpha) = \bigcap_{i=1}^n \text{Im } \mathfrak{C}(C, \alpha_i).$$

For each $i = 1, 2, \dots, n$, set

$${}^\perp(\eta_i \text{End}_{\mathfrak{C}}(C)) := \{ \underline{h} \in \underline{\mathfrak{C}}(C, Y) \mid \phi_Y(\underline{h})(\mu) = 0 \text{ for each } \mu \in \eta_i \text{End}_{\mathfrak{C}}(C) \}.$$

We observe that ${}^\perp(\eta_i \text{End}_{\mathfrak{C}}(C))$ is a right $\text{End}_{\mathfrak{C}}(C)$ -submodule of $\underline{\mathfrak{C}}(C, Y)$. Since α_i is an \mathfrak{s} -deflation, $\mathcal{P}(C, Y) \subseteq \text{Im } \mathfrak{C}(C, \alpha_i)$.

Claim. ${}^\perp(\eta_i \text{End}_{\mathfrak{C}}(C)) = \text{Im } \mathfrak{C}(C, \alpha_i) / \mathcal{P}(C, Y)$.

Let $h: C \rightarrow Y$ be a morphism in $\text{Im } \mathfrak{C}(C, \alpha_i)$. We have $h^* \eta_i$ splits. Then

$$\phi_Y(\underline{h})(\tau(\underline{f})_* \eta_i) = \gamma(h^* \tau(\underline{f})_* \eta_i) = \gamma(\tau(\underline{f})_* h^* \eta_i) = 0,$$

for each $f: C \rightarrow C$. It follows that $\text{Im } \mathfrak{C}(C, \alpha_i) / \mathcal{P}(C, Y) \subseteq {}^\perp(\eta_i \text{End}_{\mathfrak{C}}(C))$.

On the other hand, let $h \in \underline{\mathfrak{C}}(C, Y)$ such that $\underline{h} \in {}^\perp(\eta_i \text{End}_{\mathfrak{C}}(C))$. Then we have $\phi_Y(\underline{h})(\tau(\underline{f})_* \eta_i) = 0$ for each $f: C \rightarrow C$. Consider the following commutative diagram

$$\begin{array}{ccc}
 \underline{\mathfrak{C}}(C, Y) & \xrightarrow{\phi_Y} & D\mathbb{E}(Y, \tau C) \\
 \underline{\mathfrak{C}}(\underline{f}, Y) \downarrow & & \downarrow D\mathbb{E}(Y, \tau(\underline{f})) \\
 \underline{\mathfrak{C}}(C, Y) & \xrightarrow{\phi_Y} & D\mathbb{E}(Y, \tau C).
 \end{array}$$

By using diagram chasing, we have

$$\begin{aligned}
 \phi_Y(\underline{h} \circ \underline{f}) &= \phi_Y(\underline{\mathfrak{C}}(\underline{f}, Y)(\underline{h})) \\
 &= (D\mathbb{E}(Y, \tau(\underline{f})) \circ \phi_Y)(\underline{h}) \\
 &= \phi_Y(\underline{h}) \circ \mathbb{E}(Y, \tau(\underline{f})).
 \end{aligned}$$

Then for each η_i , we have

$$\phi_Y(\underline{h} \circ \underline{f})(\eta_i) = \phi_Y(\underline{h})(\tau(\underline{f})_*\eta_i) = 0.$$

It follows that

$$\phi_C(\underline{f})(h^*\eta_i) = \gamma(f^*h^*\eta_i) = \gamma((h \circ f)^*\eta_i) = \phi_Y(\underline{h \circ f})(\eta_i) = 0.$$

Observe that $\phi_C(\underline{f})$ runs over all maps in $D\mathbb{E}(C, \tau C)$, when \underline{f} runs over all morphisms in $\text{End}_{\mathfrak{C}}(C)$. It follows that $h^*\eta_i$ splits and the morphism h factors through α_i . Then we have $h \in \text{Im } \mathfrak{C}(C, \alpha_i)$ and ${}^\perp(\eta_i \text{End}_{\mathfrak{C}}(C)) \subseteq \text{Im } \mathfrak{C}(C, \alpha_i)/\mathcal{P}(C, Y)$. The claim is proved.

