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The existence of maximal n-orthogonal subcategories

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ABSTRACT

For an $(n-1)$-Auslander algebra Λ with global dimension n, we give some necessary conditions for Λ admitting a maximal $(n-1)$-orthogonal subcategory in terms of the properties of simple Λ-modules with projective dimension $n-1$ or n. For an almost hereditary algebra Λ with global dimension 2, we prove that Λ admits a maximal 1-orthogonal subcategory if and only if for any non-projective indecomposable Λ-module M, M is injective is equivalent to that the reduced grade of M is equal to 2. We give a connection between the Gorenstein Symmetric Conjecture and the existence of maximal n-orthogonal subcategories of ${}^{\perp}T$ for a cotilting module T. For a Gorenstein algebra, we prove that all non-projective direct summands of a maximal n-orthogonal module are $\Omega^n\tau$-periodic. In addition, we study the relation between the complexity of modules and the existence of maximal n-orthogonal subcategories for the tensor product of two finite-dimensional algebras.

1. Introduction

In [Iy2] and [Iy3], Iyama developed the classical 2-dimensional Auslander–Reiten theory to higher-dimensional Auslander–Reiten theory. For example, in [Iy2], the notion of almost split sequences was generalized to that of n-almost split sequences; and in [Iy3], the famous 2-dimensional Auslander correspondence was generalized to higher-dimensional Auslander correspondence. In particular, in [Iy2], Iyama introduced the notion of maximal n-orthogonal subcategories, which played a crucial role in these two papers mentioned above. In fact, Iyama's higher-dimensional Auslander–Reiten theory depends on the existence of maximal n-orthogonal subcategories. So, a natural question is: When do maximal n-orthogonal subcategories exist? Geiss, Leclerc and Schröer proved in [GLS] that

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maximal 1-orthogonal subcategories exist for preprojective algebras as well as for certain algebras of finite representation type. Erdmann and Holm gave in [EH] a necessary condition that a selfinjective algebra admits a maximal n-orthogonal subcategory. They proved that for a selfinjective finite-dimensional K-algebra Λ, the complexity of any Λ-module M is at most 1 if Λ admits a maximal n-orthogonal module for some $n \geqslant 1$, where the *complexity* of M, denoted by $\mathrm{cx}(M)$, is defined as $\inf\{b \in \mathbb{N}_0 \mid \text{there exists a } c > 0 \text{ such that } \dim_K P_n \leqslant cn^{b-1} \text{ for all } n\}$ with P_n the $(n+1)$st term in a minimal projective resolution of M. This result means that for a selfinjective algebra, maximal n-orthogonal modules rarely exist. Recently, Iyama proved in [Iy4] that if Λ is a finite-dimensional algebra of finite representation type with Auslander algebra Γ and $\mathrm{mod}\,\Gamma$ contains a maximal 1-orthogonal object, then Λ is hereditary and the dominant dimension of Λ is at least 1; furthermore, if the base field is algebraically closed, then such Λ is an upper triangular matrix ring. The aim of the paper is to study algebras of global dimension n which admit maximal $(n-1)$-orthogonal subcategories or modules, and that they are natural generalizations of Gabriel's classification theorem on path algebras of Dynkin quivers (for $n = 1$). This paper is organized as follows.

In Section 2, we give some notions and notations and collect some preliminary results.

In Section 3, we study the existence of maximal n-orthogonal subcategories for algebras with finite global dimension. We observe that an Artinian algebra Λ with global dimension n $(\geqslant 1)$ admits no maximal n-orthogonal subcategories of $\mathrm{mod}\,\Lambda$. Let Λ be an $(n-1)$-Auslander algebra with global dimension n. We prove that if Λ admits a maximal $(n-1)$-orthogonal subcategory of $\mathrm{mod}\,\Lambda$, then we can classify the simple modules with projective dimension $n-1$; furthermore, in terms of the properties of simple modules in $\mathrm{mod}\,\Lambda$ with projective dimension n, we give some necessary and sufficient conditions for Λ admitting a trivial maximal $(n-1)$-orthogonal subcategory of $\mathrm{mod}\,\Lambda$, and also give a necessary condition for Λ admitting a non-trivial maximal $(n-1)$-orthogonal subcategory of $\mathrm{mod}\,\Lambda$.

We also study the existence of maximal n-orthogonal subcategories for algebras with global dimension 2. For example, if putting $n = 1$, then the above results holds true for classical Auslander algebras with global dimension 2. In addition, for an almost hereditary algebra Λ with global dimension 2, we prove that Λ admits a maximal 1-orthogonal subcategory of $\mathrm{mod}\,\Lambda$ if and only if for any non-projective indecomposable module $M \in \mathrm{mod}\,\Lambda$, M is injective is equivalent to that the reduced grade of M is equal to 2. In particular, if Λ admits a maximal 1-orthogonal subcategory $\mathscr{C}$ of $\mathrm{mod}\,\Lambda$, then $\mathscr{C} = \mathrm{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$.

In Section 4, we study the existence of maximal n-orthogonal subcategories of ${}^{\perp}T$ for a cotilting module T. We give a connection between the Gorenstein Symmetric Conjecture with the existence of maximal n-orthogonal subcategories of ${}^{\perp}T$. As a generalization of a result of Erdmann and Holm in [EH], we prove that for a Gorenstein algebra, all non-projective direct summands of a maximal n-orthogonal module are $\Omega^n\tau$-periodic. It should be pointed out that this result can be induced by [Iy3, Theorem 2.5.1(1)].

In Section 5, both Λ and Γ are finite-dimensional K-algebras over a field K. We prove that if the selfinjective dimension of Λ is equal to n $(\geqslant 1)$ and $\mathrm{Hom}_\Gamma(\mathbb{D}\Gamma^{op}, \Gamma) \neq 0$, then $\Lambda \otimes_K \Gamma$ admits no maximal j-orthogonal subcategories of $\mathrm{mod}\,\Lambda \otimes_K \Gamma$ for any $j \geqslant n$. By this result, we can construct algebras with infinite global dimension admitting no maximal n-orthogonal subcategories for any $n \geqslant 1$. In addition, we prove that $\max\{\mathrm{cx}(M), \mathrm{cx}(N)\} \leqslant \mathrm{cx}(M \otimes_K N) \leqslant \mathrm{cx}(M) + \mathrm{cx}(N)$ for any $M \in \mathrm{mod}\,\Lambda$ and $N \in \mathrm{mod}\,\Gamma$. As an application of this result, we can construct a class of algebras Λ with selfinjective dimension n $(\geqslant 1)$, such that not all modules in $\mathrm{mod}\,\Lambda$ are of complexity at most 1, but Λ admits no maximal j-orthogonal subcategories of $\mathrm{mod}\,\Lambda$ for any $j \geqslant n$.

2. Preliminaries

In this section, we give some notions and notations in our terminology and collect some preliminary results for later use.

For a ring Λ, we use $\mathrm{mod}\,\Lambda$, $\mathrm{gl.dim}\,\Lambda$ and $J(\Lambda)$ to denote the category of finitely generated left Λ-modules, the global dimension and the Jacobson radical of Λ, respectively. We use Tr and $(-)^*$ to denote the Auslander transpose and the functor $\mathrm{Hom}_\Lambda(-, \Lambda)$, respectively.

Let M be a Λ-module. The *grade* of M, denoted by grade M, is defined as $\inf\{n \geqslant 0 \mid \mathrm{Ext}^n_\Lambda(M,\Lambda) \neq 0\}$; and the *reduced grade* of M, denoted by r.grade M, is defined as $\inf\{n \geqslant 1 \mid \mathrm{Ext}^n_\Lambda(M,\Lambda) \neq 0\}$ (see [Ho]). We use $\mathrm{pd}_\Lambda M$, $\mathrm{fd}_\Lambda M$ and $\mathrm{id}_\Lambda M$ to denote the projective, flat and injective dimensions of M, respectively. Let Λ be a left and right Noetherian ring and $M \in \mathrm{mod}\,\Lambda$. We use

$$0 \to M \to I^0(M) \to I^1(M) \to \cdots \to I^i(M) \to \cdots$$

to denote a minimal injective resolution of ${}_\Lambda M$. For positive integers m and n, recall from [Iy1] that Λ is said to satisfy the (m,n)-*condition* (resp. the $(m,n)^{op}$-*condition*) if $\mathrm{fd}_\Lambda I^i(\Lambda)$ (resp. $\mathrm{fd}_{\Lambda^{op}} I^i(\Lambda^{op}) \leqslant m-1$) for any $0 \leqslant i \leqslant n-1$.

Lemma 2.1. *(See [Iy1, Proposition 2.4].) Let Λ be a left and right Noetherian ring satisfying the (n,n)-condition and $(n,n)^{op}$-condition. Then the subcategory $\{X \mid \mathrm{grade}\, X \geqslant n\}$ of $\mathrm{mod}\,\Lambda$ is closed under submodules and factor modules.*

Recall from [FGR] that a left and right Noetherian ring Λ is called *n-Gorenstein* if $\mathrm{fd}_\Lambda I^i(\Lambda) \leqslant i$ for any $0 \leqslant i \leqslant n-1$. By [FGR, Theorem 3.7], the notion of n-Gorenstein rings is left and right symmetric. It is clear that Λ is n-Gorenstein if and only if Λ satisfies the (i,i) (or $(i,i)^{op}$)-condition for any $1 \leqslant i \leqslant n$. Recall from [Bj] that Λ is called *Auslander–Gorenstein* if Λ is n-Gorenstein for all n and both $\mathrm{id}_\Lambda \Lambda$ and $\mathrm{id}_{\Lambda^{op}} \Lambda$ are finite; and Λ is called *Auslander-regular* if Λ is n-Gorenstein for all n and gl.dim Λ is finite.

