



## Generalized umbrella rings

Gaohua Tang<sup>a</sup>, Zhaoyong Huang<sup>b,\*</sup>, Fanggui Wang<sup>c</sup>

<sup>a</sup> Department of Mathematics, Guangxi Teacher's College, Nanning 530001, PR China

<sup>b</sup> Department of Mathematics, Nanjing University, Nanjing 210093, PR China

<sup>c</sup> College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610066, PR China

Received 16 September 2006

Available online 9 February 2007

Communicated by Steven Dale Cutkosky

---

### Abstract

In this paper, the notion of generalized umbrella rings is introduced. We show that every generalized umbrella ring is a stably coherent ring. As an application, we show that every finitely generated projective module over  $R[x]$  is free if  $R$  is a generalized umbrella ring with weak global dimension two.

© 2007 Elsevier Inc. All rights reserved.

*Keywords:* Generalized umbrella rings; Stable coherence; Weak Krull dimension; (Weak) global dimension

---

### 1. Introduction

Throughout this paper it is assumed that all rings are commutative and associative with identity and all modules are unitary. The Hilbert Basis Theorem states that a polynomial ring in finitely many variables over a Noetherian ring is Noetherian. It is natural that among the first questions to be asked when studying coherence was whether this result is still valid when the Noetherian hypothesis is replaced by the coherent hypothesis. The answer is negative in general. Recall from [3] that a ring  $R$  is called a *stably coherent ring* if any polynomial ring in finitely many variables over  $R$  is a coherent ring. It is known that a coherent ring with global dimension  $\leq 2$  or weak global dimension  $\leq 1$  is stably coherent; however, a coherent ring with weak global dimension  $\leq 2$  is not necessarily stably coherent (see [3,8]).

---

\* Corresponding author.

*E-mail addresses:* [tanggh@gxtc.edu.cn](mailto:tanggh@gxtc.edu.cn) (G. Tang), [huangzy@nju.edu.cn](mailto:huangzy@nju.edu.cn) (Z. Huang).

In this paper, we will introduce the notion of generalized umbrella rings and provide examples to illustrate that the global dimension and the weak global dimension of a generalized umbrella ring can be any non-negative integer. One of the aims in this paper is to show that every generalized umbrella ring is stably coherent. As an application, we show that every finitely generated projective module over  $R[x]$  is free if  $R$  is a generalized umbrella ring with weak global dimension two, which improves the works on Bass–Quillen Conjecture over non-Noetherian rings by Lequain and Simis in [8] and Wang and Tang in [23].

In [16], Serre conjectured that every finitely generated projective module over the polynomial ring  $K[x_1, \dots, x_n]$  is free, where  $K$  is a field. Quillen in [13] and Suslin in [17] proved independently that this conjecture holds not only for fields but also for principal ideal domains. Let  $R$  be a regular Noetherian ring. So a central question concerning projective modules over polynomial  $R$ -algebras is the following famous conjecture:

**Bass–Quillen Conjecture.** (See [1,13].) *For any regular Noetherian ring  $R$ , every finitely generated projective module  $P$  over the polynomial ring  $R[x_1, \dots, x_n]$  is extended from  $R$ , i.e.  $P \cong R[x_1, \dots, x_n] \otimes_R P_0$ , where  $P_0$  is a finitely generated projective  $R$ -module.*

By Quillen’s Patching Theorem and Reduction Theorem, Bass–Quillen Conjecture is equivalent to the following: Every finitely generated projective  $R[x]$ -module is free for any regular Noetherian local ring  $R$ . From [9] and [17] we know that Bass–Quillen Conjecture holds for regular Noetherian local rings with Krull dimension  $\leq 2$ . Rao in [14] proved that this conjecture holds for regular Noetherian local rings with Krull dimension 3 but characteristic  $\neq 2, 3$ .

On the other hand, many authors have tried to investigate Bass–Quillen Conjecture over non-Noetherian rings [2,8,10,15]. Lequain and Simis in [8] proved that every finitely generated projective module over  $R[x_1, \dots, x_n]$  is extended from  $R$ , when  $R$  is a Prüfer domain. So, if  $R$  is a valuation ring or a Bézout domain, then every finitely generated projective module over  $R[x_1, \dots, x_n]$  is free. Wang and Tang in [23] proved that if  $R$  is a local ring with global dimension two then every finitely generated projective module over  $R[x]$  is free.

In Section 2, we define the notion of generalized umbrella rings and construct some examples to illustrate that the class of generalized umbrella rings is abundant enough. In Section 3, we show that generalized umbrella rings are stably coherent. As an application of this result, we show in Section 4 that if  $R$  is a generalized umbrella ring with weak global dimension two then every finitely generated projective module over the polynomial ring  $R[x]$  is free.

## 2. Definition of generalized umbrella rings and examples

First, we recall the definition of umbrella rings which was introduced by Vasconcelos in [22].

**Definition 2.1.** A local ring  $R$  is called an *umbrella ring* if  $R$  is a domain and contains a prime ideal  $P$  such that

- (a)  $P = PA_P$ , that is,  $P$  is divisible by any outside element;
- (b)  $A/P$  is a two-dimensional regular Noetherian local ring;
- (c)  $A_P$  is a valuation domain with countably generated ideals;
- (d)  $A$  has only countably many principal prime ideals.

