



Auslander-type conditions and cotorsion pairs

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Abstract

We study the properties of rings satisfying Auslander-type conditions. If an artin algebra Λ satisfies the Auslander condition (that is, Λ is an ∞ -Gorenstein artin algebra), then we construct two kinds of subcategories which form functorially finite cotorsion pairs.

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Noetherian rings satisfying ‘Auslander-type conditions’ on self-injective resolutions can be regarded as certain non-commutative analogs of commutative Gorenstein rings. Such conditions, especially dominant dimension and the n -Gorenstein condition, play a crucial role in representation theory and non-commutative algebraic geometry (e.g. [A,AR2,AR3,B,C,EHIS,FGR,FI,HN,IS,Iy4,Iy6,M,R,Sm,T,W]). They are also interesting from the viewpoint of some unsolved homological conjectures, e.g. the finitistic dimension conjecture, Nakayama conjecture, Gorenstein symmetry conjecture, and so on. It is therefore important to understand non-commutative ‘regular’ or ‘Gorenstein’ rings though it is still far from realized even for the case of finite dimensional algebras. Recently, several authors (e.g. [Hu1,Hu2,Hu3,Iy1,Iy3]) have studied some Auslander-type conditions, e.g. the quasi n -Gorenstein condition, the (l, n) -condition, and so on. This paper

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is devoted to enlarge our knowledge of the homological behavior of non-commutative rings. Especially we introduce Auslander-type conditions $G_n(k)$ and $g_n(k)$ and study their properties.

Throughout this paper, let Λ be a left and right noetherian ring (unless stated otherwise). We denote by

$$0 \rightarrow \Lambda \rightarrow I_0(\Lambda) \rightarrow I_1(\Lambda) \rightarrow \cdots \rightarrow I_i(\Lambda) \rightarrow \cdots$$

the minimal injective resolution of the left Λ -module Λ . We call Λ n -Gorenstein if $\text{fd } I_i(\Lambda) \leq i$ for any $0 \leq i \leq n - 1$,² and call Λ ∞ -Gorenstein if it is n -Gorenstein for all n . In the latter case, Λ is also said to satisfy the Auslander condition. It was proved by Auslander that the notion of n -Gorenstein rings is left–right symmetric [FGR, 3.7] (see 4.2 below). Our aim in this paper is to generalize the Auslander condition. In Section 3, we introduce Auslander-type conditions $G_n(k)$ and $g_n(k)$ and study their properties. In Section 4, we concentrate on the conditions $G_n(k)$ and $g_n(k)$ for the case $k = 0, 1$. We give a quick proof of well-known results on these conditions, then prove our main result. In Section 5, we apply our results to the finitistic dimension and the (l, n) -condition.

We denote by $\text{mod } \Lambda$ the category of finitely generated left Λ -modules, and by $\underline{\text{mod}} \Lambda$ the stable category of Λ [AB]. Put $()^* := \text{Hom}_\Lambda(, \Lambda)$ and

$$\begin{aligned} \mathbb{E}_n &:= \text{Ext}_\Lambda^n(, \Lambda) : \text{mod } \Lambda \rightarrow \text{mod } \Lambda^{op} \quad \text{for } n \geq 0, \\ \mathbb{T}_n &:= \text{Tr} \circ \Omega^{n-1} : \underline{\text{mod}} \Lambda \rightarrow \underline{\text{mod}} \Lambda^{op} \quad \text{for } n > 0, \end{aligned}$$

where $\Omega : \underline{\text{mod}} \Lambda \rightarrow \underline{\text{mod}} \Lambda$ is the syzygy functor and $\text{Tr} : \underline{\text{mod}} \Lambda \rightarrow \underline{\text{mod}} \Lambda^{op}$ is the transpose functor [AB]. Let

$$\begin{aligned} \text{grade } X &:= \inf\{i \geq 0 \mid \mathbb{E}_i X \neq 0\} \quad \text{the grade,} \\ \text{s.grade } X &:= \inf\{\text{grade } Y \mid Y \subseteq X\} \quad \text{the strong grade,} \\ \text{r.grade } X &:= \inf\{i > 0 \mid \mathbb{E}_i X \neq 0\} \quad \text{the reduced grade.} \end{aligned}$$

When Λ is an artin algebra over R , we denote by $\mathbb{D} : \text{mod } \Lambda \rightarrow \text{mod } \Lambda^{op}$ the duality induced by the Matlis duality of R . Given two homomorphisms of modules, say $f : A \rightarrow B$ and $g : B \rightarrow C$, the composition homomorphism of f and g is denoted by $fg : A \rightarrow C$.

1. Main result

1.1. For subcategories \mathcal{C}_i ($i = 1, 2$) of $\text{mod } \Lambda$, we denote by $\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)$ the subcategory of $\text{mod } \Lambda$ consisting of $C \in \text{mod } \Lambda$ such that there exists an exact sequence $0 \rightarrow C_2 \rightarrow C \rightarrow C_1 \rightarrow 0$ with $C_i \in \mathcal{C}_i$ ($i = 1, 2$). For a subcategory \mathcal{C} of $\text{mod } \Lambda$, we use $\text{add } \mathcal{C}$ to denote the subcategory of $\text{mod } \Lambda$ consisting of all Λ -modules isomorphic to direct summands of finite direct sums of modules in \mathcal{C} .

For a left Λ -module C , we use $\text{pd } C$, $\text{fd } C$ and $\text{id } C$ to denote the projective dimension, flat dimension and injective dimension of C , respectively. A module C in $\text{mod } \Lambda$ is called n -torsionfree

² Notice that there are other meanings to the notion of n -Gorenstein rings (e.g. [Iw1,Iw2,EJ,EX]).

if $\mathbb{E}_i \operatorname{Tr} C = 0$ for any $1 \leq i \leq n$ [AB]. For $n, m \geq 0$, we define several full subcategories of $\operatorname{mod} \Lambda$ as follows:

$$\begin{aligned} \mathcal{W}_n &:= \{C \in \operatorname{mod} \Lambda \mid \operatorname{r. grade} C > n\}, \\ \mathcal{F}_n &:= \{C \in \operatorname{mod} \Lambda \mid C \text{ is } n\text{-torsionfree}\}, \\ \mathcal{P}_n &:= \{C \in \operatorname{mod} \Lambda \mid \operatorname{pd}_\Lambda C < n\}, \\ \mathcal{I}_n &:= \{C \in \operatorname{mod} \Lambda \mid \operatorname{id}_\Lambda C < n\}, \\ \mathcal{X}_{n,m} &:= \mathcal{W}_n \cap \mathcal{F}_m, \\ \mathcal{Y}_{n,m} &:= \operatorname{add} \mathcal{E}(\mathcal{I}_m, \mathcal{P}_n). \end{aligned}$$

For example, we have $\mathcal{W}_0 = \mathcal{F}_0 = \operatorname{mod} \Lambda$ and $\mathcal{P}_0 = \mathcal{I}_0 = 0$. Notice that $\mathcal{E}(\mathcal{I}_m, \mathcal{P}_n)$ is not necessarily closed under direct summands (e.g. take Λ to be the path algebra of the quiver $\bullet \rightarrow \bullet \rightarrow \bullet$ and $n = m = 1$). We denote by $\mathcal{W}_n^{op}, \mathcal{F}_n^{op}, \mathcal{P}_n^{op}, \mathcal{I}_n^{op}, \mathcal{X}_{n,m}^{op}$ and $\mathcal{Y}_{n,m}^{op}$ the corresponding categories for Λ^{op} .

1.2. Following Salce [Sa], we call a pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\operatorname{mod} \Lambda$ a *cotorsion pair* if

$$\mathcal{X} = \{C \in \operatorname{mod} \Lambda \mid \operatorname{Ext}_\Lambda^1(C, \mathcal{Y}) = 0\} \quad \text{and} \quad \mathcal{Y} = \{C \in \operatorname{mod} \Lambda \mid \operatorname{Ext}_\Lambda^1(\mathcal{X}, C) = 0\}.$$

Notice that we do not assume the vanishing of higher Ext-groups $\operatorname{Ext}_\Lambda^i$ ($i > 1$).