Lemma 2.2. *(See [IS, Corollary 7].) Let Λ be an Auslander–Gorenstein ring with $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{op}} \Lambda = n \geqslant 1$. Then $\bigoplus_{i=0}^{n-1} I^i(\Lambda)$ and $I^n(\Lambda)$ have no isomorphic direct summands in common.*

The following easy observation is well known.

Lemma 2.3. *Let Λ be a left Noetherian ring and $M \in \mathrm{mod}\,\Lambda$ with $\mathrm{pd}_\Lambda M = n$ $(< \infty)$. Then $\mathrm{Ext}^n_\Lambda(M,\Lambda) \neq 0$.*

Lemma 2.4. *Let Λ be an Auslander–Gorenstein ring with $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{op}} \Lambda = n$. If $S \in \mathrm{mod}\,\Lambda$ is simple with $\mathrm{pd}_\Lambda S = n$, then $S \subseteq I^n(\Lambda)$ and $S \nsubseteq I^0(\Lambda) \oplus \cdots \oplus I^{n-1}(\Lambda)$.*

Proof. Note that $\mathrm{Ext}^i_\Lambda(S,\Lambda) \cong \mathrm{Hom}_\Lambda(S, I^i(\Lambda))$ for any $i \geqslant 0$. Because $\mathrm{pd}_\Lambda S = n$, $\mathrm{Ext}^n_\Lambda(S,\Lambda) \neq 0$ by Lemma 2.3. So $\mathrm{Hom}_\Lambda(S, I^n(\Lambda)) \neq 0$ and $S \subseteq I^n(\Lambda)$. Then by Lemma 2.2, $S \nsubseteq I^0(\Lambda) \oplus \cdots \oplus I^{n-1}(\Lambda)$. □

As a generalization of the notion of classical Auslander algebras, Iyama introduced in [Iy2] the notion of n-Auslander algebras as follows.

Definition 2.5. (See [Iy4].) For a positive integer n, an Artinian algebra Λ is called an *n-Auslander algebra* if gl.dim $\Lambda \leqslant n+1$ and $I^0(\Lambda), I^1(\Lambda), \ldots, I^n(\Lambda)$ are projective.

The notion of n-Auslander algebras is left and right symmetric by [Iy4, Theorem 1.2]. It is trivial that n-Auslander algebras with global dimension at most n are semisimple. Note that the notion of 1-Auslander algebras is just that of classical Auslander algebras. In the following, we assume that $n \geqslant 2$ when an $(n-1)$-Auslander algebra is concerned.

Proposition 2.6. *Let Λ be an $(n-1)$-Auslander algebra with gl.dim $\Lambda = n$. Then we have the following*

(1) *There does not exist an injective simple module $S \in \mathrm{mod}\,\Lambda$ with $1 \leqslant \mathrm{pd}_\Lambda S \leqslant n-1$. Dually, there does not exist a projective simple module $S \in \mathrm{mod}\,\Lambda$ with $1 \leqslant \mathrm{id}_\Lambda S \leqslant n-1$.*
(2) *For a projective simple module $S \in \mathrm{mod}\,\Lambda$, $\mathrm{Hom}_\Lambda(I^0(\Lambda), S) \neq 0$ if and only if S is injective; and $\mathrm{Hom}_\Lambda(I^0(\Lambda), S) = 0$ if and only if $\mathrm{id}_\Lambda S = n$.*

Proof. (1) Let $S \in \operatorname{mod}\Lambda$ be simple with $\operatorname{pd}_\Lambda S = i$ with $1 \leqslant i \leqslant n-1$. Then by Lemma 2.3, we have that $\operatorname{Hom}_\Lambda(S, I^i(\Lambda)) \cong \operatorname{Ext}^i_\Lambda(S, \Lambda) \neq 0$. If S is injective, then S is isomorphic to a direct summand of $I^i(\Lambda)$ and hence S is projective, which is a contradiction. Dually, we get the second assertion.

(2) Let $S \in \operatorname{mod}\Lambda$ be a projective simple module. If S is injective, then we have an epimorphism $\operatorname{Hom}_\Lambda(I^0(\Lambda), S) \to \operatorname{Hom}_\Lambda(\Lambda, S)(\cong S) \to 0$ and so $\operatorname{Hom}_\Lambda(I^0(\Lambda), S) \neq 0$. Conversely, if $\operatorname{Hom}_\Lambda(I^0(\Lambda), S) \neq 0$, then S is isomorphic to a direct summand of $I^0(\Lambda)$ and hence S is injective. The second assertion follows from the first one and (1). $\square$

Definition 2.7. (See [AR1].) Let Λ be an Artinian algebra. Assume that $\mathscr{C} \subseteq \mathscr{D}$ are full subcategories of $\operatorname{mod}\Lambda$ and $D \in \operatorname{mod}\Lambda$, $C \in \mathscr{C}$. The morphism $f : D \to C$ is said to be a *left $\mathscr{C}$-approximation* of D if $\operatorname{Hom}_\Lambda(C, C') \to \operatorname{Hom}_\Lambda(D, C') \to 0$ is exact for any $C' \in \mathscr{C}$. The morphism $f : D \to C$ is said to be *left minimal* if an endomorphism $g : C \to C$ is an automorphism whenever $f = gf$. The subcategory $\mathscr{C}$ is said to be *covariantly finite* in $\mathscr{D}$ if every module in $\mathscr{D}$ has a left $\mathscr{C}$-approximation. The notions of (*minimal) right $\mathscr{C}$-approximations* and *contravariantly finite subcategories* of $\mathscr{D}$ may be defined dually. The subcategory $\mathscr{C}$ is said to be *functorially finite* in $\mathscr{D}$ if it is both covariantly finite and contravariantly finite in $\mathscr{D}$.

The following result is due to T. Wakamatsu.

Lemma 2.8. *(See [AR1, Lemma 1.3].) Let Λ be an Artinian algebra, $\mathscr{C}$ a full subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions and $D \in \operatorname{mod}\Lambda$. If $D \xrightarrow{f} C \to Z \to 0$ is exact with f a minimal left $\mathscr{C}$-approximation of D, then* $\operatorname{Ext}^1_\Lambda(Z, \mathscr{C}) = 0$.

3. Maximal *n*-orthogonal subcategories of mod Λ

From now on, all algebras are Artinian algebras. For an Artinian R-algebra Λ, we denote by $\mathbb{D}$ the ordinary duality, that is, $\mathbb{D}(-) = \operatorname{Hom}_R(-, I(R/J(R)))$, where $I(R/J(R))$ is the injective envelope of $R/J(R)$.

In this section, we will mainly study the existence of maximal $(n-1)$-orthogonal subcategories for algebras with global dimension n, especially for $(n-1)$-Auslander algebras and for almost hereditary algebras. In particular, we will give some necessary conditions for $(n-1)$-Auslander algebras with global dimension n admitting maximal $(n-1)$-orthogonal subcategories, and give some necessary and sufficient conditions for almost hereditary algebras with global dimension 2 admitting maximal 1-orthogonal subcategories.

Let $\mathscr{C}$ be a full subcategory of $\operatorname{mod}\Lambda$ and n a positive integer. We denote ${}^{\perp_n}\mathscr{C} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, C) = 0$ for any $C \in \mathscr{C}$ and $1 \leqslant i \leqslant n\}$, and $\mathscr{C}^{\perp_n} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(C, X) = 0$ for any $C \in \mathscr{C}$ and $1 \leqslant i \leqslant n\}$.

Definition 3.1. (See [Iy2].) Let $\mathscr{C} \subseteq \mathscr{D}$ be full subcategories of $\operatorname{mod}\Lambda$ and $\mathscr{C}$ functorially finite in $\mathscr{D}$. For a positive integer n, $\mathscr{C}$ is called a *maximal n-orthogonal subcategory* of $\mathscr{D}$ if $\mathscr{C} = {}^{\perp_n}\mathscr{C} \cap \mathscr{D} = \mathscr{C}^{\perp_n} \cap \mathscr{D}$.

From the definition above, we get easily that both Λ and $\mathbb{D}\Lambda^{op}$ are in any maximal n-orthogonal subcategory of $\operatorname{mod}\Lambda$. For a module $M \in \operatorname{mod}\Lambda$, we use $\operatorname{add}_\Lambda M$ to denote the subcategory of $\operatorname{mod}\Lambda$ consisting of all modules isomorphic to direct summands of finite direct sums of copies of ${}_\Lambda M$.

Proposition 3.2. *Let n be a positive integer. If $\operatorname{id}_\Lambda \Lambda = n$ (in particular, if $\operatorname{gl.dim}\Lambda = n$), then Λ admits no maximal n-orthogonal subcategories of* $\operatorname{mod}\Lambda$.