In [21], Vasconcelos proved that if  $R$  is a local ring with global dimension two then  $R$  is one of the following cases: (1) a two-dimensional regular Noetherian local ring, (2) a valuation ring, (3) an umbrella ring. By Definition 2.1 and the proofs in [22, Corollary 4.6 and Theorem 4.10], it is not difficult to see that an umbrella ring is exactly a non-Noetherian, non-valuation local ring with global dimension two. Cohen's Lemma states that a ring  $R$  is Noetherian if and only if every prime ideal of  $R$  is finitely generated. So if  $R$  is not a Noetherian ring then there exists at least one infinitely generated prime ideal in  $R$ .

We now introduce the notion of generalized umbrella rings as follows.

**Definition 2.2.** Let  $R$  be a regular coherent local ring with finitely generated maximal ideal  $m$ .  $R$  is called a *generalized umbrella ring* if all non-finitely generated prime ideals of  $R$  are contained in every principal prime ideal of  $R$ .

In the following, we construct several examples to illustrate that the notion of generalized umbrella rings is a proper generalization of that of umbrella rings and that generalized umbrella rings are abundant enough.

**Example 2.3.** From Definition 2.2, it is clear that any regular Noetherian local ring is a generalized umbrella ring.

**Example 2.4.** Let  $R$  be a valuation ring with unique maximal ideal  $m$ . If  $m$  is finitely generated, then  $m$  is a principal ideal. So that  $m$  is the unique non-zero finitely generated prime ideal in  $R$ . Thus every valuation ring with finitely generated maximal ideal is a generalized umbrella ring, which is exactly a generalized umbrella ring with weak global dimension one. For example:

(1) Clearly any discrete valuation ring is a generalized umbrella ring.

(2) Let  $\mathbb{Z}$  denote the ring of integral numbers and  $\mathbb{Q}$  the rational field. Let  $p$  be a prime number. Set  $R = \mathbb{Z} + \mathbb{Q}[x]x$ . By [18, Theorem 1], it is not difficult to verify that  $R_{(p)}$  is a non-Noetherian valuation ring with finitely generated maximal ideal  $(p)R_{(p)}$ . Thus  $R$  is a generalized umbrella ring.

**Example 2.5.** It is obvious that every umbrella ring is a generalized umbrella ring with global dimension two and of weak global dimension two. So a local ring with finitely generated maximal ideal and global dimension  $\leq 2$  is a generalized umbrella ring.

Recall that the set of all prime ideals of  $R$  is called the *spectrum* of  $R$ , and written  $\text{Spec } R$ , the set of maximal ideals of  $R$  is called the *maximal spectrum* of  $R$ , and written  $\text{Max } R$ ; the supremum of the length  $r$ , taken over all strictly decreasing chains  $P_0 \supset P_1 \supset \cdots \supset P_r$  of prime ideals of  $R$ , is called the *Krull dimension* of  $R$ , and denoted by  $\text{k.dim } R$ .

The set of all finitely generated prime ideals of  $R$  is called the *finitely generated prime spectrum* of  $R$ , and written  $\text{f.g.Spec } R$ ; the supremum of the length  $r$ , taken over all strictly decreasing chains  $P_0 \supset P_1 \supset \cdots \supset P_r$  of finitely generated prime ideals of  $R$ , is called the *weak Krull dimension*, and denoted by  $\text{w.k.dim } R$  [19]. Obviously,  $\text{w.k.dim } R = \text{k.dim } R$  if  $R$  is a Noetherian. But the converse is not true in the general (see the following).

The following two results will provide a lot of examples of generalized umbrella rings.

**Theorem 2.6.** *For any natural numbers  $s \geq t \geq 1$  there exists a generalized umbrella ring  $R$  such that  $\text{w.gl.dim } R = \text{w.k.dim } R = t$  and  $\text{k.dim } R = s$ , where  $\text{w.gl.dim } R$  is the weak global dimension of  $R$ .*

**Proof.** If  $s = t$ , then any regular Noetherian local ring  $R$  with  $\text{k.dim } R = t$  is desired.

Now suppose  $s > t$ . Set  $l = s - t$ . Let  $D$  be a Noetherian regular local ring with unique maximal ideal  $m$  and  $\text{k.dim } R = t$ . (For example, the localization of polynomial ring  $K[x_1, \dots, x_t]$  at the maximal ideal  $(x_1, \dots, x_t)$  is such a ring, where  $K$  is a field). By [6, Theorem 169],  $m$  can be generated by a regular  $D$ -sequence of length  $t$ , that is  $m = (\alpha_1, \dots, \alpha_t)$  and  $\alpha_1, \dots, \alpha_t$  is a regular  $D$ -sequence. Set  $S = D + K[x]x$ , where  $K$  is the quotient field of  $D$ . We also denote  $mS$  by  $m$ . By [19, Lemma 2.3],  $m$  is also a maximal ideal of  $S$  and it can be generated by the regular  $S$ -sequence  $\{\alpha_1, \dots, \alpha_t\}$ . Set  $R_1 = S_m$ . By [19, Lemma 2.3(2)],  $S$  is coherent and therefore  $R_1$  is a coherent ring. The unique maximal ideal  $m_m$  of  $R_1$  is generated by the regular  $R_1$ -sequence  $\{\alpha_1, \dots, \alpha_t\}$ . The image of the infinitely generated prime ideal  $P = \{f \in K[x] \mid f(0) = 0\}$  at the natural map  $S \rightarrow S_m$  is also an infinitely generated prime ideal of  $R_1$ . By [19, Lemma 2.3], it is not difficult to see that  $\text{k.dim } R_1 = t + 1$  and  $\text{w.k.dim } R_1 = t$ .