Because

$$\underline{H} = {}^\perp(\underline{H}^\perp) = {}^\perp\left(\sum_{i=1}^n \eta_i \text{End}_{\mathfrak{C}}(C)\right) = \bigcap_{i=1}^n {}^\perp(\eta_i \text{End}_{\mathfrak{C}}(C)),$$

where the first equality follows from the isomorphism ϕ_Y , we have

$$\underline{H} = \bigcap_{i=1}^n \text{Im } \mathfrak{C}(C, \alpha_i)/\mathcal{P}(C, Y) = \text{Im } \mathfrak{C}(C, \alpha)/\mathcal{P}(C, Y).$$

Then the assertion follows since $\mathcal{P}(C, Y) \subseteq H$. \square

4.2. A characterization for \mathfrak{s} -deflations determined by objects

In this section, we give a characterization for an \mathfrak{s} -deflation being right C -determined for some object C .

Recall from [9] that two morphisms $f: X \rightarrow Y$ and $f': X' \rightarrow Y$ are called *right equivalent* if f factors through f' and f' factors through f . Assume that f and f' are right equivalent. Given an object C , we have that f is right C -determined if and only if so is f' .

In what follows, we always assume that the following *weak idempotent completeness* (WIC for short) given originally in [25, Condition 5.8] holds true on \mathfrak{C} .

WIC Condition:

- (1) Let $f \in \mathfrak{C}(A, B)$, $g \in \mathfrak{C}(B, C)$ be any composable pair of morphisms. If $g \circ f$ is an \mathfrak{s} -inflation, then so is f .
- (2) Let $f \in \mathfrak{C}(A, B)$, $g \in \mathfrak{C}(B, C)$ be any composable pair of morphisms. If $g \circ f$ is an \mathfrak{s} -deflation, then so is g .

Under this condition, we have that f is an \mathfrak{s} -deflation if and only if so is f' provided that f and f' are right equivalent.

Since \mathfrak{C} is Krull-Schmidt, each morphism $f: X \rightarrow Y$ has a right minimal version; see [7, Theorem 1]. Given a morphism $f: X \rightarrow Y$, we call a right minimal morphism $f': X' \rightarrow Y$ the *right minimal version* of f , if f and f' are right equivalent. Assume that $f: X \rightarrow Y$ is an \mathfrak{s} -deflation and there exists an \mathfrak{s} -triangle

$$K \longrightarrow X' \xrightarrow{f'} Y \dashrightarrow ,$$

then, following [26, Section 2], we call K a *intrinsic weak kernel* of f .

Dually, two morphisms $f: X \rightarrow Y$ and $f': X \rightarrow Y'$ are called *left equivalent* if f factors through f' and f' factors through f . We have that there exists some left minimal morphism $g: X \rightarrow X'$ such that f and g are left equivalent. We call g the *left minimal version* of f . Assume that $f: X \rightarrow Y$ is an \mathfrak{s} -inflation and there exists an \mathfrak{s} -triangle

$$X \xrightarrow{g} X' \longrightarrow Z \dashrightarrow ,$$

then, we call Z an *intrinsic weak cokernel* of f .

Lemma 4.8. *Let $a: X \rightarrow Y$ be an \mathfrak{s} -deflation and C an indecomposable object. If there exists a morphism $f: C \rightarrow Y$ which almost factors through a , then there exists an almost split \mathfrak{s} -triangle*

$$K \longrightarrow E \xrightarrow{r} C \xrightarrow{\rho} \gg$$

such that K is a direct summand of an intrinsic weak kernel of a .

Proof. We may assume that a is right minimal. Since $a: X \rightarrow Y$ is an \mathfrak{s} -deflation, we have an \mathfrak{s} -triangle

$$L \longrightarrow X \xrightarrow{a} Y \xrightarrow{\sigma} \gg .$$

Consider a morphism of \mathfrak{s} -triangles

$$\begin{array}{ccccc} L & \longrightarrow & Z & \xrightarrow{b} & C & \xrightarrow{\delta} & \gg \\ \parallel & & \downarrow & & \downarrow f & & \\ L & \longrightarrow & X & \xrightarrow{a} & Y & \xrightarrow{\sigma} & \gg . \end{array}$$

Then b is an \mathfrak{s} -deflation. By Proposition 4.3, b is right almost split. Let $r: E \rightarrow C$ be the right minimal version of b . Then r is a right almost split \mathfrak{s} -deflation. Let

$$K \longrightarrow E \xrightarrow{r} C \xrightarrow{\rho} \gg$$

be an \mathfrak{s} -triangle. Since b and r are right equivalent, we have the following morphisms of \mathfrak{s} -triangles

$$\begin{array}{ccccc}
 K & \longrightarrow & E & \xrightarrow{r} & C \xrightarrow{\rho} \gg \\
 \downarrow & & \downarrow s & & \parallel \\
 L & \longrightarrow & Z & \xrightarrow{b} & C \xrightarrow{\delta} \gg \\
 \downarrow & & \downarrow t & & \parallel \\
 K & \longrightarrow & E & \xrightarrow{r} & C \xrightarrow{\rho} \gg .
 \end{array}$$

Since r is right minimal, $t \circ s$ is an isomorphism, and hence K is a direct summand of L by [25, Corollary 3.6].