Proof. By the dual version of Lemma 2.3, we have that $\operatorname{Ext}^n_\Lambda(\mathbb{D}\Lambda^{op}, \Lambda) \neq 0$ and so the assertion follows. $\square$

Proposition 3.3. *Let Λ be an algebra with* $\operatorname{gl.dim}\Lambda = n \geqslant 2$. *If Λ admits a maximal $(n-1)$-orthogonal subcategory of* $\operatorname{mod}\Lambda$, *then the following statements are equivalent*:

(1) *Λ is an $(n-1)$-Auslander algebra.*
(2) $\operatorname{pd}_\Lambda \bigoplus_{i=1}^{n-1} I^i(\Lambda) \leqslant n-1$.
$(2)^{op}$ $\operatorname{pd}_{\Lambda^{op}} \bigoplus_{i=1}^{n-1} I^i(\Lambda^{op}) \leqslant n-1$.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ $(2)^{op}$ are trivial.

(2) $\Rightarrow$ (1) Because Λ admits a maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$, $\operatorname{Ext}_\Lambda^j(\mathbb{D}\Lambda^{op}, \Lambda) = 0$ for any $1 \leqslant j \leqslant n-1$. Then $\operatorname{pd}_\Lambda \bigoplus_{i=1}^{n-1} I^i(\Lambda) \leqslant n-1$ implies $\bigoplus_{i=1}^{n-1} I^i(\Lambda)$ is projective. Put $K = \operatorname{Im}(I^0(\Lambda) \to I^1(\Lambda))$. Since $\operatorname{gl.dim}\Lambda = n$, $\operatorname{pd}_\Lambda K \leqslant 1$. So $\operatorname{pd}_\Lambda I^0(\Lambda) \leqslant 1$ and hence $I^0(\Lambda)$ is projective.

$(2)^{op}$ $\Rightarrow$ (1) Note that $\mathscr{C}$ is a maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$ if and only if $\mathbb{D}\mathscr{C}$ is a maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda^{op}$. Then we get the assertion by using an argument similar to that in the proof of (2) $\Rightarrow$ (1). $\square$

Let Λ be an algebra with $\operatorname{gl.dim}\Lambda = n \geqslant 2$ admitting a maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$. Then by Proposition 3.3, we have that Λ is an $(n-1)$-Auslander algebra if and only if Λ is Auslander-regular, if and only if Λ satisfies the (n,n)-condition, if and only if Λ satisfies the $(n,n)^{op}$-condition.

Proposition 3.4. *Let Λ be an algebra with* $\operatorname{gl.dim}\Lambda = n \geqslant 2$ *and $\mathscr{C}$ a subcategory of* $\operatorname{mod}\Lambda$ *such that $\Lambda \in \mathscr{C}$ and* $\operatorname{Ext}_\Lambda^j(\mathscr{C}, \mathscr{C}) = 0$ *for any $1 \leqslant j \leqslant n-1$. Then* $\operatorname{grade} M = n$ *for any $M \in \mathscr{C}$ without projective direct summands.*

Proof. Since $\Lambda \in \mathscr{C}$ and $\operatorname{Ext}_\Lambda^j(\mathscr{C}, \mathscr{C}) = 0$ for any $1 \leqslant j \leqslant n-1$, $\mathscr{C} \subseteq {}^{\perp_{n-1}}\Lambda$. Notice that $\operatorname{gl.dim}\Lambda = n$, so $M \in \mathscr{C}$ without projective direct summands implies that $\operatorname{pd}_\Lambda M = n$. Let $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ be a minimal projective resolution of M. The induced exact sequence $0 \to M^* \to P_0^* \to P_1^* \to \cdots \to P_n^* \to \operatorname{Tr}\Omega^{n-1}M \to 0$ with $P_0^* \to P_1^* \to \cdots \to P_n^* \to \operatorname{Tr}\Omega^{n-1}M \to 0$ a minimal projective resolution of $\operatorname{Tr}\Omega^{n-1}M$ by [M, Proposition 4.2], where $\Omega^{n-1}M$ is the $(n-1)$st syzygy of M. Thus $M^* = 0$ by $\operatorname{gl.dim}\Lambda = n$ and therefore $\operatorname{grade} M = n$. $\square$

The following is an immediate consequence of Proposition 3.4.

Corollary 3.5. *Let Λ be an algebra with* $\operatorname{gl.dim}\Lambda = n \geqslant 2$ *admitting a maximal $(n-1)$-orthogonal subcategory $\mathscr{C}$. Then* $\operatorname{grade} M = n$ *for any $M \in \mathscr{C}$ without projective direct summands.*

Proposition 3.6. *Let Λ be an Auslander-regular algebra with* $\operatorname{gl.dim}\Lambda = n$. *Then there exist simple modules $S_1, S_2 \in$* $\operatorname{mod}\Lambda$ *with* $\operatorname{pd}_\Lambda S_1 = n-1$ *and* $\operatorname{pd}_\Lambda S_2 = n$.

Proof. If there do not exist simple modules with projective dimension $n-1$, then we claim that $\operatorname{Ext}_\Lambda^{n-1}(S, \Lambda) = 0$ for any simple module $S \in \operatorname{mod}\Lambda$. If $\operatorname{pd}_\Lambda S \leqslant n-2$, then $\operatorname{Ext}_\Lambda^{n-1}(S, \Lambda) = 0$. If $\operatorname{pd}_\Lambda S = n$, then $\operatorname{Hom}_\Lambda(S, I^{n-1}(\Lambda)) = 0$ by Lemma 2.4. So $\operatorname{Ext}_\Lambda^{n-1}(S, \Lambda) = 0$ and the claim is proved. It follows that $\operatorname{id}_\Lambda \Lambda \leqslant n-2$, which is a contradiction because $\operatorname{gl.dim}\Lambda = n$. The other assertion is trivial. $\square$

Recall from [AR1] that a homomorphism $f : X \to Y$ in $\operatorname{mod}\Lambda$ is called *left minimal* if an endomorphism $g : Y \to Y$ is an automorphism whenever $f = gf$. Let $\mathscr{C}$ be a subcategory of $\operatorname{mod}\Lambda$. Recall from [Iy2] that a complex:

$$M \xrightarrow{f_0} C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \tag{1}$$

with $C_i \in \mathscr{C}$ for any $i \geqslant 0$ is called *minimal* if f_i is left minimal for any $i \geqslant 0$. If the following sequence of functors:

$$\cdots \xrightarrow{\operatorname{Hom}_\Lambda(f_2,\ )} \operatorname{Hom}_\Lambda(C_1,\ ) \xrightarrow{\operatorname{Hom}_\Lambda(f_1,\ )} \operatorname{Hom}_\Lambda(C_0,\ ) \xrightarrow{\operatorname{Hom}_\Lambda(f_0,\ )} \operatorname{Hom}_\Lambda(M,\ ) \to 0$$

is exact on $\mathscr{C}$, then the complex (1) is called a *left $\mathscr{C}$-resolution*. It is trivial that if $\mathscr{C}$ is covariantly finite in mod Λ, then any $M \in \operatorname{mod}\Lambda$ has a minimal left $\mathscr{C}$-resolution.

Lemma 3.7. *Let Λ be an algebra with* gl.dim $\Lambda = n$ *admitting a maximal $(n-1)$-orthogonal subcategory $\mathscr{C}$ of* mod Λ, *and $X \in \operatorname{mod}\Lambda$. Then* $\operatorname{id}_\Lambda X \leqslant n-1$ *if and only if the injective envelope $I^0(X)$ of X gives a minimal left $\mathscr{C}$-approximation of X. In this case, the minimal injective resolution of X gives a minimal left $\mathscr{C}$-resolution of X.*

Proof. Let $X \in \operatorname{mod}\Lambda$ with $\operatorname{id}_\Lambda X = m \leqslant n-1$ and $C \in \mathscr{C}$. Since $\operatorname{Ext}^j_\Lambda(\mathbb{D}\Lambda^{op}, C) = 0$ for any $1 \leqslant j \leqslant n-1$, by applying the functor $\operatorname{Hom}_\Lambda(\ , C)$ to a minimal injective resolution of X, we get the following exact sequence:

$$0 \to \operatorname{Hom}_\Lambda\big(I^m(X), C\big) \to \cdots \to \operatorname{Hom}_\Lambda\big(I^1(X), C\big) \to \operatorname{Hom}_\Lambda\big(I^0(X), C\big) \to \operatorname{Hom}_\Lambda(X, C) \to 0.$$

The necessity is proved.

Conversely, assume that the injective envelope $0 \to X \to I^0(X) \to Y \to 0$ of X gives a minimal left $\mathscr{C}$-approximation of X. Since gl.dim $\Lambda = n$, $\operatorname{id}_\Lambda Y \leqslant n-1$. By assumption, the minimal injective resolution $0 \to Y \to I^0(Y) \to \cdots \to I^{n-1}(Y) \to 0$ of Y gives a minimal left $\mathscr{C}$-resolution of Y. Then we have a minimal left $\mathscr{C}$-resolution $0 \to X \to I^0(X) \to I^0(Y) \to \cdots \to I^{n-1}(Y) \to 0$ of X. Since any module in mod Λ has a minimal left $\mathscr{C}$-resolution of length at most n (cf. [Iy2, Theorem 2.2.3]), we have that $I^{n-1}(Y) = 0$ and $\operatorname{id}_\Lambda X \leqslant n-1$. □

Let Λ be an $(n-1)$-Auslander algebra with gl.dim $\Lambda = n$. Then by Proposition 3.6, there exist simple Λ-modules with projective dimension $n-1$. On the other hand, by Proposition 2.6, there do not exist injective simple Λ-modules with projective dimension $n-1$. In terms of the injective dimension of simple Λ-modules with projective dimension $n-1$, we give some necessary conditions for Λ admitting a maximal $(n-1)$-orthogonal subcategory of mod Λ as follows.

