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Tilting modules of finite projective dimension and a generalization of $*$-modules ☆

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Abstract

It is well known that tilting modules of projective dimension $\leqslant 1$ coincide with $*$-modules generating all injectives. This result is extended in this paper. Namely, we generalize $*$-modules to so-called $*^n$-modules and show that tilting modules of projective dimension $\leqslant n$ are $*^n$-modules which n-present all injectives.

0. Introduction

Tilting theory may be viewed as a far-reaching generalization of the Morita theory of equivalences between module categories (see [1,2,6,7] et al.). By introducing the notion of a quasi-progenerator, Fuller showed a different way of generalization of the Morita theory [5]. Later, Menini and Orsatti found a common point by discovering the general notion of $*$-modules [8]. Colpi then proved that tilting modules of projective dimension $\leqslant 1$ coincide with $*$-modules which generate all injectives [2], while quasi-progenerators are just the $*$-modules which generate all of their submodules [1]. However, tilting modules of projective dimension $\leqslant n$ are $*$-modules if and only if $n \leqslant 1$ (see Lemma 3.1, this fact was first inferred in [9]). Hence it's interesting to give some generalizations of $*$-modules and to consider the connection between them and tilting modules of finite projective dimension.

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The paper is constructed as follows. In Section 1 we introduce some notions and preliminary results. In Section 2 we generalize $*$-modules to $*^n$-modules and we give some basic properties of $*^n$-modules. As corollaries, some known results about $*$-modules are obtained. We also show that any $*^n$-module defines an equivalence between two module subcategories (Theorem 2.10). In Section 3 we first show that tilting modules of projective dimension $\leqslant n$ are $*^n$-modules (Proposition 3.4). Then we characterize $*^n$-modules which n-present the injectives (Theorem 3.5). The main result is Theorem 3.8 where a strong connection between $*^n$-modules and tilting modules of projective dimension $\leqslant n$ is given. Section 4 contains some open questions about $*^n$-modules.

1. Preliminaries

All rings have non-zero identity and all modules are unitary. For every ring R, Mod-R (R-Mod) denotes the category of all right (left) R-modules. Let $P_R \in$ Mod-R. We say that a right R-module M_R is n-presented by P_R if there exists an exact sequence $P^{(X_{n-1})} \to P^{(X_{n-2})} \to \cdots \to P^{(X_1)} \to P^{(X_0)} \to M_R \to 0$ where X_i, $0 \leqslant i \leqslant n-1$, are sets. Denote by n-$\operatorname{Pres}(P_R)$ the category of all modules n-presented by P_R. Of course, for every n we have $(n+1)$-$\operatorname{Pres}(P_R) \subseteq n$-$\operatorname{Pres}(P_R)$. We denote 2-$\operatorname{Pres}(P_R)$ by $\operatorname{Pres}(P_R)$ and 1-$\operatorname{Pres}(P_R)$ by $\operatorname{Gen}(P_R)$, as usual.

By taking a free resolution of B_A, one can prove the following result.

Lemma 1.1. *Let $P_R \in$ Mod-R and $A = \operatorname{End}(P_R)$. Then $B \otimes_A P \in \operatorname{Pres}(P_R)$ for any $B_A \in$ Mod-A. If moreover $\operatorname{Tor}_i^A(B, P) = 0$ for $1 \leqslant i \leqslant n$, then $B \otimes_A P \in (n+2)$-$\operatorname{Pres}(P_R)$.*

A right R-module P_R is selfsmall if, for any set X there is the canonical isomorphism $\operatorname{Hom}_R(P, P^{(X)}) \simeq \operatorname{Hom}_R(P, P)^{(X)}$. Namely, if $\pi_x : P^{(X)} \to P$ is the canonical xth projection, for any $f \in \operatorname{Hom}_R(P, P^{(X)})$ it turns out that $\pi_x \circ f = 0$ for almost all x of X. Clearly, every finitely generated module is selfsmall, but the converse is generally false (see [4]). Let $P_R \in$ Mod-R. We say that P_R is n-quasi-projective if for any exact sequence $0 \to M \to P^{(X)} \to N \to 0$ in Mod-R, where $M_R \in (n-1)$-$\operatorname{Pres}(P_R)$, the induced sequence $0 \to \operatorname{Hom}_R(P, M) \to \operatorname{Hom}_R(P, P^{(X)}) \to \operatorname{Hom}_R(P, N) \to 0$ is exact. Note that in case $n = 2$ it is just the familiar notion of w-Σ-quasi-projective introduced by Colpi [1].

Let A be a ring and $K_A \in$ Mod-A. A right A-module N_A is n-copresented by K_A if there exists an exact sequence $0 \to N_A \to K^{Y_0} \to K^{Y_1} \to \cdots \to K^{Y_{n-2}} \to K^{Y_{n-1}}$ where Y_i, $0 \leqslant i \leqslant n-1$, are sets. Denote by n-$\operatorname{Copres}(K_A)$ the category of all modules n-copresented by K_A. Of course, for every n we have $(n+1)$-$\operatorname{Copres}(K_A) \subseteq n$-$\operatorname{Copres}(K_A)$. We denote 2-$\operatorname{Copres}(K_A)$ by $\operatorname{Copres}(K_A)$ and 1-$\operatorname{Copres}(K_A)$ by $\operatorname{Cogen}(K_A)$, as usual.