We claim that

$$K \xrightarrow{p} E \xrightarrow{r} C \xrightarrow{\rho} \gg$$

is an almost split \mathfrak{s} -triangle. Indeed, assume $f_*\rho \neq 0$ for any non-section $f \in \mathfrak{C}(K, K')$. Let

$$K' \longrightarrow E' \xrightarrow{r'} C \xrightarrow{f_*\rho} \gg$$

be an \mathfrak{s} -triangle. Consider the following commutative diagram

$$\begin{array}{ccccc}
 K & \xrightarrow{p} & E & \xrightarrow{r} & C \xrightarrow{\rho} \gg \\
 \downarrow f & & \downarrow & & \parallel \\
 K' & \longrightarrow & E' & \xrightarrow{r'} & C \xrightarrow{f_*\rho} \gg .
 \end{array}$$

The \mathfrak{s} -deflation r' is not a retraction, and hence factors through r . We then obtain the following commutative diagram

$$\begin{array}{ccccc}
 K & \xrightarrow{p} & E & \xrightarrow{r} & C \xrightarrow{\rho} \gg \\
 \downarrow f & & \downarrow g & & \parallel \\
 K' & \longrightarrow & E' & \xrightarrow{r'} & C \xrightarrow{f_*\rho} \gg \\
 \downarrow f' & & \downarrow g' & & \parallel \\
 K & \xrightarrow{p} & E & \xrightarrow{r} & C \xrightarrow{\rho} \gg .
 \end{array}$$

Since r is right minimal, the morphism $g' \circ g$ is an isomorphism. We obtain that $f' \circ f$ is an isomorphism, which is a contradiction since f is not a section. This shows that

$f_*\rho = 0$ for any non-section $f \in \mathfrak{C}(K, K')$. By the same argument, we have $g^*\rho = 0$ for any non-retraction $g \in \mathfrak{C}(C', C)$. The claim is proved. \square

By using an argument similar to that in the proofs of [3, Proposition XI.2.4 and Lemma XI.2.1], we get that if C is a minimal right determiner of an \mathfrak{s} -deflation $\alpha: X \rightarrow Y$, then an indecomposable object C' is a direct summand of C if and only if there exists a morphism $f: C' \rightarrow Y$ which almost factors through α .

Corollary 4.9. *A minimal right determiner of an \mathfrak{s} -deflation has no non-zero \mathfrak{s} -projective direct summands and lies in \mathfrak{C}_r .*

Proof. Let $\alpha \in \mathfrak{C}(X, Y)$ be an \mathfrak{s} -deflation and C a minimal right determiner of α . It suffices to show that each indecomposable direct summand C' of C is not \mathfrak{s} -projective and lies in \mathfrak{C}_r . By assumption, there exists a morphism $f: C' \rightarrow Y$ which almost factors through α . We have that C' is not \mathfrak{s} -projective, since f does not factor through α . By Lemma 4.8, there exists an almost split \mathfrak{s} -triangle ending at C' . Then the assertion follows from Proposition 3.3. \square

Now we give a characterization for an \mathfrak{s} -deflation being right C -determined for some object C .

Theorem 4.10. *For any \mathfrak{s} -deflation $\alpha \in \mathfrak{C}(X, Y)$, the following statements are equivalent.*

- (1) α is right C -determined for some object C .
- (2) The intrinsic weak kernel of α lies in \mathfrak{C}_l .

Proof. (1) \Rightarrow (2) We may assume that C is a minimal right determiner of α . By Corollary 4.9, we have $C \in \mathfrak{C}_r$. We observe that $\text{Im } \mathfrak{C}(C, \alpha)$ is a right $\text{End}(C)$ -submodule of $\mathfrak{C}(C, Y)$. Since α is an \mathfrak{s} -deflation, we have $\mathcal{P}(C, Y) \subseteq \text{Im } \mathfrak{C}(C, \alpha)$. By Theorem 4.7, there exists an \mathfrak{s} -triangle

$$K \longrightarrow Z \xrightarrow{\beta} Y \xrightarrow{\rho} \gg$$

such that β is right C -determined, $K \in \text{add}(\tau C)$ and $\text{Im } \mathfrak{C}(C, \alpha) = \text{Im } \mathfrak{C}(C, \beta)$. For any $f \in \mathfrak{C}(C, X)$ and $g \in \mathfrak{C}(C, X')$, we have that $\alpha \circ f$ and $\beta \circ g$ factor through β and α respectively. Since α and β are right C -determined, α and β factor through each other. It follows that α and β are right equivalent.

