For any ring $R$ and any positive integer $n$, we prove that a left $R$-module is a Gorenstein $n$-syzygy if and only if it is an $n$-syzygy. Over a left and right Noetherian ring, we introduce the notion of the Gorenstein transpose of finitely generated modules. We prove that a module $M \in \text{mod } R^{op}$ is a Gorenstein transpose of a module $A \in \text{mod } R$ if and only if $M$ can be embedded into a transpose of $A$ with the cokernel Gorenstein projective. Some applications of this result are given.

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1. Introduction

Throughout this paper, $R$ is an associative ring with identity and $\text{Mod } R$ is the category of left $R$-modules.

In classical homological algebra, the notion of finitely generated projective modules is an important and fundamental research object. As a generalization of this notion, Auslander and Bridger introduced in [AB] the notion of finitely generated modules of Gorenstein dimension zero over a left and right Noetherian ring. Over a general ring, Enochs and Jenda introduced in [EJ1] the notion of Gorenstein projective modules (not necessarily finitely generated). It is well known that these two notions coincide for finitely generated modules over a left and right Noetherian ring. In particular, Gorenstein projective modules share many nice properties of projective modules (e.g. [AB,C,CFH,CI,EJ1,EJ2,H]).
The notion of a syzygy module was defined via the projective resolution of modules as follows. For a positive integer \( n \), a module \( A \in \text{Mod} \ R \) is called an \( n \)-syzygy module (of \( M \)) if there exists an exact sequence \( 0 \to A \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0 \) in \( \text{Mod} \ R \) with all \( P_i \) projective. Analogously, we call \( A \) a Gorenstein \( n \)-syzygy module (of \( M \)) if there exists an exact sequence \( 0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0 \) in \( \text{Mod} \ R \) with all \( G_i \) Gorenstein projective. It is trivial that an \( n \)-syzygy module is Gorenstein \( n \)-syzygy. In Section 2, our main result is that for every \( n \geq 1 \), a Gorenstein \( n \)-syzygy module is \( n \)-syzygy. The following auxiliary proposition plays a crucial role in proving this main result. Let \( 0 \to A \to G_1 \to G_0 \to M \to 0 \) be an exact sequence in \( \text{Mod} \ R \) with \( G_0 \) and \( G_1 \) Gorenstein projective. Then we have the following exact sequences \( 0 \to A \to P \to G \to M \to 0 \) and \( 0 \to A \to H \to Q \to M \to 0 \) in \( \text{Mod} \ R \) with \( P, Q \) projective and \( G, H \) Gorenstein projective.

In Section 3, for a left and right Noetherian ring \( R \) and a finitely generated left \( R \)-module \( A \), we introduce the notion of the Gorenstein transpose of \( A \), which is a Gorenstein version of that of the transpose of \( A \). We establish a relation between a Gorenstein transpose of a module and a transpose of the same module. We prove that a finitely generated right \( R \)-module \( M \) is a Gorenstein transpose of a finitely generated left \( R \)-module \( A \) if and only if \( M \) can be embedded into a transpose of \( A \) with the cokernel Gorenstein projective. Then we give some applications of this result: (1) The direct sum of a finitely generated Gorenstein projective right \( R \)-module and a transpose of a finitely generated left \( R \)-module \( A \) is a Gorenstein transpose of \( A \). (2) For any Gorenstein transpose and any transpose of a finitely generated left \( R \)-module, one of them is \( n \)-torsionfree if and only if so is the other. (3) A finitely generated left \( R \)-module with Gorenstein projective dimension \( n \) is a double Gorenstein transpose of a finitely generated left \( R \)-module with projective dimension \( n \).

2. Gorenstein syzygy modules

Recall from \([EJ1]\) a module \( G \in \text{Mod} \ R \) is called Gorenstein projective if there exists an exact sequence in \( \text{Mod} \ R \):

\[
\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots,
\]

such that: (1) All \( P_i \) and \( P^i \) are projective; (2) After applying the functor \( \text{Hom}_R(\ , \ P) \) the sequence is still exact for any projective module \( P \in \text{Mod} \ R \); and (3) \( G \cong \text{Im}(P_0 \to P^0) \). Let \( M \) be a module in \( \text{Mod} \ R \). The Gorenstein projective dimension of \( M \), denoted by \( \text{Gpd}_R(M) \), is defined as \( \inf \{ n \mid \text{for any exact sequence } 0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0 \text{ in } \text{Mod} \ R \text{ with all } G_i \text{ Gorenstein projective} \} \). We have \( \text{Gpd}_R(M) \geq 0 \) and we set \( \text{Gpd}_R(M) \) infinity if no such integer exists (see \([EJ1]\) or \([H]\)).