Theorem 3.8. *Let Λ be an $(n-1)$-Auslander algebra with* gl.dim $\Lambda = n$ *admitting a maximal $(n-1)$-orthogonal subcategory $\mathscr{C}$ of* mod Λ *and $S \in \operatorname{mod}\Lambda$ simple with* $\operatorname{pd}_\Lambda S = n-1$. *Then we have*

(1) *${}_\Lambda S$ is not injective.*
(2) $1 \leqslant \operatorname{id}_\Lambda S \leqslant n-1$ *if and only if* $\operatorname{Hom}_\Lambda(S, P) = 0$ *for any indecomposable projective module in* mod Λ *with* $\operatorname{id}_\Lambda P = n$.
(3) $\operatorname{id}_\Lambda S = n$ *if and only if* $\operatorname{Hom}_\Lambda(S, P) \neq 0$ *for some indecomposable projective module in* mod Λ *with* $\operatorname{id}_\Lambda P = n$.

Proof. (1) Because $\operatorname{pd}_\Lambda S = n-1$, $\operatorname{Ext}^{n-1}_\Lambda(S, \Lambda) \neq 0$ by Lemma 2.3. Notice that Λ admits a maximal $(n-1)$-orthogonal subcategory of mod Λ, so S is not injective.

(2) We first prove the sufficiency. Let $S \in \operatorname{mod}\Lambda$ be a simple module with $\operatorname{pd}_\Lambda S = n-1$ and N a module in $\mathscr{C}$ without projective direct summands. Then grade $N = n$ by Corollary 3.5. We claim that $\operatorname{Hom}_\Lambda(S, N) = 0$. Otherwise, if $\operatorname{Hom}_\Lambda(S, N) \neq 0$, then S is isomorphic to a submodule of N and so grade $S = n$ by Lemma 2.1. It follows that $\operatorname{Ext}^{n-1}_\Lambda(S, \Lambda) = 0$, which is a contradiction because $\operatorname{pd}_\Lambda S = n-1$. The claim is proved. On the other hand, $\operatorname{Hom}_\Lambda(S, P) = 0$ for any indecomposable projective module in mod Λ with $\operatorname{id}_\Lambda P = n$ by assumption.

Let $0 \to S \to C$ be a minimal left $\mathscr{C}$-approximation of S and N a non-zero indecomposable direct summand of C. Then $\operatorname{Hom}_\Lambda(S, N) \neq 0$. By the above argument, N is projective and $\operatorname{id}_\Lambda N < n$. Because

$\mathscr{C}$ is a maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$ by assumption, N is injective. So $N \cong I^0(S)$ and hence $C \cong I^0(S)$ by the minimality of the above left $\mathscr{C}$-approximation of S. Then it follows from (1) and Lemma 3.7 that $1 \leqslant \operatorname{id}_\Lambda S \leqslant n-1$.

We next prove the necessity. Let $S \in \operatorname{mod}\Lambda$ be a simple module with $\operatorname{pd}_\Lambda S = n-1$ and $P \in \operatorname{mod}\Lambda$ an indecomposable projective module with $\operatorname{id}_\Lambda P = n$. If $\operatorname{Hom}_\Lambda(S, P) \neq 0$, then there exists an epimorphism $\mathbb{D}P \to \mathbb{D}S \to 0$. Since $\mathscr{C}$ is a maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$, $\mathbb{D}\mathscr{C}$ is a maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda^{op}$. So $\operatorname{grade}\mathbb{D}P = n$ by the opposite version of Proposition 3.4 and hence $\operatorname{grade}\mathbb{D}S = n$ by the opposite version of Lemma 2.1. Since $\operatorname{gl.dim}\Lambda = n$, $\operatorname{pd}_{\Lambda^{op}} \mathbb{D}S = n$. So $\operatorname{id}_\Lambda S = n$, which is a contradiction.

(3) Since Λ is an $(n-1)$-Auslander algebra, the injective dimension of simple modules in $\operatorname{mod}\Lambda$ with projective dimension $n-1$ is situated between 1 and n by Proposition 2.6. Thus the assertion follows immediately from (2). $\square$

In the following, we will study the properties of simple modules in $\operatorname{mod}\Lambda$ with $\operatorname{pd}_\Lambda S = n$ if an $(n-1)$-Auslander algebra Λ with $\operatorname{gl.dim}\Lambda = n$ admits a maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$.

Proposition 3.9. *Let Λ be an $(n-1)$-Auslander algebra with $\operatorname{gl.dim}\Lambda = n$ admitting a maximal $(n-1)$-orthogonal subcategory $\mathscr{C}$ of $\operatorname{mod}\Lambda$ and S a simple Λ-module with $\operatorname{pd}_\Lambda S = n$. Then we have*

(1) *S is injective if and only if $S \in \mathscr{C}$ and $\operatorname{Hom}_\Lambda(S, C) = 0$ for any (non-projective) indecomposable module $C \in \mathscr{C}$ with $C \ncong I^0(S)$.*
(2) *$1 \leqslant \operatorname{id}_\Lambda S \leqslant n-1$ if and only if $S \notin \mathscr{C}$ and $\operatorname{Hom}_\Lambda(S, C) = 0$ for any (non-projective) indecomposable module $C \in \mathscr{C}$ with $C \ncong I^0(S)$.*
(3) *$\operatorname{id}_\Lambda S = n$ if and only if $\operatorname{Hom}_\Lambda(S, C) \neq 0$ for some (non-projective) indecomposable module $C \in \mathscr{C}$ with $C \ncong I^0(S)$.*

Proof. (1) The necessity is easy. Conversely, assume that $S \in \mathscr{C}$ and $\operatorname{Hom}_\Lambda(S, C) = 0$ for any non-projective indecomposable module $C \in \mathscr{C}$ with $C \ncong I^0(S)$. Because $\operatorname{Hom}_\Lambda(S, S) \neq 0$ and $\operatorname{pd}_\Lambda S = n$ by assumption, $S \cong I^0(S)$ is injective.

(2) If $1 \leqslant \operatorname{id}_\Lambda S \leqslant n-1$, then $S \notin \mathscr{C}$. Since $\mathscr{C}$ is maximal $(n-1)$-orthogonal, the minimal injective resolution of S: $0 \to S \xrightarrow{f} I^0(S) \to I^0(S)/S \to 0$ is a minimal left $\mathscr{C}$-approximation of S by Lemma 3.7. So, for any indecomposable module $C \in \mathscr{C}$ with $\operatorname{Hom}_\Lambda(S, C) \neq 0$ and $g \in \operatorname{Hom}_\Lambda(S, C)$, there exists an $h \in \operatorname{Hom}_\Lambda(I^0(S), C)$ such that $g = hf$. Since f is essential and g is a monomorphism, h is a splittable monomorphism and $C \cong I^0(S)$ is injective.

Conversely, since $\operatorname{pd}_\Lambda S = n$, $\operatorname{grade} S = n$ by Lemmas 2.3 and 2.4. So S cannot be embedded into any projective module. On the other hand, $\operatorname{Hom}_\Lambda(S, C) = 0$ for any non-projective indecomposable module $C \in \mathscr{C}$ with $C \ncong I^0(S)$ by assumption. Since $S \notin \mathscr{C}$, the injective envelope $0 \to S \to I^0(S)$ of S is a minimal left $\mathscr{C}$-approximation of S by the argument of Theorem 3.8(2). Then by Lemma 3.7, the assertion follows.

(3) It follows from (1) and (2). $\square$

For a positive integer n, we know that $\operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is contained in any maximal n-orthogonal subcategory of $\operatorname{mod}\Lambda$. On the other hand, it is easy to see that if $\operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is a maximal n-orthogonal subcategory of $\operatorname{mod}\Lambda$, then $\operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is the unique maximal n-orthogonal subcategory of $\operatorname{mod}\Lambda$. In this case, we say that Λ admits a *trivial maximal n-orthogonal subcategory* of $\operatorname{mod}\Lambda$. As an application of Proposition 3.9, we give some necessary and sufficient conditions that an $(n-1)$-Auslander algebra Λ with $\operatorname{gl.dim}\Lambda = n$ admits a trivial maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$.