1.3. Now we can state a main result in this paper, which we shall prove in 4.9.

Theorem. *Let Λ be an ∞ -Gorenstein artin algebra. Then $(\mathcal{X}_{i,j-1}, \mathcal{Y}_{i,j})$ ($i \geq 0, j \geq 1$) and $(\mathcal{Y}_{i,j}, \mathbb{D}\mathcal{X}_{j,i-1}^{op})$ ($i \geq 1, j \geq 0$) form cotorsion pairs.*

Moreover, we will show that they are functorially finite in the sense of 2.3 below. For example, we have cotorsion pairs $(\mathcal{W}_i, \mathcal{Y}_{i,1})$ ($j := 1$) and $(\mathcal{F}_{j-1}, \mathcal{I}_j)$ ($i := 0$).

2. Preliminaries

2.1. Let us start with the following simple observation, which will be used frequently in this paper.

Lemma. *Let \mathcal{Y} be a full subcategory of $\operatorname{mod} \Lambda$ which is closed under extensions, and $0 \rightarrow C_1 \rightarrow Y \rightarrow C_0 \rightarrow 0$ an exact sequence with $Y \in \mathcal{Y}$. If there exists an exact sequence $0 \rightarrow Y_0 \rightarrow X \rightarrow C_0 \rightarrow 0$ with $Y_0 \in \mathcal{Y}$, then there exist exact sequences $0 \rightarrow C_1 \rightarrow Y_1 \rightarrow X \rightarrow 0$ with $Y_1 \in \mathcal{Y}$.*

Proof. The middle row in the following pull-back diagram gives our desired exact sequence:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & C_1 & \longrightarrow & Y & \longrightarrow & C_0 \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & C_1 & \longrightarrow & Y_1 & \longrightarrow & X \longrightarrow 0. & \square \\
 & & & & \uparrow & & \uparrow & \\
 & & & & Y_0 & = & Y_0 & \\
 & & & & \uparrow & & \uparrow & \\
 & & & & 0 & & 0 &
 \end{array}$$

2.2. Assume that $\mathcal{C} \supset \mathcal{D}$ are full subcategories of $\text{mod } \Lambda$ and $C \in \mathcal{C}$, $D \in \mathcal{D}$. A morphism $f : D \rightarrow C$ is said to be a *right \mathcal{D} -approximation* of C if $\text{Hom}_\Lambda(X, f) : \text{Hom}_\Lambda(X, D) \rightarrow \text{Hom}_\Lambda(X, C) \rightarrow 0$ is exact for any $X \in \mathcal{D}$. A right \mathcal{D} -approximation $f : D \rightarrow C$ is called *minimal* if an endomorphism $g : D \rightarrow D$ is an automorphism whenever $f = gf$. The subcategory \mathcal{D} is said to be *contravariantly finite* in \mathcal{C} if any $C \in \mathcal{C}$ has a right \mathcal{D} -approximation. Dually, we define the notions of (*minimal*) *left \mathcal{D} -approximations* and *covariantly finite subcategories*. The subcategory of \mathcal{C} is said to be *functorially finite* in \mathcal{C} if it is both contravariantly finite and covariantly finite in \mathcal{C} [AR1].

2.3. Let Λ be an artin algebra. We call a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ *functorially finite* if the following equivalent conditions are satisfied:

- (1) \mathcal{X} is a contravariantly finite subcategory of $\text{mod } \Lambda$.
- (2) \mathcal{Y} is a covariantly finite subcategory of $\text{mod } \Lambda$.
- (3) For any $C \in \text{mod } \Lambda$, there exists an exact sequence $0 \rightarrow Y \rightarrow X \xrightarrow{f} C \rightarrow 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.
- (4) For any $C \in \text{mod } \Lambda$, there exists an exact sequence $0 \rightarrow C \xrightarrow{g} Y' \rightarrow X' \rightarrow 0$ with $X' \in \mathcal{X}$ and $Y' \in \mathcal{Y}$.

2.3.1 (Wakamatsu’s lemma). Let \mathcal{X} be a subcategory of $\text{mod } \Lambda$ which is closed under extensions. If $0 \rightarrow A \rightarrow B \xrightarrow{f} C$ is exact and f is a minimal right \mathcal{X} -approximation of C , then $\text{Ext}_\Lambda^1(\mathcal{X}, A) = 0$ holds.

2.3.2. Proof of 2.3. (3) \Rightarrow (1). f is a right \mathcal{X} -approximation of C by $\text{Ext}_\Lambda^1(\mathcal{X}, Y) = 0$.

(1) \Rightarrow (3). Let $f : X \rightarrow C$ be a minimal right \mathcal{X} -approximation of C . Since $\Lambda \in \mathcal{X}$, f is surjective. Since \mathcal{X} is closed under extensions, $\text{Ker } f \in \mathcal{Y}$ holds by 2.3.1.

(3) \Rightarrow (4) (cf. [Sa]). Let $0 \rightarrow C \rightarrow I \rightarrow \Omega^{-1}C \rightarrow 0$ and $0 \rightarrow Y \rightarrow X \rightarrow \Omega^{-1}C \rightarrow 0$ be exact sequences with I the injective envelope of C , $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Applying 2.1, we have the desired sequence.

(2) \Leftrightarrow (4) and (4) \Rightarrow (3) can be shown dually. \square

2.3.3. Let Λ be an artin algebra and $(\mathcal{X}, \mathcal{Y})$ a pair of subcategories of $\text{mod } \Lambda$ which are closed under direct summands. If $\text{Ext}_\Lambda^1(\mathcal{X}, \mathcal{Y}) = 0$ and the conditions 2.3(3)(4) are satisfied, then $(\mathcal{X}, \mathcal{Y})$ is a functorially finite cotorsion pair.

Proof. By the condition 2.3(3), $\text{Ext}_\Lambda^1(C, \mathcal{Y}) = 0$ implies $C \in \mathcal{X}$. By the condition 2.3(4), $\text{Ext}_\Lambda^1(\mathcal{X}, C) = 0$ implies $C \in \mathcal{Y}$. \square

For a full subcategory \mathcal{C} of $\text{mod } \Lambda$, we denote by $\underline{\mathcal{C}}$ the corresponding subcategory of $\underline{\text{mod}} \Lambda$. We denote by $\Omega^n(\text{mod } \Lambda)$ the full subcategory of $\text{mod } \Lambda$ consisting of $C \in \text{mod } \Lambda$ such that there exists an exact sequence $0 \rightarrow C \rightarrow P_0 \rightarrow \dots \rightarrow P_{n-1}$ with projective P_i .

2.4. We collect some useful results.

(1) We have the following diagram whose rows are equivalences and columns are dualities [Iy5, 1.1.1]:

$$\begin{array}{ccccccc}
 \underline{\mathcal{W}}_n = \underline{\mathcal{X}}_{n,0} & \xrightarrow{\Omega} & \underline{\mathcal{X}}_{n,1} & \xrightarrow{\Omega} & \dots & \xrightarrow{\Omega} & \underline{\mathcal{X}}_{1,n-1} & \xrightarrow{\Omega} & \underline{\mathcal{X}}_{0,n} = \underline{\mathcal{F}}_n \\
 \downarrow \text{Tr} & & \downarrow \text{Tr} & & & & \downarrow \text{Tr} & & \downarrow \text{Tr} \\
 \underline{\mathcal{F}}_n^{\text{op}} = \underline{\mathcal{X}}_{0,n}^{\text{op}} & \xleftarrow{\Omega} & \underline{\mathcal{X}}_{1,n-1}^{\text{op}} & \xleftarrow{\Omega} & \dots & \xleftarrow{\Omega} & \underline{\mathcal{X}}_{n-1,1}^{\text{op}} & \xleftarrow{\Omega} & \underline{\mathcal{X}}_{n,0}^{\text{op}} = \underline{\mathcal{W}}_n^{\text{op}}.
 \end{array}$$

In particular, we have $\mathcal{F}_n = \Omega^n \mathcal{W}_n \subseteq \Omega^n(\text{mod } \Lambda)$ for any $n \geq 1$.