Lemma 2.1. Let \( 0 \to M_3 \to M_2 \to M_1 \to 0 \) be an exact sequence in \( \text{Mod} \ R \) with \( M_3 \neq 0 \). If \( M_1 \) is Gorenstein projective, then \( \text{Gpd}_R(M_3) = \text{Gpd}_R(M_2) \).

Proof. By \([H, \text{Theorems } 2.24 \text{ and } 2.20]\), it is easy to get the assertion. □

The following result plays a crucial role in this paper.

Proposition 2.2. Let \( 0 \to A \to G_1 \xrightarrow{f} G_0 \to M \to 0 \) be an exact sequence in \( \text{Mod} \ R \) with \( G_0 \) and \( G_1 \) Gorenstein projective. Then we have the following exact sequences:

\[
0 \to A \to P \to G \to M \to 0,
\]

and

\[
0 \to A \to H \to Q \to M \to 0,
\]

in \( \text{Mod} \ R \) with \( P, Q \) projective and \( G, H \) Gorenstein projective.
Proof. Because $G_1$ is Gorenstein projective, there exists an exact sequence $0 \rightarrow G_1 \rightarrow P \rightarrow G_2 \rightarrow 0$ in $\text{Mod } R$ with $P$ projective and $G_2$ Gorenstein projective. Then we have the following push-out diagram:

\[
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & A & \rightarrow & G_1 & \rightarrow & \text{Im } f & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & A & \rightarrow & P & \rightarrow & B & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow \\
& & G_2 & \rightarrow & & G_2 & \\
\downarrow & & & & \downarrow & & \\
0 & & 0 & & & & & & \\
\end{array}
\]

Consider the following push-out diagram:

\[
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Im } f & \rightarrow & G_0 & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & B & \rightarrow & G & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow \\
& & G_2 & \rightarrow & & G_2 & \\
\downarrow & & & & \downarrow & & \\
0 & & 0 & & & & & & \\
\end{array}
\]

Because both $G_0$ and $G_2$ are Gorenstein projective, $G$ is also Gorenstein projective by Lemma 2.1. Connecting the middle rows in the above two diagrams, then we get the first desired exact sequence. Since $G_0$ is Gorenstein projective, there exists an exact sequence $0 \rightarrow G_3 \rightarrow Q \rightarrow G_0 \rightarrow 0$ in $\text{Mod } R$ with $Q$ projective and $G_3$ Gorenstein projective. Dually, taking pull-back, one gets the second desired exact sequence. □

For a positive integer $n$, recall that a module $A \in \text{Mod } R$ is called an $n$-syzygy module (of $M$) if there exists an exact sequence $0 \rightarrow A \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ in $\text{Mod } R$ with all $P_i$ projective. Analogously, we give the following

Definition 2.3. For a positive integer $n$, a module $A \in \text{Mod } R$ is called a Gorenstein $n$-syzygy module (of $M$) if there exists an exact sequence $0 \rightarrow A \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ in $\text{Mod } R$ with all $G_i$ Gorenstein projective.
The following theorem is the main result in this section.

**Theorem 2.4.** Let $n$ be a positive integer and $0 \to A \to G_{n-1} \to G_{n-2} \to \cdots \to G_0 \to M \to 0$ an exact sequence in $\text{Mod } R$ with all $G_i$ Gorenstein projective. Then we have the following:

1. There exist exact sequences $0 \to A \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to N \to 0$ and $0 \to M \to N \to G \to 0$ in $\text{Mod } R$ with all $P_i$ projective and $G$ Gorenstein projective. In particular, a module in $\text{Mod } R$ is an $n$-syzygy if and only if it is a Gorenstein $n$-syzygy.

2. There exist exact sequences $0 \to B \to Q_{n-1} \to Q_{n-2} \to \cdots \to Q_0 \to M \to 0$ and $0 \to H \to B \to A \to 0$ in $\text{Mod } R$ with all $Q_i$ projective and $H$ Gorenstein projective.