Corollary 3.10. *Let Λ be an $(n-1)$-Auslander algebra with $\operatorname{gl.dim}\Lambda = n$. Then the following statements are equivalent:*

(1) *Λ admits a trivial maximal $(n-1)$-orthogonal subcategory* $\operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ *of* $\operatorname{mod}\Lambda$.
(2) *A simple module $S \in \operatorname{mod}\Lambda$ is injective if $\operatorname{pd}_\Lambda S = n$.*
(3) *There do not exist simple modules in $\operatorname{mod}\Lambda$ with both projective and injective dimensions n; and $1 \leqslant \operatorname{pd}_\Lambda S \leqslant n-1$ if and only if $1 \leqslant \operatorname{id}_\Lambda S \leqslant n-1$ for a simple module $S \in \operatorname{mod}\Lambda$.*

Proof. (1) $\Rightarrow$ (2) Let $S \in \operatorname{mod}\Lambda$ be a simple module with $\operatorname{pd}_\Lambda S = n$. Because $\operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is a maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$, $\operatorname{Hom}_\Lambda(S, C) = 0$ for any non-projective indecomposable module $C \in \mathscr{C}$ with $C \ncong I^0(S)$. Then it follows from Proposition 3.9 that $\operatorname{id}_\Lambda S \leqslant n-1$. On the other hand, Λ is an $(n-1)$-Auslander algebra, so $\operatorname{Ext}^j_\Lambda(S, \Lambda) = 0$ for any $1 \leqslant j \leqslant n-1$ by Lemmas 2.3 and 2.4 and hence $S \in \mathscr{C}$. It follows that S is injective.

(2) $\Rightarrow$ (1) Let E be an indecomposable direct summand of $I^n(\Lambda)$. Then $E \cong I^0(S)$ for some simple module $S \in \operatorname{mod}\Lambda$. So $\operatorname{Ext}^n_\Lambda(S, \Lambda) \cong \operatorname{Hom}_\Lambda(S, I^n(\Lambda)) \neq 0$ and $\operatorname{pd}_\Lambda S = n$. Thus by (2), S is injective and $E \cong S$. Then it follows easily from Lemma 2.4 that $\operatorname{grade} E = n$, which implies that $\operatorname{grade} I^n(\Lambda) = n$. On the other hand, because Λ is an $(n-1)$-Auslander algebra, $I^0(\Lambda), \ldots, I^{n-1}(\Lambda)$ are projective. Notice that $\mathbb{D}\Lambda^{op} \in \operatorname{add}_\Lambda \bigoplus_{i=0}^n I^i(\Lambda)$, so $\operatorname{Ext}^j_\Lambda(\mathbb{D}\Lambda^{op}, \Lambda) = 0$ for any $1 \leqslant j \leqslant n-1$.

Now let $M \in {}^{\perp_{n-1}}\Lambda$ be indecomposable. If $\operatorname{pd}_\Lambda M \leqslant n-1$, then M is projective. If $\operatorname{pd}_\Lambda M = n$, then $\operatorname{grade} M = n$. By Lemma 2.1, for a simple submodule S of M, $\operatorname{grade} S = n$. So $\operatorname{pd}_\Lambda S = n$ and hence S is injective by (2). It follows that $M (\cong S)$ is injective. Thus we have that $\operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op}) = {}^{\perp_{n-1}}\Lambda$ and therefore $\operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is a maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$ by [Iy2, Proposition 2.2.2].

(3) $\Rightarrow$ (2) It is easy.

(1) + (2) $\Rightarrow$ (3) It suffices to prove the latter assertion by (2). If $S \in \operatorname{mod}\Lambda$ is simple with $1 \leqslant \operatorname{pd}_\Lambda S \leqslant n-1$, then we only have to show $\operatorname{id}_\Lambda S \neq n$ by Proposition 2.6. Otherwise, if $\operatorname{id}_\Lambda S = n$, then $\operatorname{pd}_{\Lambda^{op}} \mathbb{D}S = n$. By (1), Λ^{op} admits a trivial maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda^{op}$, so the opposite version of (2) holds true. Then it follows that $\mathbb{D}S$ is injective and S is projective, which is a contradiction. The converse can be proved dually. $\square$

We give an example to illustrate Theorem 3.8 and Corollary 3.10.

Example 3.11. Let Λ be a finite-dimensional algebra given by the quiver Q:

$$1 \xleftarrow{\beta_1} 2 \xleftarrow{\beta_2} 3 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} n+1$$

modulo the ideal generated by $\{\beta_i\beta_{i+1} \mid 1 \leqslant i \leqslant n-1\}$. Then we have

(1) Λ is an $(n-1)$-Auslander algebra with $\operatorname{gl.dim}\Lambda = n$ and admits a maximal $(n-1)$-orthogonal subcategory $\mathscr{C} = \operatorname{add}_\Lambda(P(1) \oplus P(2) \oplus P(3) \oplus \cdots \oplus P(n+1) \oplus S(n+1))$.
(2) $\operatorname{pd}_\Lambda S(n) = n-1$, $\operatorname{id}_\Lambda P(1) = n$, $\operatorname{Hom}_\Lambda(S(n), P(1)) = 0$ and $\operatorname{id}_\Lambda S(n) = 1$.
(3) $\{S \in \operatorname{mod}\Lambda \mid S$ is simple with $1 \leqslant \operatorname{pd}_\Lambda S \leqslant n-1\} = \{S \in \operatorname{mod}\Lambda \mid S$ is simple with $1 \leqslant \operatorname{id}_\Lambda S \leqslant n-1\} = \{S(i) \mid 2 \leqslant i \leqslant n\}$.
(4) $\operatorname{pd}_\Lambda S(n+1) = n$ and $S(n+1) = I(n+1)$.

As another application of Proposition 3.9, we give a necessary condition for an $(n-1)$-Auslander algebra Λ with $\operatorname{gl.dim}\Lambda = n$ admitting a non-trivial maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$.

Corollary 3.12. *Let Λ be an $(n-1)$-Auslander algebra with $\operatorname{gl.dim}\Lambda = n$. If Λ admits a non-trivial maximal $(n-1)$-orthogonal subcategory $\mathscr{C} (\neq \operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op}))$ of $\operatorname{mod}\Lambda$, then there exists a simple module $S \in \operatorname{mod}\Lambda$ such that $\operatorname{pd}_\Lambda S = n$ and $\operatorname{id}_\Lambda S = n$.*

Proof. Let $\mathscr{C} \neq \operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ be a maximal $(n-1)$-orthogonal subcategory of $\operatorname{mod}\Lambda$. Then there exists an indecomposable module $M \in \mathscr{C}$ such that $\operatorname{pd}_\Lambda M = n$ and $\operatorname{id}_\Lambda M = n$. So $\operatorname{grade} M = n$. Then by Lemma 2.1, for any simple submodule S of M, $\operatorname{grade} S = n$ and $\operatorname{pd}_\Lambda S = n$. Notice that $M \ncong I^0(S)$ and $M \in \mathscr{C}$, so $\operatorname{id}_\Lambda S = n$ by Proposition 3.9. □

In the following, we will study the existence of maximal 1-orthogonal subcategories for some kinds of algebras (Auslander algebras and almost hereditary algebras) with global dimension 2. First, if putting $n = 1$, we get the following corollary immediately from Theorem 3.8. This result means that the existence of maximal 1-orthogonal subcategories of $\operatorname{mod} \Lambda$ for an Auslander algebra Λ with $\operatorname{gl.dim} \Lambda = 2$ enables us to classify the simple modules in $\operatorname{mod} \Lambda$ with projective dimension 1.

Corollary 3.13. *Let Λ be an Auslander algebra with* $\operatorname{gl.dim} \Lambda = 2$ *admitting a maximal* 1*-orthogonal subcategory $\mathscr{C}$ of* $\operatorname{mod} \Lambda$ *and $S \in \operatorname{mod} \Lambda$ simple with* $\operatorname{pd}_\Lambda S = 1$. *Then we have*

(1) $\operatorname{id}_\Lambda S = 1$ *if and only if* $\operatorname{Hom}_\Lambda(S, P) = 0$ *for any indecomposable projective module in* $\operatorname{mod} \Lambda$ *with* $\operatorname{id}_\Lambda P = 2$.
(2) $\operatorname{id}_\Lambda S = 2$ *if and only if* $\operatorname{Hom}_\Lambda(S, P) \neq 0$ *for some indecomposable projective module in* $\operatorname{mod} \Lambda$ *with* $\operatorname{id}_\Lambda P = 2$.

Definition 3.14. (See [HRS].) An algebra Λ is called *almost hereditary* if the following conditions are satisfied: (1) $\operatorname{gl.dim} \Lambda \leqslant 2$; and (2) If $X \in \operatorname{mod} \Lambda$ is indecomposable, then either $\operatorname{id}_\Lambda X \leqslant 1$ or $\operatorname{pd}_\Lambda X \leqslant 1$.

By Proposition 3.2, if $\operatorname{gl.dim} \Lambda = 1$, then Λ admits no maximal n-orthogonal subcategories of $\operatorname{mod} \Lambda$ for any $n \geqslant 1$. For an almost hereditary algebra Λ with $\operatorname{gl.dim} \Lambda = 2$, we give some equivalent characterizations of the existence of maximal 1-orthogonal subcategories of $\operatorname{mod} \Lambda$ as follows.

Theorem 3.15. *Let Λ be an almost hereditary algebra with* $\operatorname{gl.dim} \Lambda = 2$. *Then the following statements are equivalent*:

(1) *Λ admits a maximal* 1*-orthogonal subcategory $\mathscr{C}$ of* $\operatorname{mod} \Lambda$.
(2) *The following conditions are satisfied*:
  (i) $\operatorname{r.grade} \mathbb{D}\Lambda^{op} = 2$; *and*
  (ii) *Any non-projective indecomposable module $M \in \operatorname{mod} \Lambda$ is injective if* $\operatorname{r.grade} M = 2$.
(3) *For any non-projective indecomposable module $M \in \operatorname{mod} \Lambda$, M is injective if and only if* $\operatorname{r.grade} M = 2$.

