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On pure derived categories

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ABSTRACT

We investigate the properties of pure derived categories of module categories, and show that pure derived categories share many nice properties of classical derived categories. In particular, we show that bounded pure derived categories can be realized as certain homotopy categories. We introduce the pure projective (resp. injective) dimension of complexes in pure derived categories, and give some criteria for computing these dimensions in terms of the properties of pure projective (resp. injective) resolutions and pure derived functors. As a consequence, we get some equivalent characterizations for the finiteness of the pure global dimension of rings. Finally, pure projective (resp. injective) resolutions of unbounded complexes are considered.

1. Introduction

Let $(\mathcal{A}, \mathcal{E})$ be an exact category in the sense of [17] and $\mathbf{K}(\mathcal{A})$ its homotopy category. Then one can consider the triangulated quotient of $\mathbf{K}(\mathcal{A})$ by $\mathcal{E}$, called the derived category of $(\mathcal{A}, \mathcal{E})$, which was studied by Neeman in [13]. Now let R be a ring and R-Mod the category of left R-modules. It is known that there are two interesting exact structures

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in R-Mod; one is the usual and the other is the pure exact structure. The derived category with respect to the first one is traditional which provides a broader framework for studying homological algebra, and to the second one is the pure derived category which has attracted many authors, see [5,7,9,12,15,20] for the details.

In general, triangulated quotients are not intuitive since they are usually realized as calculus of fractions. However, bounded derived categories are well understood since they are equivalent to certain homotopy categories of projective modules. It is known that pure projective modules are exactly projective objects with respect to the pure exact structure, see [11,6,16], and [21]. So, it is expected that bounded pure derived categories will share some nice properties of classical bounded derived categories. In Section 3, we show that for a ring R, R-Mod is a full subcategory of its bounded pure derived category. Moreover, we show that the bounded pure derived category of R-Mod is triangulated equivalent to a triangulated full subcategory of the bounded above (resp. below) homotopy category of pure projective (resp. injective) R-modules. Note that the results in this section are standard analogs of the corresponding classical ones.

In Section 4, we devote to building triangulated functors from (bounded) pure derived categories. A very natural choice is the right "derived" version of Hom. For this, we first establish the pure projective (resp. injective) resolutions of bounded complexes, and then use them to define right pure derived functors of Hom which preserve the corresponding triangles. As applications, we introduce and study the pure projective (resp. injective) dimension of complexes. In particular, we obtain some criteria for computing this dimension in terms of the properties of pure projective (resp. injective) resolutions and the vanishing of pure derived functors. As a consequence, we get some equivalent characterizations for the finiteness of the pure global dimension of rings. The results in this section are standard analogs of main results in [2], and generalize the corresponding ones for modules in [11] and [19].

In Section 5, pure projective (resp. injective) resolutions of certain unbounded complexes are considered. We use the technique of homotopy (co)limits to show that any bounded below (resp. above) complex admits a pure projective (resp. injective) resolution.

2. Preliminaries

Throughout this paper, R is an associate ring with identity and R-Mod is the category of left R-modules. As usual, we use $\mathbf{C}(R)$ and $\mathbf{K}(R)$ to denote the category of complexes and homotopy category of R-Mod, respectively. When we say "R-module", without an adjective, we mean left R-module. For any $X \in \mathbf{C}(R)$, we write

$$X := \cdots \longrightarrow X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \longrightarrow \cdots.$$

We regard an R-module M as the stalk complex, that is, a complex concentrated in degree 0.

We recall the bounded conditions for complexes which are standard in homological algebra, see for example [8]. Let $X \in \mathbf{C}(R)$. If $X^i = 0$ for $i \gg 0$, then X is called *bounded above* (or *bounded on the right*). If $X^i = 0$ for $i \ll 0$, then X is called *bounded below* (or *bounded on the left*). X is called *bounded* if it is bounded above and below. A cochain map $f : X \to Y$ in $\mathbf{C}(R)$ is called a *quasi-isomorphism* if it induces isomorphic homology groups; and f is called a *homotopy equivalence* if there exists a cochain map $g : Y \to X$ such that there exist homotopies $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$. For $\mathrm{Con}(f)$ we mean the *mapping cone* of a cochain map f. Let $X, Y \in \mathbf{C}(R)$. We use $\mathrm{Hom}_R(X, Y)$ to denote the *total complex*, that is, a complex of $\mathbb{Z}$-modules (where $\mathbb{Z}$ is the additive group of integers)

$$\cdots \longrightarrow \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(X^i, Y^{i+n}) \xrightarrow{d^n} \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(X^i, Y^{i+n+1}) \longrightarrow \cdots,$$

where $\prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(X^i, Y^{i+n})$ lies in degree n. For any $\varphi \in \mathrm{Hom}_R(X, Y)^n$, $d^n(\varphi) = (d_Y^{i+n} \circ \varphi^i - (-1)^n \varphi^{i+1} \circ d_X^i)_{i\in\mathbb{Z}}$. Note that this construction defines a bifunctor

$$\mathrm{Hom}_R(-,-) : \mathbf{K}(R)^{op} \times \mathbf{K}(R) \to \mathbf{K}(\mathbb{Z}).$$

Definition 2.1. (See [21].) A short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in R-Mod is called *pure exact* if for any right R-module M, the induced sequence

$$0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is exact. In this case, f is called *pure monic* and g is called *pure epic*.

Remark 2.2. Using the Cohn's theorem (see [18, Theorem 3.69]), we have that a short exact sequence

$$0 \to A \to B \to C \to 0$$

in R-Mod is pure exact if and only if

$$0 \to \mathrm{Hom}_R(F, A) \to \mathrm{Hom}_R(F, B) \to \mathrm{Hom}_R(F, C) \to 0$$

is exact for any finitely presented R-module F.

In general, the exactness of a complex of R-modules is defined “pointwise”. This definition provides convenience for understanding bounded derived categories. Let $(\mathcal{A}, \mathcal{E})$ be an exact category in the sense of [17]. Following [13], a complex X is called *exact* (or *acyclic*) with respect to the exact structure of $\mathcal{A}$ if each differential d_X^i decomposes as

$$X^i \twoheadrightarrow D^i \rightarrowtail X^{i+1},$$

where the former morphism is admissible epic and the latter one is admissible monic; furthermore, the sequence

$$D^i \rightarrowtail X^{i+1} \twoheadrightarrow D^{i+1}$$

is exact for any $i \in \mathbb{Z}$, see also [9, Section 4.2]. Now it is natural for us to propose the following definition, which provides convenience for understanding bounded pure derived categories later.

Definition 2.3. Let $X \in \mathbf{C}(R)$ and $n \in \mathbb{Z}$. Then X is called *pure exact at n*, if the differentials d_X^{n-1} and d_X^n can decompose as above, and the sequence

$$0 \to K^n \to X^n \to C^{n-1} \to 0$$

is pure exact, where $K^n = \operatorname{Ker} d_X^n$ and $C^{n-1} = \operatorname{Coker} d_X^{n-1}$. X is called *pure exact* if it is pure exact at n for all n.

Remark 2.4.

(1) $X \in \mathbf{C}(R)$ is pure exact if and only if $M \otimes_R X$ is exact for any right R-module M, and if and only if $\operatorname{Hom}_R(F, X)$ is exact for any finitely presented R-module F.
(2) A direct limit of pure exact complexes is again pure exact, since the tensor functor commutes with direct limits by [18, Theorem 5.27].
(3) By definition, pure exact complexes coincide with the exact structure in the sense of Neeman [13].

Definition 2.5. (See [21].) A module $M \in R$-Mod is called *pure projective* (resp. *injective*) if it is projective (resp. injective) with respect to every pure exact complex.