**Proof.** (1) We proceed by induction on $n$. When $n = 1$, it has been proved in the proof of Proposition 2.2. Now suppose that $n \geq 2$ and we have an exact sequence:

$$0 \to A \to G_{n-1} \to G_{n-2} \to \cdots \to G_0 \to M \to 0$$

in $\text{Mod } R$ with all $G_i$ Gorenstein projective. Put $K = \text{Coker}(G_{n-1} \to G_{n-2})$. By Proposition 2.2, we get an exact sequence:

$$0 \to A \to P_{n-1} \to G'_{n-2} \to K \to 0$$

in $\text{Mod } R$ with $P_{n-1}$ projective and $G'_{n-2}$ Gorenstein projective. Put $A' = \text{Im}(P_{n-1} \to G'_{n-2})$. Then we get an exact sequence:

$$0 \to A' \to G'_{n-2} \to G_{n-3} \to \cdots \to G_0 \to M \to 0$$

in $\text{Mod } R$. So, by the induction hypothesis, we get the assertion.

(2) The proof is dual to that of (1), so we omit it. $\square$

For a module $M \in \text{Mod } R$, we use $\text{pd}_R(M)$ to denote the projective dimension of $M$.

**Corollary 2.5.** (See [CFH, Lemma 2.17] ) Let $M \in \text{Mod } R$ and $n$ be a non-negative integer. If $\text{Gpd}_R(M) = n$, then there exists an exact sequence $0 \to M \to N \to G \to 0$ in $\text{Mod } R$ with $\text{pd}_R(N) = n$ and $G$ Gorenstein projective.

**Proof.** Let $M \in \text{Mod } R$ with $\text{Gpd}_R(M) = n$. Then one uses Theorem 2.4(1) with $A = 0$ to get an exact sequence $0 \to M \to N \to G \to 0$ in $\text{Mod } R$ with $\text{pd}_R(N) \leq n$ and $G$ Gorenstein projective. By Lemma 2.1, $\text{Gpd}_R(N) = n$, and thus $\text{pd}_R(N) = n$. $\square$

By [H, Theorem 2.20], we have that $\text{Gpd}_R(M) \leq n$ if and only if there exists an exact sequence $0 \to G_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ in $\text{Mod } R$ with all $P_i$ projective and $G_n$ Gorenstein projective. The following theorem generalizes this result. In particular, the following theorem was proved by Christensen and Iyengar in [CI, Theorem 3.1] when $R$ is a commutative Noetherian ring.

**Theorem 2.6.** Let $M \in \text{Mod } R$ and $n$ be a non-negative integer. Then the following statements are equivalent.

1. $\text{Gpd}_R(M) \leq n$.

2. For every non-negative integer $t$ such that $0 \leq t \leq n$, there exists an exact sequence $0 \to X_0 \to \cdots \to X_1 \to X_0 \to M \to 0$ in $\text{Mod } R$ such that $X_t$ is Gorenstein projective and $X_i$ is projective for $i \neq t$.

**Proof.** (2) $\Rightarrow$ (1) It is trivial.

(1) $\Rightarrow$ (2) We proceed by induction on $n$. Suppose $\text{Gpd}_R(M) \leq 1$. Then there exists an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ in $\text{Mod } R$ with $G_0$ and $G_1$ Gorenstein projective. By Proposition 2.2
with \( A = 0 \), we get the exact sequences \( 0 \to P_1 \to G'_0 \to M \to 0 \) and \( 0 \to G'_1 \to P_0 \to M \to 0 \) in Mod \( R \) with \( P_0, P_1 \) projective and \( G'_0 \) and \( G'_1 \) Gorenstein projective.

Now suppose \( n \geq 2 \). Then there exists an exact sequence \( 0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0 \) in Mod \( R \) with \( G_i \) Gorenstein projective for any \( 1 \leq i \leq n \). Set \( A = \text{Coker}(G_2 \to G_1) \). By applying Proposition 2.2 to the exact sequence \( 0 \to A \to G_1 \to G_0 \to M \to 0 \), we get an exact sequence \( 0 \to G_n \to \cdots \to G_2 \to G_1 \to P_0 \to M \to 0 \) in Mod \( R \) with \( G'_1 \) Gorenstein projective and \( P_0 \) projective. Set \( N = \text{Coker}(G_2 \to G'_1) \). Then we have \( \text{Gpd}_R(N) \leq n - 1 \). By the induction hypothesis, there exists an exact sequence \( 0 \to X_n \to \cdots \to X_t \to \cdots \to X_1 \to P_0 \to M \to 0 \) in Mod \( R \) such that \( P_0 \) is projective and \( X_t \) is Gorenstein projective. We point out the dual versions on Gorenstein injectivity.