In particular, if Λ admits a maximal 1*-orthogonal subcategory $\mathscr{C}$ of* $\operatorname{mod} \Lambda$, *then* $\mathscr{C} = \operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$. *That is, the maximal* 1*-orthogonal subcategory of* $\operatorname{mod} \Lambda$ *is trivial if it exists.*

Proof. (1) ⇒ (2) Let $\mathscr{C}$ be a maximal 1-orthogonal subcategory of $\operatorname{mod} \Lambda$ and $N \in \mathscr{C}$ an indecomposable module. Because Λ is almost hereditary, N is projective or injective. So $\mathscr{C} = \operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op}) = {}^{\perp_1}\mathscr{C} = {}^{\perp_1}\Lambda$. Since $\operatorname{pd}_\Lambda \mathbb{D}\Lambda^{op} = \operatorname{gl.dim} \Lambda = 2$, $\operatorname{Ext}^2_\Lambda(\mathbb{D}\Lambda^{op}, \Lambda) \neq 0$ by Lemma 2.3. So $\operatorname{r.grade} \mathbb{D}\Lambda^{op} = 2$. If $M \in \operatorname{mod} \Lambda$ is indecomposable with $\operatorname{r.grade} M = 2$, then $M \in \mathscr{C}$; so, if M is further non-projective, then M is injective by the above argument.

(2) ⇒ (3) It is easy.

(3) ⇒ (1) We will show that $\mathscr{C} = \operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$ is a maximal 1-orthogonal of $\operatorname{mod} \Lambda$. By [Iy2, Proposition 2.2.2], we only need to show ${}^{\perp_1}\Lambda \subseteq \operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$. Let $M \in {}^{\perp_1}\Lambda$ be an indecomposable module. If $\operatorname{pd}_\Lambda M = 2$, then $\operatorname{r.grade} M = 2$ and so M is injective by assumption. If $\operatorname{pd}_\Lambda M \leqslant 1$, then M is projective. Thus we conclude that ${}^{\perp_1}\Lambda \subseteq \operatorname{add}_\Lambda(\Lambda \oplus \mathbb{D}\Lambda^{op})$.

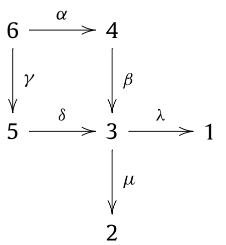
The last assertion follows from the above argument. □

At the end of this section, we give two examples related to Corollary 3.13 and Theorem 3.15 as follows.

Example 3.16. Putting $n=2$ in Example 3.11, then we have

(1) Λ is an Auslander algebra and an almost hereditary algebra with $\mathrm{gl.dim}\,\Lambda=2$ admitting a maximal 1-orthogonal subcategory $\mathscr{C}=\mathrm{add}_\Lambda(P(1)\oplus P(2)\oplus P(3)\oplus S(3))$.
(2) $\mathrm{pd}_\Lambda S(2)=1=\mathrm{id}_\Lambda S(2)$, $\mathrm{id}_\Lambda P(1)=2$ and $\mathrm{Hom}_\Lambda(S(2),P(1))=0$.
(3) $S(3)=I(3)$ and $\mathrm{pd}_\Lambda S(3)=2$.
(4) $\mathrm{r.grade}\,S(3)=2$.

Example 3.17. Let Λ be a finite-dimensional algebra given by the quiver Q:



modulo the ideal generated by $\{\beta\alpha-\delta\gamma, \mu\delta, \lambda\beta\}$. Then we have

(1) Λ is an Auslander algebra and an almost hereditary algebra with $\mathrm{gl.dim}\,\Lambda=2$.
(2) $\mathrm{pd}_\Lambda S(3)=1$, $\mathrm{id}_\Lambda S(3)=2$, $P(4)$, $P(5)$ and $P(6)$ are injective; $\mathrm{id}_\Lambda P(i)=2$ and $\mathrm{Hom}_\Lambda(S(3),P(i))=0$ for $i=1,2,3$.
(3) $\mathrm{Ext}^1_\Lambda(\mathbb{D}\Lambda^{op},\Lambda)=0$. Both $S(4)$ and $S(5)$ are of r.grade 2, and neither of them are injective.
(4) There does not exist a simple module S such that $\mathrm{pd}_\Lambda S=2$ and $\mathrm{id}_\Lambda S=2$.

So there exist no maximal 1-orthogonal subcategories of $\mathrm{mod}\,\Lambda$ by Corollary 3.13 or Theorem 3.15.

4. Maximal *n*-orthogonal subcategories of $^{\perp}T$

In this section, we will study the properties of an algebra Λ and Λ-modules if Λ admits maximal n-orthogonal subcategories of $^{\perp}T$ for a cotilting module T.

Let $\mathscr{C}$ be a full subcategory of $\mathrm{mod}\,\Lambda$. We use $\widehat{\mathscr{C}}$ to denote the subcategory of $\mathrm{mod}\,\Lambda$ consisting of the module X for which there exists an exact sequence $0\to C_n\to C_{n-1}\to\cdots\to C_0\to X\to 0$ with each C_i in $\mathscr{C}$. Denote $^{\perp}\mathscr{C}=\bigcap_{n\geqslant 1}{}^{\perp_n}\mathscr{C}$, and $\mathcal{I}^{\infty}(\Lambda)=\{X\in\mathrm{mod}\,\Lambda\mid \mathrm{id}_\Lambda X<\infty\}$.

Definition 4.1. (See [AR1].) A module $T\in\mathrm{mod}\,\Lambda$ is called a *cotilting* module if the following conditions are satisfied: (1) $\mathrm{id}_\Lambda T=n<\infty$; (2) $T\in{}^{\perp}T$; and (3) $\mathbb{D}\Lambda^{op}\in\widehat{\mathrm{add}_\Lambda T}$. A cotilting module T is called *strong cotilting* if $\mathcal{I}^{\infty}(\Lambda)=\widehat{\mathrm{add}_\Lambda T}$.

Lemma 4.2. *Let Λ be an algebra and $T\in\mathrm{mod}\,\Lambda$ a cotilting module. Then for any $M\in{}^{\perp}T$ with $\mathrm{id}_\Lambda M=n$, there exists an exact sequence $0\to M\to T_0\to T_1\to\cdots\to T_n\to 0$ with $T_i\in\mathrm{add}_\Lambda T$ for any $0\leqslant i\leqslant n$.*

Proof. For any $M\in{}^{\perp}T$, we have an exact sequence $0\to M\xrightarrow{f_0}T_0\xrightarrow{f_1}T_1\xrightarrow{f_2}\cdots\xrightarrow{f_n}T_n\xrightarrow{f_{n+1}}\cdots$ with $\mathrm{Coker}\,f_i\in{}^{\perp}T$ for any $i\geqslant 0$ by [AR1, Theorem 5.4]. Then for any $N\in{}^{\perp}T$ and $i\geqslant 1$, $0=\mathrm{Ext}^{n+i}_\Lambda(N,M)=\mathrm{Ext}^i_\Lambda(N,\mathrm{Coker}\,f_{n-1})$. So $\mathrm{Coker}\,f_{n-1}\in({}^{\perp}T)^{\perp}$. Since T is cotilting and $\mathrm{Coker}\,f_{n-1}\in{}^{\perp}T$, $\mathrm{Coker}\,f_{n-1}\in\mathrm{add}_\Lambda T$. □

Auslander and Reiten in [AR1, p. 150] posed a question: whether does $\mathrm{id}_\Lambda\Lambda<\infty$ imply $\mathrm{id}_{\Lambda^{op}}\Lambda<\infty$? This question is now referred to the Gorenstein Symmetry Conjecture, which still remains open (see [BR]). The following result gives a connection between this conjecture with the existence of maximal n-orthogonal subcategories. For a module $M \in \operatorname{mod}\Lambda$, the *basic submodule* of M, denoted by M_b, is defined as the direct sum of one copy of each non-isomorphic indecomposable direct summand of M.

Theorem 4.3. *Let Λ be an algebra with $\operatorname{id}_\Lambda \Lambda = n \geqslant 1$ and $T \in \operatorname{mod}\Lambda$ a cotilting module. If $T \in {}^{\perp_n}\Lambda$ (in particular, if Λ admits a maximal j-orthogonal subcategory of ${}^{\perp}T$ for some $j \geqslant n$), then $\operatorname{id}_{\Lambda^{op}} \Lambda = n$ and T is a strong cotilting module with $T_b = \Lambda_b$.*

Proof. Let $T \in {}^{\perp_n}\Lambda$ be a cotilting module. By Lemma 4.2, there exists an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to \cdots \to T_n \to 0$ with $T_i \in \operatorname{add}_\Lambda T$ for any $0 \leqslant i \leqslant n$, which is splittable. So we have ${}_\Lambda\Lambda \in \operatorname{add}_\Lambda T$. Note that all cotilting modules in $\operatorname{mod}\Lambda$ have the same number of non-isomorphic indecomposable direct summands which is equal to the number of non-isomorphic indecomposable projective modules in $\operatorname{mod}\Lambda$ (see [AR1] or [M]). It follows that ${}_\Lambda\Lambda_b = {}_\Lambda T_b$ and ${}_\Lambda\Lambda$ is a cotilting module. Thus $\operatorname{id}_{\Lambda^{op}} \Lambda = n$ by [AR2, Lemma 1.7]. □

From now on, Λ is a Gorenstein algebra, that is, $\operatorname{id}_\Lambda \Lambda = \operatorname{id}_{\Lambda^{op}} \Lambda < \infty$.

