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ScienceDirect
Journal of Algebra 311 (2007) 619–634
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Generalized tilting modules with finite injective dimension

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Received 25 February 2006
Available online 4 January 2007
Communicated by Luchezar L. Avramov

Abstract

Let R be a left noetherian ring, S a right noetherian ring and ${}_RU$ a generalized tilting module with $S = \mathrm{End}({}_RU)$. The injective dimensions of ${}_RU$ and U_S are identical provided both of them are finite. Under the assumption that the injective dimensions of ${}_RU$ and U_S are finite, we describe when the subcategory $\{\mathrm{Ext}^n_S(N, U) \mid N$ is a finitely generated right S-module$\}$ is submodule-closed. As a consequence, we obtain a negative answer to a question posed by Auslander in 1969. Finally, some partial answers to Wakamatsu Tilting Conjecture are given.

Keywords: Generalized tilting modules; Injective dimension; U-limit dimension; Submodule-closed; Wakamatsu Tilting Conjecture

1. Introduction

Let R be a ring. We use $\mathrm{Mod}\, R$ (respectively $\mathrm{Mod}\, R^{op}$) to denote the category of left (respectively right) R-modules, and use $\mathrm{mod}\, R$ (respectively $\mathrm{mod}\, R^{op}$) to denote the category of finitely generated left R-modules (respectively right R-modules).

We define $\mathrm{gen}^*({}_RR) = \{X \in \mathrm{mod}\, R \mid$ there exists an exact sequence $\cdots \to P_i \to \cdots \to P_1 \to P_0 \to X \to 0$ in $\mathrm{mod}\, R$ with P_i projective for any $i \geqslant 0\}$ (see [W2]). A module ${}_RU$ in $\mathrm{mod}\, R$ is called *selforthogonal* if $\mathrm{Ext}^i_R({}_RU, {}_RU) = 0$ for any $i \geqslant 1$.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.11.025

Definition 1.1. [W2] A selforthogonal module ${}_RU$ in gen*$({}_RR)$ is called a *generalized tilting module* (sometimes it is also called a *Wakamatsu tilting module*, see [BR]) if there exists an exact sequence:

$$0 \to {}_RR \to U_0 \to U_1 \to \cdots \to U_i \to \cdots$$

such that: (1) $U_i \in \operatorname{add} {}_RU$ for any $i \geqslant 0$, where $\operatorname{add} {}_RU$ denotes the full subcategory of mod R consisting of all modules isomorphic to direct summands of finite sums of copies of ${}_RU$, and (2) after applying the functor $\operatorname{Hom}_R(\ , {}_RU)$ the sequence is still exact.

Let R and S be any rings. Recall that a bimodule ${}_RU_S$ is called a *faithfully balanced bimodule* if the natural maps $R \to \operatorname{End}(U_S)$ and $S \to \operatorname{End}({}_RU)^{op}$ are isomorphisms. By [W2, Corollary 3.2], we have that ${}_RU_S$ is faithfully balanced and selforthogonal with ${}_RU \in \operatorname{gen}^*({}_RR)$ and $U_S \in \operatorname{gen}^*(S_S)$ if and only if ${}_RU$ is generalized tilting with $S = \operatorname{End}({}_RU)$ if and only if U_S is generalized tilting with $R = \operatorname{End}(U_S)$.

Let R and S be Artin algebras and ${}_RU$ a generalized tilting module with $S = \operatorname{End}({}_RU)$. Wakamatsu proved in [W1, Theorem] that the projective (respectively injective) dimensions of ${}_RU$ and U_S are identical provided both of them are finite. The result on the projective dimensions also holds true when R is a left noetherian ring and S is a right noetherian ring, by using an argument similar to that in [W1]. In this case, ${}_RU_S$ is a tilting bimodule of finite projective dimension [M, Proposition 1.6]. However, because there is no duality available, Wakamatsu's argument in [W1] does not work on the injective dimensions over noetherian rings. So, it is natural to ask the following questions: When R is a left noetherian ring and S is a right noetherian ring, (1) Do the injective dimensions of ${}_RU$ and U_S coincide provided both of them are finite? (2) If one of the injective dimensions of ${}_RU$ and U_S is finite, is the other also finite?

The answer to the first question is positive if one of the following conditions is satisfied: (1) ${}_RU_S = {}_RR_R$ [Z, Lemma A]; (2) R and S are Artin algebras [W1, Theorem]; (3) R and S are two-sided noetherian rings and ${}_RU$ is n-Gorenstein for all n [H2, Proposition 17.2.6]. In this paper, we show in Section 2 that the answer to this question is always positive.

By the positive answer to the first question, the second question is equivalent to the following question: Are the injective dimensions of ${}_RU$ and U_S identical? The above result means that the answer to this question is positive provided that both dimensions are finite. On the other hand, for Artin algebras, the positive answer to the second question is equivalent to the validity of Wakamtsu Tilting Conjecture (**WTC**). This conjecture states that every generalized tilting module with finite projective dimension is tilting, or equivalently, every generalized tilting module with finite injective dimension is cotilting. Moreover, **WTC** implies the validity of the Gorenstein Symmetry Conjecture (**GSC**), which states that the left and right self-injective dimensions of R are identical (see [BR]). In Section 4, we give some partial answers to question (2). Let R and S be two-sided artinian rings and ${}_RU$ a generalized tilting module with $S = \operatorname{End}({}_RU)$. We prove that if the injective dimension of U_S is equal to n and the U-limit dimension of each of the first $(n-1)$st terms is finite, then the injective dimension of ${}_RU$ is also equal to n. Thus it trivial that the injective dimension of U_S is at most 1 if and only if that of ${}_RU$ is at most 1. We remark that for an Artin algebra R, it is well known that the right self-injective dimension of R is at most 1 if and only if the left self-injective dimension of R is at most 1 (see [AR3, p. 121]). In addition, we prove that the left and right injective dimensions of ${}_RU$ and U_S are identical if ${}_RU$ (or U_S) is quasi-Gorenstein, that is, **WTC** holds for quasi-Gorenstein modules.

For an $(R-S)$-bimodule ${}_RU_S$ and a positive integer n, we denote $\mathcal{E}_n(U_S)=\{M\in \operatorname{mod} R \mid M=\operatorname{Ext}_S^n(N,U)$ for some $N\in \operatorname{mod} S^{op}\}$. For a two-sided noetherian ring R, Auslander showed in [A, Proposition 3.3] that any direct summand of a module in $\mathcal{E}_1(R_R)$ is still in $\mathcal{E}_1(R_R)$. He then asked whether any submodule of a module in $\mathcal{E}_1(R_R)$ is still in $\mathcal{E}_1(R_R)$. Recall that a full subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is said to be *submodule-closed* if any non-zero submodule of a module in $\mathcal{X}$ is also in $\mathcal{X}$. Then the above Auslander's question is equivalent to the following question: Is $\mathcal{E}_1(R_R)$ submodule-closed? In Section 3, under the assumption that R is a left noetherian ring, S is a right noetherian ring and ${}_RU$ is a generalized tilting module with $S=\operatorname{End}({}_RU)$ and the injective dimensions of ${}_RU$ and U_S being finite, we give some necessary and sufficient conditions for $\mathcal{E}_n(U_S)$ being submodule-closed. As a consequence, we construct some examples to illustrate that neither $\mathcal{E}_1(R_R)$ nor $\mathcal{E}_2(R_R)$ are submodule-closed in general, by which we answer the above Auslander's question negatively.

Throughout this paper, R is a left noetherian ring, S is a right noetherian ring (unless stated otherwise) and ${}_RU$ is a generalized tilting module with $S=\operatorname{End}({}_RU)$. For a module A in $\operatorname{Mod} R$ (respectively $\operatorname{Mod} S^{op}$), we use $l.\mathrm{id}_R(A)$, $l.\mathrm{fd}_R(A)$ and $l.\mathrm{pd}_R(A)$ (respectively $r.\mathrm{id}_S(A)$, $r.\mathrm{fd}_S(A)$ and $r.\mathrm{pd}_S(A)$) to denote the injective dimension, flat dimension and projective dimension of ${}_RA$ (respectively A_S), respectively.