Let $\alpha': X' \rightarrow Y$ be the right minimal version of α . Then α' and β are right equivalent. Let

$$K' \longrightarrow X' \xrightarrow{\alpha'} Y \xrightarrow{\delta} \gg$$

be an \mathfrak{s} -triangle. Then K' is an intrinsic weak kernel of α' . Since α' and β are right equivalent, there exist $s : X' \rightarrow Z$ and $t : Z \rightarrow X'$ such that $\alpha' = \beta \circ s$ and $\beta = \alpha' \circ t$, and hence we have the following morphisms of \mathfrak{s} -triangles

$$\begin{array}{ccccc}
 K' & \longrightarrow & X' & \xrightarrow{\alpha'} & Y & \xrightarrow{\delta} & \gg \\
 \downarrow s' & & \downarrow s & & \parallel & & \\
 K & \longrightarrow & Z & \xrightarrow{\beta} & Y & \xrightarrow{\rho} & \gg \\
 \downarrow t' & & \downarrow t & & \parallel & & \\
 K' & \longrightarrow & X' & \xrightarrow{\alpha'} & Y & \xrightarrow{\delta} & \gg .
 \end{array}$$

Since α' is right minimal, $t \circ s$ is isomorphic, and so is $t' \circ s'$. Thus K' is a direct summand of K . The assertion follows since $K \in \text{add}(\tau C)$ and $\tau C \in \mathfrak{C}_l$.

(2) \Rightarrow (1) Let $\alpha' : X' \rightarrow Y$ be the right minimal version of α . Then α' is an \mathfrak{s} -deflation. Let

$$K' \longrightarrow X' \xrightarrow{\alpha'} Y \xrightarrow{\delta} \gg$$

be an \mathfrak{s} -triangle. Then $K' \in \mathfrak{C}_l$ by (2). By Lemma 4.6, we have that α' is right $\tau^-(K')$ -determined. It follows that α is right $\tau^-(K')$ -determined since α and α' are right equivalent. \square

4.3. Characterizations for objects in \mathfrak{C}_r

In this subsection, we will give some characterizations for an object lying in \mathfrak{C}_r via morphisms determined by objects.

Proposition 4.11. *For an object K without non-zero \mathfrak{s} -injective direct summands, the following statements are equivalent.*

- (1) $K \in \mathfrak{C}_l$.
- (2) K is an intrinsic weak kernel of some \mathfrak{s} -deflation $\alpha : X \rightarrow Y$, which is right C -determined for some object C .

Proof. (1) \Rightarrow (2) Let $K \in \mathfrak{C}_l$. Decompose K as the direct sum of indecomposable objects K_1, K_2, \dots, K_n . We have that each K_i is non- \mathfrak{s} -injective. Then, by Proposition 3.3, for each $i = 1, 2, \dots, n$, there exists an almost split \mathfrak{s} -triangle

$$K_i \longrightarrow X_i \xrightarrow{\alpha_i} Y_i \xrightarrow{\rho_i} \gg .$$

We have that $\bigoplus_{i=1}^n \alpha_i$ is an \mathfrak{s} -deflation. It follows from Lemma 4.6 that $\bigoplus_{i=1}^n \alpha_i$ is right τ^-K -determined. Moreover, since α_i is right minimal, so is $\bigoplus_{i=1}^n \alpha_i$. It follows that K is an intrinsic weak kernel of $\bigoplus_{i=1}^n \alpha_i$.

(2) \Rightarrow (1) It follows from Theorem 4.10. \square

The following lemma is the converse of Theorem 4.7.

Lemma 4.12. *Let $C \in \mathfrak{C}$. If for each $Y \in \mathfrak{C}$ and each right $\text{End}(C)$ -submodule H of $\mathfrak{C}(C, Y)$ satisfying $\mathcal{P}(C, Y) \subseteq H$, there exists a right C -determined \mathfrak{s} -deflation $\alpha: X \rightarrow Y$ such that $\text{Im } \mathfrak{C}(C, \alpha) = H$, then $C \in \mathfrak{C}_r$.*

Proof. It suffices to show that each non- \mathfrak{s} -projective indecomposable direct summand C' of C lies in \mathfrak{C}_r .