Remark 2.7. It is known that a module \( \mathcal{A} \in \text{Mod} \) is called an \( n \)-cosyzyzy module (of \( M \)) if there exists an exact sequence \( 0 \to M \to I^0 \to I^1 \to \cdots \to I^{n-1} \to A \to 0 \) in Mod \( R \) with all \( I^i \) injective. Recall from [EJ1] that a module \( E \in \text{Mod} \) is called Gorenstein injective if there exists an exact sequence in Mod \( R \):

\[
\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots ,
\]

such that: (1) All \( I_1 \) and \( I^1 \) are injective; (2) After applying the functor \( \text{Hom}_R(1, \_ ) \) the sequence is still exact for any injective module \( I \in \text{Mod} \); and (3) \( E \cong \text{Im}(I_0 \to I^0) \). We call \( A \) a Gorenstein \( n \)-cosyzyzy module (of \( M \)) if there exists an exact sequence \( 0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to A \to 0 \) in Mod \( R \) with all \( E^i \) Gorenstein injective. We point out the dual versions on Gorenstein injectivity and (Gorenstein) \( n \)-cosyzyzy of all of the above results also hold true by using completely dual arguments.

3. Gorenstein transpose

In this section, \( R \) is a left and right Noetherian ring and mod \( R \) is the category of finitely generated left \( R \)-modules. For any \( A \in \text{mod} \) \( R \), there exists a projective presentation in mod \( R \):

\[
P_1 \xrightarrow{f} P_0 \to A \to 0.
\]

Then we get an exact sequence

\[
0 \to A^* \to P_0^* \xrightarrow{f^*} P_1^* \to \text{Coker} f^* \to 0
\]

in mod \( R^{op} \), where \( (\_)^* = \text{Hom}(\_ , R) \). Recall from [AB] that \( \text{Coker} f^* \) is called a transpose of \( A \), and denoted by \( \text{Tr} A \). We remark that the transpose of \( A \) depends on the choice of the projective presentation of \( A \), but it is unique up to projective equivalence (see [AB]).
Analogously, we introduce the notion of Gorenstein transpose of modules as follows. Let \( A \in \text{mod} \ R \). Then there exists a Gorenstein projective presentation in \( \text{mod} \ R \):

\[
\pi: X_1 \xrightarrow{g} X_0 \to A \to 0,
\]

and we get an exact sequence:

\[
0 \to A^* \to X_0^* \xrightarrow{g^*} X_1^* \to \text{Coker} \ g^* \to 0
\]

in \( \text{mod} \ R^\text{op} \). We call \( \text{Coker} \ g^* \) a Gorenstein transpose of \( A \), and denote it by \( \text{Tr}^G \pi A \). It is trivial that a transpose of \( A \) is a Gorenstein transpose of \( A \), but the converse does not hold true in general. For example, for a module \( A \) in \( \text{mod} \ R \), if \( A \) is Gorenstein projective but not projective, then some Gorenstein transpose of \( A \) is zero, and any transpose of \( A \) is Gorenstein projective (see Proposition 3.4(3) below) but non-zero (otherwise, if a transpose of \( A \) is zero, then \( A \) is projective, which is a contradiction).

Let \( A \in \text{mod} \ R \). Recall from [AB] that \( A \) is said to have Gorenstein dimension zero if \( \text{Ext}^i_R(A, R) = 0 = \text{Ext}^i_{R^\text{op}}(\text{Tr}^G A, R) \) for any \( i \geq 1 \). It is easy to see that if \( A \) has Gorenstein dimension zero, then so does \( A^* \). In addition, it is well known that \( A \) has Gorenstein dimension zero if and only if it is Gorenstein projective. Let \( \sigma_A: A \to A^{**} \) defined via \( \sigma_A(x)(f) = f(x) \) for any \( x \in A \) and \( f \in A^* \) be the canonical evaluation homomorphism. Recall that a module \( A \in \text{mod} \ R \) is called torsionless (resp. reflexive) if \( \sigma_A \) is a monomorphism (resp. an isomorphism).