(2) By (1), $\Omega^n(\text{mod } \Lambda) \subseteq \mathcal{F}_m$ if and only if $\text{Tr } \Omega^n(\text{mod } \Lambda) \subseteq \mathcal{W}_m^{\text{op}}$ if and only if $\text{r.grade } \mathbb{T}_{n+1} C \geq m + 1$ holds for any $C \in \text{mod } \Lambda$.

(3) $\text{Ext}_\Lambda^1(X, Y) = 0$ holds for any $X \in \mathcal{W}_n$ and $Y \in \mathcal{P}_n$. In fact, we take a projective resolution $0 \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow Y \rightarrow 0$ and apply the functor $\text{Hom}_\Lambda(X, _)$, then we get $\text{Ext}_\Lambda^1(X, Y) \simeq \text{Ext}_\Lambda^2(X, \Omega Y) \simeq \dots \simeq \text{Ext}_\Lambda^n(X, \Omega^{n-1} Y) = \text{Ext}_\Lambda^n(X, P_{n-1}) = 0$.

(4) Let C be in $\text{mod } \Lambda$. We let $\sigma_C : C \rightarrow C^{**}$ denote the canonical evaluation homomorphism; thus $\sigma_C(x)(f) = f(x)$ for any $x \in C$ and $f \in C^*$. Then there exists an exact sequence $0 \rightarrow \mathbb{E}_1 \text{Tr } C \rightarrow C \xrightarrow{\sigma_C} C^{**} \rightarrow \mathbb{E}_2 \text{Tr } C \rightarrow 0$ [AB]. Recall that C is called *torsionless* (respectively *reflexive*) if σ_C is a monomorphism (respectively an isomorphism). So C is torsionless (respectively reflexive) if and only if it is 1-torsionfree (respectively 2-torsionfree). In particular, we have $\mathcal{F}_1 = \Omega(\text{mod } \Lambda)$.

(5) For any $X \in \text{mod } \Lambda$ and $n > 0$, we have an exact sequence $0 \rightarrow \mathbb{E}_n X \rightarrow \mathbb{T}_n X \xrightarrow{f} \Omega \mathbb{T}_{n+1} X \rightarrow 0$ such that f^* is an isomorphism [Ho3].

(6) If $X \in \text{mod } \Lambda$ satisfies $\text{grade } \mathbb{E}_i X > i$ for any $0 \leq i \leq n$, then $\text{grade } X > n$ [Ho2, 6.2], [Iy1, 2.3].

(7) For $l, n \geq 0$, the following conditions are equivalent [Iy1, 6.1], [Hu1, 2.8].

- (i) $\text{fd } I_i(\Lambda^{\text{op}}) < l$ holds for any $0 \leq i < n$.
- (ii) $\text{s.grade } \mathbb{E}_l C \geq n$ holds for any $C \in \text{mod } \Lambda$.

In this case, we say that Λ^{op} satisfies the (l, n) -condition (or Λ satisfies the $(l, n)^{\text{op}}$ -condition).

3. The conditions $G_n(k)$ and $g_n(k)$

In this section we introduce Auslander-type conditions $G_n(k)$ and $g_n(k)$ and study their properties. Let us start with the following observation.

3.1. Lemma. *Let $C \in \mathcal{F}_m$ with $m \geq 0$.*

- (1) $\Omega C \in \mathcal{F}_{m+1}$ if and only if $\text{grade } \mathbb{E}_1 C \geq m$.
- (2) $\mathcal{E}(C, \mathcal{F}_m) \subseteq \mathcal{F}_m$ if and only if $\mathcal{E}(C, \mathcal{P}_1) \subseteq \mathcal{F}_m$ if and only if $\text{s.grade } \mathbb{E}_1 C \geq m$.

Proof. (1) We have $\text{Tr } C \in \mathcal{W}_m^{op}$ for $C \in \mathcal{F}_m$. Since we have an exact sequence $0 \rightarrow \mathbb{E}_1 C \rightarrow \text{Tr } C \xrightarrow{f} \Omega \text{Tr } \Omega C \rightarrow 0$ such that f^* is an isomorphism by 2.4(5), $\mathbb{E}_i \mathbb{E}_1 C = \mathbb{E}_{i+2} \text{Tr } \Omega C$ holds for any $0 \leq i < m$. Thus the assertion follows.

(2) See [AR3, 1.1]. \square

3.1.1. Immediately we have the following result.

Lemma. *Assume $\Omega^n(\text{mod } \Lambda) \subseteq \mathcal{F}_m$ with $n, m \geq 0$. Then for the following (1) \Leftrightarrow (2) \Leftarrow (3) \Leftrightarrow (4) holds.*

- (1) $\text{grade } \mathbb{E}_{n+1} C \geq m$ holds for any $C \in \text{mod } \Lambda$.
- (2) $\Omega^{n+1}(\text{mod } \Lambda) \subseteq \mathcal{F}_{m+1}$.
- (3) $\text{s.grade } \mathbb{E}_{n+1} C \geq m$ holds for any $C \in \text{mod } \Lambda$.
- (4) $\mathcal{E}(\Omega^n(\text{mod } \Lambda), \Omega^n(\text{mod } \Lambda)) \subseteq \mathcal{F}_m$.

Proof. We only have to check (3) \Leftrightarrow (4). This is shown by applying 3.1 as follows:

(3) $\Rightarrow \mathcal{E}(\Omega^n(\text{mod } \Lambda), \mathcal{F}_m) \subseteq \mathcal{F}_m \Rightarrow$ (4) $\Rightarrow \mathcal{E}(\Omega^n(\text{mod } \Lambda), \mathcal{P}_1) \subseteq \mathcal{F}_m \Rightarrow$ (3). \square

3.2. Proposition. *Let $0 = l_0 < l_1 < l_2 < l_3 < \dots$ be a (finite or infinite) sequence of integers. Then for the following (1) \Leftrightarrow (2) \Leftarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) holds.*

- (1) $\text{grade } \mathbb{E}_{l_i+1} C \geq i$ holds for any $C \in \text{mod } \Lambda$ and i .
- (2) $\Omega^{l_i+1}(\text{mod } \Lambda) \subseteq \mathcal{F}_{i+1}$ holds for any i .
- (3) $\text{s.grade } \mathbb{E}_{l_i+1} C \geq i$ holds for any $C \in \text{mod } \Lambda$ and i .
- (4) $\mathcal{E}(\Omega^{l_i}(\text{mod } \Lambda), \Omega^{l_i}(\text{mod } \Lambda)) \subseteq \mathcal{F}_i$ holds for any i .
- (5) $\text{fd } I_{i-1}(\Lambda^{op}) \leq l_i$ holds for any i .

Proof. (1) \Leftrightarrow (2) \Leftarrow (3) \Leftrightarrow (4). Since $\Omega^{l_i}(\text{mod } \Lambda) \subseteq \Omega^{l_{i-1}+1}(\text{mod } \Lambda)$ holds by $l_i \geq l_{i-1} + 1$, we can inductively show the assertions by 3.1.1.

(3) \Leftrightarrow (5). By 2.4(7). \square

3.2.1. Question. Put $l_i := \inf\{l \mid \Omega^{l+1}(\text{mod } \Lambda) \subseteq \mathcal{F}_{i+1}\}$. How does the sequence $(l_i)_i$ behave? Is there an example satisfying $l_i = l_{i+1}$? This equality case was excluded in 3.2.