We claim that each $f \in \mathcal{P}(C, C')$ is not a retraction. Otherwise, if some $f \in \mathcal{P}(C, C')$ is a retraction, then there exists $g \in \mathfrak{C}(C', C)$ such that $fg = \text{Id}_{C'}$. Since f is \mathfrak{s} -projective, we have that $\text{Id}_{C'} = fg$ is \mathfrak{s} -projective, and hence C' is an \mathfrak{s} -projective object, which is a contradiction. The claim is proved. Notice that $\text{rad}_{\mathfrak{C}}(C, C')$ is formed by non-retractions, so $\mathcal{P}(C, C') \subseteq \text{rad}_{\mathfrak{C}}(C, C')$. Because $\text{rad}_{\mathfrak{C}}(C, C')$ is a right $\text{End}(C)$ -submodule of $\mathfrak{C}(C, C')$, by assumption there exists a right C -determined \mathfrak{s} -deflation $\alpha: X \rightarrow C'$ such that $\text{rad}_{\mathfrak{C}}(C, C') = \text{Im } \mathfrak{C}(C, \alpha)$.

Since $\text{Im } \mathfrak{C}(C, \alpha) = \text{rad}_{\mathfrak{C}}(C, C')$ is a proper submodule of $\mathfrak{C}(C, C')$, the \mathfrak{s} -deflation α is not a retraction. Thus $\text{Id}_{C'}$ does not factor through α . Let $f \in \text{rad}_{\mathfrak{C}}(T, C')$. For each $g: C \rightarrow T$, the morphism $f \circ g$ lies in $\text{rad}_{\mathfrak{C}}(C, C') = \text{Im } \mathfrak{C}(C, \alpha)$. It follows that $f \circ g$ factors through α . Since α is right C -determined, we have that f factors through α and $\text{Id}_{C'}$ almost factors through α . By Lemma 4.8, there exists an almost split \mathfrak{s} -triangle ending at C' . Now the assertion follows from Proposition 3.3. \square

Collecting the results obtained so far, we list some characterizations for an object lying in \mathfrak{C}_r .

Theorem 4.13. *For any $C \in \mathfrak{C}$, the following statements are equivalent.*

- (1) $C \in \mathfrak{C}_r$.
- (2) For each $Y \in \mathfrak{C}$ and each right $\text{End}(C)$ -submodule H of $\mathfrak{C}(C, Y)$ satisfying $\mathcal{P}(C, Y) \subseteq H$, there exists a right C -determined \mathfrak{s} -deflation $\alpha: X \rightarrow Y$ such that $H = \text{Im } \mathfrak{C}(C, \alpha)$.

If moreover C is non- \mathfrak{s} -projective indecomposable, then all statements (1)–(6) are equivalent.

- (3) C is an intrinsic weak cokernel of some \mathfrak{s} -inflation $\alpha: X \rightarrow Y$ which is left K -determined for some object K .

- (4) *There exists an almost split \mathfrak{s} -triangle ending at C .*
- (5) *There exists a non-retraction \mathfrak{s} -deflation which is right C -determined.*
- (6) *There exists an \mathfrak{s} -deflation $\alpha: X \rightarrow Y$ and a morphism $f: C \rightarrow Y$ such that f almost factors through α .*

Proof. By Theorem 4.7 and Lemma 4.12, we have (1) \Leftrightarrow (2).

Assume that C is non- \mathfrak{s} -projective indecomposable. Then the dual of Proposition 4.11 implies (1) \Leftrightarrow (3), and Proposition 3.3 implies (1) \Leftrightarrow (4). By Lemma 4.8, we have (6) \Rightarrow (4).

It is easy to see that the right almost split \mathfrak{s} -deflation ending at C is a non-retraction and right C -determined. Then we have (4) \Rightarrow (5). Let α be a right C -determined \mathfrak{s} -deflation which is not a retraction. We have that C is a minimal right determiner of α . Thus (5) \Rightarrow (6) holds true. \square

5. Examples

5.1. In [13], Iyama, Nakaoka and Palu introduced the following notions in order to study the existence of almost split extensions. Let $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ be a k -linear extriangulated category.

- (1) A *right Auslander-Reiten-Serre duality* is a pair (τ, η) of an additive functor $\tau: \underline{\mathfrak{C}} \rightarrow \overline{\mathfrak{C}}$ and a binatural isomorphism

$$\eta_{A,B}: \underline{\mathfrak{C}}(A, B) \simeq D\mathbb{E}(B, \tau A) \text{ for any } A, B \in \underline{\mathfrak{C}}.$$

- (2) If moreover τ is an equivalence, then (τ, η) is called an *Auslander-Reiten-Serre duality*.