2. Some homological dimensions

In this section, we study the relations among the U-limit dimension (which was introduced in [H2]) of an injective module E, the flat dimension of $\operatorname{Hom}(U,E)$ and the injective dimension of U. Then we show that $l.\mathrm{id}_R(U)=r.\mathrm{id}_S(U)$ provided both of them are finite.

The following result is [W2, Corollary 3.2].

Proposition 2.1. *The following statements are equivalent.*

(1) ${}_RU$ *is a generalized tilting module with* $S=\operatorname{End}({}_RU)$.
(2) U_S *is a generalized tilting module with* $R=\operatorname{End}(U_S)$.
(3) ${}_RU_S$ *is a faithfully balanced and selforthogonal bimodule.*

We use add-lim ${}_RU$ to denote the subcategory of $\operatorname{Mod} R$ consisting of all modules isomorphic to direct summands of a direct limit of a family of modules in which each is a finite direct sum of copies of ${}_RU$ (see [H2]).

Proposition 2.2.

(1) *Let* $V\in$ add-lim ${}_RU$. *Then* $\operatorname{Ext}_R^i(U^{(I)},V)=0$ *for any index set* I *and* $i\geqslant 1$.
(2) $\operatorname{Ext}_R^i(U^{(I)},U^{(J)})=0$ *for any index sets* I, J *and* $i\geqslant 1$.

Proof. (1) It is well known that for any $i\geqslant 1$, $\operatorname{Ext}_R^i(U^{(I)},V)\cong \operatorname{Ext}_R^i(U,V)^I$. Since ${}_RU$ is finitely generated and selforthogonal and $V\in$ add-lim ${}_RU$, it follows easily from [S1, Theorem 3.2] that $\operatorname{Ext}_R^i(U,V)=0$ and so $\operatorname{Ext}_R^i(U,V)^I=0$.

(2) Because a direct sum of a family of modules is a special kind of a direct limit of these modules, (2) follows from (1) trivially. □

Definition 2.3. [H2] For a module A in $\operatorname{Mod} R$, if there exists an exact sequence $\cdots \to U_n \to \cdots \to U_1 \to U_0 \to A \to 0$ in $\operatorname{Mod} R$ with $U_i \in \text{add-lim}\,_R U$ for any $i \geqslant 0$, then we define the *U-limit dimension* of A, denoted by $U\text{-}\lim.\dim_R(A)$, as $\inf\{n \mid$ there exists an exact sequence $0 \to U_n \to \cdots \to U_1 \to U_0 \to A \to 0$ in $\operatorname{Mod} R$ with $U_i \in \text{add-lim}\,_R U$ for any $0 \leqslant i \leqslant n\}$. We set $U\text{-}\lim.\dim_R(A)$ infinity if no such an integer exists.

Remark. It is well known that a module over any ring is flat if and only if it is direct limit of a family of finitely generated free modules. So, putting $_R U = {}_R R$, a module in $\operatorname{Mod} R$ is flat if and only if it is in $\text{add-lim}\,_R R$; in this case, the dimension defined as in Definition 2.3 is just the flat dimension of modules.

For a module A in $\operatorname{Mod} R$ (respectively $\operatorname{Mod} S^{op}$), we denote either of $\operatorname{Hom}_R({}_R U_S, {}_R A)$ and $\operatorname{Hom}_S({}_R U_S, A_S)$ by ${}^*A$.

Lemma 2.4. *Let $_R E$ be an injective R-module. Then $l.\mathrm{fd}_S({}^*E) = U\text{-}\lim.\dim_R(E)$.*

Proof. The result was proved in [H2, Lemma 17.3.1] when R and S are two-sided noetherian rings. The proof in [H2] remains valid in the setting here, we omit it. □

Remark. It is not difficult to see from the proof of [H2, Lemma 17.3.1] that for any injective R-module E, there exists an exact sequence:

$$\cdots \to U_i \to \cdots \to U_1 \to U_0 \to E \to 0$$

in $\operatorname{Mod} R$ with U_i in $\text{add-lim}\,_R U$ for any $i \geqslant 0$. So $U\text{-}\lim.\dim_R(E)$ (finite or infinite) always exists for any injective R-module E.

Proposition 2.5. *Let $_R E$ be an injective R-module. If $l.\mathrm{id}_R(U) = n < \infty$ and $l.\mathrm{fd}_S({}^*E) < \infty$, then $l.\mathrm{fd}_S({}^*E) \leqslant n$.*

Proof. Suppose $l.\mathrm{fd}_S({}^*E) = m < \infty$. Then there exists an exact sequence:

$$0 \to F_m \to S^{(I_{m-1})} \to \cdots \to S^{(I_1)} \to S^{(I_0)} \to {}^*E \to 0 \tag{1}$$

in $\operatorname{Mod} S$ with F_m flat and I_i an index set for any $0 \leqslant i \leqslant m-1$. By [CE, Chapter VI, Proposition 5.3], we have that

$$\operatorname{Tor}_j^S\big(U, {}^*E\big) \cong \operatorname{Hom}_R\big(\operatorname{Ext}_S^j(U, U), E\big) = 0$$

for any $j \geqslant 1$. So, by applying the functor $U \otimes_S -$ to the exact sequence (1), we get the following exact sequence:

$$0 \to U \otimes_S F_m \to U \otimes_S S^{(I_{m-1})} \to \cdots \to U \otimes_S S^{(I_1)} \to U \otimes_S S^{(I_0)} \to U \otimes_S {}^*E \to 0.$$

By Proposition 2.1 and [S2, p. 47], we have that $U \otimes_S {}^*E \cong \operatorname{Hom}_R(\operatorname{Hom}_S(U, U), E) \cong E$. So we get the following exact sequence:

$$0 \to K_m \xrightarrow{d_m} U^{(I_{m-1})} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_2} U^{(I_1)} \xrightarrow{d_1} U^{(I_0)} \xrightarrow{d_0} E \to 0 \tag{2}$$

where $K_m = U \otimes_S F_m$. Because F_m is a flat S-module, it is a direct limit of finitely generated free S-modules. So $K_m = U \otimes_S F_m \in \text{add-lim}\,_R U$. Hence, by Proposition 2.2(1), $\mathrm{Ext}_R^j(U^{(I_i)}, K_m) = 0$ for any $j \geqslant 1$ and $0 \leqslant i \leqslant m-1$.

Since $l.\mathrm{id}_R(U) = n$, $l.\mathrm{id}_R(K_m) \leqslant n$ by [S1, Theorem 3.2]. If $m > n$, then $\mathrm{Ext}_R^m(E, K_m) = 0$. It follows from the exact sequence (2) that $\mathrm{Ext}_R^1(K_{m-1}, K_m) = 0$, where $K_{m-1} = \mathrm{Coker}\, d_m$. Thus the sequence $0 \to K_m \xrightarrow{d_m} U^{(I_{m-1})} \to K_{m-1} \to 0$ splits and $U^{(I_{m-1})} \cong K_m \oplus K_{m-1}$. In addition, we get an exact sequence:

$$0 \to K_{m-1} \to U^{(I_{m-2})} \xrightarrow{d_{m-2}} \cdots \xrightarrow{d_2} U^{(I_1)} \xrightarrow{d_1} U^{(I_0)} \xrightarrow{d_0} E \to 0$$

with $K_{m-1}, U^{(I_{m-2})}, \ldots, U^{(I_0)} \in \text{add-lim}\,_R U$. Then $U\text{-}\mathrm{lim.dim}_R(E) \leqslant m-1$. But $l.\mathrm{fd}_S(^*E) = U\text{-}\mathrm{lim.dim}_R(E)$ by Lemma 2.4. Consequently we conclude that $l.\mathrm{fd}_S(^*E) \leqslant l.\mathrm{id}_R(U)$. □

We also need the following result, which is [H2, Lemma 17.2.4].