Let $\mathcal{PP}$ (resp. $\mathcal{PI}$) be the class of all pure projective (resp. injective) R-modules. We use $\mathbf{K}^-(\mathcal{PP})$ (resp. $\mathbf{K}^+(\mathcal{PI})$) to denote the bounded above (resp. below) homotopy category of $\mathcal{PP}$ (resp. $\mathcal{PI}$).

Remark 2.6.

(1) We write $(-)^+ := \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}$ is the additive group of rational numbers. By [6, Proposition 5.3.7], we have that M^+ is a pure injective left R-module for any right R-module M. Using the fact that every R-module is a direct limit of finitely presented R-modules [18, Lemma 5.39], we have that pure projective modules are nothing but summands of direct sums of finitely presented modules.

(2) By (1), it is easy to check that a complex X is pure exact if and only if $\mathrm{Hom}_R(P, X)$ is exact for any $P \in \mathcal{PP}$, and if and only if $\mathrm{Hom}_R(X, I)$ is exact for any $I \in \mathcal{PI}$.

We need the following definition.

Definition 2.7. A cochain map $f\colon X \to Y$ in $\mathbf{C}(R)$ is called a *pure quasi-isomorphism* if its mapping cone $\mathrm{Con}(f)$ is a pure exact complex.

Remark 2.8.

(1) A cochain map $f : X \to Y$ in $\mathbf{C}(R)$ is a pure quasi-isomorphism if and only if

$$M \otimes_R f : M \otimes_R X \to M \otimes_R Y$$

is a quasi-isomorphism for any right R-module M.

(2) By Remark 2.6, a cochain map $f\colon X \to Y$ in $\mathbf{C}(R)$ is a pure quasi-isomorphism if and only if

$$\mathrm{Hom}_R(P, f) : \mathrm{Hom}_R(P, X) \to \mathrm{Hom}_R(P, Y)$$

is a quasi-isomorphism for any $P \in \mathcal{PP}$, and if and only if

$$\mathrm{Hom}_R(f, I) : \mathrm{Hom}_R(Y, I) \to \mathrm{Hom}_R(X, I)$$

is a quasi-isomorphism for any $I \in \mathcal{PI}$.

The following result concerning both pure exact complexes and pure quasi-isomorphisms is essentially contained in [3].

Lemma 2.9.

(1) *Let $X \in \mathbf{C}(R)$. Then X is pure exact if and only if $\mathrm{Hom}_R(P, X)$ is exact for any $P \in \mathbf{K}^-(\mathcal{PP})$, and if and only if $\mathrm{Hom}_R(X, I)$ is exact for any $I \in \mathbf{K}^+(\mathcal{PI})$.*

(2) *A cochain map f in $\mathbf{C}(R)$ is a pure quasi-isomorphism if and only if $\mathrm{Hom}_R(P, f)$ is a quasi-isomorphism for any $P \in \mathbf{K}^-(\mathcal{PP})$, and if and only if $\mathrm{Hom}_R(f, I)$ is a quasi-isomorphism for any $I \in \mathbf{K}^+(\mathcal{PI})$.*

Proof. The assertion (1) follows from Remark 2.6(2) and [3, Lemmas 2.4 and 2.5], and the assertion (2) follows from Remark 2.8(2) and [3, Propositions 2.6 and 2.7]. □

Lemma 2.10.

(1) *Let $f : X \to Y$ be a pure quasi-isomorphism in $\mathbf{C}(R)$ with $X, Y \in \mathbf{K}^-(\mathcal{PP})$. Then f is a homotopy equivalence.*

(2) *Let $f : X \to Y$ be a pure quasi-isomorphism in $\mathbf{C}(R)$ with $X, Y \in \mathbf{K}^+(\mathcal{PI})$. Then f is a homotopy equivalence.*

Proof. (1) Because there exists a quasi-isomorphism

$$\mathrm{Hom}_R(Y, f) : \mathrm{Hom}_R(Y, X) \to \mathrm{Hom}_R(Y, Y)$$

by Lemma 2.9, we have an isomorphism

$$\mathrm{H}^0(\mathrm{Hom}_R(Y, f)) : \mathrm{H}^0(\mathrm{Hom}_R(Y, X)) \to \mathrm{H}^0(\mathrm{Hom}_R(Y, Y)).$$

One can easily check that there exists a cochain map $g : Y \to X$ such that $f \circ g \sim \mathrm{Id}_Y$. Similarly, there exists a cochain map h such that $g \circ h \sim \mathrm{Id}_X$. As a consequence, we have that g and f are homotopy equivalences.

(2) It is the dual of (1). □

Lemma 2.11.

(1) *Let $Y \to X$ be a pure quasi-isomorphism in $\mathbf{C}(R)$ with $X \in \mathbf{K}^b(R)$ and $Y \in \mathbf{K}^+(R)$. Then there exists a pure quasi-isomorphism $X' \to Y$ with $X' \in \mathbf{K}^b(R)$.*
(2) *Let $X \to Y$ be a pure quasi-isomorphism in $\mathbf{C}(R)$ with $X \in \mathbf{K}^+(R)$ and $Y \in \mathbf{K}(R)$. Then there exists a pure quasi-isomorphism $Y \to X'$ with $X' \in \mathbf{K}^+(R)$.*

Proof. (1) We can assume that $Y^n = 0$ for any $n < 0$ and that $\mathrm{H}^i \mathrm{Hom}_R(P, Y) = 0$ for any $P \in \mathcal{PP}$ and $i \geq m + 1$. We have the following commutative diagram:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & Y^0 & \longrightarrow & \cdots & \longrightarrow & Y^{m-1} & \xrightarrow{d_Y^{m-1}} & \mathrm{Ker}\, d_Y^m & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & \downarrow{\scriptstyle \mathrm{Id}_{Y^0}} & & & & \downarrow{\scriptstyle \mathrm{Id}_{Y^{m-1}}} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & Y^0 & \longrightarrow & \cdots & \longrightarrow & Y^{m-1} & \xrightarrow{d_Y^{m-1}} & Y^m & \longrightarrow & Y^{m+1} & \longrightarrow & \cdots.
\end{array}$$

Let the upper row be the complex X'. Since $\mathrm{Hom}_R(P, -)$ preserves kernels, the cochain map is clearly a pure quasi-isomorphism by Remark 2.8(2).

(2) We can assume that $\mathrm{H}^i(M \otimes_R Y) = 0$ for any right R-module M and $i \leq -1$. We have the following commutative diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & Y^{-2} & \longrightarrow & Y^{-1} & \xrightarrow{d_Y^{-1}} & Y^0 & \longrightarrow & Y^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \widetilde{d}_Y^{-1}} & & \downarrow{\scriptstyle \mathrm{Id}_{Y^1}} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Coker}\, d_Y^{-1} & \longrightarrow & Y^1 & \longrightarrow & \cdots.
\end{array}$$

Let the lower row be the complex X'. Since $M \otimes_R -$ preserves cokernels, the cochain map is clearly a pure quasi-isomorphism by Remark 2.8(1). □

3. Pure derived categories

Put $\mathbf{K}_{\mathcal{PE}}(R) := \{X \in \mathbf{K}(R) \mid X \text{ is pure exact}\}$. Notice that pure exact complexes are closed under homotopy equivalences, so $\mathbf{K}_{\mathcal{PE}}(R)$ is well defined. If $f : X \to Y$ is a cochain map between pure exact complexes, then $\mathrm{Con}(f)$ is again pure exact. Thus $\mathbf{K}_{\mathcal{PE}}(R)$ is a triangulated subcategory of $\mathbf{K}(R)$. Because pure exact complexes are closed under summands, $\mathbf{K}_{\mathcal{PE}}(R)$ is a thick subcategory of $\mathbf{K}(R)$. Then by the Verdier's correspondence, we get the pure derived category

$$\mathbf{D}_{\mathbf{pur}}(R) := \mathbf{K}(R)/\mathbf{K}_{\mathcal{PE}}(R).$$

Similarly, we define

$$\mathbf{D}^{*}_{\mathbf{pur}}(R) := \mathbf{K}^{*}(R)/\mathbf{K}^{*}{}_{\mathcal{PE}}(R)$$

for $* \in \{+, -, b\}$. Note that the pure derived category coincides with the one given in [13] and pure exact complexes here are exactly the exact structure there.