Let R be a ring, $P_R \in$ Mod-R and let $A = \operatorname{End}(P_R)$. Take an arbitrary injective cogenerator Q_R of Mod-P and put $K_A = \operatorname{Hom}_R(P, Q)$. Denote by H_P the functor $\operatorname{Hom}_R(P, -)$ and by T_P the functor $- \otimes_A P$. It is well known that (T_P, H_P) is a pair of adjoint functors with canonical morphisms:

$$\rho_M : T_P H_P(M_R) \to M_R, \quad \text{by } f \otimes p \mapsto f(p);$$

$$\sigma_N : N_A \to H_P T_P(N_A), \quad \text{by } n \mapsto [p \mapsto n \otimes p].$$

Lemma 1.2 [1].
(a) *σ_N is a monomorphism if and only if* $N_A \in \mathrm{Cogen}(K_A)$.
(b) *ρ_M is an epimorphism if and only if* $M_R \in \mathrm{Gen}(P_R)$.

It follows that $\mathrm{Cogen}(K_A)$ does not depend on the choice of the injective cogenerator Q_R.

We say that P_R is a $*$-module if the pair (T_P, H_P) defines an equivalence:

$$T_P : \mathrm{Cogen}(K_A) \rightleftharpoons \mathrm{Gen}(P_R) : H_P.$$

In [1] the following result was proved.

Theorem 1.3. *Let $P_R \in$ Mod-R, $A = \mathrm{End}(P_R)$. Then the following conditions are equivalent:*

(1) *P_R is a $*$-module.*
(2) *P_R is selfsmall, w-Σ-quasi-projective, and* $\mathrm{Gen}(P_R) = \mathrm{Pres}(P_R)$.
(3) *P_R is selfsmall, and for any $M_R \leqslant P^{(X)}$, $M_R \in \mathrm{Gen}(P_R)$ if and only if $\mathrm{Ext}^1_R(P, M) \to \mathrm{Ext}^1_R(P, P^{(X)})$ is canonically a monomorphism.*
(4) *P_R is selfsmall and, for any exact sequence $0 \to L \to M \to N \to 0$ in Mod-R, where $M, N \in \mathrm{Gen}(P_R)$, the induced sequence $0 \to H_P(L) \to H_P(M) \to H_P(N) \to 0$ is exact if and only if $L \in \mathrm{Gen}(P_R)$.*

2. $*^n$-modules

Suggested by Theorem 1.3(2) and the ideas in [4], we give the following definition of $*^n$-modules.

Definition 2.1. Let $P_R \in$ Mod-R. P_R is a $*^n$-module if P_R is selfsmall, $(n+1)$-quasi-projective, and $(n+1)\text{-}\mathrm{Pres}(P_R) = n\text{-}\mathrm{Pres}(P_R)$.

Remark 1.
(i) When $n = 1$, $*^n$-modules are just the classical $*$-modules.
(ii) If P_R is a $*^n$-module, then it is a $*^m$-module for any $m \geqslant n$.
(iii) We will show in Section 3 that tilting modules of projective dimension $\leqslant n$ are $*^n$-modules. Hence our generalization is not trivial.

Proposition 2.2. *Let P_R be a $*^n$-module. Then ρ_N is an isomorphism and $\mathrm{Tor}^A_i(H_P(N), P) = 0$ for any $i \geqslant 1$ and any $N \in n\text{-}\mathrm{Pres}(P_R)$.*

Proof. For any $N \in n\text{-}\mathrm{Pres}(P_R)$, we have that $N \in (n+1)\text{-}\mathrm{Pres}(P_R)$ by the definition of $*^n$-modules. Hence we have an exact sequence $0 \to M \to P^{(X)} \to N \to 0$ in Mod-R where $M \in n\text{-}\mathrm{Pres}(P_R)$ and X is a set. Since P_R is $(n+1)$-quasi-projective, the induced sequence $0 \to H_P(M) \to H_P(P^{(X)}) \to H_P(N) \to 0$ is exact. We obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathrm{Tor}_1^A(H_P(N), P) & \longrightarrow & T_P H_P(M) & \longrightarrow & T_P H_P(P^{(X)}) & \longrightarrow & T_P H_P(N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_{P^{(X)}}} & & \downarrow{\scriptstyle \rho_N} & & \\ 0 & \longrightarrow & M & \longrightarrow & P^{(X)} & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

By Lemma 1.2, ρ_M is an epimorphism. Since $\rho_{P^{(X)}}$ is a natural isomorphism, ρ_N is an isomorphism. So that applying the same argument as before we can conclude that ρ_M is an isomorphism too. It follows that $\mathrm{Tor}_1^A(H_P(N), P) = 0$. Similarly, $\mathrm{Tor}_1^A(H_P(M), P) = 0$. Finally, from the fact that $\mathrm{Tor}_{i+1}^A(H_P(N), P) \simeq \mathrm{Tor}_i^A(H_P(M), P)$ for any $i \geqslant 1$ we derive that $\mathrm{Tor}_i^A(H_P(N), P) = 0$ for any $i \geqslant 1$. $\square$

We give now some characterizations of $*^n$-modules which are similar to Theorem 1.3.