Lemma 2.6.

(1) $r.\mathrm{id}_S(U) = \sup\{l.\mathrm{fd}_S(^*E) \mid {}_R E$ *is injective*$\}$. *Moreover,* $r.\mathrm{id}_S(U) = l.\mathrm{fd}_S(^*Q)$ *for any injective cogenerator* ${}_R Q$ *for* $\mathrm{Mod}\, R$.
(2) $l.\mathrm{id}_R(U) = \sup\{r.\mathrm{fd}_R(^*E') \mid E'_S$ *is injective*$\}$. *Moreover,* $l.\mathrm{id}_R(U) = r.\mathrm{fd}_R(^*Q')$ *for any injective cogenerator* Q'_S *for* $\mathrm{Mod}\, S^{op}$.

We are now in a position to prove one of the main results in this paper.

Theorem 2.7. $l.\mathrm{id}_R(U) = r.\mathrm{id}_S(U)$ *provided both of them are finite.*

Proof. Let ${}_R Q$ be an injective cogenerator for $\mathrm{Mod}\, R$. Assume that $l.\mathrm{id}_R(U) = n < \infty$ and $r.\mathrm{id}_S(U) = m < \infty$. Then $l.\mathrm{fd}_S(^*Q) = m$ by Lemma 2.6. So $m = l.\mathrm{fd}_S(^*Q) \leqslant l.\mathrm{id}_R(U) = n$ by Proposition 2.5. Dually, we may prove $n \leqslant m$. We are done. □

Definition 2.8. [AB] Let $\mathcal{X}$ be a full subcategory of $\mathrm{Mod}\, R$. For a module A in $\mathrm{Mod}\, R$, if there exists an exact sequence $\cdots \to T_n \to \cdots \to T_1 \to T_0 \to A \to 0$ in $\mathrm{Mod}\, R$ with $T_i \in \mathcal{X}$ for any $i \geqslant 0$, then we define $\mathcal{X}\text{-}\mathrm{resol.dim}_R(A) = \inf\{n \mid$ there exists an exact sequence $0 \to T_n \to \cdots \to T_1 \to T_0 \to A \to 0$ in $\mathrm{Mod}\, R$ with $T_i \in \mathcal{X}$ for any $0 \leqslant i \leqslant n\}$. We set $\mathcal{X}\text{-}\mathrm{resol.dim}_R(A)$ infinity if no such an integer exists.

We use $\mathrm{Add}\,_R U$ to denote the full subcategory of $\mathrm{Mod}\, R$ consisting of all modules isomorphic to direct summands of sums of copies of ${}_R U$. Compare the following result with Lemma 2.4.

Lemma 2.9. *Let* ${}_R E$ *be an injective* R*-module. Then* $l.\mathrm{pd}_S(^*E) = \mathrm{Add}\,_R U\text{-}\mathrm{resol.dim}_R(E)$.

Proof. We first prove that $\mathrm{Add}\,_R U\text{-}\mathrm{resol.dim}_R(E) \leqslant l.\mathrm{pd}_S(^*E)$. Without loss of generality, assume that $l.\mathrm{pd}_S(^*E) = m < \infty$. Then there exists an exact sequence:

$$0 \to Q_m \to Q_{m-1} \to \cdots \to Q_1 \to Q_0 \to {}^*E \to 0$$

in $\mathrm{Mod}\, S$ with Q_i projective for any $0 \leqslant i \leqslant m$. Then by using an argument similar to that in the proof of Proposition 2.5, we get that $\mathrm{Add}\,_RU$-$\mathrm{resol.dim}_R(E) \leqslant m$.

We next prove that $l.\mathrm{pd}_S({}^*E) \leqslant \mathrm{Add}\,_RU$-$\mathrm{resol.dim}_R(E)$. Assume that $\mathrm{Add}\,_RU$-$\mathrm{resol.dim}_R(E) = m < \infty$. Then there exists an exact sequence:

$$0 \to U_m \to \cdots \to U_1 \to U_0 \to E \to 0 \tag{3}$$

in $\mathrm{Mod}\, R$ with $U_i \in \mathrm{Add}\,_RU$ for any $0 \leqslant i \leqslant m$. By Proposition 2.2(2), we have that ${}^*U_i \in \mathrm{Add}\,_SS$ (that is, ${}^*U_i$ is a projective left S-module) and $\mathrm{Ext}^j_R(U, U_i) = 0$ for any $j \geqslant 1$ and $0 \leqslant i \leqslant m$. So by applying the functor $\mathrm{Hom}_R(_RU, -)$ to the exact sequence (3), we get the following exact sequence:

$$0 \to {}^*U_m \to \cdots \to {}^*U_1 \to {}^*U_0 \to {}^*E \to 0$$

in $\mathrm{Mod}\, S$ with ${}^*U_i$ left S-projective for any $0 \leqslant i \leqslant m$, and hence $l.\mathrm{pd}_S({}^*E) \leqslant m$. □

Remark.

(1) Put $_RU = {}_RR$. Then $\mathrm{Add}\,_RU$-$\mathrm{resol.dim}_R(A) = l.\mathrm{pd}_R(A)$ for any $A \in \mathrm{Mod}\, R$.
(2) Because a direct sum of a family of modules is a special kind of a direct limit of these modules, for any $A \in \mathrm{Mod}\, R$, we have that U-$\mathrm{lim.dim}_R(A) \leqslant \mathrm{Add}\,_RU$-$\mathrm{resol.dim}_R(A)$ if both of them exist.
(3) It is not difficult to see from the proof of Lemma 2.9 that for any injective R-module E, there exists an exact sequence:

$$\cdots \to U_i \to \cdots \to U_1 \to U_0 \to E \to 0$$

in $\mathrm{Mod}\, R$ with U_i in $\mathrm{Add}\,_RU$ for any $i \geqslant 0$. So $\mathrm{Add}\,_RU$-$\mathrm{resol.dim}_R(E)$ (finite or infinite) always exists for any injective R-module E.

The proof of Proposition 2.5 in fact proves the following more general result.

Proposition 2.10. *Let $_RE$ be an injective R-module. If $l.\mathrm{id}_R(U) = n < \infty$ and $l.\mathrm{fd}_S({}^*E) < \infty$, then $\mathrm{Add}\,_RU$-$\mathrm{resol.dim}_R(E) \leqslant n$ (equivalently, $l.\mathrm{pd}_S({}^*E) \leqslant n$).*

Theorem 2.11. *Let $_RE$ be an injective R-module. If $l.\mathrm{id}_R(U) = n < \infty$, then the following statements are equivalent.*

(1) $l.\mathrm{fd}_S({}^*E) < \infty$.
(2) $l.\mathrm{pd}_S({}^*E) < \infty$.
(3) U-$\mathrm{lim.dim}_R(E) < \infty$.
(4) $\mathrm{Add}\,_RU$-$\mathrm{resol.dim}_R(E) < \infty$.
(5) $l.\mathrm{fd}_S({}^*E) \leqslant n$.
(6) $l.\mathrm{pd}_S({}^*E) \leqslant n$.
(7) U-$\mathrm{lim.dim}_R(E) \leqslant n$.
(8) $\mathrm{Add}\,_RU$-$\mathrm{resol.dim}_R(E) \leqslant n$.

Proof. The implications that (6) ⇒ (2) ⇒ (1) and (6) ⇒ (5) ⇒ (1) are trivial. The implication of (1) ⇒ (6) follows from Proposition 2.10. By Lemma 2.4, we have (1) ⇔ (3) and (5) ⇔ (7). By Lemma 2.9, we have (2) ⇔ (4) and (6) ⇔ (8). □

As an application of the obtained results, we get the following corollary, which gives some equivalent conditions that $l.\mathrm{id}_R(U) = n$ implies $r.\mathrm{id}_S(U) = n$.