Note that, as usual, a morphism from X to Y in $\mathbf{D}_{\mathbf{pur}}(R)$ can be viewed as a graph (left roof)

$$X \overset{s}{\Longleftarrow} \bullet \overset{a}{\longrightarrow} Y$$

with s a pure quasi-isomorphism [8, Chapters III.2.8 and III.2.9]. Two roofs

$$X \overset{s}{\Longleftarrow} \bullet \overset{a}{\longrightarrow} Y \quad \text{and} \quad X \overset{s'}{\Longleftarrow} \bullet \overset{a'}{\longrightarrow} Y$$

are equivalent if there exists the following commutative diagram:

$$\begin{array}{ccccc} & & \bullet & & \\ & \overset{s}{\swarrow\!\!\swarrow} & \uparrow & \overset{a}{\searrow} & \\ X & \overset{g}{\Longleftarrow} & \bullet & \longrightarrow & Y \\ & \underset{s'}{\nwarrow\!\!\nwarrow} & \downarrow & \underset{a'}{\nearrow} & \\ & & \bullet & & \end{array}$$

with g a pure quasi-isomorphism. So, two complexes X and Y are isomorphic in $\mathbf{D}_{\mathbf{pur}}(R)$ if there exists a graph

$$X \overset{s}{\Longleftarrow} \bullet \overset{a}{\Longrightarrow} Y$$

with s and a pure quasi-isomorphisms. If either $Y \in \mathbf{K}^{+}(\mathcal{PI})$ or $X \in \mathbf{K}^{-}(\mathcal{PP})$, then morphisms in $\mathrm{Hom}_{\mathbf{D_{pur}(R)}}(X, Y)$ are easy enough as showed below.

Proposition 3.1.

(1) *Let* $X \in \mathbf{K}^-(\mathcal{PP})$ *and* $Y \in \mathbf{K}(R)$. *Then the localization functor*

$$\mathbb{F} : \mathrm{Hom}_{\mathbf{K}(R)}(X,Y) \to \mathrm{Hom}_{\mathbf{D_{pur}}(R)}(X,Y),\ \ f \mapsto f/\mathrm{Id}_X \ \ (\textit{left roof}),$$

induces an isomorphism of abelian groups.

(2) *Let* $Y \in \mathbf{K}^+(\mathcal{PI})$ *and* $X \in \mathbf{K}(R)$. *Then the localization functor*

$$\mathbb{F} : \mathrm{Hom}_{\mathbf{K}(R)}(X,Y) \to \mathrm{Hom}_{\mathbf{D_{pur}}(R)}(X,Y),\ \ f \mapsto \mathrm{Id}_Y\backslash f \ \ (\textit{right roof}),$$

induces an isomorphism of abelian groups.

Proof. We only need to prove (1). If $f/\mathrm{Id}_X = 0 = 0/\mathrm{Id}_X$, then there exists a pure quasi-isomorphism $g : Z \to X$ such that $f \circ g \sim 0$. By the proof of Lemma 2.10, there exists a pure quasi-isomorphism $h : X \to Z$ such that $g \circ h \sim \mathrm{Id}_X$. So $f \sim 0$. For any $f/s \in \mathrm{Hom}_{\mathbf{D_{pur}(R)}}(X,Y)$, since s is a pure quasi-isomorphism, again by the proof of Lemma 2.10 there exists a pure quasi-isomorphism t such that $s \circ t \sim \mathrm{Id}_X$. So we have $f/s = (f \circ t)/\mathrm{Id}_X$ in $\mathbf{D_{pur}}(R)$. □

Proposition 3.2. *For a ring* R, *we have*

(1) $\mathbf{D^b_{pur}}(R)$ *is a full subcategory of* $\mathbf{D^+_{pur}}(R)$, *and* $\mathbf{D^+_{pur}}(R)$ *is a full subcategory of* $\mathbf{D_{pur}}(R)$.
(2) $\mathbf{D^b_{pur}}(R)$ *is a full subcategory of* $\mathbf{D^-_{pur}}(R)$, *and* $\mathbf{D^-_{pur}}(R)$ *is a full subcategory of* $\mathbf{D_{pur}}(R)$.
(3) $\mathbf{D^b_{pur}}(R) = \mathbf{D^-_{pur}}(R) \cap \mathbf{D^+_{pur}}(R)$.

Proof. The assertion (1) is a consequence of [8, Proposition 3.2.10] and Lemma 2.11, and the assertion (2) is the dual of (1). The assertion (3) is an immediate consequence of (1) and (2). □

Theorem 3.3. *For a ring* R, R-Mod *is a full subcategory of* $\mathbf{D^b_{pur}}(R)$, *that is, the composition of functors*

$$R\text{-Mod} \to \mathbf{K^b}(R) \to \mathbf{D^b_{pur}}(R)$$

is fully faithful.

Proof. For any $X, Y \in R$-Mod, it suffices to prove that the morphism

$$\mathbb{F} : \mathrm{Hom}_R(X,Y) \to \mathrm{Hom}_{\mathbf{D_{pur}}(R)}(X,Y)$$

is an isomorphism.

Let $f \in \mathrm{Hom}_R(X, Y)$. If $\mathbb{F}(f) = 0$, then there exists a pure quasi-isomorphism $s : Z \to X$ such that $f \circ s \sim 0$. So $\mathrm{H}^0(f) \circ \mathrm{H}^0(s) = 0$. Since $\mathrm{H}^0(s)$ is an isomorphism, we have $f = 0$.

Let a/s be a morphism in $\mathrm{Hom}_{\mathbf{D_{pur}}(R)}(X, Y)$. Then we have a diagram

$$X \xLeftarrow{\;s\;} Z \xrightarrow{\;a\;} Y \ ,$$

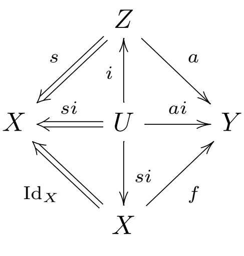
where s is a pure quasi-isomorphism, and hence a quasi-isomorphism. So $\mathrm{H}^0(s) \in \mathrm{Hom}_R(\mathrm{H}^0(Z), X)$ is an isomorphism in R-Mod (note that $\mathrm{H}^0(X) = X$). Put $f := \mathrm{H}^0(a) \circ \mathrm{H}^0(s)^{-1} \in \mathrm{Hom}_R(X, Y)$. Consider the truncation

$$U := \cdots \longrightarrow Z^{-2} \longrightarrow Z^{-1} \xrightarrow{\;d^{-1}\;} \mathrm{Ker}\, d_Z^0 \longrightarrow 0$$

of Z and the canonical map $i : U \to Z$. Note that, as in Lemma 2.11, i is a pure quasi-isomorphism. Then $s \circ i$ is also a pure quasi-isomorphism. From the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\;i\;} & Z \\ \downarrow & & \downarrow{\scriptstyle s} \\ \mathrm{H}^0(Z) & \xrightarrow{\;\mathrm{H}^0(s)\;} & X, \end{array}$$

we get $f \circ s \circ i = \mathrm{H}^0(a) \circ \mathrm{H}^0(s)^{-1} \circ s \circ i = a \circ i$. So the following diagram of complexes:



is commutative. It follows that $\mathbb{F}(f) = f/\mathrm{Id}_X = a/s$. $\square$

For any $X \in \mathbf{C}(R)$, we write

$$\mathbf{inf_p} X := \inf\{n \in \mathbb{Z} \mid X \text{ is not pure exact at } n\}, \text{ and}$$
$$\mathbf{sup_p} X := \sup\{n \in \mathbb{Z} \mid X \text{ is not pure exact at } n\}.$$

If X is not pure exact at n for any n, then we set $\mathbf{inf_p} X = -\infty$ and $\mathbf{sup_p} X = \infty$. If X is pure exact at n for all n, that is, X is a pure exact complex, then we set $\mathbf{inf_p} X = \infty$

and $\mathbf{sup_p}X = -\infty$. We will heavily rely on these two numbers in the remainder of this paper.

