Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Algebra 285 (2005) 669–681

JOURNAL OF
Algebra

www.elsevier.com/locate/jalgebra

# On $U$-dominant dimension

Zhaoyong Huang

*Department of Mathematics, Nanjing University, Nanjing 210093, PR China*

Received 20 August 2003

Available online 1 February 2005

Communicated by Kent R. Fuller

**Abstract**

Let $\Lambda$ and $\Gamma$ be artin algebras and ${}_\Lambda U_\Gamma$ a faithfully balanced selforthogonal bimodule. We show that the $U$-dominant dimensions of ${}_\Lambda U$ and $U_\Gamma$ are identical. As applications of the results obtained, we give some characterizations of the double $U$-dual functors preserving monomorphisms and being left exact respectively.


*Keywords:* $U$-dominant dimension; Flat dimension; Faithfully balanced selforthogonal bimodules; Double $U$-dual functors

## 1. Introduction

For a ring $\Lambda$, we use $\operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Lambda^{\mathrm{op}}$) to denote the category of finitely generated left $\Lambda$-modules (respectively right $\Lambda$-modules).

**Definition 1.1.** Let $\Lambda$ and $\Gamma$ be rings. A bimodule ${}_\Lambda T_\Gamma$ is called a faithfully balanced selforthogonal bimodule if it satisfies the following conditions:

(1) ${}_\Lambda T \in \operatorname{mod}\Lambda$ and $T_\Gamma \in \operatorname{mod}\Gamma^{\mathrm{op}}$.
(2) The natural maps $\Lambda \to \operatorname{End}(T_\Gamma)$ and $\Gamma \to \operatorname{End}({}_\Lambda T)^{\mathrm{op}}$ are isomorphisms.
(3) $\operatorname{Ext}^i_\Lambda({}_\Lambda T, {}_\Lambda T) = 0$ and $\operatorname{Ext}^i_\Gamma(T_\Gamma, T_\Gamma) = 0$ for any $i \geqslant 1$.

*E-mail address:* huangzy@nju.edu.cn.

0021-8693/$ – see front matter 
doi:10.1016/j.jalgebra.2004.11.008

**Definition 1.2.** Let $U$ be in mod $\Lambda$ (respectively mod $\Gamma^{\mathrm{op}}$) and $n$ a non-negative integer. For a module $M$ in mod $\Lambda$ (respectively mod $\Gamma^{\mathrm{op}}$),

(1) $M$ is said to have $U$-dominant dimension greater than or equal to $n$, written $U$-dom.dim$({}_\Lambda M)$ (respectively $U$-dom.dim$(M_\Gamma)) \geqslant n$, if each of the first $n$ terms in a minimal injective resolution of $M$ is cogenerated by ${}_\Lambda U$ (respectively $U_\Gamma$), that is, each of these terms can be embedded into a direct product of copies of ${}_\Lambda U$ (respectively $U_\Gamma$) [10].

(2) $M$ is said to have dominant dimension greater than or equal to $n$, written dom.dim$({}_\Lambda M)$ (respectively dom.dim$(M_\Gamma)) \geqslant n$, if each of the first $n$ terms in a minimal injective resolution of $M$ is $\Lambda$-projective (respectively $\Gamma^{\mathrm{op}}$-projective) [12].

Assume that $\Lambda$ is an artin algebra. By [4, Theorem 3.3], $\Lambda^I$ and each of its direct summands are projective for any index set $I$. So, when ${}_\Lambda U = {}_\Lambda\Lambda$ (respectively $U_\Gamma = \Lambda_\Lambda$), the notion of $U$-dominant dimension coincides with that of (ordinary) dominant dimension. Tachikawa in [12] showed that if $\Lambda$ is a left and right artinian ring then the dominant dimensions of ${}_\Lambda\Lambda$ and $\Lambda_\Lambda$ are identical. Hoshino then in [6] generalized this result to left and right noetherian rings. Kato in [10] characterized the modules with $U$-dominant dimension greater than or equal to one. Colby and Fuller in [5] gave some equivalent conditions of dom.dim$({}_\Lambda\Lambda) \geqslant 1$ (or 2) in terms of the properties of the double dual functors (with respect to ${}_\Lambda\Lambda_\Lambda$).

The results mentioned above motivate our interests in establishing the identity of $U$-dominant dimensions of ${}_\Lambda U$ and $U_\Gamma$ and characterizing the properties of modules with a given $U$-dominant dimension. Our characterizations will lead a better comprehension about $U$-dominant dimension and the theory of selforthogonal bimodules.

Throughout this paper, $\Lambda$ and $\Gamma$ are artin algebras and ${}_\Lambda U_\Gamma$ is a faithfully balanced selforthogonal bimodule. The main result in this paper is the following

**Theorem 1.3.** $U$-dom.dim$({}_\Lambda U) = U$-dom.dim$(U_\Gamma)$.

Put ${}_\Lambda U_\Gamma = {}_\Lambda\Lambda_\Lambda$, we immediately get the following result, which is due to Tachikawa (see [12]).

**Corollary 1.4.** dom.dim$({}_\Lambda\Lambda) =$ dom.dim$(\Lambda_\Lambda)$.

Let $M$ be in mod $\Lambda$ (respectively mod $\Gamma^{\mathrm{op}}$) and $G(M)$ the subcategory of mod $\Lambda$ (respectively mod $\Gamma^{\mathrm{op}}$) consisting of all submodules of the modules generated by $M$. $M$ is called a QF-3 module if $G(M)$ has a cogenerator which is a direct summand of every other cogenerator [13]. By [13] Proposition 2.2 we have that a finitely cogenerated $\Lambda$-module (respectively $\Gamma^{\mathrm{op}}$-module) $M$ is a QF-3 module if and only if $M$ cogenerates its injective envelope. So by Theorem 1.3 we have

**Corollary 1.5.** ${}_\Lambda U$ *is* QF-3 *if and only if* $U_\Gamma$ *is* QF-3.

We shall prove our main result in Section 2. We study the case that the double $U$-dual functors $(-)^{**}$ preserves monomorphisms by the language of Lambek torsion theory, show the left–right symmetry of the fact that $(-)^{**}$ preserves monomorphisms, and then prove the main result. It should be pointed out that this strategy is similar to that of Hoshino [7]. As applications of the results obtained in Section 2, we give in Section 3 some characterizations of the double $U$-dual functors $(-)^{**}$ preserving monomorphisms and being left exact respectively. The results of this paper are natural generalizations of (ordinary) dominant dimension and of several author's approach to dominant dimension (see Tachikawa [12], Colby–Fuller [5] and Hoshino [6,7]). In fact, most of the results here are the $U$-dual versions of the results in [6,7].