Set  $D = R_1$ . Similar to the above argument, we can get a coherent local ring  $R_2$  with maximal ideal  $m$  such that  $m = (\alpha_1, \dots, \alpha_t)R_2$  and  $\text{k.dim } R_2 = t + 2$ ,  $\text{w.k.dim } R_2 = t$ .

Continuing the above process, we finally get a coherent local ring  $R_l$  with maximal ideal  $m = (\alpha_1, \dots, \alpha_t)R_l$  is also generated by a regular  $R_l$ -sequence  $\{\alpha_1, \dots, \alpha_t\}$  and  $\text{k.dim } R_l = l + t = s$ ,  $\text{w.k.dim } R_l = t$ .

Set  $R = R_l$ . By [19, Theorem 3.7 and Corollary 3.8],  $R$  is a regular coherent local ring. Then by [19, Theorem 4.2], we have that  $\text{w.gl.dim } R = \text{w.k.dim } R = t$  and  $\text{k.dim } R = s$ . Then it follows from [19, Lemma 2.3(5)] that all non-finitely generated prime ideal of  $R$  are contained in every principal prime ideal of  $R$ . Thus  $R$  is a generalized umbrella ring, as required.  $\square$

**Theorem 2.7.** *Let  $n, t \in \mathbb{Z}$ ,  $n \geq 1$ ,  $t \geq 0$ . Then there exists a generalized umbrella ring  $R$  such that  $\text{w.gl.dim } R = n$ ,  $\text{Max}\{n, t + 2\} \leq \text{gl.dim } R \leq n + t + 1$ , where  $\text{gl.dim } R$  is the global dimension of  $R$ .*

**Proof.** Let  $\Gamma$  be a linearly ordered additive group with  $|\Gamma| = \aleph_t$ , and let  $(A, m')$  be a regular Noetherian local ring with unique maximal ideal  $m'$  and  $\text{gl.dim } A = n$ . Let  $K$  be the quotient field of  $A$  and  $x$  an indeterminate over  $K$ . Set

$$A = \left\{ f(x) = \sum_{\gamma \in \Gamma} a_\gamma x^\gamma \mid a_\gamma \in K \right\} \quad \text{and}$$

$$R = \{ f(x) \in A \mid f(0) \in A \}.$$

That is,  $A$  is the ring of all power series over  $K$  in a symbol “ $x$ ” with exponents well ordered sequences in  $\Gamma$  and  $R$  is the subring of  $A$  satisfying  $f(0) \in A$  for any  $f(x) \in A$ .

From [11, Corollary 2] we know that  $A$  is a valuation ring and  $\text{gl.dim } A = t + 2$ . By [6, Theorem 169],  $m'$  is generated by a regular  $A$ -sequence  $\alpha_1, \dots, \alpha_n$ . So  $m' = (\alpha_1, \dots, \alpha_n)$ . It is easy to verify that  $R$  is a local ring and its unique maximal ideal  $m = m'R$  and  $R/(\alpha_1, \dots, \alpha_i)R \cong A/(\alpha_1, \dots, \alpha_i)A$  for any  $i = 1, \dots, n$ . Thus  $\alpha_1, \dots, \alpha_n$  is also a regular  $R$ -sequence and  $m = (\alpha_1, \dots, \alpha_n)R$ . Set

$$P = \{ f(x) \in A \mid f(0) = 0 \}.$$

It is easy to see that  $P$  is an infinitely generated ideal of  $R$ . Since  $R/P \cong A$  and  $A$  is a domain by [6, Theorem 185],  $P$  is a prime ideal of  $R$ . For any  $g(x) \in R - P$ , assume that  $g(x) = a + g_1(x)$ , where  $0 \neq a \in A$ ,  $g_1(x) \in P$ . Then  $g(x) = a(1 + \frac{1}{a}g_1(x))$ ,  $1 + \frac{1}{a}g_1(x)$  is a unit in  $R$ . So for any  $f(x) \in P$ , we have

$$f(x) = (a + g_1(x)) \left(1 + \frac{1}{a}g_1(x)\right)^{-1} \cdot \frac{1}{a}f(x) = g(x)f_1(x),$$

where  $f_1(x) = (1 + \frac{1}{a}g_1(x))^{-1} \cdot \frac{1}{a}f(x) \in P$ . Thus  $f(x) \in g(x)P$ , which implies  $P \subseteq g(x)P$ . It is clear that  $P \supseteq g(x)P$ . So  $P = g(x)P$  for any  $g(x) \in R - P$ , which implies that  $PR_P = P$  and therefore we have the following Cartesian square which conforms the general format.