3.3. Definition. Let $n, k \geq 0$. We say that Λ is $G_n(k)$ if $\text{s.grade } \mathbb{E}_{i+k} C \geq i$ holds for any $C \in \text{mod } \Lambda$ and $1 \leq i \leq n$.

Similarly, we say that Λ is $g_n(k)$ if $\text{grade } \mathbb{E}_{i+k} C \geq i$ holds for any $C \in \text{mod } \Lambda$ and $1 \leq i \leq n$. We say that Λ is $G_n(k)^{op}$ (respectively $g_n(k)^{op}$) if Λ^{op} is $G_n(k)$ (respectively $g_n(k)$). By 2.4(7), Λ is $G_n(k)$ if and only if $\text{fd } I_i(\Lambda^{op}) \leq i + k$ holds for any $0 \leq i < n$. Thus $G_n(0)$ is the n -Gorenstein condition, and $G_n(1)^{op}$ is the quasi n -Gorenstein condition in [Hu2] (see 4.2, 4.3 below).

We have the following obvious relations for any $n \geq n'$ and $k \leq k'$:

$$\begin{array}{ccc} G_n(k) & \implies & G_{n'}(k') \\ \Downarrow & & \Downarrow \\ g_n(k) & \implies & g_{n'}(k'). \end{array}$$

3.4. Theorem. *The conditions (1) and (2) below are equivalent for any $n, k \geq 0$. If $k > 0$, then (1)–(5) are equivalent.*

- (1) Λ is $g_n(k)$.
- (2) For any monomorphism $A \xrightarrow{f} B$ with $A, B \in \Omega^{k+1}(\text{mod } \Lambda)$, $\mathbb{E}_i \mathbb{E}_i f$ is a monomorphism for any $0 \leq i < n$.
- (3) $\Omega^{i+k}(\text{mod } \Lambda) \subseteq \mathcal{F}_{i+1}$ holds for any $1 \leq i \leq n$.
- (4) For any $C \in \text{mod } \Lambda$ and $0 \leq i \leq n$, there exists an exact sequence $0 \rightarrow Y \rightarrow X \rightarrow \Omega^{k-1} C \rightarrow 0$ with $X \in \mathcal{W}_{i+1}$ and $Y \in \mathcal{P}_{i+1}$.
- (5) For any $C \in \text{mod } \Lambda$ and $0 \leq i \leq n$, there exists an exact sequence $0 \rightarrow \Omega^k C \rightarrow Y' \rightarrow X' \rightarrow 0$ with $X' \in \mathcal{W}_{i+1}$ and $Y' \in \mathcal{P}_{i+1}$.

3.4.1. Remark. By 2.4(3), the sequence in (4) gives a right \mathcal{W}_{i+1} -approximation of $\Omega^{k-1} C$, and the sequence in (5) gives a left \mathcal{P}_{i+1} -approximation of $\Omega^k C$.

3.4.2. Lemma. *Assume that $n \geq 1$ and $C \in \mathcal{W}_{n-1}$ satisfies $\text{grade } \mathbb{E}_n C \geq n - 1$. Then there exists an exact sequence $0 \rightarrow Y \rightarrow X \rightarrow C \rightarrow 0$ with $Y \in \mathcal{P}_n$ and $X \in \mathcal{W}_n$.*

Proof. Let $0 \rightarrow \Omega^n C \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow C \rightarrow 0$ be a projective resolution of C . Take the following commutative diagram, where the lower sequence is a projective resolution of $\mathbb{E}_n C$:

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & \mathbb{E}_n C & \longleftarrow & (\Omega^n C)^* & \longleftarrow & P_{n-1}^* & \longleftarrow & \dots & \longleftarrow & P_1^* & \longleftarrow & P_0^* \\ & & \parallel & & \uparrow & & \uparrow & & & & \uparrow & & \\ 0 & \longleftarrow & \mathbb{E}_n C & \longleftarrow & Q_0 & \longleftarrow & Q_1 & \longleftarrow & \dots & \longleftarrow & Q_{n-1} & & \end{array}$$

Taking the mapping cone of the above commutative diagram, we get an exact sequence $0 \leftarrow (\Omega^n C)^* \leftarrow P_{n-1}^* \oplus Q_0 \leftarrow \dots \leftarrow P_0^* \oplus Q_{n-1}$. Since $\text{grade } \mathbb{E}_n C \geq n - 1$, we have an exact

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & Y_i & \longrightarrow & X_i & \longrightarrow & \Omega^{k-1}C \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \parallel \\
 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & \Omega^{k-1}C \longrightarrow 0. \quad \square \\
 & & \uparrow & & \uparrow & & \\
 & & Y' & \xlongequal{\quad} & Y' & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

3.5. Theorem. *The conditions (1) and (2) below are equivalent for any $n, k \geq 0$. If $k > 0$, then (1)–(3) are equivalent.*

- (1) Λ is $G_n(k)$.
- (2) For any exact sequence $0 \rightarrow A \xrightarrow{f} B \rightarrow C \rightarrow 0$ with $C \in \Omega^k(\text{mod } \Lambda)$, $\mathbb{E}_i \mathbb{E}_i f$ is a monomorphism for any $0 \leq i < n$.
- (3) $\mathcal{E}(\Omega^{i+k}(\text{mod } \Lambda), \Omega^{i+k}(\text{mod } \Lambda)) \subseteq \mathcal{F}_{i+1}$ holds for any $0 \leq i < n$.

Proof. (1) \Rightarrow (2). Let $C = \Omega^k C'$ with $C' \in \text{mod } \Lambda$ and $g := \mathbb{E}_i f$. We have an exact sequence $\mathbb{E}_{i+k} C' \rightarrow \mathbb{E}_i B \xrightarrow{g} \mathbb{E}_i A \rightarrow \mathbb{E}_{i+k+1} C'$. Thus we have exact sequences $\mathbb{E}_i \text{Cok } g \rightarrow \mathbb{E}_i \mathbb{E}_i A \xrightarrow{a} \mathbb{E}_i \text{Im } g$ and $\mathbb{E}_{i-1} \text{Ker } g \rightarrow \mathbb{E}_i \text{Im } g \xrightarrow{b} \mathbb{E}_i \mathbb{E}_i B$. Since $\mathbb{E}_i \text{Cok } g = 0 = \mathbb{E}_{i-1} \text{Ker } g$ holds by (1), $\mathbb{E}_i g = ab$ is monic.

(2) \Rightarrow (1). For any $C \in \text{mod } \Lambda$, fix i ($0 \leq i < n$) and a Λ^{op} -submodule D of $\mathbb{E}_{i+k+1} C$. Take an exact sequence $Q \xrightarrow{a} D \rightarrow 0$ in $\text{mod } \Lambda^{op}$ with Q projective and a' the composition $Q \xrightarrow{a} D \hookrightarrow \mathbb{E}_{i+k+1} C$. We lift a' to $b: Q \rightarrow (\Omega^{i+k+1} C)^*$. Take the following push-out diagram, where b' is the composition $\Omega^{i+k+1} C \xrightarrow{\sigma_{\Omega^{i+k+1} C}} (\Omega^{i+k+1} C)^{**} \xrightarrow{b^*} Q^*$ and P_{i+k} is a projective module in $\text{mod } \Lambda$:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega^{i+k+1}C & \longrightarrow & P_{i+k} & \longrightarrow & \Omega^{i+k}C \longrightarrow 0 \\
 & & \downarrow b' & & \downarrow & & \parallel \\
 0 & \longrightarrow & Q^* & \xrightarrow{c} & X & \xrightarrow{d} & \Omega^{i+k}C \longrightarrow 0.
 \end{array}$$

We then have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 0 & \longleftarrow & \mathbb{E}_{i+k+1}C & \longleftarrow & (\Omega^{i+k+1}C)^* & \longleftarrow & P_{i+k}^* & \longleftarrow & (\Omega^{i+k}C)^* & \longleftarrow & 0 \\
 & & \cup & & \uparrow b & & \uparrow & & \parallel & & \\
 0 & \longleftarrow & D & \xleftarrow{a} & Q & \xleftarrow{c^*} & X^* & \xleftarrow{d^*} & (\Omega^{i+k}C)^* & \longleftarrow & 0.
 \end{array}$$

Let $i = 0$. Since c^{**} is monic by (2), we obtain $D^* = 0$. Thus $\text{s.grade } \mathbb{E}_{1+k}C \geq 1$.