Following [13, Proposition 3.5 and Theorem 3.4], we have

Proposition 5.1. *Let \mathfrak{C} be a k -linear Ext-finite Krull-Schmidt extriangulated category.*

- (1) $\mathfrak{C} = \mathfrak{C}_r$ if and only if \mathfrak{C} has a right Auslander-Reiten-Serre duality (τ, η) , in which τ is fully faithful.
- (2) $\mathfrak{C} = \mathfrak{C}_r = \mathfrak{C}_l$ if and only if \mathfrak{C} has an Auslander-Reiten-Serre duality. In this case, the Auslander-Reiten-Serre duality is exactly the pair (τ, ψ) defined in Section 3.2.

Let A be a finite-dimensional algebra over a field k and $A\text{-mod}$ the category of finitely generated left A -modules. We use $\mathcal{P}(A)$ to denote the subcategory of $A\text{-mod}$ consisting of projective modules, and use $\mathcal{GP}(A)$ to denote the subcategory of $A\text{-mod}$ consisting of Gorenstein projective modules.

Example 5.2.

- (1) It is well known that $A\text{-mod}$ has an Auslander-Reiten-Serre duality. Moreover, if A is self-injective, then the stable category $A\text{-}\underline{\text{mod}}$ has an Auslander-Reiten-Serre duality ([11]).
- (2) If A is Gorenstein (that is, the left and right self-injective dimensions of A are finite), then the stable category $\underline{\mathcal{GP}}(A)$ has an Auslander-Reiten-Serre duality. In fact, since $\mathcal{GP}(A)$ is an extension-closed functorially finite subcategory of $A\text{-mod}$, $\mathcal{GP}(A)$ has almost split sequences, and they induce almost split triangle in $\underline{\mathcal{GP}}(A)$. Moreover, if A is Gorenstein and $\mathcal{GP}(A)$ is of finite type, then the Gorenstein derived category $D_{gp}^b(A)$ has an Auslander-Reiten-Serre duality ([10]).
- (3) If the global dimension of A is finite, then the bounded derived category $D^b(A)$ has an Auslander-Reiten-Serre duality ([11]).
- (4) If \mathcal{X} is an extension-closed functorially finite subcategory of the bounded homotopy category $K^b(\mathcal{P}(A))$, then \mathcal{X} has an Auslander-Reiten-Serre duality [13, Proposition 6.1].

5.2. Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 consists of all vertices, and Q_1 consists of all arrows. For $x \in Q_0$, we use the symbol x^+ (resp. x^-) to denote the set of arrows starting (resp. ending) with x . For $x, y \in Q_0$, let $Q(x, y)$ stand for the set of paths in Q from x to y . Then

- Q is called *locally finite* if x^+ and x^- are finite for each $x \in Q_0$.
- Q is called *interval-finite* if $Q(x, y)$ is finite for each pair $x, y \in Q_0$.
- Q is called *strongly locally finite* if it is locally finite and interval-finite.

Let k be a field. A k -representation M of a quiver Q consists of a family of k -spaces M_x with $x \in Q_0$, and a family of k -maps $M_\alpha : M_x \rightarrow M_y$ with $\alpha : x \rightarrow y$ in Q_1 . A k -representation M is called *locally finite-dimensional* if M_x is of finite k -dimensional for each $x \in Q_0$, and called *finite-dimensional* if $\sum_{x \in Q_0} \dim M_x < \infty$.

In the following, we will use frequently three kinds of k -representations S_a, P_a and I_a which are defined by the following way:

- S_a is the k -representation such that $S_a(a) = k$ and $S_a(x) = 0$ for any $x \in Q_0 \setminus \{a\}$.
- P_a is the k -representation such that $P_a(x)$, for any $x \in Q_0$, is the k -space spanned by $Q(a, x)$ and $P_a(\alpha) : P_a(x) \rightarrow P_a(y)$, for any $\alpha : x \rightarrow y \in Q_1$, is the k -map sending every path p to αp .
- I_a is the k -representation such that $I_a(x)$, for any $x \in Q_0$, is the k -space spanned by $Q(x, a)$ and $I_a(\alpha) : I_a(x) \rightarrow I_a(y)$, for any $\alpha : x \rightarrow y \in Q_1$, is the k -map sending every $p\alpha$ to p and vanishing on the paths which do not factor through α .