## 2. The proof of main result

Let $E_0$ be the injective envelope of ${}_\Lambda U$. Then $E_0$ defines a torsion theory in $\operatorname{mod}\Lambda$. The torsion class $\mathcal{T}$ is the subcategory of $\operatorname{mod}\Lambda$ consisting of the modules $X$ satisfying $\operatorname{Hom}_\Lambda(X, E_0) = 0$, and the torsionfree class $\mathcal{F}$ is the subcategory of $\operatorname{mod}\Lambda$ consisting of the modules $Y$ cogenerated by $E_0$ (equivalently, $Y$ can be embedded in $E_0^I$ for some index set $I$). A module in $\operatorname{mod}\Lambda$ is called torsion (respectively torsionfree) if it is in $\mathcal{T}$ (respectively $\mathcal{F}$). The injective envelope $E_0'$ of $U_\Gamma$ also defines a torsion theory in $\operatorname{mod}\Gamma^{\mathrm{op}}$ and we may give in $\operatorname{mod}\Gamma^{\mathrm{op}}$ the corresponding notions as above. Let $X$ be in $\operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Gamma^{\mathrm{op}}$) and $t(X)$ the torsion submodule, that is, $t(X)$ is the submodule $X$ such that $\operatorname{Hom}_\Lambda(t(X), E_0) = 0$ (respectively $\operatorname{Hom}_\Gamma(t(X), E_0') = 0$) and $E_0$ (respectively $E_0'$) cogenerates $X/t(X)$ (cf. [9]).

Let $A$ be in $\operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Gamma^{\mathrm{op}}$). We call $\operatorname{Hom}_\Lambda({}_\Lambda A, {}_\Lambda U_\Gamma)$ (respectively $\operatorname{Hom}_\Gamma(A_\Gamma, {}_\Lambda U_\Gamma)$) the dual module of $A$ with respect to ${}_\Lambda U_\Gamma$, and denote either of these modules by $A^*$. For a homomorphism $f$ between $\Lambda$-modules (respectively $\Gamma^{\mathrm{op}}$-modules), we put $f^* = \operatorname{Hom}(f, {}_\Lambda U_\Gamma)$. Let $\sigma_A : A \to A^{**}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$ be the canonical evaluation homomorphism. $A$ is called $U$-torsionless (respectively $U$-reflexive) if $\sigma_A$ is a monomorphism (respectively an isomorphism).

The following result is analogous to [7, Lemma 4].

**Lemma 2.1.** *For a module $X$ in* $\operatorname{mod}\Lambda$ (*respectively* $\operatorname{mod}\Gamma^{\mathrm{op}}$), $t(X) = \operatorname{Ker}\sigma_X$ *if and only if* $\operatorname{Hom}_\Lambda(\operatorname{Ker}\sigma_X, E_0) = 0$ (*respectively* $\operatorname{Hom}_\Gamma(\operatorname{Ker}\sigma_X, E_0') = 0$).

**Proof.** The necessity is trivial. Now we prove the sufficiency.

We have the following commutative diagram with the upper row exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t(X) & \longrightarrow & X & \xrightarrow{\pi} & X/t(X) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \sigma_X} & & \downarrow{\scriptstyle \sigma_{X/t(X)}} & & \\
 & & & & X^{**} & \xrightarrow{\pi^{**}} & [X/t(X)]^{**} & &
\end{array}$$

Since $\operatorname{Hom}_\Lambda(t(X), E_0) = 0$, $[t(X)]^* = 0$ and $\pi^*$ is an isomorphism. So $\pi^{**}$ is also an isomorphism and hence $t(X) \subset \operatorname{Ker}\sigma_X$. On the other hand, $\operatorname{Hom}_\Lambda(\operatorname{Ker}\sigma_X, E_0) = 0$ by assumption, which implies that $\operatorname{Ker}\sigma_X$ is a torsion module and contained in $X$. So we conclude that $\operatorname{Ker}\sigma_X \subset t(X)$ and $\operatorname{Ker}\sigma_X = t(X)$. □

**Remark.** From the above proof we always have $t(X) \subset \operatorname{Ker}\sigma_X$.

Suppose that $A \in \operatorname{mod}\Lambda$ (respectively $\operatorname{mod}\Gamma^{\mathrm{op}}$) and $P_1 \xrightarrow{f} P_0 \to A \to 0$ is a (minimal) projective resolution of $A$. Then we have an exact sequence

$$0 \to A^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{Coker} f^* \to 0.$$

We call $\operatorname{Coker} f^*$ the transpose (with respect to ${}_\Lambda U_\Gamma$) of $A$, and denote it by $\operatorname{Tr}_U A$.

The following result is the $U$-dual version of [7, Theorem A].

**Proposition 2.2.** *The following statements are equivalent.*

(1) $t(X) = \operatorname{Ker}\sigma_X$ *for every* $X \in \operatorname{mod}\Lambda$.
(2) $f^{**}$ *is monic for every monomorphism* $f : A \to B$ *in* $\operatorname{mod}\Lambda$.
$(1)^{\mathrm{op}}$ $t(Y) = \operatorname{Ker}\sigma_Y$ *for every* $Y \in \operatorname{mod}\Gamma^{\mathrm{op}}$.
$(2)^{\mathrm{op}}$ $g^{**}$ *is monic for every monomorphism* $g : C \to D$ *in* $\operatorname{mod}\Gamma^{\mathrm{op}}$.

**Proof.** By symmetry, it suffices to prove the implications of $(1) \Rightarrow (2)^{\mathrm{op}} \Rightarrow (1)^{\mathrm{op}}$.

$(1) \Rightarrow (2)^{\mathrm{op}}$. Let $g : C \to D$ be monic in $\operatorname{mod}\Gamma^{\mathrm{op}}$. Set $X = \operatorname{Coker} g$. We have that $\operatorname{Ker}\sigma_{\operatorname{Tr}_U X} \cong \operatorname{Ext}^1_\Gamma(X, U)$ and $\operatorname{Tr}_U X \in \operatorname{mod}\Lambda$ by [8, Lemma 2.1]. By (1) and Lemma 2.1, $\operatorname{Hom}_\Lambda(\operatorname{Ext}^1_\Gamma(X, U), E_0) = 0$. Since $\operatorname{Coker} g^*$ can be imbedded in $\operatorname{Ext}^1_\Gamma(X, U)$, $\operatorname{Hom}_\Lambda(\operatorname{Coker} g^*, E_0) = 0$. But $(\operatorname{Coker} g^*)^* \subset \operatorname{Hom}_\Lambda(\operatorname{Coker} g^*, E_0)$, so $(\operatorname{Coker} g^*)^* = 0$ and hence $\operatorname{Ker} g^{**} \cong (\operatorname{Coker} g^*)^* = 0$, which implies that $g^{**}$ is monic.

$(2)^{\mathrm{op}} \Rightarrow (1)^{\mathrm{op}}$. Let $Y$ be in $\operatorname{mod}\Gamma^{\mathrm{op}}$ and $X$ any submodule of $\operatorname{Ker}\sigma_Y$ and $f_1 : X \to \operatorname{Ker}\sigma_Y$ the inclusion. Assume that $f$ is the composition:

$$X \xrightarrow{f_1} \operatorname{Ker}\sigma_Y \to Y.$$

Then $\sigma_Y f = 0$ and $f^*\sigma_Y^* = (\sigma_Y f)^* = 0$. But $\sigma_Y^*$ is epic by [1, Proposition 20.14], so $f^* = 0$ and $f^{**} = 0$. By $(2)^{\mathrm{op}}$, $f^{**}$ is monic, so $X^{**} = 0$ and $X^{***} = 0$. Since $X^*$ is isomorphic to a submodule of $X^{***}$ by [1, Proposition 20.14], $X^* = 0$.