Put

$$\mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP}) := \{X \in \mathbf{K}^{-}(\mathcal{PP}) \mid \mathbf{inf_p}X \text{ is finite}\}, \text{ and}$$
$$\mathbf{K}^{+,\mathbf{pb}}(\mathcal{PI}) := \{X \in \mathbf{K}^{+}(\mathcal{PI}) \mid \mathbf{sup_p}X \text{ is finite}\}.$$

Proposition 3.4. *Let $X \in \mathbf{C}(R)$. Then the following hold:*

(1) *X is pure exact in degree $\leq n$ if and only if $M \otimes_R X$ is exact in degree $\leq n$ for any right R-module M.*
(2) *X is pure exact in degree $\geq n$ if and only if $\mathrm{Hom}_R(P, X)$ is exact in degree $\geq n$ for any $P \in \mathcal{PP}$.*
(3) *The numbers $\mathbf{inf_p}X$ and $\mathbf{sup_p}X$ are well defined for any $X \in \mathbf{D_{pur}}(R)$, that is, if $X \cong Y$ in $\mathbf{D_{pur}}(R)$, then $\mathbf{inf_p}X = \mathbf{inf_p}Y$ and $\mathbf{sup_p}X = \mathbf{sup_p}Y$.*
(4) *$\mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$ and $\mathbf{K}^{+,\mathbf{pb}}(\mathcal{PI})$ are triangulated subcategories of $\mathbf{K}^{-}(\mathcal{PP})$ and $\mathbf{K}^{+}(\mathcal{PI})$, respectively.*

Proof. (1) Consider the following commutative diagram (tensor products act on R):

$$\begin{array}{ccccccccc}
\cdots \to M\otimes X^{n-1} & & \xrightarrow{M\otimes d_X^{n-1}} & & M\otimes X^{n} & & \xrightarrow{M\otimes d_X^{n}} & & M\otimes X^{n+1} \to \cdots, \\
& \searrow^{M\otimes \widetilde{d}_X^{n-2}} & & \nearrow_{M\otimes \iota^{n}} & & \searrow^{M\otimes \widetilde{d}_X^{n-1}} & & \nearrow_{M\otimes \iota^{n+1}} & \\
& & M\otimes C^{n-2} & & & & M\otimes C^{n-1} & &
\end{array}$$

where $\widetilde{d}_X^{n-2}$ (resp. $\widetilde{d}_X^{n-1}$) denotes the cokernel of d_X^{n-2} (resp. d_X^{n-1}) and ι^n (resp. ι^{n+1}) denotes the kernel of d_X^n (resp. d_X^{n+1}). Then the assertion follows standardly.

(2) The proof is similar to that of (1).

(3) We only need to prove the assertion whenever both $\mathbf{inf_p}X$ (resp. $\mathbf{sup_p}X$) and $\mathbf{inf_p}Y$ (resp. $\mathbf{sup_p}Y$) are finite. By Remark 2.8, it is an immediate consequence of (1) and (2).

(4) We only prove that $\mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$ is a triangulated subcategory of $\mathbf{K}^{-}(\mathcal{PP})$ and the proof of the other assertion is similar. Observe that $\mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$ is closed under shifts. So it suffices to show that $\mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$ is closed under extensions. Let

$$X \to Y \to Z \to X[1]$$

be a triangle in $\mathbf{K}^{-}(\mathcal{PP})$ with $X, Z \in \mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$. Then we have a triangle

$$X \otimes_R M \to Y \otimes_R M \to Z \otimes_R M \to X[1] \otimes_R M$$

in $\mathbf{K}(\mathbb{Z})$ for any right R-module M. It induces a long exact sequence of homological groups since $\mathrm{H}^0(-)$ is cohomological by [8, Chapter IV.1.6]. By (1) there exists $n \in \mathbb{Z}$ such that both $X \otimes_R M$ and $Z \otimes_R M$ are exact in degree $\leq n$, so $Y \otimes_R M$ is also exact in degree $\leq n$. Thus we have $Y \in \mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$. $\square$

Proposition 3.5.

(1) *There exist a functor $P : \mathbf{K}^{\mathbf{b}}(R) \to \mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$ and a pure quasi-isomorphism $f_X : P_X \to X$ for any $X \in \mathbf{K}^{\mathbf{b}}(R)$, which is functorial in X.*
(2) *There exist a functor $I : \mathbf{K}^{\mathbf{b}}(R) \to \mathbf{K}^{+,\mathbf{pb}}(\mathcal{PI})$ and a pure quasi-isomorphism $g_X : X \to I_X$ for any $X \in \mathbf{K}^{\mathbf{b}}(R)$, which is functorial in X.*

Proof. We first prove that for any $X \in \mathbf{K}^{\mathbf{b}}(R)$, there exists a pure quasi-isomorphism $P_X \to X$ with $P_X \in \mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$. We proceed by induction on the cardinal of the finite set $\mathcal{W}(X) := \{i \in \mathbb{Z} \mid X^i \neq 0\}$.

If $\mathcal{W}(X) = 1$, then the assertion follows from the fact that every module admits a pure projective precover (see [6, Example 8.3.2]).

Now suppose that $\mathcal{W}(X) \geq 2$ with $X^j \neq 0$ and $X^i = 0$ for any $i < j$. Then we have a distinguished triangle

$$X_1 \xrightarrow{u} X_2 \longrightarrow X \longrightarrow X_1[1]$$

in $\mathbf{K}^{\mathbf{b}}(R)$, where $X_1 = X^j[-j-1]$ and $X_2 = X^{>j}$. By the induction hypothesis, there exist pure quasi-isomorphisms $f_{X_1} : P_{X_1} \to X_1$ and $f_{X_2} : P_{X_2} \to X_2$ with $P_{X_1}, P_{X_2} \in \mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$. Then by Lemma 2.9, f_{X_2} induces an isomorphism

$$\mathrm{Hom}_{\mathbf{K}(R)}(P_{X_1}, P_{X_2}) \cong \mathrm{Hom}_{\mathbf{K}(R)}(P_{X_1}, X_2).$$

So there exists a morphism $f : P_{X_1} \to P_{X_2}$, which is unique up to homotopy, such that $f_{X_2} \circ f = u \circ f_{X_1}$. We have the distinguished triangle

$$P_{X_1} \xrightarrow{f} P_{X_2} \longrightarrow \mathrm{Con}(f) \longrightarrow P_{X_1}[1]$$

in $\mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$. Then there exists a morphism $f_X : \mathrm{Con}(f) \to X$ such that the following diagram:

$$\begin{array}{ccccccc}
P_{X_1} & \xrightarrow{f} & P_{X_2} & \longrightarrow & \mathrm{Con}(f) & \longrightarrow & P_{X_1}[1] \\
{\scriptstyle f_{X_1}}\downarrow & & {\scriptstyle f_{X_2}}\downarrow & & {\scriptstyle f_X}\downarrow & & {\scriptstyle f_{X_1}[1]}\downarrow \\
X_1 & \xrightarrow{u} & X_2 & \longrightarrow & X & \longrightarrow & X_1[1]
\end{array}$$

in $\mathbf{K}(R)$ commutes. For any $P \in \mathcal{PP}$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(P, P_{X_1}) & \longrightarrow & \mathrm{Hom}_R(P, P_{X_2}) & \longrightarrow & \mathrm{Hom}_R(P, \mathrm{Con}(f)) & \longrightarrow & \mathrm{Hom}_R(P, P_{X_1}[1]) \\
{\scriptstyle (f_{X_1})_*}\downarrow & & {\scriptstyle (f_{X_2})_*}\downarrow & & {\scriptstyle (f_X)_*}\downarrow & & {\scriptstyle (f_{X_1}[1])_*}\downarrow \\
\mathrm{Hom}_R(P, X_1) & \longrightarrow & \mathrm{Hom}_R(P, X_2) & \longrightarrow & \mathrm{Hom}_R(P, X) & \longrightarrow & \mathrm{Hom}_R(P, X_1[1])
\end{array}$$

in $\mathbf{K}(\mathbb{Z})$, where both rows are exact triangles and $(-)_*$ denotes the functor $\mathrm{Hom}_R(P, -)$. Since both f_{X_1} and f_{X_2} are pure quasi-isomorphisms, we have that both $(f_{X_1})_*$ and $(f_{X_2})_*$ are quasi-isomorphisms. Passing to homology we get that $(f_X)_*$ is a quasi-isomorphism, so f_X is a pure quasi-isomorphism by Remark 2.8(2).