We use the following notations:

- $\text{Rep}(Q)$:=the category of all k -representations of Q .
- $\text{rep}(Q)$:=the full subcategory of $\text{Rep}(Q)$ consisting of all locally finite-dimensional k -representations of Q .
- $\text{Inj}(Q)$:=the full additive subcategory of $\text{Rep}(Q)$ generated by the objects isomorphic to $I_a \otimes V_a$ with $a \in Q_0$ and V_a some k -space.
- $\text{Proj}(Q)$:=the full additive subcategory of $\text{Rep}(Q)$ generated by the objects isomorphic to $P_a \otimes U_a$ with $a \in Q_0$ and U_a some k -space.
- $\text{inj}(Q) := \text{Inj}(Q) \cap \text{rep}(Q)$ and $\text{proj}(Q) := \text{Proj}(Q) \cap \text{rep}(Q)$.
- $\text{rep}^+(Q)$:=the full subcategory of $\text{rep}(Q)$ consisting of all finitely presented k -representations of Q .
- $\text{rep}^-(Q)$:=the full subcategory of $\text{rep}(Q)$ consisting of all finitely co-presented k -representations of Q .

Here a k -representation M is called *finitely presented* if there exists an exact sequence

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

with $P_0, P_1 \in \text{proj}(Q)$, and *finitely co-presented* if there exists an exact sequence

$$0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow 0$$

with $I_0, I_1 \in \text{inj}(Q)$.

By [6, (1.15)], $\text{rep}^+(Q)$ and $\text{rep}^-(Q)$ are Hom-finite k -linear abelian categories, and hence they are extriangulated categories in which the corresponding \mathbb{E} and \mathfrak{s} were given in [25, Example 2.13].

Following [19, Theorem 5.2] and Proposition 3.3, we have

Theorem 5.3. *Let Q be a strongly locally finite quiver (for example, type of \mathbb{A}_∞ or \mathbb{A}_∞^∞) and M an indecomposable k -representation in $\text{rep}(Q)$.*

- (1) *If $M \in \text{rep}^+(Q)$ is not projective, then $M \in \text{rep}^+(Q)_r$ if and only if τM is finite-dimensional.*
- (2) *If $N \in \text{rep}^-(Q)$ is not injective, then $N \in \text{rep}^-(Q)_l$ if and only if $\tau^- M$ is finite-dimensional.*

Here τ and τ^- are defined to be the Auslander-Reiten translations $D\text{Tr}$ and $\text{Tr}D$ (see [6, Definition 2.4]).

Now let Q be a quiver of type \mathbb{A}_∞ as follows:

$$1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow \dots$$

and $\mathfrak{C} = \text{rep}^+(Q)$.

- (1) Let $C = S_1$ be non-projective indecomposable. Since $\tau S_1 = S_2$ is finite-dimensional, we have $S_1 \in \text{rep}^+(Q)_r$. Moreover, there exists an almost split sequence

$$0 \longrightarrow S_2 \xrightarrow{\alpha} P_1 \longrightarrow S_1 \longrightarrow 0,$$

and α is left S_2 -determined by the dual of Lemma 4.6.

- (2) Clearly, P_4 is not injective in $\text{rep}^+(Q)$. By [6, Proposition 3.6], there exists no almost split sequence starting with P_4 in $\text{rep}^+(Q)$. Thus, by Proposition 3.3, $P_4 \notin \text{rep}^+(Q)_l$. Consider the canonical epimorphism $p : P_4 \rightarrow S_4$. It is easy to see that the intrinsic weak kernel of p is P_5 and $P_5 \notin \text{rep}^+(Q)_l$. Thus, by Theorem 4.10, p could not be right determined by any object in $\text{rep}^+(Q)$.

Now we let $\mathfrak{C} = D^b(\text{rep}^+(Q))$ be the derived category of the bounded complexes in $\text{rep}^+(Q)$. Following [6, Theorem 7.11], we have

Theorem 5.4.

- (1) $D^b(\text{rep}^+(Q))_l = D^b(\text{rep}^+(Q))$ if and only if Q has no left infinite path.
- (2) $D^b(\text{rep}^+(Q))_r = D^b(\text{rep}^+(Q))$ if and only if Q has no right infinite path.
- (3) $D^b(\text{rep}^+(Q))_l = D^b(\text{rep}^+(Q)) = D^b(\text{rep}^+(Q))_r$ if and only if Q has no infinite path.