We claim: $\operatorname{Hom}_\Gamma(\operatorname{Ker}\sigma_Y, E_0') = 0$. Otherwise, there exists $0 \neq \alpha \in \operatorname{Hom}_\Gamma(\operatorname{Ker}\sigma_Y, E_0')$. Then $\operatorname{Im}\alpha \cap U_\Gamma \neq 0$ since $U_\Gamma$ is an essential submodule of $E_0'$. So $\alpha^{-1}(\operatorname{Im}\alpha \cap U_\Gamma)$ is a non-zero submodule of $\operatorname{Ker}\sigma_Y$ and there exists a non-zero map $\alpha^{-1}(\operatorname{Im}\alpha \cap U_\Gamma) \to U_\Gamma$, which implies that $(\alpha^{-1}(\operatorname{Im}\alpha \cap U_\Gamma))^* \neq 0$, a contradiction with the former argument. Hence we conclude that $t(Y) = \operatorname{Ker}\sigma_Y$ by Lemma 2.1. □

Let $A$ be a $\Lambda$-module (respectively a $\Gamma^{\mathrm{op}}$-module). Denote either of $\operatorname{Hom}_\Lambda({}_\Lambda U_\Gamma, {}_\Lambda A)$ and $\operatorname{Hom}_\Gamma({}_\Lambda U_\Gamma, A_\Gamma)$ by ${}^*A$, and the left (respectively right) flat dimension of $A$ by $\mathrm{l.fd}_\Lambda(A)$ (respectively $\mathrm{r.fd}_\Gamma(A)$). We give a remark as follows. For an artin algebra $R$ and a left (respectively right) $R$-module $A$, we have that the left (respectively right) flat dimension of $A$ and its left (respectively right) projective dimension are identical; especially, $A$ is left (respectively right) flat if and only if it is left (respectively right) projective.

**Lemma 2.3.** *Let* ${}_\Lambda E$ (*respectively* $E_\Gamma$) *be injective and* $n$ *a non-negative integer. Then* $\mathrm{l.fd}_\Gamma({}^*E)$ (*respectively* $\mathrm{r.fd}_\Lambda({}^*E) \leqslant n$) *if and only if* $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^{n+1}(A, U), E)$ (*respectively* $\mathrm{Hom}_\Gamma(\mathrm{Ext}_\Lambda^{n+1}(A, U), E) = 0$) *for any* $A \in \mathrm{mod}\,\Gamma^{\mathrm{op}}$ (*respectively* $\mathrm{mod}\,\Lambda$).

**Proof.** It is trivial by [3, Chapter VI, Proposition 5.3]. □

The following result is similar to [7, Proposition B]. In fact, we obtain the first two statements of this result by replacing "$E({}_R R)$ is flat" and "$E$ is flat" of [7, Proposition B] by "${}^*E_0$ is flat" and "${}^*E$ is flat" respectively. The third statement is analogous to the corresponding one of [7, Proposition B].

**Proposition 2.4.** *The following statements are equivalent.*

(1) ${}^*E_0$ *is flat.*
(2) *There is an injective* $\Lambda$*-module* $E$ *such that* ${}^*E$ *is flat and* $E$ *cogenerates* $E_0$.
(3) $t(X) = \mathrm{Ker}\,\sigma_X$ *for any* $X \in \mathrm{mod}\,\Lambda$.

**Proof.** (1) $\Rightarrow$ (2). It is trivial.

(2) $\Rightarrow$ (3). Let $X \in \mathrm{mod}\,\Lambda$. Since $\mathrm{Ker}\,\sigma_X \cong \mathrm{Ext}_\Gamma^1(\mathrm{Tr}_U X, U)$ with $\mathrm{Tr}_U X \in \mathrm{mod}\,\Gamma^{\mathrm{op}}$ by [8, Lemma 2.1]. By (2) and Lemma 2.3, $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^1(\mathrm{Tr}_U X, U), E) = 0$.

Since $E$ cogenerates $E_0$, there is an exact sequence $0 \to E_0 \to E^I$ for some index set $I$. So

$$\begin{aligned}\mathrm{Hom}_\Lambda\big(\mathrm{Ext}_\Gamma^1(\mathrm{Tr}_U X, U), E_0\big) &\subset \mathrm{Hom}_\Lambda\big(\mathrm{Ext}_\Gamma^1(\mathrm{Tr}_U X, U), E^I\big)\\ &\cong \big[\mathrm{Hom}_\Lambda\big(\mathrm{Ext}_\Gamma^1(\mathrm{Tr}_U X, U), E\big)\big]^I = 0 \quad \text{and}\end{aligned}$$

$$\mathrm{Hom}_\Lambda\big(\mathrm{Ext}_\Gamma^1(\mathrm{Tr}_U X, U), E_0\big) = 0.$$

By Lemma 2.1, $t(X) = \mathrm{Ker}\,\sigma_X$.

(3) $\Rightarrow$ (1). Let $N \in \mathrm{mod}\,\Gamma^{\mathrm{op}}$. Since $\mathrm{Ker}\,\sigma_{\mathrm{Tr}_U N} \cong \mathrm{Ext}_\Gamma^1(N, U)$ with $\mathrm{Tr}_U N \in \mathrm{mod}\,\Lambda$ by [8, Lemma 2.1], By (3) and Lemma 2.1 we have $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^1(N, U), E_0) \cong \mathrm{Hom}_\Lambda(\mathrm{Ker}\,\sigma_{\mathrm{Tr}_U N}, E_0) = 0$, and so ${}^*E_0$ is flat by Lemma 2.3. □

Dually, we have the following

**Proposition 2.4′.** *The following statements are equivalent.*

(1) ${}^*E_0'$ *is flat.*
(2) *There is an injective* $\Gamma^{\mathrm{op}}$*-module* $E'$ *such that* ${}^*E'$ *is flat and* $E'$ *cogenerates* $E_0'$.
(3) $t(Y) = \mathrm{Ker}\,\sigma_Y$ *for any* $Y \in \mathrm{mod}\,\Gamma^{\mathrm{op}}$.

**Corollary 2.5.** ${}^*E_0$ *is flat if and only if* ${}^*E_0'$ *is flat.*

**Proof.** By Propositions 2.2, 2.4 and 2.4′. □

Let $A \in \operatorname{mod} \Lambda$ (respectively $\operatorname{mod} \Gamma^{\mathrm{op}}$) and $i$ a non-negative integer. We say that the grade of $A$ with respect to ${}_\Lambda U_\Gamma$, written $\operatorname{grade}_U A$, is greater than or equal to $i$ if $\operatorname{Ext}^j_\Lambda(A, U) = 0$ (respectively $\operatorname{Ext}^j_\Gamma(A, U) = 0$) for any $0 \leqslant j < i$.