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_P \\ \downarrow j_1 & & \downarrow j_2 \\ R/P & \xrightarrow{i_2} & R_P/P \end{array}$$

Since  $R_P \cong A$  is a valuation ring,  $R_P$  is a coherent ring. The fact  $R/P \cong A$  implies that  $R/P$  is a Noetherian ring. Then by [3, Theorem 5.1.3], we have that  $R$  is a coherent ring. Since the unique maximal ideal  $m$  of  $R$  is generated by a regular  $R$ -sequence of  $n$  elements, by [19, Theorem 3.7 and Corollary 3.8],  $R$  is a regular coherent local ring and  $\text{w.gl.dim } R = n$ .

Since the elements  $\frac{b}{a}x^\gamma$  ( $\gamma \neq 0$ ,  $a, b \in A - \{0\}$ ) are reducible in  $R$  and every element in  $P$  has a divisor  $x^\gamma$  (for some  $\gamma \neq 0$ ), there does not exist any irreducible element in  $P$  and then there does not exist any principal prime element in  $P$ . By the above argument, for any  $g(x) \in R - P$ , we have  $P \subset (g(x))$  and  $P$  is contained in every principal prime ideal of  $R$ .

Now, let  $I$  be any ideal in  $R$  satisfying  $I \not\subseteq P$ . Then there exists  $\alpha \in I - P \subseteq R - P$ , that is, there exists  $\alpha \in I$  such that  $P \subset (\alpha)$ , which implies that  $P \subset I$ . Since  $R/P \cong A$  is a Noetherian ring,  $\bar{I} = I/P$  is a finitely generated ideal of  $R/P$ . Assume  $\bar{I} = (\bar{\alpha}_1, \dots, \bar{\alpha}_s)$ , where  $\alpha_1, \dots, \alpha_s \in R - P$ . From the above argument,  $P \subset (\alpha_i)$  for any  $i = 1, \dots, s$ , so  $I = (\alpha_1, \dots, \alpha_s, P) = (\alpha_1, \dots, \alpha_s)$  is a finitely generated ideal of  $R$ . Thus all infinitely generated ideals of  $R$  are contained in  $P$  and therefore contained in every principal prime ideal of  $R$ . It follows that  $R$  is a generalized umbrella ring.

By [12, Corollary 2.47], we have

$$\text{gl.dim } R \leq \text{w.gl.dim } R + t + 1 = n + t + 1$$

and by [3, Theorem 1.3.13], we have

$$\text{gl.dim } R \geq \text{gl.dim } R_P = \text{gl.dim } A = t + 2.$$

Thus

$$\text{Max}\{n, t + 2\} \leq \text{gl.dim } R \leq n + t + 1. \quad \square$$

### 3. Stable coherence of generalized umbrella rings

Recall from [3] that a ring  $R$  is called *stably coherent* if for every positive integer  $n$  the polynomial ring in  $n$  variables over  $R$  is a coherent ring, and that  $R$  is called *regular* if every finitely generated ideal of  $R$  has finite projective dimension.

By using the results in [5], Glaz proved that every coherent ring with global dimension  $\leq 2$  is stably coherent and provided examples of coherent rings with weak global dimension two where stable coherence does not hold [3]. In this section we will prove that every generalized umbrella ring is stably coherent.

From now on, for a ring  $R$ , we use  $\mathfrak{P}$  to denote the union of all non-finitely generated prime ideals of  $R$ . Because these prime ideals are linearly ordered,  $\mathfrak{P}$  is also a non-finitely generated prime ideal.

**Lemma 3.1.** *Let  $R$  be a generalized umbrella ring with maximal ideal  $m$ .*

- (1) *If  $(p)$  is a principal prime ideal of  $R$  then  $R_{(p)}$  is a valuation ring.*
- (2) *The non-finitely generated prime ideals of  $R$  are linearly ordered.*
- (3)  *$R_{\mathfrak{P}}$  is a valuation ring.*
- (4)  *$R/\mathfrak{P}$  is a regular Noetherian local ring and  $\text{gl.dim } R/\mathfrak{P} = \text{w.gl.dim } R$ .*

**Proof.** (1) Since  $R$  is coherent,  $R_{(p)}$  is also coherent. The unique maximal ideal  $(p)R_{(p)}$  of  $R_{(p)}$  is generated by a regular  $R_{(p)}$ -sequence of one element, so by [19, Theorem 3.7 and Corollary 3.8],  $R_{(p)}$  is a regular coherent local ring and  $\text{w.gl.dim } R_{(p)} = 1$ . So  $R_{(p)}$  is a Prüfer local ring and hence a valuation ring.

(2) Let  $p \in m - m^2$ . From the proof of [19, Theorem 3.7], we know that  $(p)$  is a principal prime ideal of  $R$ . By (1),  $R_{(p)}$  is a valuation ring and the prime ideals of  $R_{(p)}$  are linearly ordered, which implies that all prime ideals of  $R$  contained in  $(p)$  are linearly ordered. Since  $R$  is a generalized umbrella ring, all non-finitely generated prime ideals are contained in  $(p)$ . So the non-finitely generated prime ideals of  $R$  are linearly ordered.