Corollary 2.12. *Let* $_RQ$ *be an injective cogenerator for* Mod R. *If* $l.\mathrm{id}_R(U) = n$ $(< \infty)$, *then the following statements are equivalent.*

(1) $r.\mathrm{id}_S(U) = n$.
(2) *One of* $l.\mathrm{fd}_S({}^*Q)$, $l.\mathrm{pd}_S({}^*Q)$, $U\text{-}\lim.\dim_R(Q)$ *and* $\mathrm{Add}\,{}_RU\text{-resol.dim}_R(Q)$ *is finite.*

Proof. Let $_RQ$ be an injective cogenerator for Mod R. Then by Lemmas 2.6(1), 2.4 and 2.9, we have that $r.\mathrm{id}_S(U) = l.\mathrm{fd}_S({}^*Q) = U\text{-}\lim.\dim_R(Q) \leqslant \mathrm{Add}\,{}_RU\text{-resol.dim}_R(Q) = l.\mathrm{pd}_S({}^*Q)$. Now the equivalence of (1) and (2) follows easily from Theorems 2.11 and 2.7. □

3. Submodule-closure of $\mathcal{E}_n(U_S)$

In this section, we study Auslander's question mentioned in Section 1 in a more general situation.

Lemma 3.1. *For any injective module* $_RE$ *and any non-negative integer* t, $l.\mathrm{fd}_S({}^*E) \leqslant t$ *if and only if* $\mathrm{Hom}_R(\mathrm{Ext}_S^{t+1}(N, U), E) = 0$ *for any module* $N \in \mathrm{mod}\, S^{op}$.

Proof. It is easy by [CE, Chapter VI, Proposition 5.3]. □

For a module $A \in \mathrm{mod}\, R$ and a non-negative integer n, we say that the *grade* of A with respect to $_RU$, written as $\mathrm{grade}_U\, A$, is at least n if $\mathrm{Ext}_R^i(A, U) = 0$ for any $0 \leqslant i < n$. We say that the *strong grade* of A with respect to $_RU$, written as $\mathrm{s.grade}_U\, A$, is at least n if $\mathrm{grade}_U\, B \geqslant n$ for all submodules B of A (see [H2]). Assume that

$$0 \to {}_RU \to E_0 \xrightarrow{\alpha_0} E_1 \xrightarrow{\alpha_1} \cdots \to E_i \xrightarrow{\alpha_i} \cdots$$

is a minimal injective resolution of $_RU$.

Lemma 3.2. *Let* n *be a positive integer and* m *an integer with* $m \geqslant -n$. *Then the following statements are equivalent.*

(1) $U\text{-}\lim.\dim_R(\bigoplus_{i=0}^{n-1} E_i) \leqslant n + m$.
(2) $\mathrm{s.grade}_U\, \mathrm{Ext}_S^{n+m+1}(N, U) \geqslant n$ *for any* $N \in \mathrm{mod}\, S^{op}$.
(3) $l.\mathrm{fd}_S({}^*E_i) \leqslant n + m$ *for any* $0 \leqslant i \leqslant n - 1$.

Proof. This conclusion has been proved in [H2, Lemma 17.3.2] when R and S are two-sided noetherian rings. The argument there remains valid in the setting here, we omit it. □

For a module A in $\operatorname{mod} R$ (respectively $\operatorname{mod} S^{op}$), we call $\operatorname{Hom}_R({}_RA, {}_RU_S)$ (respectively $\operatorname{Hom}_S(A_S, {}_RU_S)$) the dual module of A with respect to ${}_RU_S$, and denote either of these modules by A^*. For a homomorphism f between R-modules (respectively S^{op}-modules), we put $f^* = \operatorname{Hom}(f, {}_RU_S)$. We use $\sigma_A : A \to A^{**}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$ to denote the canonical evaluation homomorphism. A is called *U-torsionless* (respectively *U-reflexive*) if σ_A is a monomorphism (respectively an isomorphism).

Definition 3.3. [H3] Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod} R$. $\mathcal{X}$ is said to have the *U-torsionless property* (respectively the *U-reflexive property*) if each module in $\mathcal{X}$ is U-torsionless (respectively U-reflexive).

We denote ${}^{\perp}_{R}U = \{M \in \operatorname{mod} R \mid \operatorname{Ext}^i_R(M, {}_RU) = 0$ for any $i \geqslant 1\}$ and ${}^{\perp_n}_{R}U = \{M \in \operatorname{mod} R \mid \operatorname{Ext}^i_R(M, {}_RU) = 0$ for any $1 \leqslant i \leqslant n\}$ (where n is a positive integer). A module M in $\operatorname{mod} R$ is said to have *generalized Gorenstein dimension zero* (with respect to ${}_RU_S$), denoted by $\operatorname{G-dim}_U(M) = 0$, if the following conditions are satisfied: (1) M is U-reflexive, and (2) $M \in {}^{\perp}_{R}U$ and $M^* \in {}^{\perp}U_S$. Symmetrically, we may define the notion of a module in $\operatorname{mod} S^{op}$ having generalized Gorenstein dimension zero (with respect to ${}_RU_S$) (see [AR2]). We use $\mathcal{G}_U$ to denote the full subcategory of $\operatorname{mod} R$ consisting of the modules with generalized Gorenstein dimension zero. It is trivial that ${}^{\perp}_{R}U \supseteq \mathcal{G}_U$.

Proposition 3.4. [H3, Proposition 2.3] *The following statements are equivalent.*

(1) *${}^{\perp}_{R}U$ has the U-torsionless property.*
(2) *${}^{\perp}_{R}U$ has the U-reflexive property.*
(3) *${}^{\perp}_{R}U = \mathcal{G}_U$.*

For any $n \geqslant 0$, we denote $\mathcal{H}_n({}_RU) = \{M \in \operatorname{mod} R \mid \operatorname{Ext}^i_R(M, {}_RU) = 0$ for any $i \geqslant 0$ with $i \neq n\}$ [W2].

Lemma 3.5. *If ${}^{\perp}_{R}U$ has the U-torsionless property, then $\mathcal{H}_n({}_RU) \subseteq \mathcal{E}_n(U_S)$ for any $n \geqslant 1$.*

Proof. It follows from Proposition 3.4 and [H4, Lemma 3.3]. □

Lemma 3.6. *Assume that ${}^{\perp_n}_{R}U$ has the U-torsionless property, where n is a positive integer. If A is a non-zero module in* $\operatorname{mod} R$ *with* $\operatorname{grade}_U A \geqslant n$, *then* $\operatorname{grade}_U A = n$.

Proof. Let $0 \neq A \in \operatorname{mod} R$ with $\operatorname{grade}_U A \geqslant n$. If $\operatorname{grade}_U A > n$, then $A^* = 0$ and $A \in {}^{\perp_n}_{R}U$. Since ${}^{\perp_n}_{R}U$ has the U-torsionless property, A is U-torsionless and $A \hookrightarrow A^{**} = 0$, which is a contradiction. Thus $\operatorname{grade}_U A = n$. □

For any $n \geqslant 0$, we denote $\overline{\mathcal{H}_n({}_RU)} = \{M \in \operatorname{mod} R \mid$ any non-zero submodule of M is in $\mathcal{H}_n({}_RU)\}$. It is clear that $\overline{\mathcal{H}_n({}_RU)} \subseteq \mathcal{H}_n({}_RU)$. We are now in a position to give the main result in this section.