**Lemma 2.6.** *Let $X$ be in $\operatorname{mod} \Gamma^{\mathrm{op}}$ and $n$ a non-negative integer. If $\operatorname{grade}_U X \geqslant n$ and $\operatorname{grade}_U \operatorname{Ext}^n_\Gamma(X, U) \geqslant n+1$, then $\operatorname{Ext}^n_\Gamma(X, U) = 0$.*

**Proof.** Since $X^*$ is $U$-torsionless, $X^{**} = 0$ if and only if $X^* = 0$. Then the case $n = 0$ follows.

Now let $n \geqslant 1$ and

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to X \to 0$$

be a projective resolution of $X$ in $\operatorname{mod} \Gamma^{\mathrm{op}}$. Put $X_n = \operatorname{Coker}(P_{n+1} \to P_n)$. Then we have an exact sequence

$$0 \to P_0^* \to \cdots \to P_{n-1}^* \xrightarrow{f} X_n^* \to \operatorname{Ext}^n_\Gamma(X, U) \to 0$$

in $\operatorname{mod} \Lambda$ with each $P_i^* \in \operatorname{add} {}_\Lambda U$. Since $\operatorname{grade}_U \operatorname{Ext}^n_\Gamma(X, U) \geqslant n+1$,

$$\operatorname{Ext}^i_\Lambda\big(\operatorname{Ext}^n_\Gamma(X, U), U\big) = 0 \quad \text{for any } 0 \leqslant i \leqslant n.$$

So $\operatorname{Ext}^i_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), P_j^*) = 0$ for any $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n-1$, and hence $\operatorname{Ext}^1_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), \operatorname{Im} f) \cong \operatorname{Ext}^n_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), P_0^*) = 0$, which implies that we have an exact sequence $\operatorname{Hom}_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), X_n^*) \to \operatorname{Hom}_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), \operatorname{Ext}^n_\Gamma(X, U)) \to 0$. Notice that $X_n^*$ is $U$-torsionless and $\operatorname{Hom}_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), U) = 0$. So $\operatorname{Hom}_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), X_n^*) = 0$ and $\operatorname{Hom}_\Lambda(\operatorname{Ext}^n_\Gamma(X, U), \operatorname{Ext}^n_\Gamma(X, U)) = 0$, which implies that $\operatorname{Ext}^n_\Gamma(X, U) = 0$. □

**Remark.** We point out that all of the above results (from 2.1 to 2.6) in this section also hold in the case $\Lambda$ and $\Gamma$ are left and right noetherian rings.

For a module $T$ in $\operatorname{mod} \Lambda$ (respectively $\operatorname{mod} \Gamma^{\mathrm{op}}$), we use $\operatorname{add} {}_\Lambda T$ (respectively $\operatorname{add} T_\Gamma$) to denote the subcategory of $\operatorname{mod} \Lambda$ (respectively $\operatorname{mod} \Gamma^{\mathrm{op}}$) consisting of all modules isomorphic to direct summands of finite direct sums of copies of ${}_\Lambda T$ (respectively $T_\Gamma$). Let $A$ be in $\operatorname{mod} \Lambda$. If there is an exact sequence $\cdots \to U_n \to \cdots \to U_1 \to U_0 \to A \to 0$ in $\operatorname{mod} \Lambda$ with each $U_i \in \operatorname{add} {}_\Lambda U$ for any $i \geqslant 0$, then we define $U\text{-resol.dim}_\Lambda(A) = \inf\{n \mid$ there is an exact sequence $0 \to U_n \to \cdots \to U_1 \to U_0 \to A \to 0$ in $\operatorname{mod} \Lambda$ with each $U_i \in \operatorname{add} {}_\Lambda U$ for any $0 \leqslant i \leqslant n\}$. We set $U\text{-resol.dim}_\Lambda(A)$ infinity if no such an integer exists. Dually, for a module $B$ in $\operatorname{mod} \Gamma^{\mathrm{op}}$, we may define $U\text{-resol.dim}_\Gamma(B)$ (see [2]).

**Lemma 2.7.** *Let $E$ be injective in $\operatorname{mod} \Lambda$ (respectively $\operatorname{mod} \Gamma^{\mathrm{op}}$). Then $\operatorname{l.fd}_\Gamma({}^*E)$ (respectively $\operatorname{r.fd}_\Lambda({}^*E) \leqslant n$) if and only if $U\text{-resol.dim}_\Lambda(E)$ (respectively $U\text{-resol.dim}_\Gamma(E) \leqslant n$).*

**Proof.** Assume that $E$ is injective in $\operatorname{mod}\Lambda$ and $\text{l.fd}_\Gamma({}^*E) \leqslant n$. Then there is an exact sequence $0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to {}^* E \to 0$ with each $Q_i$ flat (and hence projective) in $\operatorname{mod}\Gamma$ for any $0 \leqslant i \leqslant n$. By [3, Chapter VI, Proposition 5.3] $\operatorname{Tor}_j^\Gamma(U, {}^* E) \cong \operatorname{Hom}_\Lambda(\operatorname{Ext}_\Gamma^j(U,U), E) = 0$ for any $j \geqslant 1$. Then we easily have an exact sequence:

$$0 \to U \otimes_\Gamma Q_n \to \cdots \to U \otimes_\Gamma Q_1 \to U \otimes_\Gamma Q_0 \to U \otimes_\Gamma^* E \to 0.$$

It is clear that $U \otimes_\Gamma Q_i \in \operatorname{add} {}_\Lambda U$ for any $0 \leqslant i \leqslant n$. By [11, p. 47], $U \otimes_\Gamma^* E \cong \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Gamma(U,U), E) \cong E$. Hence we conclude that $U\text{-resol.dim}_\Lambda(E) \leqslant n$.

Conversely, if $U\text{-resol.dim}_\Lambda(E) \leqslant n$ then there is an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to E \to 0$ with each $X_i$ in $\operatorname{add} {}_\Lambda U$ for any $0 \leqslant i \leqslant n$. Since $\operatorname{Ext}_\Lambda^j(U, X_i) = 0$ for any $j \geqslant 1$ and $0 \leqslant i \leqslant n$, $0 \to {}^*X_n \to \cdots \to {}^*X_1 \to {}^*X_0 \to {}^* E \to 0$ is exact with each ${}^*X_i$ $(0 \leqslant i \leqslant n)$ $\Gamma$-projective. Hence we are done. □

**Corollary 2.8.** *Let $E$ be injective in* $\operatorname{mod}\Lambda$ (*respectively* $\operatorname{mod}\Gamma^{\text{op}}$). *Then ${}^*E$ is flat in* $\operatorname{mod}\Gamma$ (*respectively* $\operatorname{mod}\Lambda^{\text{op}}$) *if and only if* ${}_\Lambda E \in \operatorname{add} {}_\Lambda U$ (*respectively* $E_\Gamma \in \operatorname{add} U_\Gamma$).

From now on, assume that

$$0 \to {}_\Lambda U \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} E_i \xrightarrow{f_{i+1}} \cdots$$

is a minimal injective resolution of ${}_\Lambda U$.

The following result is the $U$-dual version of [6, Lemma 2.2].