Put $P_X := \mathrm{Con}(f)$. Then we have a pure quasi-isomorphism $f_X : P_X \to X$ with $P_X \in \mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$. In the following we prove that f_X is functorial in X.

Let $X, Y \in \mathbf{K}^b(R)$. Then we have two pure quasi-isomorphisms $f_X : P_X \to X$ and $f_Y : P_Y \to Y$. These induce an isomorphism

$$\mathrm{Hom}_{\mathbf{K}(R)}(P_X, P_Y) \cong \mathrm{Hom}_{\mathbf{K}(R)}(P_X, Y).$$

Let $f : X \to Y$ be a cochain map. Then there exists a cochain map $f \circ f_X : P_X \to Y$. Using the above isomorphism, we have that there exists a unique cochain map $f' : P_X \to P_Y$ such that the following diagram:

$$\begin{array}{ccc}
P_X & \xrightarrow{f_X} & X \\
{\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\
P_Y & \xrightarrow{f_Y} & Y
\end{array}$$

commutes up to homotopy. This completes the proof by putting $Y = X$. □

(2) It is the dual of (1) just using the fact that every module admits a pure injective preenvelope by [6, Proposition 5.3.9].

Theorem 3.6. *For a ring R, there exist triangle-equivalences as follows:*

(1) $\mathbf{D}^{\mathbf{b}}_{\mathbf{pur}}(R) \simeq \mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$.
(2) $\mathbf{D}^{\mathbf{b}}_{\mathbf{pur}}(R) \simeq \mathbf{K}^{+,\mathbf{pb}}(\mathcal{PI})$.

Proof. We only need to prove (1). Let $\mathbb{H}$ be the composition of the embedding

$$\mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP}) \hookrightarrow \mathbf{K}^{-}(R)$$

and the localization functor

$$\mathbb{F} : \mathbf{K}^{-}(R) \to \mathbf{D}^{-}_{\mathbf{pur}}(R).$$

For any $X \in \mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$, there exists $n \in \mathbb{Z}$ such that $\mathbf{inf_p} X = n$. So X is pure exact in degree $\leq n-1$ and the following cochain map f is a pure quasi-isomorphism:

$$\begin{array}{ccccccccccc} X := \cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow & \cdots \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X^{\supset n} := \cdots & \longrightarrow & 0 & \longrightarrow & \operatorname{Coker} d^{n-1} & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow & \cdots. \end{array}$$

It follows that $\mathbb{H}(X) \cong X^{\supset n}$ in $\mathbf{D_{pur}}(R)$. So $\mathbb{H}(X) \in \mathbf{D^b_{pur}}(R)$ and hence $\mathbb{H}$ induces a functor from $\mathbf{K}^{-,\mathbf{pb}}(\mathcal{PP})$ to $\mathbf{D^b_{pur}}(R)$, again denoted by $\mathbb{H}$. By Propositions 3.1 and 3.5, $\mathbb{H}$ is fully faithful and dense. This completes the proof. □

4. Derived functors and dimensions

In this section, we introduce and investigate the pure projective and injective dimensions of complexes based on pure derived functors of Hom in $\mathbf{D^b_{pur}}(R)$. For the pure projective and injective dimensions of modules and pure derived functors in R-Mod, we refer to [11] and [19].

We have already known that $\mathrm{Hom}_R(P, -)$ transforms pure quasi-isomorphisms to quasi-isomorphisms for any $P \in \mathbf{K}^-(\mathcal{PP})$. In order to define pure projective (resp. injective) resolutions of complexes in $\mathbf{D_{pur}}(R)$, we need the following lemma.

Lemma 4.1. *Let X be a pure exact complex of R-modules. Then we have*

(1) *$M \otimes_R X$ is a pure exact complex for any right R-module M.*
(2) *$\mathrm{Hom}_R(P, X)$ is a pure exact complex for any $P \in \mathcal{PP}$.*
(3) *$\mathrm{Hom}_R(X, I)$ is a pure exact complex for any $I \in \mathcal{PI}$.*

Proof. (1) It is obvious by the associativity of tensor products.

(2) We will prove that $\mathrm{Hom}_{\mathbb{Z}}(F, \mathrm{Hom}_R(P, X))$ is exact for any finitely presented $\mathbb{Z}$-module F and $P \in \mathcal{PP}$.

By Remark 2.6, $\mathcal{PP}$ consists of summands of direct sums of finitely presented R-modules. So we may assume that P is finitely presented. Note that $P \otimes_{\mathbb{Z}} F$ is a finitely presented R-module. So by the adjoint isomorphism $\mathrm{Hom}_{\mathbb{Z}}(F, \mathrm{Hom}_R(P, X)) \cong \mathrm{Hom}_R(P \otimes_{\mathbb{Z}} F, X)$, we have that $\mathrm{Hom}_R(P, X)$ is pure exact.

(3) Let $I \in \mathcal{PI}$. Then I is a direct summand of I^{++} by [6, Proposition 5.3.9]. We may assume $I = M^+$ for some right R-module M. Let F be a finitely presented $\mathbb{Z}$-module. By the adjoint isomorphism theorem, we have the isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}}(F, \mathrm{Hom}_R(X, M^+)) \cong \mathrm{Hom}_{\mathbb{Z}}(F, (M \otimes_R X)^+) \cong (F \otimes_{\mathbb{Z}} M \otimes_R X)^+.$$

By (1), $F \otimes_{\mathbb{Z}} M \otimes_R X$ is pure exact. So $(F \otimes_{\mathbb{Z}} M \otimes_R X)^+$ is exact, and hence $\mathrm{Hom}_R(X, I)$ is pure exact. □

Remark 4.2. By [3, Lemmas 2.4 and 2.5] and Lemma 4.1, after a standard computation we have

(1) $\mathrm{Hom}_R(P,-)$ preserves pure exact complexes for any $P \in \mathbf{K}^-(\mathcal{PP})$.
(2) $\mathrm{Hom}_R(-,I)$ preserves pure exact complexes for any $I \in \mathbf{K}^+(\mathcal{PI})$.

Definition 4.3. Let $X \in \mathbf{D_{pur}}(R)$.

(1) A *pure projective resolution* of X is a pure quasi-isomorphism $f : P \to X$ with P a complex of pure projective R-modules, such that $\mathrm{Hom}_R(P,-)$ preserves pure exact complexes. Dually, a *pure injective resolution* of X is defined.
(2) X is said to have *pure projective dimension* at most n, written $\mathrm{p.pd}_R\, X \leq n$, if there exists a pure projective resolution $P \to X$ with $P^i = 0$ for any $i < -n$. If $\mathrm{p.pd}_R\, X \leq n$ for all n, then we write $\mathrm{p.pd}_R\, X = -\infty$; and if there exists no n such that $\mathrm{p.pd}_R\, X \leq n$, then we write $\mathrm{p.pd}_R\, X = \infty$. Dually, the *pure injective dimension* $\mathrm{p.id}_R\, X$ of X is defined.