Theorem 2.3. *Let* $P_R \in \text{Mod-}R$ *and* $A = \mathrm{End}(P_R)$. *Then the following conditions are equivalent*:

(1) P_R *is a* $*^n$*-module.*
(2) P_R *is selfsmall and for any exact sequence* $0 \to M \to P^{(X)} \to N \to 0$ *in* Mod-R *where* $N \in n\text{-}\mathrm{Pres}(P_R)$ *and* X *is a set,* $M \in n\text{-}\mathrm{Pres}(P_R)$ *if and only if* $\mathrm{Ext}_R^1(P, M) \to \mathrm{Ext}_R^1(P, P^{(X)})$ *is canonically a monomorphism.*
(3) P_R *is selfsmall and for any epimorphism* $\phi : P^{(X)} \to N$ *where* $N \in n\text{-}\mathrm{Pres}(P_R)$ *and* X *is a set, say* $\phi = (\phi_x)_x$, *we have* $\mathrm{Ker}\,\phi \in n\text{-}\mathrm{Pres}(P_R)$ *if and only if* $\mathrm{Hom}_R(P, N) = \sum_x \phi_x A$.

Proof. (1) $\Rightarrow$ (2). First assume that $M \in n\text{-}\mathrm{Pres}(P_R)$. Since P_R is $(n+1)$-quasi-projective and $(n+1)\text{-}\mathrm{Pres}(P_R) = n\text{-}\mathrm{Pres}(P_R)$, the canonical morphism $\mathrm{Ext}_R^1(P, M) \to \mathrm{Ext}_R^1(P, P^{(X)})$ is clearly a monomorphism.

On the other hand, assume that the canonical morphism $\mathrm{Ext}_R^1(P, M) \to \mathrm{Ext}_R^1(P, P^{(X)})$ is a monomorphism for the exact sequence $0 \to M \to P^{(X)} \to N \to 0$. It follows that the induced sequence $0 \to H_P(M) \to H_P(P^{(X)}) \to H_P(N) \to 0$ is exact. Now consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_P H_P(M) & \longrightarrow & T_P H_P(P^{(X)}) & \longrightarrow & T_P H_P(N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_{P^{(X)}}} & & \downarrow{\scriptstyle \rho_N} & & \\ 0 & \longrightarrow & M & \longrightarrow & P^{(X)} & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

By Proposition 2.2 ρ_N is an isomorphism and $\mathrm{Tor}_i^A(H_P(N), P) = 0$ for any $i \geqslant 1$. Therefore the above diagram is exact, so that ρ_M is an isomorphism and $\mathrm{Tor}_i^A(H_P(M), P) = 0$ for any $i \geqslant 1$. Hence $M \in n\text{-}\mathrm{Pres}(P_R)$ by Lemma 1.1.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) are similar to the proof of (5) $\Rightarrow$ (4) $\Rightarrow$ (3) in [1, Theorem 4.1]. □

Proposition 2.4. *Let P_R be a $*^n$-module. Then T_P is an exact functor in $H_P(n\text{-}\mathrm{Pres}(P_R))$. Moreover, $H_P(n\text{-}\mathrm{Pres}(P_R)) = {}_A^{\perp}P := \{M_A \mid \mathrm{Tor}_i^A(M, P) = 0$ for all $i \geqslant 1\}$, where $A = \mathrm{End}(P_R)$.*

Proof. By Proposition 2.2 we have that $H_P(n\text{-}\mathrm{Pres}(P_R)) \subseteq {}_A^{\perp}P$. In particular the functor T_P is exact in $H_P(n\text{-}\mathrm{Pres}(P_R))$. On the other hand, we have that $T_P(M) \in n\text{-}\mathrm{Pres}(P_R)$ for any $M_A \in {}_A^{\perp}P$ by Lemma 1.1. Therefore given the exact sequence $0 \to L_A \to A^{(X)} \to M_A \to 0$ where X is a set, we have $L_A \in {}_A^{\perp}P$ and $T_P(L) \in n\text{-}\mathrm{Pres}(P_R)$. Consider the induced exact sequence $0 \to T_P(L) \to T_P(A^{(X)}) \to T_P(M) \to 0$ (note that $\mathrm{Tor}_1^A(M, P) = 0$). Since P_R is a $*^n$-module and $T_P(L) \in n\text{-}\mathrm{Pres}(P_R)$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & A^{(X)} & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_L} & & \downarrow{\scriptstyle \sigma_{A^{(X)}}} & & \downarrow{\scriptstyle \sigma_M} & & \\
0 & \longrightarrow & H_PT_P(L) & \longrightarrow & H_PT_P(A^{(X)}) & \longrightarrow & H_PT_P(M) & \longrightarrow & 0
\end{array}$$

Note that σ_M is an epimorphism since $\sigma_{A^{(X)}}$ is a natural isomorphism. The same argument proves that σ_L is an epimorphism too. It follows that σ_M is an isomorphism. Therefore $M_A \simeq H_PT_P(M) \in H_P(n\text{-}\mathrm{Pres}(P_R))$. So that the inclusion ${}_A^{\perp}P \subseteq H_P(n\text{-}\mathrm{Pres}(P_R))$ is proved. □

As an application, we immediately obtain a new proof of the following result in [3].

Corollary 2.5. *Let P_R be a $*$-module, $A = \mathrm{End}(P_R)$ and let $K_A = \mathrm{Hom}_R(P, Q)$ where Q_R is an arbitrary injective cogenerator of* Mod-R. *Then*

(1) *T_P is an exact functor in* $\mathrm{Cogen}(K_A)$.
(2) $\mathrm{Cogen}(K_A) = {}_A^{\perp_1}P := \{M_A \mid \mathrm{Tor}_1^A(M, P) = 0\}$.