**Lemma 2.9.** *Suppose* $U\text{-dom.dim}({}_\Lambda U) \geqslant 1$. *Then, for any $n \geqslant 2$,* $U\text{-dom.dim}({}_\Lambda U) \geqslant n$ *if and only if* $\operatorname{grade}_U M \geqslant n$ *for any $M \in \operatorname{mod}\Lambda$ with $M^* = 0$.*

**Proof.** For any $M \in \operatorname{mod}\Lambda$ and $i \geqslant 1$, we have an exact sequence

$$\operatorname{Hom}_\Lambda(M, E_{i-1}) \to \operatorname{Hom}_\Lambda(M, \operatorname{Im} f_i) \to \operatorname{Ext}_\Lambda^i(M, U) \to 0. \qquad (\dagger)$$

Suppose $U\text{-dom.dim}({}_\Lambda U) \geqslant n$. Then $E_i$ is cogenerated by ${}_\Lambda U$ for any $0 \leqslant i \leqslant n-1$. So, for a given $M \in \operatorname{mod}\Lambda$ with $M^* = 0$ we have that $\operatorname{Hom}_\Lambda(M, E_i) = 0$ and $\operatorname{Hom}_\Lambda(M, \operatorname{Im} f_i) = 0$ for any $0 \leqslant i \leqslant n-1$. Then by the exactness of (†), $\operatorname{Ext}_\Lambda^i(M, U) = 0$ for any $1 \leqslant i \leqslant n-1$, and so $\operatorname{grade}_U M \geqslant n$.

Now we prove the converse, that is, we will prove that $E_i \in \operatorname{add} {}_\Lambda U$ for any $0 \leqslant i \leqslant n-1$.

First, $E_0 \in \operatorname{add} {}_\Lambda U$ by assumption. We next prove $E_1 \in \operatorname{add} {}_\Lambda U$. For any $0 \neq x \in \operatorname{Im} f_1$, we claim that $M^* = \operatorname{Hom}_\Lambda(M, U) \neq 0$, where $M = \Lambda x$. Otherwise, we have $\operatorname{Ext}_\Lambda^i(M, U) = 0$ for any $0 \leqslant i \leqslant n-1$ by assumption. Since $E_0 \in \operatorname{add} {}_\Lambda U$, $\operatorname{Hom}_\Lambda(M, E_0) = 0$. So from the exactness of (†) we know that $\operatorname{Hom}_\Lambda(M, \operatorname{Im} f_1) = 0$, which is a contradiction. Then we conclude that $\operatorname{Im} f_1$, and hence $E_1$, is cogenerated by ${}_\Lambda U$. Notice that $E_1$ is finitely cogenerated, so $E_1 \in \operatorname{add} {}_\Lambda U$. Finally, suppose that $n \geqslant 3$ and $E_i \in \operatorname{add} {}_\Lambda U$ for any $0 \leqslant i \leqslant n-2$. Then by using a similar argument to that above we have $E_{n-1} \in \operatorname{add} {}_\Lambda U$. The proof is finished. □

Dually, we have the following

**Lemma 2.9′.** *Suppose* $U$-dom.dim$(U_\Gamma) \geqslant 1$. *Then, for any* $n \geqslant 2$, $U$-dom.dim$(U_\Gamma) \geqslant n$ *if and only if* $\mathrm{grade}_U N \geqslant n$ *for any* $N \in \mathrm{mod}\, \Gamma^{\mathrm{op}}$ *with* $N^* = 0$.

We now are in a position to prove the main result in this paper.

**Proof of Theorem 1.3.** We only need to prove $U$-dom.dim$({}_\Lambda U) \leqslant U$-dom.dim$(U_\Gamma)$. Without loss of generality, suppose $U$-dom.dim$({}_\Lambda U) = n$.

The case $n = 1$ follows from Corollaries 2.5 and 2.8. Let $n \geqslant 2$. Notice that $U$-dom.dim$({}_\Lambda U) \geqslant 1$ and $U$-dom.dim$(U_\Gamma) \geqslant 1$. By Lemma 2.9′ it suffices to show that $\mathrm{grade}_U N \geqslant n$ for any $N \in \mathrm{mod}\, \Gamma^{\mathrm{op}}$ with $N^* = 0$. By Lemmas 2.3 and 2.7, for any $i \geqslant 1$, $\mathrm{Hom}_\Lambda(\mathrm{Ext}^i_\Gamma(N, U), E_0) \cong \mathrm{Tor}^\Gamma_i(N, {}^*E_0) = 0$, so $[\mathrm{Ext}^i_\Gamma(N, U)]^* = 0$. Then by assumption and Lemma 2.9, $\mathrm{grade}_U \mathrm{Ext}^i_\Gamma(N, U) \geqslant n$ for any $i \geqslant 1$. It follows from Lemma 2.6 that $\mathrm{grade}_U N \geqslant n$. □

## 3. Some applications

As applications of the results in above section, we give in this section some characterizations of $(-)^{**}$ preserving monomorphisms and being left exact respectively.

Assume that

$$0 \to U_\Gamma \xrightarrow{f'_0} E'_0 \xrightarrow{f'_1} E'_1 \xrightarrow{f'_2} \cdots \xrightarrow{f'_i} E'_i \xrightarrow{f'_{i+1}} \cdots$$

is a minimal injective resolution of $U_\Gamma$. We first have the following

**Proposition 3.1.** *The following statements are equivalent for any positive integer* $k$.

(1) $U$-dom.dim$({}_\Lambda U) \geqslant k$.
(2) $0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**} \xrightarrow{f_1^{**}} E_1^{**} \xrightarrow{f_2^{**}} \cdots \xrightarrow{f_{k-1}^{**}} E_{k-1}^{**}$ *is exact.*
$(1)^{\mathrm{op}}$ $U$-dom.dim$(U_\Gamma) \geqslant k$.
$(2)^{\mathrm{op}}$ $0 \to (U_\Gamma)^{**} \xrightarrow{(f'_0)^{**}} (E'_0)^{**} \xrightarrow{(f'_1)^{**}} (E'_1)^{**} \xrightarrow{(f'_2)^{**}} \cdots \xrightarrow{(f'_{k-1})^{**}} (E'_{k-1})^{**}$ *is exact.*

**Proof.** By Theorem 1.3 we have $(1) \Leftrightarrow (1)^{\mathrm{op}}$. By symmetry, we only need to prove $(1) \Leftrightarrow (2)$.

If $U$-dom.dim$({}_\Lambda U) \geqslant k$, then $E_i$ is in $\mathrm{add}_\Lambda U$ for any $1 \leqslant i \leqslant k-1$. Notice that ${}_\Lambda U$ and each $E_i$ $(0 \leqslant i \leqslant k-1)$ are $U$-reflexive and hence we have that

$$0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**} \xrightarrow{f_1^{**}} E_1^{**} \xrightarrow{f_2^{**}} \cdots \xrightarrow{f_{k-1}^{**}} E_{k-1}^{**}$$

is exact. Assume that (2) holds. We proceed by induction on $k$. By assumption we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & {}_\Lambda U & \xrightarrow{f_0} & E_0 & \xrightarrow{f_1} & E_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{k-1}} & E_{k-1} \\ & & \downarrow{\scriptstyle \sigma_U} & & \downarrow{\scriptstyle \sigma_{E_0}} & & \downarrow{\scriptstyle \sigma_{E_1}} & & & & \downarrow{\scriptstyle \sigma_{E_{k-1}}} \\ 0 & \longrightarrow & ({}_\Lambda U)^{**} & \xrightarrow{f_0^{**}} & E_0^{**} & \xrightarrow{f_1^{**}} & E_1^{**} & \xrightarrow{f_2^{**}} & \cdots & \xrightarrow{f_{k-1}^{**}} & E_{k-1}^{**} \end{array}$$