Remark 4.4.

(1) Let X be an R-module (viewed as a complex concentrated in degree 0), then these definitions coincide with the usual ones, see [11] and [19].
(2) In the above definition, $\mathrm{p.pd}_R\, X = -\infty$ means that X is a pure exact complex.

These dimensions can be also expressed by the following equalities:

$$\mathrm{p.pd}_R\, X = -\sup\{\inf\{n \in \mathbb{Z} \mid P^n \neq 0\} \mid P \to X \text{ is a pure projective resolution}\}, \text{ and}$$
$$\mathrm{p.id}_R\, X = \inf\{\sup\{n \in \mathbb{Z} \mid I^n \neq 0\} \mid X \to I \text{ is a pure injective resolution}\}.$$

Let $X \in \mathbf{D^b_{pur}}(R)$. Then by Proposition 3.5, there exists a complex $P \in \mathbf{K}^-(\mathcal{PP})$ such that $P \cong X$ in $\mathbf{D^b_{pur}}(R)$. By Remark 4.2, $\mathrm{Hom}_R(P,-)$ preserves pure exact complexes, and hence preserves pure quasi-isomorphisms. Then after an easy computation we get a pure quasi-isomorphism from P to X. The statements for the pure injective version are dual. Thus, if $X \in \mathbf{D^b_{pur}}(R)$, then X admits pure projective (resp. injective) resolutions.

Now we may define a functor

$$\mathbf{R}\,\mathrm{Hom}_R(-,-) : \mathbf{D^b_{pur}}(R)^{op} \times \mathbf{D^b_{pur}}(R) \to \mathbf{D_{pur}}(\mathbb{Z})$$

using either the pure projective resolution of the first variable or the pure injective resolution of the second variable. More precisely, let P_X be a pure projective resolution of X and I_Y a pure injective resolution of Y. Then we have a diagram of pure quasi-isomorphisms

$$\mathbf{R}\operatorname{Hom}_R(X,Y) := \operatorname{Hom}_R(P_X,Y) \to \operatorname{Hom}_R(P_X,I_Y) \leftarrow \operatorname{Hom}_R(X,I_Y) := \mathbf{R}\operatorname{Hom}_R(X,Y).$$

It follows that $\mathbf{R}\operatorname{Hom}_R(-,-)$ is well defined, and we call it the *right pure derived functor* of Hom.

Let $P \to X$ be a pure projective resolution of X and $Y \to I$ a pure injective resolution of Y. In order to coincide with the classical ones in [11] and [19], we put

$$\operatorname{Pext}^i_R(X,Y) := \mathrm{H}^i\,\mathbf{R}\operatorname{Hom}_R(X,Y) = \mathrm{H}^i\operatorname{Hom}_R(P,Y), \text{ and}$$
$$\operatorname{Pext}^i_R(X,Y) := \mathrm{H}^i\,\mathbf{R}\operatorname{Hom}_R(X,Y) = \mathrm{H}^i\operatorname{Hom}_R(X,I).$$

Recall that $X \in \mathbf{C}(R)$ is called *contractible* if it is isomorphic to the zero object in $\mathbf{K}(R)$, equivalently, the identical map Id_X is homotopic to zero. That is to say, X is splitting exact (see [22, Exercise 1.4.3]).

Theorem 4.5. *For any $X \in \mathbf{D}^{\mathbf{b}}_{\mathbf{pur}}(R)$ and $n \in \mathbb{Z}$, the following statements are equivalent:*

(1) $\mathrm{p.pd}_R X \leq n$.
(2) $\mathbf{inf_p} X \geq -n$, *and if $f' : P' \to X$ is a pure projective resolution of X, then the R-module* $\operatorname{Coker} d^{-n-1}_{P'}$ *is pure projective.*
(3) *If $f' : P' \to X$ is a pure projective resolution of X, then $P' = P_1 \bigoplus P_2$, where $P_1^i = 0$ for any $i < -n$ and P_2 is contractible.*
(4) $\operatorname{Pext}^i_R(X,Y) = 0$ *for any $Y \in \mathbf{D_{pur}}(R)$ and $i > n + \mathbf{sup_p} Y$.*
(5) $\mathbf{inf_p} X \geq -n$ *and* $\operatorname{Pext}^{n+1}_R(X,N) = 0$ *for any $N \in R$-Mod.*

Proof. (1) $\Rightarrow$ (2) Let $\mathrm{p.pd}_R X \leq n$. Then there exists a pure projective resolution $f : P \to X$ with $P^i = 0$ for any $i < -n$. By Proposition 3.4, we have $\mathbf{inf_p} X \geq -n$. Let $f' : P' \to X$ be another pure projective resolution of X. Then there exists a quasi-isomorphism of complexes

$$\operatorname{Hom}_R(P,f') : \operatorname{Hom}_R(P,P') \to \operatorname{Hom}_R(P,X).$$

Thus there exists a cochain map $g : P \to P'$ such that $f' \circ g = f$, and therefore

$$\operatorname{Hom}_R(F,f') \circ \operatorname{Hom}_R(F,g) = \operatorname{Hom}_R(F,f)$$

for any finitely presented R-module F. It follows from Remark 2.8(2) that g is a pure quasi-isomorphism. Then g is a homotopy equivalence by Lemma 2.10. It is easy to check that the exact sequence

$$\cdots \longrightarrow P'^{-n-1} \xrightarrow{d^{-n-1}_{P'}} P'^{-n} \xrightarrow{\widetilde{d}^{-n-1}_{P'}} \operatorname{Coker} d^{-n-1}_{P'} \longrightarrow 0$$

is contractible. So $\operatorname{Coker} d^{-n-1}_{P'}$ is pure projective.

(2) ⇒ (3) Let $f' : P' \to X$ be a pure projective resolution of X. Because $\mathbf{inf_P} P' = \mathbf{inf_p} X \geq -n$, we have that the sequence

$$\cdots \longrightarrow P'^{\,-n-1} \xrightarrow{d_{P'}^{-n-1}} P'^{\,-n} \xrightarrow{\widetilde{d}_{P'}^{-n-1}} \operatorname{Coker} d_{P'}^{-n-1} \longrightarrow 0 \tag{4.1}$$

is pure exact. Because $\operatorname{Coker} d_{P'}^{-n-1}$ is pure projective by assumption, (4.1) is contractible. Now let $P'^{\,-n} = M \oplus \operatorname{Coker} d_{P'}^{-n-1}$, and put

$$P_1 := \cdots \to 0 \to \operatorname{Coker} d_{P'}^{-n-1} \to P'^{\,-n+1} \to P'^{\,-n+2} \to \cdots, \text{ and}$$
$$P_2 := \cdots \to P'^{\,-n-2} \to P'^{\,-n-1} \to M \to 0 \to \cdots.$$

Then we have $P' = P_1 \bigoplus P_2$, where $P_1^i = 0$ for any $i < -n$ and P_2 is contractible.

(3) ⇒ (1) By (3), we have that the embedding $P_1 \hookrightarrow P'$ is clearly a pure quasi-isomorphism. This implies that X admits a pure projective resolution $P_1 \hookrightarrow P' \to X$ with $P_1^i = 0$ for any $i < -n$.

(3) ⇒ (4) We only need to consider the situation when $\mathbf{sup_p} Y = m < \infty$. Let $P \to X$ be a pure projective resolution of X. Then $P = P_1 \bigoplus P_2$, where $P_1^i = 0$ for any $i < -n$ and P_2 is contractible. So we have

$$\operatorname{Pext}_R^i(X, Y) = \mathrm{H}^i \operatorname{Hom}_R(P, Y) = \mathrm{H}^i \operatorname{Hom}_R(P_1, Y).$$

As in Lemma 2.11, let Y' be the right canonical truncation complex of Y at degree m. Then the embedding $Y' \hookrightarrow Y$ is a pure quasi-isomorphism. So we have

$$\mathrm{H}^i \operatorname{Hom}_R(P_1, Y) = \mathrm{H}^i \operatorname{Hom}_R(P_1, Y') = 0$$

for any $i > n + m$. Thus $\operatorname{Pext}_R^i(X, Y) = 0$ for any $Y \in \mathbf{D_{pur}}(R)$ and $i > n + \mathbf{sup_p} Y$.