(3) If  $\mathfrak{P} = 0$ , then  $R_{\mathfrak{P}}$  is a field, clearly a valuation ring. Now suppose  $\mathfrak{P} \neq 0$ . We need to prove  $R_{\mathfrak{P}}$  is a valuation ring. This can be induced by the fact that  $R_{\mathfrak{P}}$  is a localization of a valuation ring  $R_{(p)}$  for any  $p \in m - m^2$ .

(4) If  $\mathfrak{P} = 0$ , then  $R$  is a Noetherian ring and  $\text{gl.dim } R/\mathfrak{P} = \text{w.gl.dim } R$ . Now suppose  $\mathfrak{P} \neq 0$ . Because  $\mathfrak{P}$  is the union of all non-finitely generated prime ideals of  $R$ , there does not exist any non-finitely generated prime ideal of  $R/\mathfrak{P}$ . Thus, by Cohen’s Lemma,  $R/\mathfrak{P}$  is a Noetherian ring. Since  $R$  is a regular coherent local ring and the maximal ideal  $m$  is finitely generated,  $m$  is generated by a regular  $R$ -sequence  $\alpha_1, \dots, \alpha_n$  and  $\text{w.gl.dim } R = n$  by [19, Theorem 3.7]. From the proof of [19, Theorem 3.7] we know that  $(\alpha_i)$  is a principal prime ideal of  $R$  for any  $1 \leq i \leq n$ , which implies that  $\mathfrak{P} \subset (\alpha_i)$  (for any  $1 \leq i \leq n$ ) since  $R$  is a generalized umbrella ring. Set  $\bar{R} = R/\mathfrak{P}$ ,  $\bar{m} = (\bar{\alpha}_1, \dots, \bar{\alpha}_n)$ . It is easy to verify that  $\bar{\alpha}_1, \dots, \bar{\alpha}_n$  is also a regular  $\bar{R}$ -sequence. By [19, Theorem 3.7] again, we have that  $R/\mathfrak{P}$  is a regular Noetherian ring and  $\text{gl.dim } R/\mathfrak{P} = n = \text{w.gl.dim } R$ .  $\square$

In the rest of this section, we always assume that  $R$  is a generalized umbrella ring,  $m$  is the unique maximal ideal of  $R$ .

**Lemma 3.2.** *For any  $\alpha \in R - \mathfrak{P}$ , if  $\alpha$  is not a unit, then  $\alpha$  has a unique factorization.*

**Proof.** If  $\mathfrak{P} = 0$ , then  $R$  is a regular Noetherian local ring. By [6, Theorem 185],  $R$  is a unique factorization domain (abbreviated by UFD).

Now suppose  $\mathfrak{P} \neq 0$ . Let  $a \in R - \mathfrak{P}$  and  $a$  is not a unit. By [6, Theorem 3], we need only to prove that  $a$  can be written as a product  $p_1 p_2 \cdots p_n$  of principal prime elements. Set  $\bar{R} = R/\mathfrak{P}$ . By Lemma 3.1,  $\bar{R}$  is a regular Noetherian local ring, so it is a UFD. Thus  $\bar{a}$  has a unique factorization:

$$\bar{a} = \bar{p}_1 \bar{p}_2 \cdots \bar{p}_s,$$

where  $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_s$  are principal prime elements in  $\bar{R}$ . We claim that every  $p_i$  ( $1 \leq i \leq s$ ) is also a principal prime element in  $R$ . In fact, if  $p_i = ab$ , then  $\bar{p}_i = \bar{a}\bar{b}$ , which implies that  $\bar{a}$  or  $\bar{b}$  is a unit in  $\bar{R}$ . So  $a$  or  $b$  is a unit in  $R$  and  $p_i$  is an irreducible element in  $R$ . It follows from [24, Corollary 11] that  $R$  is a greatest common divisor domain (abbreviated by GCD domain). Thus  $p_i$  is a principal prime element of  $R$  for any  $1 \leq i \leq s$ . Set  $b = a - p_1 p_2 \cdots p_s$ . It is clear that  $b \in \mathfrak{P}$  and  $\mathfrak{P} \subset (p_i)$  ( $1 \leq i \leq s$ ). So  $b \in \mathfrak{P} \subset (p_1 p_2 \cdots p_s)$  and we get  $b = p_1 p_2 \cdots p_s \cdot c$  for some  $c \in \mathfrak{P}$ . Then  $a = (1 + c)p_1 p_2 \cdots p_s$  as required.  $\square$

**Lemma 3.3.** For any  $b \in m - \mathfrak{P}$ , we have  $b\mathfrak{P} = \mathfrak{P}$  and so  $\mathfrak{P} = \mathfrak{P}R_{\mathfrak{P}}$ .