Since $\sigma_U$ is an isomorphism, $\sigma_{E_0} f_0 = f_0^{**}\sigma_U$ is a monomorphism. But $f_0$ is essential, so $\sigma_{E_0}$ is monic, that is, $E_0$ is $U$-torsionless and $E_0$ is cogenerated by ${}_\Lambda U$. Moreover, $E_0$ is finitely cogenerated, so we have that $E_0 \in \operatorname{add} {}_\Lambda U$ (and hence $\sigma_{E_0}$ is an isomorphism). The case $k = 1$ is proved. Now suppose that $k \geqslant 2$ and $E_i \in \operatorname{add} {}_\Lambda U$ (and then $\sigma_{E_i}$ is an isomorphism) for any $0 \leqslant i \leqslant k-2$ . Put $A_0 = {}_\Lambda U$, $B_0 = ({}_\Lambda U)^{**}$, $g_0 = f_0$, $g_0' = f_0^{**}$ and $h_0 = \sigma_U$. Then, for any $0 \leqslant i \leqslant k-2$, we get the following commutative diagrams with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_i & \xrightarrow{g_i} & E_i & \longrightarrow & A_{i+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle \sigma_{E_i}} & & \downarrow{\scriptstyle h_{i+1}} & & \\ 0 & \longrightarrow & B_i & \xrightarrow{g_i'} & E_i^{**} & \longrightarrow & B_{i+1} & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccc} 0 & \longrightarrow & A_{i+1} & \xrightarrow{g_{i+1}} & E_{i+1} \\ & & \downarrow{\scriptstyle h_{i+1}} & & \downarrow{\scriptstyle \sigma_{E_{i+1}}} \\ 0 & \longrightarrow & B_{i+1} & \xrightarrow{g_{i+1}'} & E_{i+1}^{**} \end{array}$$

where $A_i = \operatorname{Im} f_i$ and $A_{i+1} = \operatorname{Im} f_{i+1}$, $B_i = \operatorname{Im} f_i^{**}$ and $B_{i+1} = \operatorname{Im} f_{i+1}^{**}$, $g_i$ and $g_{i+1}$ are essential monomorphisms, $h_i$ and $h_{i+1}$ are induced homomorphisms. We may get inductively that each $h_j$ is an isomorphism for any $0 \leqslant j \leqslant k-1$. Because $\sigma_{E_{k-1}} g_{k-1} = g_{k-1}' h_{k-1}$ is a monomorphism, by using a similar argument to that above we have $E_{k-1} \in \operatorname{add} {}_\Lambda U$. Hence we conclude that $U$-dom.dim$({}_\Lambda U) \geqslant k$. □

The following result develops [5, Theorem 1] and [6, Proposition 3.1].

**Proposition 3.2.** *The following statements are equivalent.*

(1) $U$-dom.dim$({}_\Lambda U) \geqslant 1$.

(2) $(-)^{**} : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda$ *preserves monomorphisms.*

(3) $0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**}$ *is exact.*

$(1)^{\mathrm{op}}$ $U$-dom.dim$(U_\Gamma) \geqslant 1$.

$(2)^{\mathrm{op}}$ $(-)^{**}: \mathrm{mod}\,\Gamma^{\mathrm{op}} \to \mathrm{mod}\,\Gamma^{\mathrm{op}}$ *preserves monomorphisms.*

$(3)^{\mathrm{op}}$ $0 \to (U_\Gamma)^{**} \xrightarrow{(f'_0)^{**}} (E'_0)^{**}$ *is exact.*

**Proof.** By Theorem 1.3 we have $(1) \Leftrightarrow (1)^{\mathrm{op}}$. By symmetry, we only need to prove that the conditions of (1), (2) and (3) are equivalent.

$(1) \Rightarrow (2)$. If $U$-dom.dim$({}_\Lambda U) \geqslant 1$ then $t(X) = \mathrm{Ker}\,\sigma_X$ for any $X \in \mathrm{mod}\,\Lambda$ by Corollary 2.8 and Proposition 2.4. So $(-)^{**}$ preserves monomorphisms by Proposition 2.2.

$(2) \Rightarrow (3)$ is trivial and $(3) \Rightarrow (1)$ follows from Proposition 3.1. □

The following result except (3) and $(3)^{\mathrm{op}}$ is the $U$-dual version of [7, Proposition E], which develops [5, Theorem 2].

**Proposition 3.3.** *The following statements are equivalent.*

(1) $U$-dom.dim$({}_\Lambda U) \geqslant 2$.

(2) $(-)^{**}: \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ *is left exact.*

(3) $0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**} \xrightarrow{f_1^{**}} E_1^{**}$ *is exact.*

(4) $(-)^{**}: \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ *preserves monomorphisms and* $\mathrm{Ext}^1_\Gamma(\mathrm{Ext}^1_\Lambda(X, U), U) = 0$ *for any* $X \in \mathrm{mod}\,\Lambda$.

$(1)^{\mathrm{op}}$ $U$-dom.dim$(U_\Gamma) \geqslant 2$.

$(2)^{\mathrm{op}}$ $(-)^{**}: \mathrm{mod}\,\Gamma^{\mathrm{op}} \to \mathrm{mod}\,\Gamma^{\mathrm{op}}$ *is left exact.*

$(3)^{\mathrm{op}}$ $0 \to (U_\Gamma)^{**} \xrightarrow{(f'_0)^{**}} (E'_0)^{**} \xrightarrow{(f'_1)^{**}} (E'_1)^{**}$ *is exact.*

$(4)^{\mathrm{op}}$ $(-)^{**}: \mathrm{mod}\,\Gamma^{\mathrm{op}} \to \mathrm{mod}\,\Gamma^{\mathrm{op}}$ *preserves monomorphisms and* $\mathrm{Ext}^1_\Lambda(\mathrm{Ext}^1_\Gamma(Y, U), U) = 0$ *for any* $Y \in \mathrm{mod}\,\Gamma^{\mathrm{op}}$.

**Proof.** By Theorem 1.3 we have $(1) \Leftrightarrow (1)^{\mathrm{op}}$ and by Proposition 3.1 we have $(1) \Leftrightarrow (3)$. So, by symmetry we only need to prove that $(1) \Leftrightarrow (2)$ and $(1) \Rightarrow (4) \Rightarrow (1)^{\mathrm{op}}$.

$(1) \Leftrightarrow (2)$. Assume that $(-)^{**}: \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ is left exact. Then, by Proposition 3.2, we have that $U$-dom.dim$({}_\Lambda U) \geqslant 1$ and $E_0 \in \mathrm{add}\,{}_\Lambda U$.