Fix i ($0 < i < n$) and assume that Λ is $G_i(k)$. By 3.2, $\mathcal{E}(\Omega^{i+k}(\text{mod } \Lambda), \Omega^{i+k}(\text{mod } \Lambda)) \subseteq \mathcal{F}_i$ holds. In particular, X in the first diagram is contained in \mathcal{F}_i . Take the following commutative diagram with exact rows, where the upper sequence is still exact by taking $()^*$:

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & X & \longrightarrow & Q_0 & \longrightarrow & \cdots & \longrightarrow & Q_{i-1} & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow & & & & \downarrow & & \downarrow g & & \\
 0 & \longrightarrow & \Omega^{i+k}C & \longrightarrow & P_{i+k-1} & \longrightarrow & \cdots & \longrightarrow & P_k & \longrightarrow & \Omega^k C & \longrightarrow & 0.
 \end{array}$$

We can assume that g is epic by adding a projective direct summand to Y and Q_{i-1} . Thus we obtain an exact sequence $0 \rightarrow Z \xrightarrow{h} Y \xrightarrow{g} \Omega^k C \rightarrow 0$ such that $\Omega^i g = d$. Since $\text{Cok } \mathbb{E}_i h \simeq \text{Ker } \mathbb{E}_{i+1} g \simeq \text{Ker } \mathbb{E}_1 d \simeq \text{Cok } c^* \simeq D$ and $\mathbb{E}_i Y = 0$ hold, we obtain $D \simeq \mathbb{E}_i Z$. Since $\mathbb{E}_i \mathbb{E}_i h$ is monic by (2), we have $\mathbb{E}_i D = 0$. Thus $\text{s.grade } \mathbb{E}_{i+k+1}C \geq i + 1$ holds, and Λ is $G_{i+1}(k)$. Inductively, we obtain (1).

(1) \Leftrightarrow (3). Put $l_i := i + k - 1$ in 3.2. \square

4. The conditions $G_n(k)$ and $g_n(k)$ for $k = 0, 1$

In this section we concentrate on the conditions $G_n(k)$ and $g_n(k)$ for the case $k = 0, 1$. Let us start with giving a quick proof of the following remarkable ‘left–right symmetry,’ where (1) is well known in [FGR, 3.7], (2) is in [HN, 4.7], [Hu3, 2,4] and (3) is in [AR3], [HN, 4.1].

4.1. Theorem.

- (1) $G_n(0) \Leftrightarrow G_n(0)^{op}$.
- (2) $g_n(1) \Leftrightarrow g_n(1)^{op}$.
- (3) $g_n(0) \Leftrightarrow G_n(1)^{op}$.

Proof. We shall proceed by using induction on n . The case $n = 0$ is obvious. Now assume that $n \geq 1$ and the assertions hold for $n - 1$. Thus we can assume that $g_{n-1}(1)^{op}$ holds for each case, so $\Omega^i(\text{mod } \Lambda^{op}) \subseteq \mathcal{F}_i^{op}$ holds for any $1 \leq i \leq n$ by 3.4. Thus $\mathbb{T}_{i+1}D \in \mathcal{W}_i$ holds for any $D \in \text{mod } \Lambda^{op}$ and $1 \leq i \leq n$.

(1) We will show the ‘only if’ part. Take $D \subseteq \mathbb{E}_n C$ for $C \in \text{mod } \Lambda^{op}$. We have an exact sequence $0 \rightarrow D \rightarrow \mathbb{T}_n C \xrightarrow{f} D' \rightarrow 0$ such that f^* is an isomorphism by 2.4(5). Since $\mathbb{T}_n C \in \mathcal{W}_{n-1}$ holds, we have $\mathbb{E}_i D \subseteq \mathbb{E}_{i+1} D'$ for any $0 \leq i < n$. Since $\text{grade } \mathbb{E}_i D > i$ holds for any $0 \leq i < n$ by $G_n(0)$, we have $\text{grade } D \geq n$ by 2.4(6).

(2) We will show the ‘only if’ part. Put $D := \mathbb{E}_{n+1} C$ for $C \in \text{mod } \Lambda^{op}$. We have an exact sequence $0 \rightarrow D \rightarrow \mathbb{T}_{n+1} C \xrightarrow{f} \Omega \mathbb{T}_{n+2} C \rightarrow 0$ such that f^* is an isomorphism by 2.4(5). Since $\mathbb{T}_{n+1} C \in \mathcal{W}_n$ holds, we have $\mathbb{E}_i D \simeq \mathbb{E}_{i+1} \Omega \mathbb{T}_{n+2} C \simeq \mathbb{E}_{i+2} \mathbb{T}_{n+2} C$ for any $0 \leq i < n$. Since $\text{grade } \mathbb{E}_i D > i$ holds for any $0 \leq i < n$ by $g_n(1)$, we have $\text{grade } D \geq n$ by 2.4(6).

(3) We will show the ‘only if’ part. Take $D \subseteq \mathbb{E}_{n+1} C$ for $C \in \text{mod } \Lambda^{op}$. We have an exact sequence $0 \rightarrow D \rightarrow \mathbb{T}_{n+1} C \xrightarrow{f} D' \rightarrow 0$ such that f^* is an isomorphism by 2.4(5). Since $\mathbb{T}_{n+1} C \in \mathcal{W}_n$ holds, we have $\mathbb{E}_i D \simeq \mathbb{E}_{i+1} D'$ for any $0 \leq i < n$. Since $\text{grade } \mathbb{E}_i D > i$ holds for any $0 \leq i < n$ by $g_n(0)$, we have $\text{grade } D \geq n$ holds by 2.4(6).

We will show the ‘if’ part. Put $D := \mathbb{E}_n C$ for $C \in \text{mod } \Lambda$. We have an exact sequence $0 \rightarrow D \rightarrow \mathbb{T}_n C \xrightarrow{f} \Omega \mathbb{T}_{n+1} C \rightarrow 0$ such that f^* is an isomorphism by 2.4(5). Since $\mathbb{T}_n C \in \mathcal{W}_{n-1}^{op}$ holds, we have $\mathbb{E}_i D \subseteq \mathbb{E}_{i+1} \Omega \mathbb{T}_{n+1} C \simeq \mathbb{E}_{i+2} \mathbb{T}_{n+1} C$ for any $0 \leq i < n$. Since $\text{grade } \mathbb{E}_i D > i$ holds for any $0 \leq i < n$ by $G_n(1)^{op}$, we have $\text{grade } D \geq n$ by 2.4(6). \square

4.1.1. Question. It is natural to ask for the existence of a common generalization of the conditions $G_n(k)$ and $g_n(k)$ satisfying certain ‘left–right symmetry.’ For example, is there some natural condition $G_n(k, l)$ for each triple (n, k, l) of non-negative integers with the following properties?

- (i) $G_n(k, 0) = G_n(k)$, and $G_n(k, 1) = g_n(k)$.
- (ii) $G_n(k, l) \Leftrightarrow G_n(l, k)^{op}$.
- (iii) $G_n(k, l) \Rightarrow G_{n'}(k', l')$ if $n \geq n', k \leq k'$ and $l \leq l'$.

We give characterizations of the conditions $G_n(k)$ and $g_n(k)$ for $k = 0, 1$. We will prove them in 4.6.