(4) ⇒ (5) For any $N \in R$-Mod, we have $\mathbf{sup_p} N = 0$ and $\operatorname{Pext}_R^{n+1}(X, N) = 0$ by (4). Let M be a right R-module. Then

$$\mathrm{H}^i((M \otimes_R X)^+) = \mathrm{H}^i(\operatorname{Hom}_R(X, M^+)) = \operatorname{Pext}_R^i(X, M^+) = 0$$

for any $i > n$ by the adjoint isomorphism theorem and (4). So $M \otimes_R X$ is exact in degree $<-n$, and hence X is pure exact in degree $<-n$ by Proposition 3.4. It implies that $\mathbf{inf_p} X \geq -n$.

(5) ⇒ (3) Let P' be a pure projective resolution of X and $N \in R$-Mod. Then we have $\mathbf{inf_p} P' = \mathbf{inf_p} X \geq -n$. So P' is pure exact in degree $\leq -n-1$, and hence the sequence

$$\cdots \longrightarrow P'^{\,-n-2} \longrightarrow P'^{\,-n-1} \longrightarrow P'^{\,-n} \longrightarrow \operatorname{Coker}_{P'}^{-n-1} \longrightarrow 0$$

is pure exact and it is a pure projective resolution of $\operatorname{Coker}_{P'}^{-n-1}$. We have the following equalities:

$$\mathrm{Pext}^1_R(\mathrm{Coker}^{-n-1}_{P'}, N) = \mathrm{H}^{n+1}\,\mathrm{Hom}_R(P', N) = \mathrm{Pext}^{n+1}_R(X, N) = 0.$$

It implies that $\mathrm{Coker}^{-n-1}_{P'}$ is pure projective. Thus the above pure exact complex is contractible, and therefore $P' = P_1 \bigoplus P_2$, where $P^i_1 = 0$ for any $i < -n$. □

Dually, we have the following

Theorem 4.6. *For any $Y \in \mathbf{D}^{\mathbf{b}}_{\mathbf{pur}}(R)$ and $n \in \mathbb{Z}$, the following statements are equivalent:*

(1) $\mathrm{p.id}_R\, Y \leq n$.
(2) $\mathbf{sup_p} Y \leq n$, *and if $f' : Y \to I'$ is a pure injective resolution of X, then the R-module* $\mathrm{Ker}\, d^n_{I'}$ *is pure injective.*
(3) *If $f' : Y \to I'$ is a pure injective resolution of X, then $I' = I_1 \bigoplus I_2$, where $I^i_1 = 0$ for any $i > n$ and I_2 is contractible.*
(4) $\mathrm{Pext}^i_R(X, Y) = 0$ *for any* $X \in \mathbf{D_{pur}}(R)$ *and* $i > n - \mathbf{inf_p} X$.
(5) $\mathbf{sup_p} Y \leq n$ *and* $\mathrm{Pext}^{n+1}_R(M, Y) = 0$ *for any* $M \in R$-Mod.

By the above two theorems and Proposition 3.4, for any complex $X \in \mathbf{D}^{\mathbf{b}}_{\mathbf{pur}}(R)$, we have the following characterizations of $\mathrm{p.pd}_R\, X$ and $\mathrm{p.id}_R\, X$ via the pure derived functor $\mathbf{R}\,\mathrm{Hom}$:

$$\mathrm{p.pd}_R\, X = \sup\{i \in \mathbb{Z} \mid \mathrm{Pext}^i_R(X, N) \neq 0 \text{ for some } N \text{ in } R\text{-Mod}\}, \text{ and}$$
$$\mathrm{p.id}_R\, Y = \sup\{i \in \mathbb{Z} \mid \mathrm{Pext}^i_R(M, Y) \neq 0 \text{ for some } M \text{ in } R\text{-Mod}\}.$$

Recall that the *left pure global dimension* of R, written $\mathrm{p.gldim}\, R$, is the supremum of the pure projective dimension of all modules in R-Mod. It is also equals to the supremum of the pure injective dimension of all modules in R-Mod. It is well known that $\mathrm{p.gldim}\, R \leq n$ if and only if $\mathrm{Pext}^i_R(M, N) = 0$ for any $M, N \in R$-Mod and $i > n$, see for example [19, p. 95]. We have the cochain complex version of this result.

Theorem 4.7. *For any $n \in \mathbb{Z}$, the following statements are equivalent:*

(1) $\mathrm{p.gldim}\, R \leq n$.
(2) $\mathrm{p.pd}_R\, X \leq n - \mathbf{inf_p} X$ *for any* $X \in \mathbf{D}^{\mathbf{b}}_{\mathbf{pur}}(R)$.
(3) $\mathrm{p.id}_R\, Y \leq n + \mathbf{sup_p} Y$ *for any* $Y \in \mathbf{D}^{\mathbf{b}}_{\mathbf{pur}}(R)$.
(4) $\mathrm{Pext}^i_R(X, Y) = 0$ *for any* $X, Y \in \mathbf{D}^{\mathbf{b}}_{\mathbf{pur}}(R)$ *and* $i > n + \mathbf{sup_p} Y - \mathbf{inf_p} X$.

Proof. The implications (2) ⇒ (4) and (3) ⇒ (4) follow from Theorems 4.5 and 4.6, respectively. The implication (4) ⇒ (1) is obvious just letting both X and Y be R-modules. The implication (1) ⇒ (2) is the dual of (1) ⇒ (3). So it remains to prove the implication (1) ⇒ (3).

Let $\mathbf{sup_p}Y = m$ and $Y \to I$ be a pure injective resolution of Y. Then $\mathbf{sup_p}I = m$ and I is pure exact in degree $\geq m+1$. So

$$0 \to \operatorname{Ker} d_I^m \to I^m \to I^{m+1} \to \cdots$$

is a pure injective resolution of $\operatorname{Ker} d_I^m$. By (1), we have $\text{p.id}_R \operatorname{Ker} d_I^m \leq n$. Let

$$0 \to \operatorname{Ker} d_I^m \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

be a pure injective resolution of $\operatorname{Ker} d_I^m$. Then it is easy to check that

$$\cdots \to I^{m-2} \to I^{m-1} \to K^0 \to \cdots \to K^{n-1} \to K^n \to 0 \to \cdots$$

is a pure injective resolution of Y and $\text{p.id}_R Y \leq n+m$. □

5. The case of unbounded complexes

In this section, we study the existence of pure projective resolutions of unbounded complexes. We need the tools of homotopy colimits and limits [14].

Let

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} \cdots \tag{5.1}$$

be a sequence in $\mathbf{K}^b(R)$, where j_i is a morphism of complexes for any $i > 0$. Then we can form the *homotopy colimit* of this sequence, written $\operatorname{Ho}\varinjlim X_i$, by the triangle

$$\bigoplus_{i=0}^{\infty} X_i \xrightarrow{\text{1-shift}} \bigoplus_{i=0}^{\infty} X_i \longrightarrow \operatorname{Ho}\varinjlim X_i \longrightarrow (\bigoplus_{i=0}^{\infty} X_i)[1]$$

in $\mathbf{K}(R)$. The notion of *homotopy limits* is defined dually, and denoted by $\operatorname{Ho}\varprojlim$.