Let $K = \mathrm{Im}(E_0 \to E_1)$ and $v: K \to E_1$ be the essential monomorphism. By assumption and the exactness of the sequences $0 \to U \to E_0 \to K \to 0$ and $0 \to K \xrightarrow{v} E_1$, we have the following exact commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & E_0 & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_U} & & \downarrow{\scriptstyle \sigma_{E_0}} & & \downarrow{\scriptstyle \sigma_K} & & \\
0 & \longrightarrow & U^{**} & \longrightarrow & E_0^{**} & \longrightarrow & K^{**} & &
\end{array}$$

and

$$\begin{array}{ccccc} 0 \longrightarrow & K & \xrightarrow{v} & E_1 \\ & \downarrow{\sigma_K} & & \downarrow{\sigma_{E_1}} \\ 0 \longrightarrow & K^{**} & \xrightarrow{v^{**}} & E_1^{**} \end{array}$$

where $\sigma_U$ and $\sigma_{E_0}$ are isomorphisms. By applying the snake lemma to the first diagram we have that $\sigma_K$ is monic. Then we know from the second diagram that $\sigma_{E_1} v = v^{**}\sigma_K$ is a monomorphism. However, $v$ is essential, so $\sigma_{E_1}$ is monic, that is, $E_1$ is $U$-torsionless and $E_1$ is cogenerated by ${}_\Lambda U$. Moreover, $E_1$ is finitely cogenerated, so we conclude that $E_1 \in \operatorname{add} {}_\Lambda U$.

Conversely, assume that $U$-dom.dim$({}_\Lambda U) \geqslant 2$ and $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence in $\operatorname{mod} \Lambda$. By Proposition 3.2, $\alpha^{**}$ is monic. By assumption, Corollary 2.8 and Lemma 2.3 we have $\operatorname{Hom}_\Gamma(\operatorname{Ext}^1_\Lambda(C, U), E_0) = 0$. Since $\operatorname{Coker}\alpha^*$ is isomorphic to a submodule of $\operatorname{Ext}^1_\Lambda(C, U)$, $\operatorname{Hom}_\Gamma(\operatorname{Coker}\alpha^*, E_0) = 0$ and $\operatorname{Hom}_\Gamma(\operatorname{Coker}\alpha^*, U) = 0$. Then, by Theorem 1.3 and Lemma 2.9′, $\operatorname{grade}_U \operatorname{Coker}\alpha^* \geqslant 2$. It follows easily that $0 \to A^{**} \xrightarrow{\alpha^{**}} B^{**} \xrightarrow{\beta^{**}} C^{**}$ is exact.

(1) ⇒ (4). Suppose $U$-dom.dim$({}_\Lambda U) \geqslant 2$. By Proposition 3.2, $(-)^{**} : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda$ preserves monomorphisms. On the other hand, we have that $U$-dom.dim$(U_\Gamma) \geqslant 2$ by Theorem 1.3. It follows from Corollary 2.8 and Lemma 2.3 that $\operatorname{Hom}_\Gamma(\operatorname{Ext}^1_\Lambda(X, U), E_0') = 0$ for any $X \in \operatorname{mod} \Lambda$. So $[\operatorname{Ext}^1_\Lambda(X, U)]^* = 0$ and hence $\operatorname{Ext}^1_\Gamma(\operatorname{Ext}^1_\Lambda(X, U), U) = 0$ by Lemma 2.9′.

(4) ⇒ $(1)^{\mathrm{op}}$. Suppose that (4) holds. Then $U$-dom.dim$(U_\Gamma) \geqslant 1$ by Proposition 3.2.

Let $A$ be in $\operatorname{mod} \Lambda$ and $B$ any submodule of $\operatorname{Ext}^1_\Lambda(A, U)$ in $\operatorname{mod} \Gamma^{\mathrm{op}}$. Since $U$-dom.dim$(U_\Gamma) \geqslant 1$, $\operatorname{Hom}_\Gamma(\operatorname{Ext}^1_\Lambda(A, U), E_0') = 0$ by Corollary 2.8 and Lemma 2.3. So $\operatorname{Hom}_\Gamma(B, E_0') = 0$ and hence $\operatorname{Hom}_\Gamma(B, E_0'/U) \cong \operatorname{Ext}^1_\Gamma(B, U)$. On the other hand, $\operatorname{Hom}_\Gamma(B, E_0') = 0$ implies $B^* = 0$. Then by [8, Lemma 2.1] we have that $B \cong \operatorname{Ext}^1_\Lambda(\operatorname{Tr}_U B, U)$ with $\operatorname{Tr}_U B$ in $\operatorname{mod} \Lambda$. By (4), $\operatorname{Hom}_\Gamma(B, E_0'/U) \cong \operatorname{Ext}^1_\Gamma(B, U) \cong \operatorname{Ext}^1_\Gamma(\operatorname{Ext}^1_\Lambda(\operatorname{Tr}_U B, U), U) = 0$. Then by using a similar argument to that in the proof $(2)^{\mathrm{op}} \Rightarrow (1)^{\mathrm{op}}$ in Proposition 2.2, we have that $\operatorname{Hom}_\Gamma(\operatorname{Ext}^1_\Lambda(A, U), E_1') = 0$ (note: $E_1'$ is the injective envelope of $E_0'/U$). Thus $E_1' \in \operatorname{add} U_\Gamma$ by Lemma 2.3 and Corollary 2.8, and therefore $U$-dom.dim$(U_\Gamma) \geqslant 2$. □

Finally we give some equivalent characterizations of $U$-resol.dim$_\Lambda(E_0) \leqslant 1$ as follows, which is the $U$-dual version of [7, Proposition D].

**Proposition 3.4.** *The following statements are equivalent.*

(1) $U$-resol.dim$_\Lambda(E_0) \leqslant 1$.
(2) $\sigma_X$ *is an essential monomorphism for any $U$-torsionless module $X$ in* $\operatorname{mod} \Lambda$.
(3) $f^{**}$ *is a monomorphism for any monomorphism $f : X \to Y$ in* $\operatorname{mod} \Lambda$ *with $Y$ $U$-torsionless.*

(4) $\mathrm{grade}_U \operatorname{Ext}^1_\Lambda(X, U) \geqslant 1$ (*that is*, $[\operatorname{Ext}^1_\Lambda(X, U)]^* = 0$) *for any* $X$ *in* $\operatorname{mod}\Lambda$.

**Proof.** (1) $\Rightarrow$ (2). Assume that $X$ is $U$-torsionless in $\operatorname{mod}\Lambda$. Then $\operatorname{Coker}\sigma_X \cong \operatorname{Ext}^2_\Gamma(\mathrm{Tr}_U X, U)$ by [8, Lemma 2.1]. By Lemmas 2.7 and 2.3 we have

$$\operatorname{Hom}_\Lambda(\operatorname{Coker}\sigma_X, E_0) = \operatorname{Hom}_\Lambda\big(\operatorname{Ext}^2_\Gamma(\mathrm{Tr}_U X, U), E_0\big) = 0.$$

Then $\operatorname{Hom}_\Lambda(A, {}_\Lambda U) = 0$ for any submodule $A$ of $\operatorname{Coker}\sigma_X$, which implies that any nonzero submodule of $\operatorname{Coker}\sigma_X$ is not $U$-torsionless.

Let $B$ be a submodule of $X^{**}$ with $X \cap B = 0$. Then $B \cong B/(X \cap B) \cong (X + B)/X$ is isomorphic to a submodule of $\operatorname{Coker}\sigma_X$. On the other hand, $B$ is clearly $U$-torsionless. So $B = 0$ and hence $\sigma_X$ is essential.