Theorem 3.7. *If $l.\operatorname{id}_R(U) \leqslant n$ and ${}^{\perp_n}_{R}U$ has the U-torsionless property, where n is a positive integer, then the following statements are equivalent.*

(1) U-$\mathrm{lim.dim}_R(\bigoplus_{i=0}^{n-1} E_i) \leqslant n-1$.
(2) $\mathcal{E}_n(U_S)$ *is submodule-closed and* $\mathcal{E}_n(U_S) = \underline{\mathcal{H}_n({}_RU)}$.
(3) $\mathcal{E}_n(U_S)$ *is submodule-closed and* $\mathcal{E}_n(U_S) = \overline{\mathcal{H}_n({}_RU)}$.

Proof. Since ${}^{\perp_n}_R U$ has the U-torsionless property, $\overline{\mathcal{H}_n({}_RU)} \subseteq \mathcal{H}_n({}_RU) \subseteq \mathcal{E}_n(U_S)$ by Lemma 3.5. So the implication of (3) $\Rightarrow$ (2) is trivial.

(1) $\Rightarrow$ (3). Assume that U-$\mathrm{lim.dim}_R(\bigoplus_{i=0}^{n-1} E_i) \leqslant n-1$ and M is any non-zero module in $\mathcal{E}_n(U_S)$. Then $\mathrm{s.grade}_U M \geqslant n$ by Lemma 3.2.

Let A be any non-zero submodule of M in $\mathrm{mod}\, R$. Then $\mathrm{grade}_U A \geqslant n$. By Lemma 3.6, $\mathrm{grade}_U A = n$. In addition, $l.\mathrm{id}_R(U) \leqslant n$, so $A \in \mathcal{H}_n({}_RU)$ and $M \in \overline{\mathcal{H}_n({}_RU)}$. It follows that $\mathcal{E}_n(U_S) \subseteq \overline{\mathcal{H}_n({}_RU)}$ and $\mathcal{E}_n(U_S) = \overline{\mathcal{H}_n({}_RU)}$.

Notice that $\overline{\mathcal{H}_n({}_RU)}$ is clearly submodule-closed, so $\mathcal{E}_n(U_S)$ is also submodule-closed.

(2) $\Rightarrow$ (1). We first prove U-$\mathrm{lim.dim}_R(E_0) \leqslant n-1$. If U-$\mathrm{lim.dim}_R(E_0) > n-1$, then $l.\mathrm{fd}_S({}^*E_0) > n-1$ by Lemma 2.4. So by Lemma 3.1, there exists a module $N \in \mathrm{mod}\, S^{op}$ such that $\mathrm{Hom}_R(\mathrm{Ext}_S^n(N,U), E_0) \neq 0$. Hence there exists a non-zero homomorphism $f : \mathrm{Ext}_S^n(N,U) \to E_0$. Since ${}_RU$ is essential in E_0, $f^{-1}({}_RU)$ is a non-zero submodule of $\mathrm{Ext}_S^n(N,U)$. By assumption, $\mathcal{H}_n({}_RU) = \mathcal{E}_n(U_S)$ and $\mathcal{E}_n(U_S)$ is submodule-closed. So $f^{-1}({}_RU) \in \mathcal{E}_n(U_S)$ $(= \mathcal{H}_n({}_RU))$ and hence $[f^{-1}({}_RU)]^* = 0$, which is a contradiction. Consequently, U-$\mathrm{lim.dim}_R(E_0) \leqslant n-1$.

We next prove U-$\mathrm{lim.dim}_R(E_1) \leqslant n-1$ (note: at this moment, $n \geqslant 2$). If U-$\mathrm{lim.dim}_R(E_1) > n-1$, then $l.\mathrm{fd}_S({}^*E_1) > n-1$ by Lemma 2.4. So by Lemma 3.1, there exists a module $N_1 \in \mathrm{mod}\, S^{op}$ such that $\mathrm{Hom}_R(\mathrm{Ext}_S^n(N_1,U), E_1) \neq 0$. Hence there exists a non-zero homomorphism $f_1 : \mathrm{Ext}_S^n(N_1,U) \to E_1$. Since $\mathrm{Ker}\,\alpha_1$ is essential in E_1, $f_1^{-1}(\mathrm{Ker}\,\alpha_1)$ is a non-zero submodule of $\mathrm{Ext}_S^n(N_1,U)$. By assumption, $f_1^{-1}(\mathrm{Ker}\,\alpha_1) \in \mathcal{E}_n(U_S)$. Since $l.\mathrm{fd}_S({}^*E_0) = U$-$\mathrm{lim.dim}_R(E_0) \leqslant n-1$, $\mathrm{Hom}_R(f_1^{-1}(\mathrm{Ker}\,\alpha_1), E_0) = 0$ by Lemma 3.1.

From the exact sequence $0 \to {}_RU \to E_0 \to \mathrm{Ker}\,\alpha_1 \to 0$ we get the following exact sequence:

$$\mathrm{Hom}_R\big(f_1^{-1}(\mathrm{Ker}\,\alpha_1), E_0\big) \to \mathrm{Hom}_R\big(f_1^{-1}(\mathrm{Ker}\,\alpha_1), \mathrm{Ker}\,\alpha_1\big) \to \mathrm{Ext}_R^1\big(f_1^{-1}(\mathrm{Ker}\,\alpha_1), {}_RU\big).$$

Since $f_1^{-1}(\mathrm{Ker}\,\alpha_1) \in \mathcal{E}_n(U_S)$ $(= \mathcal{H}_n({}_RU))$, $\mathrm{Ext}_R^1(f_1^{-1}(\mathrm{Ker}\,\alpha_1), {}_RU) = 0$. So $\mathrm{Hom}_R(f_1^{-1}(\mathrm{Ker}\,\alpha_1), \mathrm{Ker}\,\alpha_1) = 0$, which is a contradiction. Hence we conclude that U-$\mathrm{lim.dim}_R(E_1) \leqslant n-1$.

Continuing this process, we get that U-$\mathrm{lim.dim}_R(E_i) \leqslant n-1$ for any $0 \leqslant i \leqslant n-1$. □

If $r.\mathrm{id}_S(U) \leqslant n$, then ${}^{\perp_n}_R U$ has the U-reflexive property by [HT, Theorem 2.2]. So by Theorem 3.7, we have the following

Corollary 3.8. *If* $l.\mathrm{id}_R(U) = r.\mathrm{id}_S(U) \leqslant n$, *then the following statements are equivalent.*

(1) U-$\mathrm{lim.dim}_R(\bigoplus_{i=0}^{n-1} E_i) \leqslant n-1$.
(2) $\mathcal{E}_n(U_S)$ *is submodule-closed and* $\mathcal{E}_n(U_S) = \underline{\mathcal{H}_n({}_RU)}$.
(3) $\mathcal{E}_n(U_S)$ *is submodule-closed and* $\mathcal{E}_n(U_S) = \overline{\mathcal{H}_n({}_RU)}$.

Let R be a two-sided noetherian ring. Recall that R is called an *Iwanaga–Gorenstein ring* if the injective dimensions of ${}_RR$ and R_R are finite. Also recall that R is said to satisfy the *Auslander condition* if the flat dimension of the $(i+1)$st term in a minimal injective resolution of

${}_RR$ is at most i for any $i \geqslant 0$, and R is called *Auslander–Gorenstein* if it is Iwanaga–Gorenstein and satisfies the Auslander condition (see [Bj]). It is well known that any commutative Iwanaga–Gorenstein ring is Auslander–Gorenstein.

The following corollary gives a positive answer to the Auslander's question for Auslander–Gorenstein rings and so in particular for commutative Iwanaga–Gorenstein rings.

Corollary 3.9. *If R is an Auslander–Gorenstein ring with self-injective dimension n, then $\mathcal{E}_n(R_R)$ is submodule-closed.*

Proof. Notice that $R\text{-}\lim.\dim_R(A) = l.\mathrm{fd}_R(A)$ for any $A \in \operatorname{Mod} R$, so our assertion follows from Corollary 3.8. □

Assume that $X \in \operatorname{mod} S^{op}$ and there exists an exact sequence $H_1 \xrightarrow{g} H_0 \to X \to 0$ in $\operatorname{mod} S^{op}$. We denote $A = \operatorname{Coker} g^*$. The following result is a generalization of [HT, Lemma 2.1]. The proof here is similar to that in [HT], we omit it.