Proof. By Proposition 2.4, the functor T_P is exact in $H_P(\mathrm{Gen}(P_R))$. Since P_R is a $*$-module, $H_P(\mathrm{Gen}(P_R)) = \mathrm{Cogen}(K_A)$. Hence (1) holds true.

By [9, Lemma 2.1] the flat dimension of ${}_AP \leqslant 1$, so ${}_A^{\perp_1}P = \{M \mid \mathrm{Tor}_1^A(M, P) = 0\} = \{M \mid \mathrm{Tor}_i^A(M, P) = 0$ for all $i \geqslant 1\} = {}_A^{\perp}P$. Finally, thanks to Proposition 2.4 we see that (2) holds true. □

Proposition 2.6. *Let P_R be a $*^n$-module, $A = \mathrm{End}(P_R)$. Then H_P preserves any exact sequence in $n\text{-}\mathrm{Pres}(P_R)$.*

Proof. Consider any exact sequence $0 \to M \to N \to L \to 0$ in $n\text{-}\mathrm{Pres}(P_R)$ and the induced exact sequence $0 \to H_P(M) \to H_P(N) \to H_P(L) \to D_A \to 0$, where $D_A = \mathrm{Im}(H_P(L) \to \mathrm{Ext}_R^1(P, M))$. Let $C_A = \mathrm{Im}(H_P(N) \to H_P(L))$. Applying the functor T_P

to the exact sequence $0 \to H_P(M) \to H_p(N) \to C \to 0$, thanks to Proposition 2.2, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \operatorname{Tor}_1^A(C,P) & \longrightarrow & T_P H_P(M) & \longrightarrow & T_P H_P(N) & \longrightarrow & T_P(C) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_N} & & \downarrow & & \\ & & 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

where ρ_M and ρ_N are isomorphisms and $\operatorname{Tor}_i^A(H_P(M),P) = 0 = \operatorname{Tor}_i^A(H_P(N),P)$ for any $i \geqslant 1$. Then $\operatorname{Tor}_i^A(C,P) = 0$ for any $i \geqslant 1$, and $T_P(C) \simeq L$. By Proposition 2.4 we have $C_A = H_P(X)$ for some $X_R \in n\text{-}\operatorname{Pres}(P_R)$. Then

$$C_A = H_P(X) \simeq H_P\big(T_P H_P(X)\big) \simeq H_P T_P\big(H_P(X)\big) = H_P T_P(C).$$

It follows that

$$D_A = \operatorname{Coker}\big(C \to H_P(L)\big) \simeq \operatorname{Coker}\big(H_P T_P(C) \to H_P T_P H_P(L)\big) = 0.$$

Hence $0 \to H_P(M) \to H_P(N) \to H_P(L) \to 0$ is exact. □

In particular, we obtain the following corollary.

Corollary 2.7 [1]. *Let P_R be a $*$-module. Then H_P is an exact functor in* $\operatorname{Gen}(P_R)$.

Thanks to Proposition 2.6, we are able to give the following characterization of $*^n$-modules which generalizes (4) in Theorem 1.3.

Theorem 2.8. *Let $P_R \in$ Mod-R. Then the following conditions are equivalent*:

(1) *P_R is a $*^n$-module.*
(2) *P_R is selfsmall and for any exact sequence $0 \to M \to N \to L \to 0$ in Mod-R where $N, L \in n\text{-}\operatorname{Pres}(P_R)$, we have $M \in n\text{-}\operatorname{Pres}(P_R)$ if and only if the induced sequence $0 \to H_P(M) \to H_P(N) \to H_P(L) \to 0$ is exact.*

Proof. (1) ⇒ (2). The necessity follows from Proposition 2.6 and the sufficiency from a similar proof as in (1) ⇒ (2) in Theorem 2.3.

(2) ⇒ (1). It follows from (2) ⇒ (1) in Theorem 2.3. □

Proposition 2.9. *Let P_R be a $*^n$-module. Then $n\text{-}\operatorname{Pres}(P_R)$ is extension closed if and only if $n\text{-}\operatorname{Pres}(P_R) \subseteq P_R^{\perp 1} := \{M_R \mid \operatorname{Ext}_R^1(P,M) = 0\}$.*

Proof. The necessity. For any $M \in n\text{-}\operatorname{Pres}(P_R)$ and any extension of M by $P_R : 0 \to M \to N \to^f P_R \to 0$, we have that $N \in n\text{-}\operatorname{Pres}(P_R)$ by assumption. Thanks to Proposition 2.6, the induced sequence $0 \to \operatorname{Hom}_R(P,M) \to \operatorname{Hom}_R(P,N) \to \operatorname{Hom}_R(P,P) \to 0$ is exact.

Hence there is a morphism $g : P_R \to N$ such that $fg = 1_{P_R}$. This proves that n-$\mathrm{Pres}(P_R) \subseteq P_R^{\perp_1}$.

The sufficiency. For any $M, L \in n$-$\mathrm{Pres}(P_R)$ and any extension of M by $L : 0 \to M \to N \to L \to 0$ we get that the induced sequence $0 \to \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, L) \to 0$ is exact by assumption. Thank to Proposition 2.2, both ρ_M and ρ_L are isomorphisms and both $H_P(M)$ and $H_P(L)$ are in ${}_A^{\perp}P$. It follows that ρ_N is an isomorphism and $H_P(N) \in {}_A^{\perp}P$. Thanks to Lemma 1.1, we obtain that $N \in n$-$\mathrm{Pres}(P_R)$, i.e., n-$\mathrm{Pres}(P_R)$ is closed under extensions. $\square$

We conclude this section with the following category-theoretical characterization of $*^n$-modules.