(2) $\Rightarrow$ (3). Let $f : X \to Y$ be monic in $\operatorname{mod}\Lambda$ with $Y$ $U$-torsionless. Then $f^{**}\sigma_X = \sigma_Y f$ is monic. By (2), $\sigma_X$ is an essential monomorphism, so $f^{**}$ is monic.

(3) $\Rightarrow$ (4). Let $X$ be in $\operatorname{mod}\Lambda$ and $0 \to Y \xrightarrow{g} P \to X \to 0$ an exact sequence in $\operatorname{mod}\Lambda$ with $P$ projective. It is easy to see that $[\operatorname{Ext}^1_\Lambda(X, U)]^* \cong \operatorname{Ker} g^{**}$. On the other hand, $g^{**}$ is monic by (3). So $\operatorname{Ker} g^{**} = 0$ and $[\operatorname{Ext}^1_\Lambda(X, U)]^* = 0$.

(4) $\Rightarrow$ (1). Let $M$ be in $\operatorname{mod}\Gamma^{\mathrm{op}}$ and $\cdots \to P_1 \to P_0 \to M \to 0$ a projective resolution of $M$ in $\operatorname{mod}\Gamma^{\mathrm{op}}$. Put $N = \operatorname{Coker}(P_2 \to P_1)$. By [8, Lemma 2.1], $\operatorname{Ext}^2_\Gamma(M, U) \cong \operatorname{Ext}^1_\Gamma(N, U) \cong \operatorname{Ker}\sigma_{\mathrm{Tr}_U N}$. On the other hand, since $N$ is $U$-torsionless, $\operatorname{Ext}^1_\Lambda(\mathrm{Tr}_U N, U) \cong \operatorname{Ker}\sigma_N = 0$.

Let $X$ be any finitely generated submodule of $\operatorname{Ext}^2_\Gamma(M, U)$ and $f_1 : X \to \operatorname{Ext}^2_\Gamma(M, U)$ ($\cong \operatorname{Ker}\sigma_{\mathrm{Tr}_U N}$) the inclusion, and let $f$ be the composition:

$$X \xrightarrow{f_1} \operatorname{Ext}^2_\Gamma(M, U) \xrightarrow{g} \mathrm{Tr}_U N,$$

where $g$ is a monomorphism. By using the same argument as that in the proof of $(2)^{\mathrm{op}} \Rightarrow (1)^{\mathrm{op}}$ in Proposition 2.2, we get that $f^* = 0$. Hence, by applying $\operatorname{Hom}_\Lambda(-, U)$ to the exact sequence

$$0 \to X \xrightarrow{f} \mathrm{Tr}_U N \to \operatorname{Coker} f \to 0,$$

we have $X^* \cong \operatorname{Ext}^1_\Lambda(\operatorname{Coker} f, U)$. Then $X^{**} \cong [\operatorname{Ext}^1_\Lambda(\operatorname{Coker} f, U)]^* = 0$ by (4), which implies that $X^* = 0$ since $X^*$ is a direct summand of $X^{***} (= 0)$ by [1, Proposition 20.24]. Also by using the same argument as that in the proof of $(2)^{\mathrm{op}} \Rightarrow (1)^{\mathrm{op}}$ in Proposition 2.2, we get that $\operatorname{Hom}_\Lambda(\operatorname{Ext}^2_\Gamma(M, U), E_0) = 0$. It follows from Lemma 2.3 that $\mathrm{l.fd}_\Gamma({}^*E_0) \leqslant 1$. Therefore $U\text{-resol.dim}_\Lambda(E_0) \leqslant 1$ by Lemma 2.7. □

**Remark.** By Theorem 1.3, we have that $E_0 \in \operatorname{add}{}_\Lambda U$ if and only if $E'_0 \in \operatorname{add} U_\Gamma$, that is, $U\text{-resol.dim}_\Lambda(E_0) = 0$ if and only if $U\text{-resol.dim}_\Gamma(E'_0) = 0$. However, in general, we don't have the fact that $U\text{-resol.dim}_\Lambda(E_0) \leqslant 1$ if and only if $U\text{-resol.dim}_\Gamma(E'_0) \leqslant 1$ even when ${}_{\Lambda}U_{\Gamma} = {}_{\Lambda}\Lambda_{\Lambda}$. We use $I_0$ and $I_0'$ to denote the injective envelope of ${}_{\Lambda}\Lambda$ and $\Lambda_{\Lambda}$, respectively. Consider the following example. Let $K$ be a field and $\Delta$ the quiver:

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \xrightarrow{\gamma} 3.$$

(1) If $\Lambda = K\Delta/(\alpha\beta\alpha)$. Then $\mathrm{l.fd}_{\Lambda}(I_0) = 1$ and $\mathrm{r.fd}_{\Lambda}(I_0') \geqslant 2$. (2) If $\Lambda = K\Delta/(\gamma\alpha, \beta\alpha)$. Then $\mathrm{l.fd}_{\Lambda}(I_0) = 2$ and $\mathrm{r.fd}_{\Lambda}(I_0') = 1$.

## Acknowledgments

The author thanks Professor Kent R. Fuller and the referee for their helpful comments. The research of the author was partially supported by National Natural Science Foundation of China (Grant No. 10001017) and Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20030284033).

## References

[1] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, second ed., Grad. Texts in Math., vol. 13, Springer-Verlag, Berlin–Heidelberg–New York, 1992.
[2] M. Auslander, R.O. Buchweitz, The homological theory of maximal Cohen–Macaulay approximations, Soc. Math. France 38 (1989) 5–37.
[3] H. Cartan, S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, NJ, 1956.
[4] S.U. Chase, Direct products of modules, Trans. Amer. Math. Soc. 97 (1960) 457–473.
[5] R.R. Colby, K.R. Fuller, Exactness of the double dual, Proc. Amer. Math. Soc. 82 (1981) 521–526.
[6] M. Hoshino, On dominant dimension of Noetherian rings, Osaka J. Math. 26 (1989) 275–280.
[7] M. Hoshino, On Lambek torsion theories, Osaka J. Math. 29 (1992) 447–453.
[8] Z.Y. Huang, G.H. Tang, Self-orthogonal modules over coherent rings, J. Pure Appl. Algebra 161 (2001) 167–176.
[9] J. Lambek, Torsion Theories, Additive Semantics, and Rings of Quotients (with an Appendix by H.H. Storrer on Torsion Theories and Dominant Dimension), Lecture Notes in Math., vol. 177, Springer-Verlag, Berlin–Heidelberg–New York, 1971.
[10] T. Kato, Rings of $U$-dominant dimension $\geqslant 1$, Tôhoku Math. J. 21 (1969) 321–327.
[11] B. Stenström, Rings of Quotients, Grundlehren Math. Wiss. Einz., vol. 217, Springer-Verlag, Berlin–Heidelberg–New York, 1975.
[12] H. Tachikawa, Quasi-Frobenius Rings and Generalizations (QF-3 and QF-1 Rings), Lecture Notes in Math., vol. 351, Springer-Verlag, Berlin–Heidelberg–New York, 1973.
[13] R. Wisbauer, Decomposition properties in module categories, Acta Univ. Carolin. Math. Phys. 26 (1985) 57–68.