Lemma 3.10. *Let X, A, H_0 and H_1 be as above. Assume that H_0 and H_1 are U-reflexive.*

(1) *If $H_i^* \in {}_R^{\perp_{i+1}}U$ for $i = 0, 1$, then we have the following exact sequence*:

$$0 \to \operatorname{Ext}^1_R(A, U) \to X \xrightarrow{\sigma_X} X^{**} \to \operatorname{Ext}^2_R(A, U) \to 0.$$

(2) *If $H_i \in {}^{\perp_{2-i}}U_S$ for $i = 0, 1$, then we have the following exact sequence*:

$$0 \to \operatorname{Ext}^1_S(X, U) \to A \xrightarrow{\sigma_A} A^{**} \to \operatorname{Ext}^2_S(X, U) \to 0.$$

Lemma 3.11. *Let $\mathcal{X}$ be a full subcategory of ${}^{\perp}U_S$ which has the U-reflexive property and X a module in* $\operatorname{mod} S^{op}$. *If $\mathcal{X}\text{-}\operatorname{resol.dim}_S(X) = n$ $(\geqslant 1)$, then* $\operatorname{grade}_U \operatorname{Ext}^n_S(X, U) \geqslant 1$; *if furthermore $n \geqslant 2$ and $Y^* \in {}_R^{\perp_2}U$ for any $Y \in \mathcal{X}$, then* $\operatorname{grade}_U \operatorname{Ext}^n_S(X, U) \geqslant 2$.

Proof. Assume that $\mathcal{X}\text{-}\operatorname{resol.dim}_S(X) = n$ $(\geqslant 1)$. Then there exists an exact sequence:

$$0 \to X_n \xrightarrow{d_n} \cdots \to X_1 \to X_0 \to X \to 0$$

in $\operatorname{mod} S^{op}$ with $X_i \in \mathcal{X}$ $(\subseteq {}^{\perp}U_S)$ for any $0 \leqslant i \leqslant n$. Set $N = \operatorname{Coker} d_n$. Then $\operatorname{Ext}^1_S(N, U) \cong \operatorname{Ext}^n_S(X, U)$.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & X_n & \xrightarrow{d_n} & X_{n-1} & \longrightarrow & N & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \sigma_{X_n}} & & \downarrow{\scriptstyle \sigma_{X_{n-1}}} & & & & \\ 0 & \longrightarrow & [\operatorname{Ext}^n_S(X, U)]^* & \longrightarrow & X_n^{**} & \xrightarrow{d_n^{**}} & X_{n-1}^{**}. & & & & \end{array}$$

Because $\mathcal{X}$ has the U-reflexive property, both σ_{X_n} and $\sigma_{X_{n-1}}$ are isomorphisms. So we have that $[\operatorname{Ext}^n_S(X, U)]^* = 0$ and $\operatorname{grade}_U \operatorname{Ext}^n_S(X, U) \geqslant 1$.

If $n \geqslant 2$ and $Y^* \in {}_R^{\perp_2}U$ for any $Y \in \mathcal{X}$, by applying Lemma 3.10(1) to the exact sequence $0 \to X_n \xrightarrow{d_n} X_{n-1} \to N \to 0$, we then get the following exact sequence:

$$0 \to \mathrm{Ext}_R^1\big(\mathrm{Ext}_S^n(X, U), U\big) \to N \xrightarrow{\sigma_N} N^{**} \to \mathrm{Ext}_R^2\big(\mathrm{Ext}_S^n(X, U), U\big) \to 0.$$

Because $X_{n-2} \in \mathcal{X}$ and $\mathcal{X}$ has the U-reflexive property, X_{n-2} is U-reflexive. Then N is U-torsionless for it is isomorphic to a submodule of X_{n-2}. So σ_N is monic and $\mathrm{Ext}_R^1(\mathrm{Ext}_S^n(X, U), U) = 0$. Hence we conclude that $\mathrm{grade}_U \mathrm{Ext}_S^n(X, U) \geqslant 2$. □

For a non-negative integer t, a module N in $\mathrm{mod}\, S^{op}$ is said to have *generalized Gorenstein dimension* at most t (with respect to ${}_RU_S$), denoted by $\text{G-dim}_U(N) \leqslant t$, if there exists an exact sequence $0 \to N_t \to \cdots \to N_1 \to N_0 \to N \to 0$ in $\mathrm{mod}\, S^{op}$ with $\text{G-dim}_U(N_i) = 0$ for any $0 \leqslant i \leqslant t$ (see [AR2]).

Lemma 3.12. *Let $n \leqslant 2$. If $l.\mathrm{id}_R(U) = r.\mathrm{id}_S(U) \leqslant n$, then $\mathcal{E}_n(U_S) = \mathcal{H}_n({}_RU)$.*

Proof. Because $l.\mathrm{id}_R(U) = r.\mathrm{id}_S(U) \leqslant n$, both ${}_R^{\perp_n}U$ and ${}^{\perp_n}U_S$ have the U-reflexive property by [HT, Theorem 2.2]. It follows from Lemma 3.5 that $\mathcal{H}_n({}_RU) \subseteq \mathcal{E}_n(U_S)$.

Assume that $n = 1$ and $0 \neq M \in \mathcal{E}_1(U_S)$. Let E_0' be the injective envelope of U_S. Because $r.\mathrm{fd}_R({}^*E_0') \leqslant l.\mathrm{id}_R(U) \leqslant 1$ by assumption and Lemma 2.6, it follows from the symmetric statements of [H2, Theorem 17.5.5] that $\mathrm{grade}_U M \geqslant 1$. By Lemma 3.6, $\mathrm{grade}_U M = 1$. Thus $M \in \mathcal{H}_1({}_RU)$ and $\mathcal{E}_1(U_S) \subseteq \mathcal{H}_1({}_RU)$. The case $n = 1$ follows.

Assume that $n = 2$ and $0 \neq M \in \mathcal{E}_2(U_S)$. Then there exists a module $N \in \mathrm{mod}\, S^{op}$ such that $M = \mathrm{Ext}_S^2(N, U)$ $(\neq 0)$. Because $l.\mathrm{id}_R(U) = r.\mathrm{id}_S(U) \leqslant 2$, by [HT, Theorem 3.5] we have that $\text{G-dim}_U(N) \leqslant 2$. If $\text{G-dim}_U(N) < 2$, then there exists an exact sequence $0 \to N_1 \to N_0 \to N \to 0$ in $\mathrm{mod}\, S^{op}$ with $\text{G-dim}_U(N_1) = \text{G-dim}_U(N_0) = 0$. So $\mathrm{Ext}_S^2(N, U) \cong \mathrm{Ext}_S^1(N_1, U) = 0$, which is a contradiction. Hence we conclude that $\text{G-dim}_U(N) = 2$. Then by Lemma 3.11, $\mathrm{grade}_U M = \mathrm{grade}_U \mathrm{Ext}_S^2(N, U) \geqslant 2$. By Lemma 3.6, $\mathrm{grade}_U M = 2$. Thus $M \in \mathcal{H}_2({}_RU)$ and $\mathcal{E}_2(U_S) \subseteq \mathcal{H}_2({}_RU)$. The case $n = 2$ follows. □

Proposition 3.13. *Let $n \leqslant 2$. Assume that $l.\mathrm{id}_R(U) = r.\mathrm{id}_S(U) \leqslant n$.*