For the sequence (5.1), we can also form the *direct limit*, written $\varinjlim X_i$, in $\mathbf{C}(R)$. We have the following exact sequence of complexes (note: the morphism 1-shift is monic):

$$0 \longrightarrow \bigoplus_{i=0}^{\infty} X_i \xrightarrow{\text{1-shift}} \bigoplus_{i=0}^{\infty} X_i \xrightarrow{\iota} \varinjlim X_i \longrightarrow 0. \tag{5.2}$$

Then it is easy to check that there exists a morphism $\alpha : \operatorname{Ho}\varinjlim X_i \to \varinjlim X_i$, since $\iota \circ (\text{1-shift}) = 0$. Passing to homology we conclude that α is a quasi-isomorphism, since $\mathrm{H}^0(-)$ is cohomological by [8, Chapter IV.1.6].

Because $\operatorname{Hom}_R(F, -)$ commutes with direct limits in $\mathbf{C}(R)$ for any finitely presented R-module F by [1, Corollary 1.54] (see also [20, Remark 4.13] or [4, Corollary 4.6]), we have that (5.2) is pure exact in $\mathbf{C}(R)$, that is, it is pure exact in each degree. So after applying the functor $\operatorname{Hom}_R(P, -)$ for any $P \in \mathcal{PP}$, we get the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(P, \bigoplus_{i=0}^{\infty} X_i) \xrightarrow{\mathrm{Hom}_R(P,1\text{-shift})} \mathrm{Hom}_R(P, \bigoplus_{i=0}^{\infty} X_i) \longrightarrow \mathrm{Hom}_R(P, \varinjlim X_i) \longrightarrow 0,$$

and the following exact triangle:

$$\mathrm{Hom}_R(P, \bigoplus_{i=0}^{\infty} X_i) \xrightarrow{\mathrm{Hom}_R(P,1\text{-shift})} \mathrm{Hom}_R(P, \bigoplus_{i=0}^{\infty} X_i) \longrightarrow \mathrm{Hom}_R(P, \mathrm{Ho}\varinjlim X_i) \longrightarrow \mathrm{Hom}_R(P, \bigoplus_{i=0}^{\infty} X_i)[1]$$

in $\mathbf{K}(R)$. When passing to homology we see that α is a pure quasi-isomorphism.

Theorem 5.1. *Let $X \in \mathbf{C}(R)$ be a bounded below complex. Then there exists a complex P consisting of pure projective R-modules satisfying the following properties:*

(1) *There exists a pure quasi-isomorphism $f : P \to X$.*
(2) $\mathrm{Hom}_R(P, -)$ *preserves pure exact complexes.*

That is, $f : P \to X$ is a pure projective resolution of X.

Proof. (1) Write $X := \varinjlim X_i$ with the structure map $j_{i+1} : X_i \to X_{i+1}$, where X_i is a bounded complex for any $i \geq 0$. By Proposition 3.5, for any X_i there exists a pure quasi-isomorphism $f_i : P_i \longrightarrow X_i$ with $P_i \in \mathbf{K}^-(\mathcal{PP})$ for any $i \geq 0$. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} P_i & \xrightarrow{\overline{j_{i+1}}} & P_{i+1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i+1}} \\ X_i & \xrightarrow{j_{i+1}} & X_{i+1} \end{array}$$

in $\mathbf{K}(R)$, where $\overline{j_{i+1}}$ is induced by j_{i+1}. So there exists a morphism of exact triangles

$$\begin{array}{ccccccc} \bigoplus_{i=0}^{\infty} P_i & \xrightarrow{1\text{-shift}} & \bigoplus_{i=0}^{\infty} P_i & \longrightarrow & \mathrm{Ho}\varinjlim P_i & \longrightarrow & (\bigoplus_{i=0}^{\infty} P_i)[1] \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \bigoplus_{i=0}^{\infty} X_i & \xrightarrow{1\text{-shift}} & \bigoplus_{i=0}^{\infty} X_i & \longrightarrow & \mathrm{Ho}\varinjlim X_i & \longrightarrow & (\bigoplus_{i=0}^{\infty} X_i)[1] \end{array}$$

in $\mathbf{K}(R)$. After applying the localization functor, it is a morphism of exact triangles in $\mathbf{D}_{\mathbf{pur}}(R)$. Since pure quasi-isomorphisms are closed under coproducts by Remark 2.8(2), we have that the first two vertical maps in the above diagram are pure quasi-isomorphisms. So f and

$$\alpha \circ f : P = \mathrm{Ho}\varinjlim P_i \to \mathrm{Ho}\varinjlim X_i \to \varinjlim X_i$$

are also pure quasi-isomorphisms. By the construction, we have that $\operatorname{Ho}\varinjlim P_i$ is the mapping cone of some cochain map between complexes consisting of pure projective R-modules. Thus $\operatorname{Ho}\varinjlim P_i$ is also a complex consisting of pure projective R-modules.

(2) We will prove that $\operatorname{Hom}_{\mathbb{Z}}(F, \operatorname{Hom}_R(P, X))$ is exact for any pure exact complex X of R-modules and any finitely presented $\mathbb{Z}$-module F. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\operatorname{Hom}_R(P,X) & \longrightarrow & \operatorname{Hom}_R(\bigoplus_{i=0}^{\infty} P_i, X) & \longrightarrow & \operatorname{Hom}_R(\bigoplus_{i=0}^{\infty} P_i, X) & \longrightarrow & (\operatorname{Hom}_R(P,X))[1] \\
= \downarrow & & \cong \downarrow & & \cong \downarrow & & \downarrow \\
\operatorname{Hom}_R(P,X) & \longrightarrow & \prod_{i=0}^{\infty} \operatorname{Hom}_R(P_i, X) & \longrightarrow & \prod_{i=0}^{\infty} \operatorname{Hom}_R(P_i, X) & \longrightarrow & (\operatorname{Hom}_R(P,X))[1]
\end{array}$$

in $\mathbf{K}(\mathbb{Z})$, where both rows are exact triangles. We have the following isomorphisms:

$$\operatorname{Hom}_{\mathbb{Z}}(F, \operatorname{Hom}_R(\bigoplus_{i=0}^{\infty} P_i, X)) \cong \prod_{i=0}^{\infty} \operatorname{Hom}_{\mathbb{Z}}(F, \operatorname{Hom}_R(P_i, X)).$$

Note that the latter one is exact by Remark 4.2. Because $\operatorname{Hom}_{\mathbb{Z}}(F, -)$ is a triangulated functor, the assertion follows standardly. □

Theorem 5.2. *Let $X \in \mathbf{C}(R)$ be a bounded above complex. Then there exists a complex I consisting of pure injective R-modules satisfying the following properties:*

(1) *There exists a pure quasi-isomorphism $f : X \to I$.*
(2) $\operatorname{Hom}_R(-, I)$ *preserves pure exact complexes.*

That is, $f : X \to I$ is a pure injective resolution of X.

Proof. Write $X := \varprojlim X_i$ with X_i a bounded complex for any $i \leq 0$. Then by [10, Lemma 2.6], we have $X \cong \operatorname{Ho}\varprojlim X_i$ in $\mathbf{K}(R)$. Note that pure quasi-isomorphisms are closed under products by Remark 2.8(2). Now by using an argument similar to that in the proof of Theorem 5.1, we get the assertion. □

Remark 5.3. One can find that the derived functor

$$\mathbf{R}\operatorname{Hom}_R(-,-) : \mathbf{D}^{\mathbf{b}}_{\mathbf{pur}}(R)^{op} \times \mathbf{D}^{\mathbf{b}}_{\mathbf{pur}}(R) \to \mathbf{D}_{\mathbf{pur}}(\mathbb{Z})$$

may be extended to

$$\mathbf{R}\operatorname{Hom}_R(-,-) : \mathbf{D}^{+}_{\mathbf{pur}}(R)^{op} \times \mathbf{D}^{-}_{\mathbf{pur}}(R) \to \mathbf{D}_{\mathbf{pur}}(\mathbb{Z}).$$

The corresponding characterizations of dimensions in Section 4 also hold in this situation.

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