4.2. Theorem. (Cf. [FGR, 3.7].) *The following conditions are equivalent.*

- (1) Λ is $G_n(0)$, i.e. n -Gorenstein.
- (2) $\mathbb{E}_i \mathbb{E}_i$ preserves monomorphisms in $\text{mod } \Lambda$ for any $0 \leq i < n$.
- (3) $\text{fd } I_i(\Lambda) \leq i$ holds for any $0 \leq i < n$.
- (i)^{op} Opposite side version of (i) ($1 \leq i \leq 3$).

4.3. Theorem. (Cf. [AB], [Ho1, 2.1], [HN, 4.7], [Hu2, 3.3], [Hu3, 2.4].) *The following conditions are equivalent.*

- (1) Λ is $g_n(1)$.
- (2) For any monomorphism $A \xrightarrow{f} B$ with $A, B \in \Omega^2(\text{mod } \Lambda)$, $\mathbb{E}_i \mathbb{E}_i f$ is a monomorphism for any $0 \leq i < n$.
- (3) $\Omega^i(\text{mod } \Lambda) = \mathcal{F}_i$ holds for any $1 \leq i \leq n + 1$.
- (4) For any $C \in \text{mod } \Lambda$ and $0 \leq i \leq n$, there exists an exact sequence $0 \rightarrow Y \rightarrow X \rightarrow C \rightarrow 0$ with $X \in \mathcal{W}_{i+1}$ and $Y \in \mathcal{P}_{i+1}$.
- (5) For any $C \in \text{mod } \Lambda$ and $0 \leq i \leq n$, there exists an exact sequence $0 \rightarrow \Omega C \rightarrow Y \rightarrow X \rightarrow 0$ with $X \in \mathcal{W}_{i+1}$ and $Y \in \mathcal{P}_{i+1}$.
- (i)^{op} Opposite side version of (i) ($1 \leq i \leq 5$).

4.4. Theorem. (Cf. [Ho1, 2.4], [AR3], [IST, 2.1], [HN, 4.1], [Hu2, 3.6].) *The following conditions (1)–(7) are equivalent. If Λ is an artin algebra, then (8) is also equivalent.*

- (1) Λ is $G_n(1)$.
- (2) For any exact sequence $0 \rightarrow A \xrightarrow{f} B \rightarrow C \rightarrow 0$ with $C \in \Omega(\text{mod } \Lambda)$, $\mathbb{E}_i \mathbb{E}_i f$ is a monomorphism for any $0 \leq i < n$.
- (3) $\mathcal{E}(\Omega^{i+1}(\text{mod } \Lambda), \Omega^{i+1}(\text{mod } \Lambda)) \subseteq \mathcal{F}_{i+1}$ holds for any $0 \leq i < n$.
- (4) $\Omega^i(\text{mod } \Lambda)$ is closed under extensions for any $1 \leq i \leq n$.
- (4') $\text{add } \Omega^i(\text{mod } \Lambda)$ is closed under extensions for any $1 \leq i \leq n$.
- (5) $\text{fd } I_i(\Lambda^{op}) \leq i + 1$ holds for any $0 \leq i < n$.

Then we obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 0 \rightarrow & X & \rightarrow & \bigoplus_{j=0}^{i-1} P_{j+1,j} & \rightarrow \cdots \rightarrow & P_{i-1,0} \oplus P_{i,1} & \rightarrow & P_{i,0} & \rightarrow & \Omega^{-i-1}C \rightarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 0 \rightarrow & Y & \rightarrow & I_0 \oplus \bigoplus_{j=0}^{i-1} P_{j+1,j} & \rightarrow \cdots \rightarrow & I_{i-2} \oplus P_{i-1,0} \oplus P_{i,1} & \rightarrow & I_{i-1} \oplus P_{i,0} & \rightarrow & I_i \rightarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 0 \rightarrow & C & \rightarrow & I_0 & \rightarrow \cdots \rightarrow & I_{i-2} & \rightarrow & I_{i-1} & \rightarrow & \Omega^{-i}C \rightarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & 0 & & 0 & & 0 & & 0 & & 0
 \end{array}$$

where $\Omega^{-i-1}C = \text{Cok}(I_{i-1} \rightarrow I_i)$ and $\Omega^{-i}C = \text{Cok}(I_{i-2} \rightarrow I_{i-1})$. Since $\text{id}(I_l \oplus \bigoplus_{j=0}^{i-l-1} P_{j+l+1,j}) \leq i-l$ for any $0 \leq l \leq i-1$ holds by (5), we have $Y \in \mathcal{I}_{i+1}$. Thus we get one of the desired sequences $0 \rightarrow C \rightarrow Y \rightarrow X \rightarrow 0$.

Now we take exact sequences $0 \rightarrow \Omega C \rightarrow P \rightarrow C \rightarrow 0$ with projective P and $0 \rightarrow \Omega C \rightarrow Y' \rightarrow X' \rightarrow 0$ with $X' \in \Omega^i(\text{mod } \Lambda)$ and $Y' \in \mathcal{I}_{i+1}$. Since $\Omega^i(\text{mod } \Lambda)$ is closed under extensions by (4), we have the desired sequence $0 \rightarrow Y' \rightarrow X'' \rightarrow C \rightarrow 0$ by applying the dual of 2.1. \square

4.7. Corollary.

- (1) If Λ is $g_n(1)$, then \mathcal{P}_{i+1} is a covariantly finite subcategory of $\text{mod } \Lambda$ for any $0 \leq i \leq n$.
- (2) [Hu2, 3.6] If an artin algebra Λ is $g_n(0)$, then \mathcal{P}_{i+1} is a functorially finite subcategory of $\text{mod } \Lambda$ for any $0 \leq i \leq n$.

Proof. (1) For any $C \in \text{mod } \Lambda$, let C' be a maximal factor module of C such that any homomorphism from C to \mathcal{P}_{i+1} factors through C' . We can take an exact sequence $0 \rightarrow C' \rightarrow Y \rightarrow C'' \rightarrow 0$ with $Y \in \mathcal{P}_{i+1}$. Take the sequence $0 \rightarrow Y' \rightarrow X' \rightarrow C'' \rightarrow 0$ with $X' \in \mathcal{W}_{i+1}$ and $Y' \in \mathcal{P}_{i+1}$ by 4.3(4). Applying 2.1, we have an exact sequence $0 \rightarrow C' \xrightarrow{f} Y'' \rightarrow X' \rightarrow 0$ with $Y'' \in \mathcal{P}_{i+1}$. Then f is a left \mathcal{P}_{i+1} -approximation of C' , and so the composition $C \rightarrow C' \xrightarrow{f} Y''$ gives a left \mathcal{P}_{i+1} -approximation of C .

(2) \mathcal{P}_{i+1} is covariantly finite by (1). Since \mathcal{I}_{i+1}^{op} is covariantly finite by 4.4(8), \mathcal{P}_{i+1} is contravariantly finite. \square

4.8. Our main result in this paper is the following.

Theorem. Assume that an artin algebra Λ is $G_n(1)$. For any $i \geq 0$ and $n+1 \geq j \geq 1$ satisfying $i+j \leq n+2$, $(\mathcal{X}_{i,j-1}, \mathcal{Y}_{i,j})$ forms a functorially finite cotorsion pair.