(1) *If $n = 1$, then $\mathcal{E}_1(U_S)$ is submodule-closed if and only if $E_0 \in \text{add-lim}\, {}_RU$ (that is, $U\text{-}\mathrm{lim.dim}_R(E_0) = 0$) if and only if $U\text{-}\mathrm{lim.dim}_R(E_i) \leqslant i$ for $i = 0, 1$.*
(2) *If $n = 2$, then $\mathcal{E}_2(U_S)$ is submodule-closed if and only if $U\text{-}\mathrm{lim.dim}_R(E_0 \oplus E_1) \leqslant 1$.*

Proof. The former equivalence in (1) and the equivalence in (2) follow from Lemma 3.12 and Theorem 3.7. Notice that $U\text{-}\mathrm{lim.dim}_R(E_1) = l.\mathrm{fd}_S({}^*E_1) \leqslant r.\mathrm{id}_S(U)$ by Lemmas 2.4 and 2.6, then the latter equivalence in (1) follows. □

Let

$$0 \to {}_RR \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$$

be a minimal injective resolution of ${}_RR$. Putting ${}_RU_S = {}_RR_R$, by Proposition 3.13 we immediately have the following

Corollary 3.14. *Let $n \leqslant 2$. Assume that $l.\mathrm{id}_R(R) = r.\mathrm{id}_R(R) \leqslant n$.*

(1) *If $n = 1$, then $\mathcal{E}_1(R_R)$ is submodule-closed if and only if I_0 is flat if and only if R is Auslander–Gorenstein.*
(2) *If $n = 2$, then $\mathcal{E}_2(R_R)$ is submodule-closed if and only if $l.\mathrm{fd}_R(I_0 \oplus I_1) \leqslant 1$.*

In the following, we give some examples to illustrate that neither $\mathcal{E}_1(R_R)$ nor $\mathcal{E}_2(R_R)$ are submodule-closed in general.

Example 3.15. Let K be a field and R a finite dimensional K-algebra which is given by the quiver:

$$2 \longleftarrow 1 \longrightarrow 3.$$

Then R is Iwanaga–Gorenstein with $l.\mathrm{id}_R(R) = r.\mathrm{id}_R(R) = 1$ and $l.\mathrm{fd}_R(I_0) = 1$. By Corollary 3.14, $\mathcal{E}_1(R_R)$ is not submodule-closed.

Example 3.16. Let K be a field and Δ the quiver:

$$\begin{array}{ccc} 1 & \xrightarrow{\gamma} & 3 \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\delta} \\ 2 & \xrightarrow{\beta} & 4. \end{array}$$

If $R = K\Delta/(\beta\alpha)$, then R is Iwanaga–Gorenstein with $l.\mathrm{id}_R(R) = r.\mathrm{id}_R(R) = 2$ and $l.\mathrm{fd}_R(E(P_4)) = 2$, where $E(P_4)$ is the injective envelope of the indecomposable projective module corresponding to the vertex 4. Since P_4 is a direct summand of ${}_RR$, $l.\mathrm{fd}_R(I_0) = 2$. By Corollary 3.14, $\mathcal{E}_2(R_R)$ is not submodule-closed.

It is clear that $\operatorname{mod} R \supseteq \mathcal{E}_1(R_R) \supseteq \mathcal{E}_2(R_R) \supseteq \cdots \supseteq \mathcal{E}_i(R_R) \supseteq \cdots$. From the above argument we know that $\mathcal{E}_n(R_R)$ is submodule-closed for an Auslander–Gorenstein ring R with self-injective dimension n for any $n \geqslant 1$, and neither $\mathcal{E}_1(R_R)$ nor $\mathcal{E}_2(R_R)$ are submodule-closed in general. However, we do not know whether $\mathcal{E}_n(R_R)$ (where $n \geqslant 3$) is submodule-closed or not in general.

4. Wakamatsu tilting conjecture and (quasi-)Gorenstein modules

Let R be an Artin algebra. Recall that a module ${}_RT$ in $\operatorname{mod} R$ is called a *tilting module* of finite projective dimension if the following conditions are satisfied: (1) $l.\mathrm{pd}_R(T) < \infty$; (2) ${}_RT$ is selforthogonal; and (3) there exists an exact sequence $0 \to R \to T_0 \to T_1 \to \cdots \to T_t \to 0$ in $\operatorname{mod} R$ with $T_i \in \operatorname{add} {}_RT$ for any $0 \leqslant i \leqslant t$. The notion of *cotilting modules* of finite injective dimension may be defined dually. A generalized tilting module is not necessarily tilting or cotilting. The following conjecture is called Wakamatsu Tilting Conjecture (**WTC**): Every generalized tilting module with finite projective dimension is tilting, or equivalently, every generalized tilting module with finite injective dimension is cotilting (see [BR]). For Artin algebras R and S and a generalized tilting module ${}_RU$ with $S = \mathrm{End}({}_RU)$, by Theorem 2.7 and the dual results of [M, Theorem 1.5 and Proposition 1.6], we easily get the following equivalent statements:

(1) **WTC** holds.
(2) If one of $l.\mathrm{id}_R(U)$ and $r.\mathrm{id}_S(U)$ is finite, then the other is also finite.
(3) $l.\mathrm{id}_R(U) = r.\mathrm{id}_S(U)$.

The Gorenstein Symmetry Conjecture (**GSC**) states that the left and right self-injective dimensions of R are identical for an Artin algebra R (see [BR]). It is trivial from the above equivalent conditions that **WTC** $\Rightarrow$ **GSC**. As an application of the results obtained in Section 2, we now give some sufficient conditions for the validity of statement (2). In other words, we establish some cases in which **WTC** holds true.

Theorem 4.1. [H1, Theorem] *Let R and S be two-sided artinian rings and n and m positive integers. If $r.\mathrm{id}_S(U) \leqslant n$ and $\mathrm{grade}_U \mathrm{Ext}_R^m(M, U) \geqslant n-1$ for any $M \in \mathrm{mod}\, R$, then $l.\mathrm{id}_R(U) \leqslant n+m-1$.*

The following corollary is an immediate consequence of Theorems 4.1 and 2.7.

Corollary 4.2. *Let R and S be two-sided artinian rings. Then $r.\mathrm{id}_S(U) \leqslant 1$ if and only if $l.\mathrm{id}_R(U) \leqslant 1$.*

Let

$$0 \to {}_RU \to E_0 \to E_1 \to \cdots \to E_i \to \cdots$$

and

$$0 \to U_S \to E'_0 \to E'_1 \to \cdots \to E'_i \to \cdots$$

be minimal injective resolutions of ${}_RU$ and U_S, respectively. The following Propositions 4.3 and 4.6 generalize some results in [H1] and [AR4].

Proposition 4.3. *Let R and S be two-sided artinian rings and n a positive integer. If $r.\mathrm{id}_S(U) = n$ and $U\text{-}\mathrm{lim.dim}_S(\bigoplus_{i=0}^{n-2} E'_i) < \infty$, then $l.\mathrm{id}_R(U) = n$.*

Proof. Assume that $U\text{-}\mathrm{lim.dim}_S(\bigoplus_{i=0}^{n-2} E'_i) = r$ $(< \infty)$. It follows from [H2, Lemma 17.3.2] that $\mathrm{s.grade}_U \mathrm{Ext}_R^{r+1}(M, U) \geqslant n-1$ for any $M \in \mathrm{mod}\, R$. By Theorem 4.1, $l.\mathrm{id}_R(U) \leqslant r+n (< \infty)$. Thus $l.\mathrm{id}_R(U) = r.\mathrm{id}_S(U) = n$ by Theorem 2.7. □

The following two results are cited from [H2].

Theorem 4.4. [H2, Theorem 17.1.11] *Let R and S be two-sided noetherian rings. Then, for a positive integer n, the following statements are equivalent.*

(1) $\mathrm{s.grade}_U \mathrm{Ext}_R^i(M, U) \geqslant i$ *for any $M \in \mathrm{mod}\, R$ and $1 \leqslant i \leqslant n$.*
$(1)^{op}$ $\mathrm{s.grade}_U \mathrm{Ext}_S^i(N, U) \geqslant i$ *for any $N \in \mathrm{mod}\, S^{op}$ and $1 \leqslant i \leqslant n$.*

${}_RU$ (symmetrically U_S) is called an n-Gorenstein module if one of the above equivalent conditions is satisfied, and ${}_RU$ (symmetrically U_S) is called a Gorenstein module if it is n-Gorenstein for all n.