Following [Iy3], assume that the Abelian category $\mathscr{A} = \operatorname{mod}\Lambda$ and $\mathscr{B} = {}^{\perp}\Lambda$. Then the categories ${}^{\perp_n}\Lambda \cap {}^{\perp}\Lambda = \mathscr{B}$ and $\Lambda^{\perp_n} \cap {}^{\perp}\Lambda = \mathscr{B}$ for any $n \geqslant 1$. Denote by $\underline{\mathscr{B}}$ (resp. $\overline{\mathscr{B}}$) the stable category $\mathscr{B}$ modulos relative projectives (resp. injectives) in $\mathscr{B}$. We remark that $\mathscr{B}$ forms a Frobenius category, so the relative projectives in $\mathscr{B}$ coincide with relative injectives in $\mathscr{B}$, and that we have $\underline{\mathscr{B}} = \overline{\mathscr{B}}$.

We use $\operatorname{Proj}(\operatorname{mod}\overline{\mathscr{B}})$ to denote the subcategory of $\operatorname{mod}\overline{\mathscr{B}}$ consisting of projective objects. The functors $\tau : \underline{\mathscr{B}} \to \overline{\mathscr{B}}$ and $\tau^- : \overline{\mathscr{B}} \to \underline{\mathscr{B}}$ are quasi-inverse equivalences, where $\tau = F^- \circ G$ and $\tau^- = G^- \circ F$ with $F : \overline{\mathscr{B}} \to \operatorname{Proj}(\operatorname{mod}\overline{\mathscr{B}})$ via $X \to \underline{\mathscr{B}}(\ , X)$ and $G : \underline{\mathscr{B}} \to \operatorname{Proj}(\operatorname{mod}\overline{\mathscr{B}})$ via $X \to \mathbb{D}\operatorname{Ext}^1_{\mathscr{A}}(X,\ )$. Let Ω and Ω^- be the relative syzygy and cosyzygy functors in ${}^{\perp}\Lambda$.

Lemma 4.4.

(1) *(See [Iy3, Corollary 2.3.2].) The functors τ and τ^- give mutually-inverse equivalences $\tau : \underline{\mathscr{B}} \to \underline{\mathscr{B}}$ and $\tau^- : \underline{\mathscr{B}} \to \underline{\mathscr{B}}$.*

(2) *(See [Iy3, Theorem 2.3.1].) There exist functorial isomorphisms $\underline{\operatorname{Hom}}_{\mathscr{B}}(Y, \tau X) \cong \mathbb{D}\operatorname{Ext}^1_{\mathscr{A}}(X, Y) \cong \underline{\operatorname{Hom}}_{\mathscr{B}}(\tau^- Y, X)$ for any $X, Y \in \underline{\mathscr{B}}$.*

The following result is a generalization of [EH, Lemma 3.2].

Lemma 4.5. *For any $M, N \in {}^{\perp}\Lambda$ and $i \geqslant 1$, $\operatorname{Ext}^i_{\mathscr{A}}(M, N) \cong \mathbb{D}\operatorname{Ext}^1_{\mathscr{A}}(N, \Omega^i \tau M)$.*

Proof. Note that the functors $\Omega : \underline{\mathscr{B}} \to \underline{\mathscr{B}}$ and $\Omega^- : \underline{\mathscr{B}} \to \underline{\mathscr{B}}$ are mutually-inverse equivalences by [AR2, Proposition 3.1]. Then by Lemma 4.4(2), we have that $\operatorname{Ext}^i_{\mathscr{A}}(M, N) \cong \underline{\operatorname{Hom}}_{\mathscr{B}}(\Omega^i M, N) \cong \underline{\operatorname{Hom}}_{\mathscr{B}}(\Omega M, \Omega^{-i+1} N) \cong \operatorname{Ext}^1_{\mathscr{A}}(M, \Omega^{-i+1} N) \cong \mathbb{D}\underline{\operatorname{Hom}}_{\mathscr{B}}(\tau^{-1}\Omega^{-i+1} N, M) \cong \mathbb{D}\underline{\operatorname{Hom}}_{\mathscr{B}}(\Omega^{-i+1} N, \tau M) \cong \mathbb{D}\underline{\operatorname{Hom}}_{\mathscr{B}}(\Omega^1 N, \Omega^i \tau M) \cong \mathbb{D}\operatorname{Ext}^1_{\mathscr{A}}(N, \Omega^i \tau M)$. □

Let M be a module in $\operatorname{mod}\Lambda$. For a positive integer n, $M \in \operatorname{mod}\Lambda$ is called *$\Omega^n\tau$-periodic* if there exists a positive integer t, such that $(\Omega^n \tau)^t M \cong M$. Recall from [EH] that M is called a *maximal n-orthogonal module* if $\operatorname{add}_\Lambda M$ is a maximal n-orthogonal subcategory of $\operatorname{mod}\Lambda$. The following result generalizes [EH, Theorem 3.1]. It should be pointed out that this result can be induced by [Iy3, Theorem 2.5.1(1)].

Theorem 4.6. *Let $X \in {}^{\perp}\Lambda$ be a maximal n-orthogonal module. If Y is a direct summand of X then so is $\Omega^n \tau Y$ for some $n \geqslant 1$. Hence the non-projective direct summand of X is $\Omega^n\tau$-periodic.*

Proof. By Lemma 4.4(1), $\tau X \in {}^{\perp}\Lambda$ and so $\Omega^n \tau X \in {}^{\perp}\Lambda$. Since X is maximal n-orthogonal, for any $1 \leqslant i \leqslant n$, we have that $\mathrm{Ext}^i_{\mathscr{A}}(X, \Omega^n \tau X) \cong \underline{\mathrm{Hom}}_{\mathscr{B}}(\Omega^i X, \Omega^n \tau X) \cong \underline{\mathrm{Hom}}_{\mathscr{B}}(\Omega^1 X, \Omega^{-i+n+1} \tau X) \cong \mathrm{Ext}^1_{\mathscr{A}}(X, \Omega^{-i+n+1} \tau X) \cong \mathbb{D}\,\mathrm{Ext}^{-i+n+1}_{\mathscr{A}}(X, X) = 0$ by Lemma 4.5. Also since X is maximal n-orthogonal, $\Omega^n \tau X \in \mathrm{add}_\Lambda X$. Notice that both Ω and τ are equivalences in ${}^{\perp}\Lambda$, it follows that if Y is a direct summand of X, then $\Omega^n \tau Y$ is also a direct summand of X. Since X has only finitely many indecomposable direct summands, some power of $\Omega^n \tau$ is identity on the non-projective direct summands of X. □

5. The complexity of modules

In this section, both Λ and Γ are finite-dimensional K-algebras over a field K. We will study the relation between the complexity of modules and the existence of maximal n-orthogonal subcategories of $\mathrm{mod}\,\Lambda \otimes_K \Gamma$.

From the following result, we can construct algebras with infinite global dimension admitting no maximal n-orthogonal subcategories for any $n \geqslant 1$.

Proposition 5.1. *If* $\mathrm{id}_\Lambda \Lambda = n \geqslant 1$ *and* $\Lambda \otimes_K \Gamma$ *admits a maximal* j*-orthogonal subcategory of* $\mathrm{mod}\,\Lambda \otimes_K \Gamma$ *for some* $j \geqslant n$*, then* $\mathrm{Hom}_\Gamma(\mathbb{D}\Gamma^{op}, \Gamma) = 0$.

Proof. Suppose that $\mathscr{C}$ is a maximal j-orthogonal subcategory of $\mathrm{mod}\,\Lambda \otimes_K \Gamma$ for some $j \geqslant n$. Then both $\Lambda \otimes_K \Gamma$ and $\mathbb{D}\Lambda^{op} \otimes_K \mathbb{D}\Gamma^{op}$ are in $\mathscr{C}$. Thus by [CE, Chapter XI, Theorem 3.1], $0 = \mathrm{Ext}^i_{\Lambda \otimes_K \Gamma}(\mathbb{D}\Lambda^{op} \otimes_K \mathbb{D}\Gamma^{op}, \Lambda \otimes_K \Gamma) = \bigoplus_{r+s=i} \mathrm{Ext}^r_\Lambda(\mathbb{D}\Lambda^{op}, \Lambda) \otimes_K \mathrm{Ext}^s_\Gamma(\mathbb{D}\Gamma^{op}, \Gamma)$ for any $1 \leqslant i \leqslant n$. So $\mathrm{Ext}^i_\Lambda(\mathbb{D}\Lambda^{op}, \Lambda) \otimes_K \mathrm{Hom}_\Gamma(\mathbb{D}\Gamma^{op}, \Gamma) = 0$ for any $1 \leqslant i \leqslant n$.