Theorem 2.10. *Let $P_R \in$ Mod-R, $A = \mathrm{End}(P_R)$. Then the following conditions are equivalent*:

(1) *P_R is a $*^n$-module.*
(2) *P_R induces an equivalence*: $T_P : {}_A^{\perp}P \rightleftharpoons n$-$\mathrm{Pres}(P_R) : H_P$, *where ${}_A^{\perp}P$ is defined as in Proposition* 2.4.

Proof. (1) $\Rightarrow$ (2). By Propositions 2.2 and 2.4.

(2) $\Rightarrow$ (1). Since $A \in {}_A^{\perp}P$, we have that $\mathrm{Hom}_R(P, P)^{(X)} = A^{(X)} \simeq H_P T_P(A^{(X)}) = H_P(T_P(A^{(X)})) \simeq H_P(P^{(X)}) = \mathrm{Hom}_R(P, P^{(X)})$ canonically. Hence P_R is selfsmall. Since $H_P(N) \in {}_A^{\perp}P$ and $T_P H_P(N) \simeq N$ for any $N \in n$-$\mathrm{Pres}(P_R)$, we get that $N \in (n+1)$-$\mathrm{Pres}(P_R)$ by Lemma 1.1. So that $(n+1)$-$\mathrm{Pres}(P_R) = n$-$\mathrm{Pres}(P_R)$. Finally, for any exact sequence $0 \to M \to P^{(X)} \to N \to 0$ where $M \in n$-$\mathrm{Pres}(P_R)$, we have an induced exact sequence $0 \to H_P(M) \to H_P(P^{(X)}) \to H_P(N) \to D_A \to 0$ where $D_A = \mathrm{Im}(H_P(N) \to \mathrm{Ext}_R^1(P, M))$. A similar proof as in Proposition 2.6 shows that $D_A = 0$, i.e., P_R is $(n+1)$-quasi-projective. $\square$

3. Tilting modules

In this section we study the connection between tilting modules of projective dimension $\leqslant n$ and $*^n$-modules. In particular, we characterize tilting modules of projective dimension $\leqslant n$ as a subclass of $*^n$-modules. The results in this section generalize the case $n = 1$ in [2,3], etc.

Following Miyashita [7], we say that P_R is a tilting module of projective dimension $\leqslant n$ if it satisfies the following three conditions:

(1) P_R has a projective resolution $0 \to F_n \to \cdots \to F_0 \to P_R \to 0$ such that each F_i is finitely generated.
(2) $\mathrm{Ext}_R^i(P, P) = 0$ if $1 \leqslant i \leqslant n$.
(3) There exists an exact sequence $0 \to R \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that each P_i is a direct summand of a finite direct sum of copies of P_R.

Assume that P_R has a finitely generated projective resolution. Following Wakamatsu [11,12], we say that P_R is a Wakamatsu-tilting module if it satisfies the following two conditions:

(1) $R \simeq \operatorname{End}({}_A P)$ where $A = \operatorname{End}(P_R)$.
(2) $\operatorname{Ext}^i_R(P, P) = 0 = \operatorname{Ext}^i_A(P, P)$ for all $i \geqslant 1$.

By [11] these conditions are equivalent to the following:

(i) $\operatorname{Ext}^i_R(P, P) = 0$ for all $i \geqslant 1$.
(ii) There is an infinite exact sequence $0 \to R \to^i P_0 \to^{f_0} P_1 \to \cdots$, where each P_i is a direct summand of a finite direct sum of copies of P_R, and $\operatorname{Ext}^1_R(\operatorname{Ker} f_i, P) = 0$ for any $i \geqslant 0$.

Note that both tilting modules of finite projective dimension and Wakamatsu-tilting modules are left–right symmetric [7,11].

We first prove the following fact.

Lemma 3.1. *Let P_R be a tilting module of projective dimension $\leqslant n$. The following conditions are equivalent*:

(1) *P_R is a $*$-module.*
(2) $n \leqslant 1$.

Proof. (1) $\Rightarrow$ (2). By [2, Theorem 3] it is sufficient to prove that the injective envelope E of R_R is generated by P_R. Since $\operatorname{Ext}^i_R(P, E)$ is clearly zero for all $i \geqslant 1$, the map ρ_E is an isomorphism by [7, Lemma 1.8]. This shows that $E \in \operatorname{Gen}(P_R)$.

(2) $\Rightarrow$ (1) is well known. □

The proof of the following crucial lemma is essentially due to an idea which comes from [8, Theorem 4.3].

Lemma 3.2. *Assume that P_R has a finitely generated projective resolution. The following conditions are equivalent*:

(1) $\operatorname{Ext}^n_R(P, P) = 0$.
(2) $\operatorname{Ext}^n_R(P, P^{(X)}) = 0$ *for any set X.*