It follows from [H2, Corollary 17.1.12] that a two-sided noetherian ring R satisfies the Auslander condition if and only if $_RR$ is a Gorenstein module.

Theorem 4.5. [H2, Theorem 17.5.4] *Let R and S be two-sided noetherian rings. Then, for a positive integer n, the following statements are equivalent.*

(1) $\text{s.grade}_U \operatorname{Ext}_S^{i+1}(N, U) \geqslant i$ *for any* $N \in \operatorname{mod} S^{op}$ *and* $1 \leqslant i \leqslant n$.
(2) $\text{grade}_U \operatorname{Ext}_R^{i}(M, U) \geqslant i$ *for any* $M \in \operatorname{mod} R$ *and* $1 \leqslant i \leqslant n$.

$_RU$ is called a quasi-n-Gorenstein module if one of the above equivalent conditions is satisfied, and $_RU$ is called a quasi-Gorenstein module if it is quasi-n-Gorenstein for all n.

An (n-)Gorenstein module is clearly quasi-(n-)Gorenstein. But the conserve does not hold in general because the notion of (n)-Gorenstein modules is left–right symmetric by Theorem 4.4, and that of quasi-(n)-modules is not left–right symmetric even in the case $_RU_S = {_RR_R}$ (see [H2, Example 17.5.2]).

Proposition 4.6. *Let R and S be two-sided artinian rings. Then $l.\text{id}_R(U) = r.\text{id}_S(U)$ provided that $_RU$ (or U_S) is quasi-Gorenstein.*

Proof. Let $_RU$ be a quasi-Gorenstein module. By Theorem 4.5, for any $i \geqslant 1$, we have that $\text{grade}_U \operatorname{Ext}_R^{i}(M, U) \geqslant i$ for any $M \in \operatorname{mod} R$ and $\text{s.grade}_U \operatorname{Ext}_S^{i+1}(N, U) \geqslant i$ for any $N \in \operatorname{mod} S^{op}$. Then it is easy to see from Theorem 4.1 that $l.\text{id}_R(U) < \infty$ if and only if $r.\text{id}_S(U) < \infty$. Thus $l.\text{id}_R(U) = r.\text{id}_S(U)$ by Theorem 2.7. □

Note that Proposition 4.6 generalizes [AR4, Corollary 5.5(b)] which asserts that $l.\text{id}_R(R) = r.\text{id}_R(R)$ if R is an Artin algebra satisfying the Auslander condition.

Conjecture 4.7. *Let R and S be Artin algebras and $_RU$ a generalized tilting module with $S = \operatorname{End}(_RU)$. If $_RU$ is (quasi-)Gorenstein, then $l.\text{id}_R(U) = r.\text{id}_S(U) < \infty$ (in fact, under our assumption it has been proved in Proposition 4.6 that $l.\text{id}_R(U) = r.\text{id}_S(U)$).*

Auslander and Reiten in [AR4] raised the following conjecture, which we call Auslander Gorenstein Conjecture (**AGC**): An Artin algebra is Iwanaga–Gorenstein if it satisfies the Auslander condition (in other words, an Artin algebra R satisfies $l.\text{id}_R(R) = r.\text{id}_R(R) < \infty$ provided $_RR$ is a Gorenstein module). It is trivial that this conjecture is situated between **Conjecture 4.7** and the famous Nakayama Conjecture (**NC**), which states that an Artin algebra R is self-injective if each term in a minimal injective resolution of $_RR$ is projective. That is, we have the following implications: **Conjecture 4.7** ⇒ **AGC** ⇒ **NC**.

Recall moreover the Generalized Nakayama Conjecture (**GNC**): Every indecomposable injective R-module occurs as the direct summand of some term in a minimal injective resolution of $_RR$ for an Artin algebra R. An equivalent version of **GNC** is: For an Artin algebra R and every simple module $T \in \operatorname{mod} R$, there exists a non-negative integer k such that $\operatorname{Ext}_R^k(T, R) \neq 0$ (see [AR1]). It is well known that **GNC** implies **AGC**. We now show the corresponding result for **Conjecture 4.7**.

Proposition 4.8. *Let R and S be Artin algebras and ${}_RU$ a generalized tilting module with $S = \mathrm{End}({}_RU)$. If the following condition is satisfied: for every simple module $T \in \operatorname{mod} R$, there exists a negative-integer k such that $\mathrm{Ext}_R^k(T, {}_RU) \neq 0$, then* **Conjecture 4.7** *holds* for R.

Proof. Let $\{T_1, \ldots, T_t\}$ be the set of all non-isomorphic simple modules in $\operatorname{mod} R$. By assumption, for each T_i $(1 \leqslant i \leqslant t)$, there exists a non-negative integer k_i such that $\mathrm{Ext}_R^{k_i}(T_i, {}_RU) \neq 0$. It is easy to verify that $\mathrm{Hom}_R(T, E_j) \cong \mathrm{Ext}_R^j(T, {}_RU)$ for any simple R-module T and $j \geqslant 0$. So $\mathrm{Hom}_R(T_i, E_{k_i}) \neq 0$ for any $1 \leqslant i \leqslant t$ and hence $E(T_i)$ (the injective envelope of T_i) is isomorphic to a direct summand of E_{k_i} for any $1 \leqslant i \leqslant t$.

Now suppose ${}_RU$ is quasi-Gorenstein. Then by Theorem 4.5 and Lemma 3.2, we have that $l.\mathrm{fd}_S({}^*E_i) \leqslant i + 1$ for any $i \geqslant 0$. So $l.\mathrm{fd}_S({}^*[E(T_i)]) \leqslant l.\mathrm{fd}_S({}^*E_{k_i}) \leqslant k_i + 1$ for any $1 \leqslant i \leqslant t$. Put $E = \bigoplus_{i=1}^t E(T_i)$ and $k = \max\{k_1, \ldots, k_t\}$. Then E is an injective cogenerator for $\operatorname{Mod} R$ and $l.\mathrm{fd}_S({}^*E) \leqslant k + 1$. It follows from Lemma 2.6(1) that $r.\mathrm{id}_S(U) \leqslant k + 1$. We are done. □

Let N be a module in $\operatorname{mod} S^{op}$. Recall that an injective resolution:

$$0 \to N \xrightarrow{\delta_0} V_0 \xrightarrow{\delta_1} V_1 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_i} V_i \xrightarrow{\delta_{i+1}} \cdots$$

is called *ultimately closed* if there exists a positive integer n such that $\mathrm{Im}\,\delta_n = \bigoplus_{j=0}^m W_j$, where each W_j is a direct summand of $\mathrm{Im}\,\delta_{i_j}$ with $i_j < n$. By [HT, Theorem 2.4], if U_S has a ultimately closed injective resolution (especially, if $r.\mathrm{id}_S(U) < \infty$), then ${}^{\perp}_{R}U$ has the U-reflexive property and the condition in Proposition 4.8 is satisfied.

Acknowledgments

Part of the paper was written while the author was staying at Universität Bielefeld supported by the SFB 701 “Spectral Structures and Topological Methods in Mathematics.” The author is grateful to Prof. Claus M. Ringel for his warm hospitality and useful comments on this paper. The research of the author was partially supported by Specialized Research Fund for the Doctoral Program of Higher Education (Grant Nos. 20030284033, 20060284002) and NSF of Jiangsu Province of China (Grant No. BK2005207). The author is grateful to the referee for the careful reading and the valuable and detailed suggestions in shaping this paper into its present version.

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