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References

- [1] N. Abe, H. Nakaoka, General heart construction on a triangulated category (II): associated homological functor, *Appl. Categ. Struct.* 20 (2012) 161–174.
- [2] M. Auslander, Functors and morphisms determined by objects, in: *Representation Theory of Algebras, Proc. Conf., Temple Univ., Philadelphia, Pa., 1976*, in: *Lecture Notes in Pure Appl. Math.*, vol. 37, Dekker, New York, 1978, pp. 1–244.
- [3] M. Auslander, I. Reiten, S.O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Adv. Math., vol. 36, Cambridge Univ. Press, Cambridge, 1995.
- [4] M. Auslander, I. Reiten, Representation theory of Artin algebras III: almost split sequences, *Commun. Algebra* 3 (1975) 239–294.
- [5] M. Auslander, I. Reiten, Representation theory of Artin algebras IV: invariants given by almost split sequences, *Commun. Algebra* 5 (1977) 443–518.
- [6] R. Bautista, S. Liu, C. Paquette, Representation theory of strongly locally finite quivers, *Proc. Lond. Math. Soc.* (3) 106 (2013) 97–162.

- [7] N. Bian, Right minimal morphisms in arbitrary Krull-Schmidt categories, *J. Math.* 29 (2009) 577–580.
- [8] X.-W. Chen, Generalized Serre duality, *J. Algebra* 328 (2011) 268–286.
- [9] X.-W. Chen, J. Le, A note on morphisms determined by objects, *J. Algebra* 428 (2015) 138–148.
- [10] N. Gao, Auslander-Reiten triangles on Gorenstein derived categories, *Commun. Algebra* 40 (2012) 3912–3919.
- [11] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*, London Math. Soc. Lecture Note Ser., vol. 119, Cambridge Univ. Press, Cambridge, 1988.
- [12] J. Hu, D. Zhang, P. Zhou, Proper classes and Gorensteinness in extriangulated categories, *J. Algebra* 551 (2020) 23–60.
- [13] O. Iyama, H. Nakaoka, Y. Palu, Auslander-Reiten theory in extriangulated categories, Preprint available at [arXiv:1805.03776](https://arxiv.org/abs/1805.03776).
- [14] P. Jiao, The generalized Auslander-Reiten duality on an exact category, *J. Algebra Appl.* 17 (12) (2018) 1850227, 14 pp.
- [15] P. Jiao, J. Le, The Auslander-Reiten duality via morphisms determined by objects, *J. Pure Appl. Algebra* 222 (2018) 807–817.
- [16] S. Koenig, B. Zhu, From triangulated categories to abelian categories: cluster tilting in a general framework, *Math. Z.* 258 (2008) 143–160.
- [17] H. Krause, Krull-Schmidt categories and projective covers, *Expo. Math.* 33 (2015) 535–549.
- [18] H. Lenzing, R. Zuazua, Auslander-Reiten duality for abelian categories, *Bol. Soc. Mat. Mexicana* 10 (2004) 169–177.
- [19] S. Liu, P. Ng, C. Paquette, Almost split sequences and approximations, *Algebr. Represent. Theory* 16 (2013) 1809–1827.
- [20] Y. Liu, Hearts of twin cotorsion pairs on exact categories, *J. Algebra* 394 (2013) 245–284.
- [21] Y. Liu, H. Nakaoka, Hearts of twin cotorsion pairs on extriangulated categories, *J. Algebra* 528 (2019) 96–149.
- [22] D. Miličić, Lectures on derived categories, Preprint available at <https://www.math.utah.edu/~milicic/Eprints/dercat.pdf>.
- [23] H. Nakaoka, General heart construction on a triangulated category (I): unifying t-structures and cluster tilting subcategories, *Appl. Categ. Struct.* 19 (2011) 879–899.
- [24] H. Nakaoka, General heart construction for twin torsion pairs on triangulated categories, *J. Algebra* 374 (2013) 195–215.
- [25] H. Nakaoka, Y. Palu, Mutation via Hovey twin cotorsion pairs and model structures in extriangulated categories, *Cah. Topol. Géom. Différ. Catég.* LX-2 (2019) 117–193.
- [26] C.M. Ringel, Morphisms determined by objects: the case of modules over Artin algebras, *Ill. J. Math.* 56 (2012) 981–1000.
- [27] C.M. Ringel, The Auslander bijections: how morphisms are determined by modules, *Bull. Math. Sci.* 3 (2013) 409–484.
- [28] T. Zhao, Z. Huang, Phantom ideals and cotorsion pairs in extriangulated categories, *Taiwan. J. Math.* 23 (2019) 29–61.
- [29] P. Zhou, B. Zhu, Triangulated quotient categories revisited, *J. Algebra* 502 (2018) 196–232.