The following result establishes a relation between a Gorenstein transpose of a module with a transpose of the same module.

**Theorem 3.1.** Let \( M \in \text{mod} \ R^\text{op} \) and \( A \in \text{mod} \ R \). Then \( M \) is a Gorenstein transpose of \( A \) if and only if \( M \) can be embedded into a transpose \( \text{Tr}^G A \) of \( A \) with the cokernel Gorenstein projective, that is, there exists an exact sequence \( 0 \to M \to \text{Tr}^G A \to H \to 0 \) in \( \text{mod} \ R^\text{op} \) with \( H \) Gorenstein projective.

**Proof.** We first prove the necessity. Assume that \( M(\cong \text{Tr}^G \pi A) \) is a Gorenstein transpose of \( A \). Then there exists an exact sequence \( \pi: X_1 \xrightarrow{g} X_0 \to A \to 0 \) in \( \text{mod} \ R \) with \( X_0 \) and \( X_1 \) Gorenstein projective such that \( \text{Tr}^G \pi A = \text{Coker} \ g^* \). So there exists an exact sequence \( 0 \to H_1' \to P_0' \to X_0 \to 0 \) in \( \text{mod} \ R \) with \( P_0' \) projective and \( H_1' \) Gorenstein projective. Let \( K_1' = \text{Im} \ g \) and \( g = i \alpha \) be the natural epic-monic decomposition of \( g \). Then we have the following pull-back diagram:

```
\[
\begin{array}{ccccccccc}
0 & \quad & 0 & \downarrow & \quad & \downarrow & \quad & 0 \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
H_1' & \quad & H_1' & \downarrow & \quad & \downarrow & \quad & \downarrow \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
0 & \quad & K_1' & \xrightarrow{i} & P_0' & \to & A & \to & 0 \\
\downarrow & \quad & \downarrow & \quad & \quad & \quad & \quad & \quad & \downarrow \\
0 & \quad & K_1 & \xrightarrow{i} & X_0 & \to & A & \to & 0 \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
0 & \quad & 0 & \quad & \quad & \quad & \quad & \quad & 0
\end{array}
\]```
Now consider the following pull-back diagram:

\[
\begin{array}{c}
0 & \rightarrow & K_2 & \rightarrow & G & \rightarrow & K_1' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & X_1 & \rightarrow & \alpha & & K_1 & \rightarrow & 0 \\
\end{array}
\]

where $K_2 = \text{Ker} \ g$. Because both $X_1$ and $H_1'$ are Gorenstein projective, $G$ is Gorenstein projective by Lemma 2.1. So there exists an exact sequence $0 \rightarrow G_1 \rightarrow P_0 \rightarrow G \rightarrow 0$ in mod $R$ with $P_0$ projective and $G_1$ Gorenstein projective. Consider the following pull-back diagram:

\[
\begin{array}{c}
0 & \rightarrow & K_2' & \rightarrow & P_0 & \rightarrow & K_1' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K_2 & \rightarrow & G & \rightarrow & K_1 & \rightarrow & 0 \\
\end{array}
\]

So we get the following commutative diagram with exact rows:

\[
\begin{array}{c}
0 & \rightarrow & K_2' & \rightarrow & P_0 & \rightarrow & K_1' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K_2 & \rightarrow & G & \rightarrow & K_1 & \rightarrow & 0 \\
\end{array}
\]
It yields the following commutative diagram with exact columns and rows:

\[
\begin{array}{c}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\text{Ker } \beta & H_1 & H'_1 \\
\downarrow & \downarrow & \downarrow \\
0 & K'_2 & P_0 & K'_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & K_2 & X_1 & K_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

where \( H_1 = \text{Ker}(P_0 \to X_1) \). By the snake lemma, we get the exact sequence \( 0 \to \text{Ker } \beta \to H_1 \xrightarrow{h} H'_1 \to 0 \). By Lemma 2.1, \( H_1 \) is Gorenstein projective and hence \( \text{Ker } \beta \) is also Gorenstein projective. Combining the above diagram with the first one in this proof, we get the following commutative diagram with exact columns and rows:

\[
\begin{array}{c}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \text{Ker } \beta & H_1 & \overset{h}{\longrightarrow} H'_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & K'_2 & P_0 & K'_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & K_2 & X_1 & K_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

By applying the functor \((\ )^*\) to the above diagram, we get the following commutative diagram with exact columns and rows:

\[
\begin{array}{c}
0 & 0 \\
\uparrow & \uparrow \\
H_1^* & H'_1^* \\
\uparrow & \uparrow \\
P_0^* & P_0'^* \\
\uparrow & \uparrow \\
X_1^* & X_0^* \\
\uparrow & \uparrow \\
0 & 0
\end{array}
\]
By the snake lemma, we get an exact sequence:

\[ 0 \to \text{Tr}_{\mathbb{C}} A(= \text{Coker } g^*) \to \text{Tr } A \to \text{Coker } h^* \to 0 \]

in \( \text{mod } R^{op} \) with \( \text{Coker } h^*(\cong (\text{Ker } h)^* \cong (\text{Ker } \beta)^*) \) Gorenstein projective.

We next prove the sufficiency. Let \( P_1 \xrightarrow{f} P_0 \to A \to 0 \) be a projective presentation of \( A \) in \( \text{mod } R \). Then we have the following pull-back diagram:

\[
\begin{array}{ccccccc}
0 & \downarrow & 0 \\
\downarrow & & \downarrow \\
A^* & \xrightarrow{h^*} & A^* \\
\downarrow & & \downarrow \\
P_0^* & \xrightarrow{f^*} & P_0^* \\
\downarrow h & & \downarrow f^* \\
0 & \xrightarrow{g} & K & \xrightarrow{g^*} & P_1^* & \xrightarrow{f^*} & H & \xrightarrow{g^*} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{M} & \text{Tr } A & \xrightarrow{H} & H & \xrightarrow{0} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

Because \( H \) is Gorenstein projective and \( P_1^* \) is projective, \( K \) is Gorenstein projective by Lemma 2.1. Again because \( H \) is Gorenstein projective, by applying the functor \((\quad)^*\) to the above commutative diagram, we get the following commutative diagram with exact columns and rows:

\[
\begin{array}{ccccccc}
0 & \downarrow & 0 \\
\downarrow & & \downarrow \\
0 & \xrightarrow{H^*} & (\text{Tr } A)^* & \xrightarrow{g^*} & M^* & \xrightarrow{0} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{H^*} & P_1^{**} & \xrightarrow{f^{**}} & K^* & \xrightarrow{h^*} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P_0^{**} & \xrightarrow{f^{**}} & P_0^{**} & \xrightarrow{h^*} & A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
\]

By the snake lemma, we have \( \text{Im } h^* \cong \text{Im } f^{**} \). Thus we get \( \text{Coker } h^* = P_0^{**} / \text{Im } h^* \cong P_0^{**} / \text{Im } f^{**} \cong A \), and therefore we get a Gorenstein projective presentation of \( A \) in \( \text{mod } R \):

\[ K^* \xrightarrow{h^*} P_0^{**} \to A \to 0. \]
Because both $K$ and $P_0^*$ are reflexive, we get an exact sequence $0 \to A^* \to P_0^{**} \xrightarrow{h^{**}} K^{**} \to M \to 0$ in mod $R^{op}$ and $M$ is a Gorenstein transpose of $A$. □

As a consequence of Theorem 3.1, we get the following