Proof. (1) $\Rightarrow$ (2). By assumption we have an exact sequence $\cdots \to R^{m_{i+1}} \to^{f_{i+1}} R^{m_i} \to^{f_i} \cdots \to R^{m_0} \to^{f_0} P_R \to 0$ where each $m_j \in \mathbb{N}$. Let $L_j = \operatorname{Im} f_j$ for all $j \geqslant 0$. Therefore $L_0 = P_R$ and each L_j is a finitely generated right R-module. Note $\operatorname{Ext}^k_R(R^{m_j}, P) = 0$ for all $k \geqslant 1$ and all $j \geqslant 1$, so that $\operatorname{Ext}^1_R(L_{n-1}, P) \simeq \operatorname{Ext}^n_R(P, P) = 0$. Now applying the functor $\operatorname{Hom}_R(-, P)$ to the exact sequence $0 \to L_n \to R^{m_{n-1}} \to L_{n-1} \to 0$ we get the induced exact sequence $0 \to \operatorname{Hom}_R(L_{n-1}, P) \to \operatorname{Hom}_R(R^{m_{n-1}}, P) \to \operatorname{Hom}_R(L_n, P) \to 0 = \operatorname{Ext}^1_R(L_{n-1}, P)$. It follows that every morphism $L_n \to P_R$ can be extended to a morphism $R^{m_{n-1}} \to P_R$. Consider now a morphism $g: L_n \to P^{(X)}$. As L_n is finitely generated, g is a diagonal morphism of finite family of morphisms from L_n into P. Hence g extends to a morphism from $R^{m_{n-1}}$ into $P_R^{(X)}$. Therefore the induced sequence $0 \to \mathrm{Hom}_R(L_{n-1}, P^{(X)}) \to \mathrm{Hom}_R(R^{m_{n-1}}, P^{(X)}) \to \mathrm{Hom}_R(L_n, P^{(X)}) \to 0$ is exact. As $\mathrm{Ext}_R^1(R^{m_{n-1}}, P^{(X)}) = 0$ we get $\mathrm{Ext}_R^1(L_{n-1}, P^{(X)}) = 0$. It follows that $\mathrm{Ext}_R^n(P, P^{(X)}) \simeq \mathrm{Ext}_R^1(L_{n-1}, P^{(X)}) = 0$.

(2) $\Rightarrow$ (1) is clear. □

To study the connection between tilting modules of projective dimension $\leqslant n$ and $*^n$-modules, we need the following lemma.

Lemma 3.3. *Let P_R be a selfsmall right R-module. Assume that n-$\mathrm{Pres}(P_R) = P_R^{\perp} := \{M_R \mid \mathrm{Ext}_R^i(P, M) = 0$ for all $i \geqslant 1\}$. Then P_R is a $*^n$-module.*

Proof. For any exact sequence $0 \to M \to P^{(X)} \to N \to 0$ where $N \in n$-$\mathrm{Pres}(P_R)$, the induced sequence $0 \to H_P(M) \to H_P(P^{(X)}) \to H_P(N) \to \mathrm{Ext}_R^1(P, M) \to 0$ is exact. Note that $N, P^{(X)} \in n$-$\mathrm{Pres}(P_R) = P_R^{\perp}$, so that $\mathrm{Ext}_R^i(P, M) = 0$ for $i \geqslant 2$. Therefore $\mathrm{Ext}_R^1(P, M) \to 0$ is canonically a monomorphism if and only if $\mathrm{Ext}_R^1(P, M) = 0$ if and only if $M \in P_R^{\perp} = n$-$\mathrm{Pres}(P_R)$. It follows that P_R is a $*^n$-module by Theorem 2.3. □

We are now ready to prove that a tilting module of projective dimension $\leqslant n$ is a $*^n$-module.

Proposition 3.4. *Suppose that P_R is a tilting module of projective dimension $\leqslant n$. Then n-$\mathrm{Pres}(P_R) = P_R^{\perp}$, so that P_R is a $*^n$-module.*

Proof. For any $N \in n$-$\mathrm{Pres}(P_R)$, there exists an exact sequence $0 \to M \to P^{(X_{n-1})} \to P^{(X_{n-2})} \to \cdots \to P^{(X_0)} \to N \to 0$ for some $M_R \in$ Mod-R where X_i, $0 \leqslant i \leqslant n-1$, are sets. Thanks to Lemma 3.2, we have that $\mathrm{Ext}_R^i(P, N) \simeq \mathrm{Ext}_R^{i+n}(P, M) = 0$ for all $i \geqslant 1$ by assumption. It follows that n-$\mathrm{Pres}(P_R) \subseteq P_R^{\perp}$.

Now let $M \in P_R^{\perp}$ and $A = \mathrm{End}(P_R)$. Let $0 \to M \to I_0 \to I_1 \to \cdots \to I_n$ be an injective resolution of M. Then the induced sequence $0 \to H_P(M) \to H_P(I_0) \to H_P(I_1) \to \cdots \to H_P(I_n) \to C \to 0$ is exact for some $C \in$ Mod-A. Moreover, $\mathrm{Tor}_i^A(H_P(I), P) = 0$ for all $i \geqslant 1$ and any injective module $I \in$ Mod-R by [7, Lemma 1.7]. It follows that $\mathrm{Tor}_i^A(H_P(M), P) \simeq \mathrm{Tor}_{i+n}^A(C, P) = 0$ for all $i \geqslant 1$. By [7, Lemma 1.8] $T_P H_P(M) \simeq M$. Thus $M \in n$-$\mathrm{Pres}(P_R)$ by Lemma 1.1. □

In fact, the condition n-$\mathrm{Pres}(P_R) = P_R^{\perp}$ characterizes the $*^n$-modules P_R such that every injective module is n-presented by P_R, as the following theorem shows.

Theorem 3.5. *Let P_R be a right R-module. Denote by Inj. the class of all injective right R-modules. Then the following conditions are equivalent*:

(1) *P_R is a $*^n$-module and Inj. $\subseteq n$-$\mathrm{Pres}(P_R)$.*

(2) *P_R is selfsmall and n-*Pres$(P_R) = P_R^{\perp}$.