**Proof.** If  $\mathfrak{P}$  is a principal prime element in  $R$ , then  $\mathfrak{P} \subset (p)$ . So it is trivial that  $\mathfrak{P} = p\mathfrak{P}$ . Now let  $b \in m - \mathfrak{P}$ . By Lemma 3.2,  $b$  can be expressed as a product of finite principal prime elements of  $R$ :  $b = p_1 p_2 \cdots p_s$ . So we may get that  $b\mathfrak{P} = (p_1 p_2 \cdots p_s)\mathfrak{P} = (p_1 p_2 \cdots p_{s-1})p_s \mathfrak{P} = (p_1 p_2 \cdots p_{s-1})\mathfrak{P} = \mathfrak{P}$  by induction on  $s$ .  $\square$

**Theorem 3.4.** Any generalized umbrella ring is stably coherent.

**Proof.** Let  $R$  be a generalized umbrella ring. If  $\mathfrak{P} = 0$ , then  $R$  is a Noetherian ring. By Hilbert’s Basis Theorem, the assertion holds. Now suppose  $\mathfrak{P} \neq 0$ . By Lemma 3.3,  $\mathfrak{P} = \mathfrak{P}R_{\mathfrak{P}}$ . From the Cartesian square

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_{\mathfrak{P}} \\ \downarrow j_1 & & \downarrow j_2 \\ R/\mathfrak{P} & \xrightarrow{i_2} & R_{\mathfrak{P}}/\mathfrak{P} \end{array}$$

we get the following Cartesian square:

$$\begin{array}{ccc} R[x_1, \dots, x_n] & \longrightarrow & R_{\mathfrak{P}}[x_1, \dots, x_n] \\ \downarrow & & \downarrow \\ R/\mathfrak{P}[x_1, \dots, x_n] & \longrightarrow & R_{\mathfrak{P}}/\mathfrak{P}[x_1, \dots, x_n]. \end{array}$$

Set  $Q = \mathfrak{P}[x_1, \dots, x_n]$ . Then  $Q$  is a prime ideal of  $R[x_1, \dots, x_n]$  and  $R[x_1, \dots, x_n]/Q \cong R/\mathfrak{P}[x_1, \dots, x_n]$ ,  $Q \cdot R_{\mathfrak{P}}[x_1, \dots, x_n] = \mathfrak{P}[x_1, \dots, x_n] \cdot R_{\mathfrak{P}}[x_1, \dots, x_n] = (\mathfrak{P} \cdot R_{\mathfrak{P}})[x_1, \dots, x_n] =$

$\mathfrak{P}[x_1, \dots, x_n] = Q$ . Thus  $(R_{\mathfrak{P}}/\mathfrak{P})[x_1, \dots, x_n] = R_{\mathfrak{P}}[x_1, \dots, x_n]/Q$  and therefore the last Cartesian square diagram conforms the general format. By Lemma 3.1,  $R/\mathfrak{P}$  is a Noetherian ring and  $R/\mathfrak{P}[x_1, \dots, x_n]$  is also a Noetherian ring. By Lemma 3.1 again,  $R_{\mathfrak{P}}$  is a valuation ring. It follows from [3, Theorem 7.3.3] that  $R_{\mathfrak{P}}[x_1, \dots, x_n]$  is coherent. Then by [3, Theorem 5.1.5],  $R[x_1, \dots, x_n]$  is also coherent, which completes the proof.  $\square$

**4. Finitely generated projective modules over  $R[x]$**

In this section, we will show that if  $R$  is a generalized umbrella ring with  $\text{w.gl.dim } R = 2$ , then every finitely generated projective module over  $R[x]$  is free. We first give several notations and lemmas.

Let  $R$  be a ring and  $R[x]$  the polynomial ring in one variable over  $R$ . Set

$$\Phi = \{f(x) \in R[x] \mid \text{the leading coefficient of } f(x) \text{ is the identity } 1\}.$$

It is a multiplicative closed subset of  $R[x]$ . Set

$$R(x) = R[x]_{\Phi} = \left\{ \frac{f(x)}{g(x)} \mid f(x) \in R[x], g(x) \in \Phi \right\},$$

which is the localization of  $R[x]$  at  $\Phi$ .

**Lemma 4.1.** *Let  $R$  be a generalized umbrella ring. Then  $R(x)$  is a coherent GCD domain.*

**Proof.** Since  $R$  is a regular coherent local ring,  $R$  is a GCD domain by [24, Corollary 11], which implies that both  $R[x]$  and  $R(x)$  are GCD domains. By Theorem 3.4,  $R[x]$  is coherent. So  $R(x)$  is also coherent by [4, Theorem 1].  $\square$

**Lemma 4.2.** *Let  $R$  be a generalized umbrella ring with maximal ideal  $m$ . If  $\text{w.gl.dim } R = 2$ , then for any non-zero prime ideal  $Q (\neq m)$  of  $R$ , the localization ring  $R_Q$  is a valuation ring.*

**Proof.** It follows from the definition of generalized umbrella rings and [20, Corollary 2.12].  $\square$

**Lemma 4.3.** *Let  $R$  be a generalized umbrella ring with maximal ideal  $m$ . If  $\text{w.gl.dim } R = 2$  and  $u \in m - m^2$ , then*

- (1)  $u$  is a principal prime element.
- (2)  $R/(u)$  is a discrete valuation ring.
- (3)  $R_u$  is a Bézout domain, where  $R_u$  denote the localization of  $R$  at the multiplicative closed subset  $\{u^k \mid k \geq 0\}$  of  $R$ .