**Corollary 3.2.** Let $A \in \text{mod } R$. Then for any Gorenstein projective module $H \in \text{mod } R^{op}$ and any transpose $\text{Tr} A$ of $A$, $H \oplus \text{Tr} A$ is a Gorenstein transpose of $A$.

**Proof.** Assume that $H \in \text{mod } R^{op}$ is a Gorenstein projective module. Then there exists an exact sequence $0 \to H \to P \to H' \to 0$ in mod $R^{op}$ with $P$ projective and $H'$ Gorenstein projective, which induces an exact sequence $0 \to H \oplus \text{Tr} A \to P \oplus \text{Tr} A \to H' \to 0$. Because $P \oplus \text{Tr} A$ is again a transpose of $A$, $H \oplus \text{Tr} A$ is a Gorenstein transpose of $A$ by Theorem 3.1. □

It is clear that the Gorenstein transpose of a module $A$ in mod $R$ depends on the choice of the Gorenstein projective presentation of $A$. Corollary 3.2 provides a method to construct a Gorenstein transpose of a module from a transpose of the same module. It is interesting to ask the following

**Question 3.3.** Is any Gorenstein transpose obtained in this way?

If the answer to this question is positive, then we can conclude that the Gorenstein transpose of a module is unique up to Gorenstein projective equivalence.

Let $A \in \text{mod } R$. By [A, Proposition 6.3] (or [AB, Proposition 2.6]), there exists an exact sequence:

$$0 \to \text{Ext}^1_{R^{op}}(\text{Tr} A, R) \to A \xrightarrow{\sigma A} A^{**} \to \text{Ext}^2_{R^{op}}(\text{Tr} A, R) \to 0$$

(\*)

in mod $R$. For a positive integer $n$, recall from [AB] that $A$ is called $n$-torsionfree if $\text{Ext}^i_{R^{op}}(\text{Tr} A, R) = 0$ for any $1 \leq i \leq n$. From the exact sequence (\*), it is easy to see that $A$ is torsionless (resp. reflexive) if and only if it is 1-torsionfree (resp. 2-torsionfree).

The following result shows that some homological properties of any Gorenstein transpose and any transpose of a given module are identical.

**Proposition 3.4.** Let $A \in \text{mod } R$. Then for any Gorenstein transpose $\text{Tr}^G_A$ and any transpose $\text{Tr} A$ of $A$, we have

1. $\text{Ext}^i_{R^{op}}(\text{Tr}^G_A, R) \cong \text{Ext}^i_{R^{op}}(\text{Tr} A, R)$ for any $i \geq 1$.
2. For any $n \geq 1$, $\text{Tr}^G_A$ is $n$-torsionfree if and only if so is $\text{Tr} A$.
3. Some Gorenstein transpose of $A$ is zero if and only if $A$ is Gorenstein projective, if and only if any (Gorenstein) transpose of $A$ is Gorenstein projective.
4. $\text{Gpd}_{R^{op}}(\text{Tr}^G_A) = \text{Gpd}_{R^{op}}(\text{Tr} A)$.

**Proof.**

(1) It is an immediate consequence of Theorem 3.1.

(2) Let $\text{Tr}^G_A$ be any Gorenstein transpose of $A$. By Theorem 3.1, there exists a transpose $\text{Tr} A$ of $A$ satisfying the exact sequence $0 \to \text{Tr}^G_A \to \text{Tr} A \to H \to 0$ in mod $R^{op}$ with $H$ Gorenstein projective.

If $\text{Ext}^1_R(\text{Tr}(\text{Tr} A), R) = 0$, then $\text{Tr} A$ is torsionless. So $\text{Tr}^G_A$ is also torsionless and $\text{Ext}^i_R(\text{Tr}(\text{Tr}^G_A), R) = 0$. Because $H$ is Gorenstein projective, we get an exact sequence $0 \to \text{Tr} H \to \text{Tr}(\text{Tr} A) \to \text{Tr}(\text{Tr}^G_A) \to 0$ in mod $R$ with $\text{Tr} H$ Gorenstein projective. So we have that $\text{Ext}^i_R(\text{Tr}(\text{Tr}^G_A), R) \cong \text{Ext}^i_R(\text{Tr}(\text{Tr} A), R)$ for any $i \geq 2$, and $\text{Ext}^i_R(\text{Tr}(\text{Tr}^G_A), R) \to \text{Ext}^i_R(\text{Tr}(\text{Tr} A), R) \to 0$ is exact. So for any $i \geq 1$, $\text{Ext}^i_R(\text{Tr}(\text{Tr}^G_A), R) = 0$ if and only if $\text{Ext}^i_R(\text{Tr}(\text{Tr} A), R) = 0$, and thus we conclude that for any $n \geq 1$, $\text{Tr}^G_A$ is $n$-torsionfree if and only if so is $\text{Tr} A$.

(3) Because $A$ is a (Gorenstein) transpose of any (Gorenstein) transpose of $A$, it is not difficult to verify the assertion by (1) and (2).