Proof. $\text{Ext}_\Lambda^1(\mathcal{W}_i, \mathcal{P}_i) = 0$ by 2.4(3), and $\text{Ext}_\Lambda^1(\mathcal{F}_{j-1}, \mathcal{I}_j) = 0$ by $\mathcal{F}_{j-1} \subseteq \Omega^{j-1}(\text{mod } \Lambda)$, so $\text{Ext}_\Lambda^1(\mathcal{X}_{i,j-1}, \mathcal{Y}_{i,j}) = 0$ holds. Since $\mathcal{X}_{i,j-1}$ and $\mathcal{Y}_{i,j}$ are closed under direct summands, we only have to show that $(\mathcal{X}_{i,j-1}, \mathcal{Y}_{i,j})$ satisfies the conditions 2.3(3)(4) by 2.3.3. For any $C \in \text{mod } \Lambda$, we can take an exact sequence $0 \rightarrow I \rightarrow \Omega^{j-1}C' \rightarrow C \rightarrow 0$ with $I \in \mathcal{I}_j$ and $C' \in \text{mod } \Lambda$ by 4.4(8). By 4.3(4), we can take an exact sequence $0 \rightarrow P \rightarrow X \rightarrow C' \rightarrow 0$

with $X \in \mathcal{W}_{i+j-1}$ and $P \in \mathcal{P}_{i+j-1}$. Taking $(j - 1)$ th syzygies, we have an exact sequence $0 \rightarrow P' \rightarrow X' \rightarrow \Omega^{j-1}C' \rightarrow 0$ with $X' \in \mathcal{X}_{i,j-1}$ and $P' \in \mathcal{P}_i$. Consider the following pull-back diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & I & \longrightarrow & \Omega^{j-1}C' & \longrightarrow & C \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \parallel \\
 0 & \longrightarrow & Y & \longrightarrow & X' & \longrightarrow & C \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & P' & \xlongequal{\quad} & P' & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Since the middle row $0 \rightarrow Y \rightarrow X' \rightarrow C \rightarrow 0$ satisfies $Y \in \mathcal{Y}_{i,j}$ and $X' \in \mathcal{X}_{i,j-1}$, the condition 2.3(3) holds.

Moreover, take an exact sequence $0 \rightarrow C \rightarrow I \rightarrow \Omega^{-1}C \rightarrow 0$ with injective I and $0 \rightarrow Y_0 \rightarrow X_0 \rightarrow \Omega^{-1}C \rightarrow 0$ with $X_0 \in \mathcal{X}_{i,j-1}$ and $Y_0 \in \mathcal{Y}_{i,j}$. Applying the proof of 2.1 and using $\mathcal{E}(\mathcal{I}_1, \mathcal{Y}_{i,j}) \subseteq \mathcal{Y}_{i,j}$, we have an exact sequence $0 \rightarrow C \rightarrow Y_1 \rightarrow X_1 \rightarrow 0$ with $X_1 \in \mathcal{X}_{i,j-1}$ and $Y_1 \in \mathcal{Y}_{i,j}$. Thus the condition 2.3(4) holds. \square

4.9. Our Theorem 1.3 is a special case of the following result.

Corollary. *Let Λ be an artin algebra which is $G_\infty(1)$ and $G_\infty(1)^{op}$. Then $(\mathcal{X}_{i,j-1}, \mathcal{Y}_{i,j})$ ($i \geq 0, j \geq 1$) and $(\mathcal{Y}_{i,j}, \mathbb{D}\mathcal{X}_{j,i-1}^{op})$ ($i \geq 1, j \geq 0$) form functorially finite cotorsion pairs.*

Proof. This is immediate from 4.8 and the fact that $\mathcal{Y}_{i,j} = \mathbb{D}\mathcal{Y}_{j,i}^{op}$. \square

4.10. As an application of 1.3, we have the following

Corollary. *Let Λ be an ∞ -Gorenstein artin algebra. Then $\mathcal{W}_1 \supseteq \mathcal{W}_2 \supseteq \dots \supseteq$ is a chain of contravariantly finite subcategories of $\text{mod } \Lambda$.*

5. Finitistic dimension and the (l, n) -condition

In this final section, we give some results on finitistic dimension and left–right symmetry of the (l, n) -condition.

5.1. Recall that the finitistic dimension of Λ , denoted by $\text{fin.dim } \Lambda$, is defined as $\sup\{\text{pd } X \mid X \in \text{mod } \Lambda \text{ and } \text{pd } X < \infty\}$.

Lemma. Assume that Λ is $g_{n+1}(k)$ with $n \geq 0$ and $k > 0$. If $\text{fin.dim } \Lambda = n$, then $\text{id } \Lambda \leq n + k$.

Proof. Let $C \in \text{mod } \Lambda$. By 3.4, there exists an exact sequence $0 \rightarrow Y \rightarrow X \rightarrow \Omega^{k-1}C \rightarrow 0$ with $X \in \mathcal{W}_{n+2}$ and $\text{pd } Y \leq n + 1$. Then $\text{pd } Y \leq n$ because $\text{fin.dim } \Lambda = n$. So $\mathbb{E}_{n+k+1}C \simeq \mathbb{E}_{n+2}\Omega^{k-1}C \simeq \mathbb{E}_{n+1}Y = 0$ and hence $\text{id } \Lambda \leq n + k$. \square

5.2. Theorem. If Λ is $g_\infty(k)$ with $k \geq 0$, then $\text{fin.dim } \Lambda \leq \text{id } \Lambda \leq \text{fin.dim } \Lambda + k$.

Proof. It is well known that $\text{fin.dim } \Lambda \leq \text{id } \Lambda$. So it suffices to prove $\text{id } \Lambda \leq \text{fin.dim } \Lambda + k$. The case $k > 0$ follows easily from 5.1. Now suppose $k = 0$. Then Λ is $g_\infty(0)$, and so $\text{fd } I_i(\Lambda) \leq i + 1$ for any $i \geq 0$ by [HN, 4.1] (see 4.4 above). It follows from [Hu3, 2.15] that $\text{fin.dim } \Lambda = \text{id } \Lambda$. \square

5.3. Corollary.

- (1) If Λ is $G_\infty(0)$, then $\text{fin.dim } \Lambda = \text{id } \Lambda$ and $\text{fin.dim } \Lambda^{op} = \text{id } \Lambda^{op}$.
- (2) If Λ is $G_\infty(1)$, then $\text{fin.dim } \Lambda \leq \text{id } \Lambda \leq \text{fin.dim } \Lambda + 1$ and $\text{fin.dim } \Lambda^{op} = \text{id } \Lambda^{op}$.

Proof. (1) follows from the symmetry of $G_\infty(0)$ and 5.2.

(2) Because Λ is $G_\infty(1)$ if and only if Λ is $g_\infty(0)^{op}$ by [AR3, 0.1] and [HN, 4.1] (see 4.4 above), our conclusion follows from 5.2 and its dual. \square

5.4. Recall that we say that Λ satisfies the (l, n) -condition (or Λ^{op} satisfies the $(l, n)^{op}$ -condition) if $\text{s.grade } \mathbb{E}_l C \geq n$ holds for any $C \in \text{mod } \Lambda^{op}$ (see 2.4(7)). Similarly, we say that Λ satisfies the weak (l, n) -condition (or Λ^{op} satisfies the weak $(l, n)^{op}$ -condition) if $\text{grade } \mathbb{E}_l C \geq n$ holds for any $C \in \text{mod } \Lambda^{op}$. For example, Λ is $G_n(k)$ if and only if Λ satisfies the $(k + i, i)^{op}$ -condition for any $1 \leq i \leq n$, and Λ is $g_n(k)$ if and only if Λ satisfies the weak $(k + i, i)^{op}$ -condition for any $1 \leq i \leq n$.

Lemma.

- (1) $(k, l) + \text{weak } (l, n) \Rightarrow (k, n)$.
- (2) $\text{weak } (k, l) + \text{weak } (l, n) \Rightarrow \text{weak } (k, n)$.
- (3) $(k, l) + \text{weak } (l, n)^{op} \Rightarrow (k, n)$.
- (4) $\text{weak } (k, l) + \text{weak } (l, n)^{op} \Rightarrow \text{weak } (k, n)$.

Proof. Using [Iy1, 2.3(2)], we can show all assertions in a manner similar to the proof of [Iy1, 2.3(3)]. \square

5.5. We call $l \in \mathbb{N}$ a dominant number of Λ if $\text{fd } I_i(\Lambda) < \text{fd } I_l(\Lambda)$ holds for any $i < l$ [Iy1]. The following result generalizes a theorem in [Iy1, 1.1].

Theorem. Any dominant number l of Λ satisfies $\text{fd } I_l(\Lambda) \geq l$.