**Proof.** (1) Clearly,  $u$  is an irreducible element of  $R$ . Since  $R$  is a GCD domain,  $u$  is a principal prime element.

(2) Because  $\mathfrak{P} \subset (u)$ ,  $R/(u)$  is a Noetherian local ring. From the proof of [6, Theorem 2.6], we have that  $\text{gl.dim } R/(u) = \text{w.gl.dim } R/(u) = \text{w.gl.dim } R - 1 = 1$  and so  $R/(u)$  is a discrete valuation ring.

(3) Let  $Q'$  be any maximal ideal of  $R_u$ . Then there exists a prime ideal  $Q$  of  $R$  such that  $Q' = Q_u$  and  $Q \cap \{u^k \mid k \geq 0\} = \emptyset$ . Clearly,  $Q \neq m$  and  $(R_u)_{Q'} = (R_u)_{Q_u} = (R_Q)_u$ . By



Lemma 4.2,  $R_Q$  is a valuation ring. Thus  $(R_u)_Q$  is a valuation ring and therefore  $R_u$  is a Prüfer ring. Since  $R$  is a GCD domain,  $R_u$  is also a GCD domain. Thus  $R_u$  is a Bézout domain.  $\square$

We use  $\mathbb{S}\mathbb{L}_n(R)$  and  $\mathbb{E}_n(R)$  to denote the special linear group and the elementary linear group of  $R$ , respectively. Recall from [7] that a principal ideal domain  $R$  is called a *special principal ideal domain* if  $\mathbb{S}\mathbb{L}_n(R) = \mathbb{E}_n(R)$  for any  $n \geq 1$ .

**Lemma 4.4.** (See [7, p. 119, Corollary 5.3].) *Let  $R$  be a discrete valuation ring. Then  $R(x)$  is a special principal ideal domain.*

**Lemma 4.5.** (See [23, Theorem 2.13].) *Let  $R$  be a coherent GCD domain and  $u \in R$  a prime element. If  $\bar{R} = R/(u)$  is a special principal ideal domain and  $T$  a finitely generated reflexive  $R$ -module with  $T_u$  a free  $R_u$ -module, then  $T$  is a free  $R$ -module.*

**Lemma 4.6.** (See [7, p. 102, Horrocks Theorem].) *Let  $R$  be a local ring and  $F$  a finitely generated  $R[x]$ -module. If  $F(x) = F \otimes_{R[x]} R(x)$  is a free  $R(x)$ -module, then  $F$  is a free  $R$ -module.*

We are now in a position to give the main result in this section.

**Theorem 4.7.** *Let  $R$  be a generalized umbrella ring with  $\text{w.gl.dim } R = 2$ . Then every finitely generated projective  $R[x]$ -module is free.*

**Proof.** Let  $F$  be a finitely generated projective  $R[x]$ -module. By Lemma 4.6, we need only to prove that  $F(x) (= F \otimes_{R[x]} R(x))$  is a free  $R(x)$ -module.

Let  $m$  be the unique maximal ideal of  $R$  and  $u \in m - m^2$ . Then  $F_u = F \otimes_{R[x]} (R[x])_u = F \otimes_{R[x]} R_u[x]$  is a finitely generated  $R_u[x]$ -module. By Lemma 4.3,  $R_u$  is a Bézout domain. So by [8, Theorem B],  $F_u$  is a free  $R_u[x]$ -module. Because

$$\begin{aligned} F(x)_u &= F(x) \otimes_{R(x)} (R(x))_u \\ &= F \otimes_{R[x]} R(x) \otimes_{R(x)} (R(x))_u \\ &= F \otimes_{R[x]} (R(x))_u \\ &= F \otimes_{R[x]} R_u[x] \otimes_{R_u[x]} (R(x))_u \\ &= F_u \otimes_{R_u[x]} (R(x))_u \end{aligned}$$

and  $F_u$  is a free  $R_u[x]$ -module,  $F(x)_u$  is a finitely generated free  $R(x)_u$ -module. Set  $\bar{R} = R/(u)$ . By Lemma 4.3,  $\bar{R}$  is a discrete valuation ring and it is easy to verify that  $\bar{R}(x) \cong R(x)/uR(x)$ . It follows from Lemma 4.4 that  $R(x)/uR(x)$  is a special principal ideal domain. Then by Lemma 4.1,  $R(x)$  is a GCD domain and clearly  $u$  is a prime element in  $R(x)$ . Consequently  $F(x)$  is a free  $R(x)$ -module by Lemma 4.5.  $\square$

**Corollary 4.8.** *Let  $R$  be a ring with  $\text{w.gl.dim } R = 2$ . If  $R$  satisfies the condition that  $R_m$  is a generalized umbrella ring with  $\text{w.gl.dim } R_m = 2$  for any  $m \in \text{Max}(R)$ , then every finitely generated projective  $R[x]$ -module can be extended from  $R$ .*

**Proof.** By the assumption,  $R_m$  is a generalized umbrella ring or a valuation ring for any  $m \in \text{Max}(R)$ . Then the assertion follows from Theorem 4.7, [8, Theorem B] and [7, p. 125, Theorem 1.6].  $\square$