(4) Let $\text{Tr}_G^\pi A$ be any Gorenstein transpose of $A$. If $\text{Tr}_G^\pi A = 0$, then the assertion follows from (3). Now suppose $\text{Tr}_G^\pi A \neq 0$. By Theorem 3.1, there exists a transpose $\text{Tr} A$ of $A$ satisfying the exact sequence $0 \rightarrow \text{Tr}_G^\pi A \rightarrow \text{Tr} A ightarrow H \rightarrow 0 \pmod{R^{\text{op}}}$ with $H$ Gorenstein projective. Then we have that $\text{Gpd}_{R^{\text{op}}} (\text{Tr}_G^\pi A) = \text{Gpd}_{R^{\text{op}}} (\text{Tr} A)$ by Lemma 2.1. □

Let $A \in \text{mod } R$. By Proposition 3.4(1), we have that $A$ is $n$-torsionfree if and only if $\text{Ext}_{R^{\text{op}}}^i(\text{Tr}_G^\pi A, R) = 0$ for any (or some) Gorenstein transpose $\text{Tr}_G^\pi A$ of $A$ and $1 \leq i \leq n$. On the other hand, also by Proposition 3.4(1), we get a Gorenstein version of the formula (∗) as follows. For any Gorenstein transpose $\text{Tr}_G^\pi A$ of $A$, we have the following exact sequence:

$$0 \rightarrow \text{Ext}_{R^{\text{op}}}^1(\text{Tr}_G^\pi A, R) \rightarrow A \xrightarrow{\sigma A} A^{**} \rightarrow \text{Ext}_{R^{\text{op}}}^2(\text{Tr}_G^\pi A, R) \rightarrow 0$$

in mod $R$. It is easy to see that $A$ is a Gorenstein transpose of $\text{Tr}_G^\pi A$. So we also get the following exact sequence:

$$0 \rightarrow \text{Ext}_{R}^1(A, R) \rightarrow \text{Tr}_G^\pi A \xrightarrow{\sigma \text{Tr}_G^\pi A} (\text{Tr}_G^\pi A)^{**} \rightarrow \text{Ext}_{R}^2(A, R) \rightarrow 0$$

in mod $R^{\text{op}}$.

The following result shows that any double Gorenstein transpose of $A$ shares some homological properties of $A$.

**Corollary 3.5.** Let $A \in \text{mod } R$. Then for any Gorenstein transpose $\text{Tr}_G^\pi A$ of $A$ and any Gorenstein transpose $\text{Tr}_G^\pi (\text{Tr}_G^\pi A)$ of $\text{Tr}_G^\pi A$, we have

1. $\text{Ext}_{R}^i(\text{Tr}_G^\pi (\text{Tr}_G^\pi A), R) \cong \text{Ext}_{R}^i(A, R)$ for any $i \geq 1$.
2. For any $n \geq 1$, $\text{Tr}_G^\pi (\text{Tr}_G^\pi A)$ is $n$-torsionfree if and only if so is $A$.
3. $\text{Gpd}_{R} (\text{Tr}_G^\pi (\text{Tr}_G^\pi A)) = \text{Gpd}_{R} (A)$.

**Proof.** Note that $A$ is a Gorenstein transpose of any Gorenstein transpose $\text{Tr}_G^\pi A$ of $A$. So all of the assertions follow from Proposition 3.4. □

Note that a transpose of a module is a special Gorenstein transpose of the same module. The following result shows that a module with Gorenstein projective dimension $n$ is a double Gorenstein transpose of a module with projective dimension $n$.

**Proposition 3.6.** Let $A \in \text{mod } R$ and $n$ be a non-negative integer. Then $\text{Gpd}_{R} (A) = n$ if and only if there exists a module $B \in \text{mod } R$ with $\text{pd}_{R} (B) = n$ such that $A$ is a Gorenstein transpose of some transpose $\text{Tr} B$ of $B$ (that is, $A = \text{Tr}_G^\pi (\text{Tr} B)$, where $\text{Tr}_G^\pi (\text{Tr} B)$ is a Gorenstein transpose of some transpose $\text{Tr} B$ of $B$).

**Proof.** Assume that $A \in \text{mod } R$ with $\text{Gpd}_{R} (A) = n$. By Corollary 2.5, there exists an exact sequence $0 \rightarrow A \rightarrow B \rightarrow H \rightarrow 0 \pmod{R}$ with $\text{pd}_{R} (B) = n$ and $H$ Gorenstein projective. Note that $B$ is a transpose of some transpose $\text{Tr} B$ of $B$. By Theorem 3.1, $A$ is a Gorenstein transpose of $\text{Tr} B$.

Conversely, if $A$ is a Gorenstein transpose of some transpose $\text{Tr} B$ of a module $B \in \text{mod } R$ with $\text{pd}_{R} (B) = n$, then $\text{Gpd}_{R} (A) = \text{Gpd}_{R} (B) = \text{pd}_{R} (B) = n$ by Corollary 3.5. □

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