Proof. (1) $\Rightarrow$ (2). P_R is clearly selfsmall. For any $M \in n$-Pres(P_R), let E be the injective envelope of M with the exact sequence $0 \to M \to E \to N \to 0$. We derive the induced exact sequence $0 \to H_P(M) \to H_P(E) \to H_P(N) \to \mathrm{Ext}_R^1(P, M) \to 0$. Let $X_A = \mathrm{Im}(H_P(E) \to H_P(N))$, where $A = \mathrm{End}(P_R)$. Applying T_P to the exact sequence $0 \to H_P(M) \to H_P(E) \to X \to 0$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Tor}_1^A(X, P) & \to & T_P H_P(M) & \to & T_P H_P(E) & \to & T_P(X) & \to 0 \\
 & & & \downarrow \rho_M & & \downarrow \rho_E & & \downarrow & \\
 & 0 & \to & M & \to & E & \to & N & \to 0
\end{array}$$

By assumption, both ρ_M and ρ_E are isomorphisms, and $\mathrm{Tor}_i^A(H_P(M), P) = 0 = \mathrm{Tor}_i^A(H_P(E), P)$ for all $i \geqslant 1$, thanks to Proposition 2.2. It follows that $\mathrm{Tor}_i^A(X, P) = 0$ for all $i \geqslant 1$ and that $T_P(X) \simeq N$. Hence $N \simeq T_P(X) \in n$-Pres(P_R) by Lemma 1.1. Therefore the induced sequence $0 \to H_P(M) \to H_P(E) \to H_P(N) \to 0$ is exact by Proposition 2.6. So that $\mathrm{Ext}_R^1(P, M) = 0$. Similarly, $\mathrm{Ext}_R^1(P, N) = 0$. Since $\mathrm{Ext}_R^i(P, N) \simeq \mathrm{Ext}_R^{i+1}(P, M)$ for all $i \geqslant 1$, from the arbitrarity of $M \in n$-Pres(P_R) it follows that $\mathrm{Ext}_R^i(P, M) = 0$ for all $i \geqslant 1$. This proves that n-Pres$(P_R) \subseteq P_R^{\perp}$. The opposite inclusion can be proved by an argument similar to the second part of the proof 3.4.

(2) $\Rightarrow$ (1). It follows from Lemma 3.3. □

Proposition 3.6. *Assume that one of the conditions in Theorem* 3.5 *holds and that P_R has a finitely generated projective resolution. Then P_R is a Wakamatsu-tilting module.*

Proof. Let E be the injective envelope of R_R. Since $E \in n$-Pres(P_R) and R is projective, we obtain the following commutative diagram where X is a set:

$$\begin{array}{ccccc}
 & & R & & \\
 & \swarrow & \downarrow & & \\
P^{(X)} & \to & E & \to & 0
\end{array}$$

This shows that P_R is faithful. Hence there is an exact sequence $0 \to R \to \mathrm{Hom}_A(P, P) \to X \to 0$ for some $X_R \in$ Mod-R, where $A = \mathrm{End}(P_R)$. Let $E(X)$ be the injective envelope of X. Then the induced sequence $0 \to \mathrm{Hom}_R(X, E(X)) \to \mathrm{Hom}_R(\mathrm{Hom}_A(P, P), E(X)) \to \mathrm{Hom}_R(R, E(X)) \to 0$ is exact. Since P_R has a finitely generated projective resolution, $\mathrm{Hom}_R(R, E(X)) \simeq E(X) \simeq T_P H_P(E(X)) = \mathrm{Hom}_R(P, E(X)) \otimes_A P \simeq \mathrm{Hom}_R(\mathrm{Hom}_A(P, P), E(X))$ canonically. It follows that $\mathrm{Hom}_R(X, E(X)) = 0$, i.e., $X = 0$. Hence $R \simeq \mathrm{End}({}_A P)$.

It is clear that $\mathrm{Ext}^i_R(P,P)=0$ for all $i \geqslant 1$. Moreover, by Proposition 2.4 we have $\mathrm{Tor}^A_i(H_P(I),P)=0$ for all $i \geqslant 1$ and any injective module $I \in \text{Mod-}R$. It follows that $\mathrm{Ext}^i_A(P,P)=0$ for all $i \geqslant 1$ by [7, Lemma 1.7]. □

Lemma 3.7. *Assume that P_R has a finitely generated projective resolution. Denote by* $P_R^{\perp_n} := \{M_R \mid \mathrm{Ext}^n_R(P,M)=0\}$.

(1) *If* $\mathrm{Ext}^n_R(P,P)=0$ *and* $\operatorname{proj\,dim}(P_R) \leqslant n$, *then* $\mathrm{Gen}(P_R) \subseteq P_R^{\perp_n}$.
(2) *If Inj.* $\subseteq \mathrm{Gen}(P_R) \subseteq P_R^{\perp_n}$, *then* $\operatorname{proj\,dim}(P_R) \leqslant n$.

Proof. (1) For any $M \in \mathrm{Gen}(P_R)$, from an exact sequence $0 \to N \to P^{(X)} \to M \to 0$ we get the induced exact sequence $\mathrm{Ext}^n_R(P,P^{(X)}) \to \mathrm{Ext}^n_R(P,M) \to \mathrm{Ext}^{n+1}_R(P,N)$. By assumption and Lemma 3.2 we get $\mathrm{Ext}^n_R(P,P^{(X)}) = 0 = \mathrm{Ext}^{n+1}_R(P,N)$. Hence $\mathrm{Ext}^n_R(P,M)=0$. This proves the thesis.