If $0 \neq \mathrm{Hom}_\Gamma(\mathbb{D}\Gamma^{op}, \Gamma)$, then $\mathrm{Hom}_\Gamma(\mathbb{D}\Gamma^{op}, \Gamma) \cong K^m$ as K-vector spaces for some $m \geqslant 1$. Thus $[\mathrm{Ext}^i_\Lambda(\mathbb{D}\Lambda^{op}, \Lambda)]^m \cong \mathrm{Ext}^i_\Lambda(\mathbb{D}\Lambda^{op}, \Lambda) \otimes_K K^m \cong \mathrm{Ext}^i_\Lambda(\mathbb{D}\Lambda^{op}, \Lambda) \otimes_K \mathrm{Hom}_\Gamma(\mathbb{D}\Gamma^{op}, \Gamma) = 0$ and $\mathrm{Ext}^i_\Lambda(\mathbb{D}\Lambda^{op}, \Lambda) = 0$ for any $1 \leqslant i \leqslant n$. Since $\mathbb{D}\Lambda^{op}$ is an injective cogenerator for $\mathrm{mod}\,\Lambda$ and $\mathrm{id}_\Lambda \Lambda = n$, it is easy to see that Λ is selfinjective by applying the functor $\mathrm{Hom}_\Lambda(\mathbb{D}\Lambda^{op}, -)$ to the minimal injective resolution of ${}_\Lambda\Lambda$, which is a contradiction. □

Assume that $\mathrm{id}_\Lambda \Lambda = 1$. If there exists a torsionless injective module in $\mathrm{mod}\,\Gamma$ (that is, there exists an injective module $Q \in \mathrm{mod}\,\Gamma$ such that the canonical evaluation homomorphism $Q \to \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Gamma(Q, \Gamma), \Gamma)$ is monomorphic), especially, if there exists a projective–injective module in $\mathrm{mod}\,\Gamma$ (for example, Γ is 1-Gorenstein), then $\Lambda \otimes_K \Gamma$ admits no maximal n-orthogonal subcategories of $\mathrm{mod}\,\Lambda \otimes_K \Gamma$ for any $n \geqslant 1$ by Proposition 5.1.

Definition 5.2. (See [EH].) Let

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

be a minimal projective resolution of a module $M \in \mathrm{mod}\,\Lambda$. The *complexity* of M is defined as $\mathrm{cx}(M) = \inf\{b \geqslant 0 \mid \text{there exists a } c > 0 \text{ such that } \dim_K P_n \leqslant cn^{b-1} \text{ for all } n\}$ if it exists, otherwise $\mathrm{cx}(M) = \infty$.

It is easy to see that $\mathrm{cx}(M) = 0$ implies M is of finite projective dimension, and $\mathrm{cx}(M) \leqslant 1$ if and only if the dimensions of P_n are bounded. Erdmann and Holm proved in [EH, Theorem 1.1] that if Λ is selfinjective and there exists a maximal n-orthogonal module in $\mathrm{mod}\,\Lambda$ for some $n \geqslant 1$, then all modules in $\mathrm{mod}\,\Lambda$ have complexity at most 1. At the end of [EH], Erdmann and Holm posed a question: Whether there can exist maximal n-orthogonal modules for non-selfinjective algebras Λ for which not all modules in $\mathrm{mod}\,\Lambda$ are of complexity at most 1? In the following, we will give some properties of the complexity of the tensor product of modules. Then we construct a class of algebras Λ with $\mathrm{id}_\Lambda \Lambda = n$ ($\geqslant 1$), such that not all modules in $\mathrm{mod}\,\Lambda$ are of complexity at most 1, but Λ admits no maximal j-orthogonal subcategories of $\mathrm{mod}\,\Lambda$ for any $j \geqslant n$.

Lemma 5.3. *Let*

$$\cdots \to P_i \xrightarrow{f_i} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

and

$$\cdots \to Q_j \xrightarrow{g_j} \cdots \to Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{g_0} N \to 0$$

be minimal projective resolutions of $M \in \operatorname{mod} \Lambda$ and $N \in \operatorname{mod} \Gamma$, respectively. Then the following is a minimal projective resolution of $M \otimes_K N$ as a $\Lambda \otimes_K \Gamma$-module:

$$\cdots \to R_n \to \cdots \to R_1 \xrightarrow{(1_{P_0} \otimes_K g_1) \oplus (f_1 \otimes_K 1_{Q_0})} R_0 \xrightarrow{f_0 \otimes_K g_0} M \otimes_K N \to 0 \tag{2}$$

where $R_n = \bigoplus_{i+j=n} P_i \otimes_K Q_j$ for any $n \geqslant 0$.

Proof. It is well known that the sequence (2) is a projective resolution of $M \otimes_K N$ *as a* $\Lambda \otimes_K \Gamma$-module. So it suffices to prove the minimality.

It is straightforward to verify that $\operatorname{Ker} f_0 \otimes_K g_0 \cong (\operatorname{Ker} f_0 \otimes_K Q_0) + (P_0 \otimes_K \operatorname{Ker} g_0)$. On the other hand, note that $\operatorname{Ker} f_0 \subseteq J(\Lambda)P_0$, $\operatorname{Ker} g_0 \subseteq J(\Gamma)Q_0$ and the nilpotent ideal $(J(\Lambda) \otimes_K \Gamma) + (\Lambda \otimes_K J(\Gamma)) \subseteq J(\Lambda \otimes_K \Gamma)$. So we have that $(\operatorname{Ker} f_0 \otimes_K Q_0) + (P_0 \otimes_K \operatorname{Ker} g_0) \subseteq J(P_0 \otimes_K Q_0)$ and hence $f_0 \otimes_K g_0$ is minimal. By using an argument similar to above, we get the desired assertion. □

Proposition 5.4. $\max\{\operatorname{cx}(M), \operatorname{cx}(N)\} \leqslant \operatorname{cx}(M \otimes_K N) \leqslant \operatorname{cx}(M) + \operatorname{cx}(N)$ *for any $M \in \operatorname{mod} \Lambda$ and $N \in \operatorname{mod} \Gamma$.*

Proof. By Lemma 5.3, we have that $\dim_K R_n = \sum_{i+j=n} \dim_K P_i \dim_K Q_j \geqslant \dim_K P_n \dim_K Q_0 \geqslant \dim_K P_n$. Similarly, $\dim_K R_n \geqslant \dim_K Q_n$. So $\max\{\operatorname{cx}(M), \operatorname{cx}(N)\} \leqslant \operatorname{cx}(M \otimes_K N)$.

To prove the second inequality, without loss of generality, assume that both $\operatorname{cx}(M)$ and $\operatorname{cx}(N)$ are finite. Then there exist $c, c' > 0$ such that $\dim_K P_i \leqslant c i^{\operatorname{cx}(M)-1}$ and $\dim_K Q_j \leqslant c' j^{\operatorname{cx}(N)-1}$ for any $i, j \geqslant 0$. Thus by Lemma 5.3, we have that $\dim_K R_n = \sum_{i+j=n} \dim_K P_i \dim_K Q_j \leqslant \sum_{i+j=n} (c i^{\operatorname{cx}(M)-1}) \times (c' j^{\operatorname{cx}(N)-1}) \leqslant (n+1)(cc') n^{\operatorname{cx}(M)+\operatorname{cx}(N)-2} \leqslant (2cc') n^{\operatorname{cx}(M)+\operatorname{cx}(N)-1}$, which implies $\operatorname{cx}(M \otimes_K N) \leqslant \operatorname{cx}(M) + \operatorname{cx}(N)$. □

Now we are in a position to give the following example.

Example 5.5. Let $\operatorname{id}_\Lambda \Lambda = n \geqslant 1$ and Γ be selfinjective of infinite representation type with $J(\Gamma)^3 = 0$. Then we have

(1) $\operatorname{id}_{\Lambda \otimes_K \Gamma} \Lambda \otimes_K \Gamma = n$.
(2) $\sup\{\operatorname{cx}(X) \mid X \in \operatorname{mod} \Lambda \otimes_K \Gamma\} \geqslant 2$.
(3) $\Lambda \otimes_K \Gamma$ admits no maximal j-orthogonal subcategories of $\operatorname{mod} \Lambda \otimes_K \Gamma$ for any $j \geqslant n$.

Proof. Because Γ is selfinjective, $\operatorname{Hom}_\Gamma(\mathbb{D}\Gamma^{op}, \Gamma) \neq 0$ and then the assertion (3) follows from Proposition 5.1. It is well known that $\max\{\operatorname{id}_\Lambda \Lambda, \operatorname{id}_\Gamma \Gamma\} \leqslant \operatorname{id}_{\Lambda \otimes_K \Gamma} \Lambda \otimes_K \Gamma \leqslant \operatorname{id}_\Lambda \Lambda + \operatorname{id}_\Gamma \Gamma$ (cf. [AR2, Proposition 2.2]). By assumption, $\operatorname{id}_\Lambda \Lambda = n$ and Γ is selfinjective, so $\operatorname{id}_{\Lambda \otimes_K \Gamma} \Lambda \otimes_K \Gamma = n$ and the assertion (1) follows. By [GW, Theorem 6.1], $\sup\{\operatorname{cx}(N) \mid N \in \operatorname{mod} \Gamma\} \geqslant 2$. Then it follows from Proposition 5.4 that the assertion (2) holds true. □

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