**Corollary 4.9.** *Let  $R$  be a ring with  $\text{gl.dim } R = 2$ . Then every finitely generated projective  $R[x]$ -module can be extended from  $R$ . Particularly, every finitely generated projective  $R[x]$ -module is free if  $R$  is a local ring with  $\text{gl.dim } R = 2$ .*

**Proof.** For any  $m \in \text{Max}(R)$ , if  $\text{w.gl.dim } R_m = 2$ , then we can get that  $\text{gl.dim } R_m = \text{w.gl.dim } R_m = 2$  for  $\text{gl.dim } R = 2$ . By [3, Theorem 6.2.15], the unique maximal ideal of  $R_m$  is finitely generated. From Example 2.5, we know that  $R_m$  is a generalized umbrella ring. Now the assertion follows from Corollary 4.8.  $\square$

## Acknowledgments

The first author was partially supported by Guangxi Science Foundation (Grant No. 0575052), the Support Program for 100 Young and Middle-aged Disciplinary Leaders in Guangxi Higher Education Institutions and Scientific Reserch Foundation of Guangxi Educational Committee. The second author was partially supported by Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20060284002) and NSF of Jiangsu Province of China (Grant No. BK2005207).

## References

- [1] H. Bass, Some problems in “classical” algebraic  $K$ -theory, in: Algebraic  $K$ -Theory, in: Lecture Notes in Math., vol. 342, Springer-Verlag, Berlin, 1973, pp. 1–70.
- [2] S.M. Bhatwadekar, P.L.N. Varma, Extensibility criterion for a projective module of rank one over  $R[T]$  and  $R[T, T^{-1}]$ , Proc. Amer. Math. Soc. 119 (1993) 1069–1075.
- [3] S. Glaz, Commutative Coherent Rings, Lecture Notes in Math., vol. 1371, Springer-Verlag, Berlin, 1989.
- [4] S. Glaz, On coherence and weak dimension of the rings  $R(x)$  and  $R(x)$ , Proc. Amer. Math. Soc. 106 (1989) 579–587.
- [5] B. Greenberg, Coherence in Cartesian squares, J. Algebra 50 (1978) 12–25.
- [6] I. Kaplansky, Commutative Rings, Univ. of Chicago Press, 1974.
- [7] T.Y. Lam, Serre’s Conjecture, Lecture Notes in Math., vol. 635, Springer-Verlag, Berlin, 1978.
- [8] Y. Lequain, A. Simis, Projective modules over  $R[x_1, \dots, x_n]$ ,  $R$  a Prüfer domain, J. Pure Appl. Algebra 18 (1980) 165–171.
- [9] M.P. Murthy, Projective  $A[x]$  modules, J. London Math. Soc. 41 (1966) 453–456.
- [10] C.G. Naude, G. Naude, Stably free modules of big rank over polynomial rings are free, J. Pure Appl. Algebra 66 (1990) 311–314.
- [11] B.L. Osofsky, Global dimension of valuation rings, Trans. Amer. Math. Soc. 127 (1967) 136–149.
- [12] B.L. Osofsky, Homological Dimensions of Modules, CBMS Reg. Conf. Ser. Math., vol. 12, Amer. Math. Soc., Providence, RI, 1973.
- [13] D. Quillen, Projective modules over polynomial rings, Invent. Math. 36 (1976) 167–171.
- [14] R.A. Rao, The Bass–Quillen conjecture in dimension three but characteristic  $\neq 2, 3$  via a question of A. Suslin, Invent. Math. 93 (1988) 609–618.
- [15] R.A. Rao, Stability theorems for overrings of polynomial rings, J. Algebra 78 (1982) 437–444.
- [16] J.P. Serre, Faisceaux algébriques cohérents, Ann. Math. 61 (1955) 197–278.
- [17] A.A. Suslin, Projective modules over a polynomial ring are free, Dokl. Akad. Nauk SSSR 229 (1976) 1163–1166.
- [18] G.H. Tang, Spectrum and homological dimension of a subring of  $Q[x]$ , J. Nanjing Univ. Math. Biquarterly 18 (2001) 41–45.

- [19] G.H. Tang, Weak Krull dimension over commutative rings, in: J.L. Chen, N.Q. Ding, H. Marubayashi (Eds.), *Advances in Ring Theory, Proc. of the 4th China–Japan–Korea International Conference*, World Scientific Press, 2005, pp. 215–224.
- [20] G.H. Tang, X.B. Yin, W.T. Tong, A generalization of Auslander–Buchsbaum theorem, *Guangxi Sciences* 12 (2005) 97–101.
- [21] W.V. Vasconcelos, The local rings of global dimension two, *Proc. Amer. Math. Soc.* 35 (1972) 381–385.
- [22] W.V. Vasconcelos, *The Rings of Dimension Two*, Marcel Dekker, Inc., New York, 1976.
- [23] F.G. Wang, G.H. Tang, Reflexive modules over coherent GCD-domains, *Comm. Algebra* 33 (2005) 4383–4392.
- [24] Y.C. Zhao, On commutative indecomposable coherent regular rings, *Comm. Algebra* 20 (1992) 1389–1394.