Proof. Put $m := \text{fd } I_l(\Lambda)$ and assume $m < l$. Since l is a dominant number of Λ , Λ satisfies the (m, l) and $(m + 1, l + 1)$ -condition. Since Λ satisfies the $(m, m + 1)$ -condition by $m < l$, Λ satisfies the $(m, l + 1)$ -condition by the dual of 5.4(1). This is in contradiction to $m = \text{fd } I_l(\Lambda)$. \square

5.6. The following result is an analog of 5.4(2) and 4.1.

Theorem. Let $k, l, n \geq 0$. Assume $k \geq n - 1$.

- (1) If $\text{s.grade } \mathbb{E}_1 \mathcal{F}_k \geq l$ and $\text{s.grade } \mathbb{E}_1 \mathcal{F}_l^{op} \geq n$, then $\text{s.grade } \mathbb{E}_1 \mathcal{F}_k \geq n$.
- (2) If $\text{grade } \mathbb{E}_1 \mathcal{F}_k \geq l - 1$ and $\text{grade } \mathbb{E}_1 \mathcal{F}_l^{op} \geq n - 1$, then $\text{grade } \mathbb{E}_1 \mathcal{F}_k \geq n - 1$.

Proof. We can assume $n > l$.

(1) Fix $C \in \mathcal{F}_k$ and $D \subseteq \mathbb{E}_1 C$. Then $\text{grade } D \geq l$ holds by our first assumption. By 2.4(5), we have an exact sequence $0 \rightarrow D \rightarrow \text{Tr } C \xrightarrow{f} D' \rightarrow 0$ such that $\text{Tr } C \in \mathcal{W}_k^{op}$ and f^* is an isomorphism. Thus $D' \in \mathcal{W}_l^{op}$ by $k \geq l$, and $\Omega^l D' \in \mathcal{F}_l^{op}$. Using 3.1(1) and $\text{s.grade } \mathbb{E}_1 \mathcal{F}_l^{op} \geq n > l$, we have $\Omega^i D' \in \mathcal{F}_l^{op}$ for any $i \geq l$ inductively. On the other hand, applying $(\)^*$ to the exact sequence above, we have an inclusion $\mathbb{E}_i D \subseteq \mathbb{E}_{i+1} D' = \mathbb{E}_1 \Omega^i D'$ for any $0 \leq i \leq k$. Thus our second assumption implies $\text{grade } \mathbb{E}_i D \geq n$ for any $l \leq i \leq k$. Since $\text{grade } D \geq l$, we obtain $\text{grade } D \geq n$ by 2.4(6).

(2) Fix $C \in \mathcal{F}_k$ and put $D := \mathbb{E}_1 C$ and $D' := \Omega \text{Tr } \Omega C$. Then $\text{grade } D \geq l - 1$ and $\Omega C \in \mathcal{F}_l$ hold by 3.1(1). We have $\text{Tr } \Omega C \in \mathcal{W}_l^{op}$ and $\Omega^{l-1} D' = \Omega^l \text{Tr } \Omega C \in \mathcal{F}_l^{op}$. Using 3.1(1) and $\text{grade } \mathbb{E}_1 \mathcal{F}_l^{op} \geq n - 1 > l - 1$, we have $\Omega^i D' \in \mathcal{F}_l^{op}$ for any $i \geq l - 1$ inductively. By 2.4(5), we have an exact sequence $0 \rightarrow D \rightarrow \text{Tr } C \xrightarrow{f} D' \rightarrow 0$ such that $\text{Tr } C \in \mathcal{W}_k^{op}$ and f^* is an isomorphism. Applying $(\)^*$, we have $\mathbb{E}_i D \simeq \mathbb{E}_{i+1} D' \simeq \mathbb{E}_1 \Omega^i D'$ for any $0 \leq i < k$. Thus our second assumption implies $\text{grade } \mathbb{E}_i D \geq n - 1$ for any $l - 1 \leq i < k$. Since $\text{grade } D \geq l - 1$, we obtain $\text{grade } D \geq n - 1$ by 2.4(6). \square

5.7. Recall that Λ is $G_n(0)$ if and only if Λ satisfies the $(i, i)^{op}$ -condition for any $1 \leq i \leq n$. It is known that Λ is $G_n(0)$ if and only if so is Λ^{op} [FGR, 3.7] (see 4.2 above). However, the (i, i) -condition does not possess such a symmetric property in general [Hu1, p. 1460], [Iy2, 2.1.1]. For example, the algebra given by the quiver $\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \leftarrow \bullet$ modulo the ideal ab satisfies exactly one of the $(2, 2)$ and $(2, 2)^{op}$ -conditions.

As an application of 5.6, we give a sufficient condition that the (i, i) -condition implies the $(i, i)^{op}$ -condition as follows.

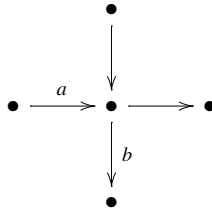
Corollary. $G_{n-1}(1) + (n, n) \Rightarrow (n, n)^{op}$.

Proof. The case $n = 0$ is trivial. Now suppose $n \geq 1$. Then $\text{s.grade } \mathbb{E}_1 \mathcal{F}_{n-1} \geq n - 1$ and $\text{s.grade } \mathbb{E}_1 \mathcal{F}_{n-1}^{op} \geq n$. So $\text{s.grade } \mathbb{E}_1 \mathcal{F}_{n-1} \geq n$ by 5.6(1). On the other hand, $\Omega^i(\text{mod } \Lambda) = \mathcal{F}_i$ for any $1 \leq i \leq n$ by [HN, 4.7] (see 3.4 above). Thus $\text{s.grade } \mathbb{E}_n C = \text{s.grade } \mathbb{E}_1 \Omega^{n-1} C \geq n$ for any $C \in \text{mod } \Lambda$ and Λ satisfies the $(n, n)^{op}$ -condition. \square

5.7.1. Corollary. $(2, 2)^{op} + (3, 3) \Rightarrow (3, 3)^{op}$.

Proof. This is immediate from 5.7 for $n = 3$. \square

5.7.2. Example. We give an example satisfying the conditions in 5.7.1. Let K be a field and Λ a finite dimensional K -algebra given by the quiver



modulo the ideal ab . Then $\text{fd } I_0(\Lambda) = \text{fd } I_1(\Lambda) = \text{fd } I_0(\Lambda^{op}) = \text{fd } I_1(\Lambda^{op}) = 1$, and $\text{fd } I_2(\Lambda) = \text{fd } I_2(\Lambda^{op}) = 2$.

5.8. Corollary. *The following conditions are equivalent.*

- (1) $\text{s.grade } \mathbb{E}_i C \geq i$ holds for any $C \in \text{mod } \Lambda$ and $1 \leq i \leq n$ (i.e. Λ is $G_n(0)$).
- (2) $\text{s.grade } \mathbb{E}_1 \mathcal{F}_{i-1} \geq i$ holds for any $1 \leq i \leq n$.
- (i)^{op} The opposite side version of (i) ($i = 1, 2$).

Proof. (1) \Rightarrow (2) is trivial.

(2) \Rightarrow (1). By 3.1(1), $\Omega \mathcal{F}_{i-1} \subseteq \mathcal{F}_i$ holds for any $1 \leq i \leq n$. This implies inductively that $\Omega^i(\text{mod } \Lambda) = \mathcal{F}_i$ for any $1 \leq i \leq n$. So the assertion follows easily.

(2)^{op} \Rightarrow (2). The case $n = 0$ is obvious. Now suppose $n \geq 1$. By 5.6(1), $\text{s.grade } \mathbb{E}_1 \mathcal{F}_{n-1} \geq n - 1$ and $\text{s.grade } \mathbb{E}_1 \mathcal{F}_{n-1}^{op} \geq n$ imply $\text{s.grade } \mathbb{E}_1 \mathcal{F}_{n-1} \geq n$. Thus the assertion follows inductively. \square

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