(2) For any $M \in \text{Mod-}R$, consider the exact sequence $0 \to M \to E \to L \to 0$ where E is the injective envelope of M. By assumption $E \in \mathrm{Gen}(P_R)$, so $L \in \mathrm{Gen}(P_R)$ too. Hence $\mathrm{Ext}^n_R(P,L)=0$ by assumption. From the induced exact sequence $0=\mathrm{Ext}^n_R(P,L) \to \mathrm{Ext}^{n+1}_R(P,M) \to \mathrm{Ext}^{n+1}_R(P,E)=0$ we derive that $\mathrm{Ext}^{n+1}_R(P,M)=0$. This proves the thesis. □

We give now a characterization of tilting modules of projective dimension $\leqslant n$ in term of $*^n$-modules.

Theorem 3.8. *Assume that P_R has a finitely generated projective resolution. Then the following conditions are equivalent*:

(1) *P_R is a tilting module of projective dimension $\leqslant n$.*
(2) $n\text{-}\mathrm{Pres}(P_R) = P_R^{\perp}$ *and* $\mathrm{Gen}(P_R) \subseteq P_R^{\perp_n}$.
(3) *P_R is a $*^n$-module, Inj.* $\subseteq n\text{-}\mathrm{Pres}(P_R)$ *and* $\mathrm{Gen}(P_R) \subseteq P_R^{\perp_n}$.

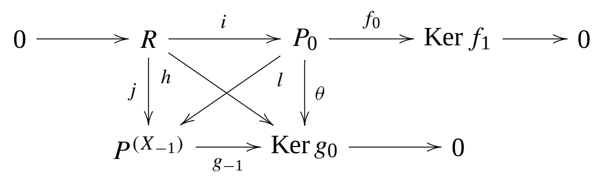
Proof. We already know that (1) ⇒ (2) ⇒ (3) hold true.

(2) ⇒ (1). It remains to be proved that there is an exact sequence $0 \to R \to P_0 \to P_1 \to \cdots \to P_n \to 0$ where each P_i, $0 \leqslant i \leqslant n$, is a direct summand of a finite direct sum of copies of P_R. By Proposition 3.6 P_R is a Wakamatsu-tilting module, so that there is an infinite exact sequence $0 \to R \to^{i} P_0 \to^{f_0} P_1 \to^{f_1} \cdots$, where P_i's are finite direct sums of copies of P_R and $\mathrm{Ext}^1_R(\mathrm{Ker}\, f_i, P)=0$ for all $i \geqslant 0$. Let $X = \mathrm{Ker}\, f_n$. Then $X \in n\text{-}\mathrm{Pres}(P_R)$. Note that $(n+1)\text{-}\mathrm{Pres}(P_R) = n\text{-}\mathrm{Pres}(P_R)$, so that we have an exact sequence $P^{(X_{-1})} \to^{g_{-1}} P^{(X_0)} \to^{g_0} \cdots \to^{g_{n-2}} P^{(X_{n-1})} \to^{g_{n-1}} X \to 0$ where $\mathrm{Ker}\, g_i \in n\text{-}\mathrm{Pres}(P_R)$ and all X_i, $-1 \leqslant i \leqslant n-1$, are finite sets. We claim that $\mathrm{Ext}^1_R(X, \mathrm{Ker}\, g_{n-1})=0$. Therefore X is just a summand of $P^{(X_{n-1})}$ and the result follows.

In fact, we can show, by induction on k, that $\mathrm{Ext}_R^1(\mathrm{Ker}\, f_k, \mathrm{Ker}\, g_{k-1}) = 0$ for $k \geqslant 1$. In case $k = 1$, note that $\mathrm{Ker}\, g_i \in n\text{-}\mathrm{Pres}(P_R)$, so that $\mathrm{Ext}_R^1(P_j, \mathrm{Ker}\, g_i) = 0$ for $-1 \leqslant i \leqslant n-1$ and $j \geqslant 0$. It follows that $\mathrm{Ext}_R^1(\mathrm{Ker}\, f_1, \mathrm{Ker}\, g_0) = 0$ if and only if

$$\mathrm{Hom}_R(P_0, \mathrm{Ker}\, g_0) \to^{\sigma} \mathrm{Hom}_R(R, \mathrm{Ker}\, g_0) \to 0$$

is exact. To show that σ is epic, let $h \in \mathrm{Hom}_R(R, \mathrm{Ker}\, g_0)$. Consider the following diagram:



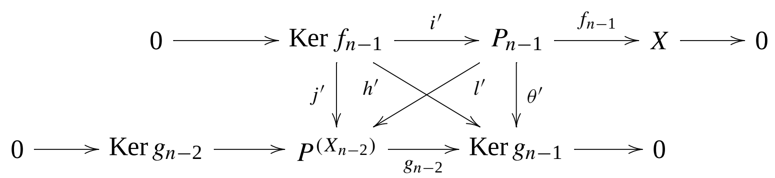
Since R is projective, there exists $j \in \mathrm{Hom}_R(R, P^{(X_{-1})})$ such that $h = g_{-1} \circ j$. Then the induced sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_R\big(\mathrm{Ker}\, f_1, P^{(X_{-1})}\big) \to \mathrm{Hom}_R\big(P_0, P^{(X_{-1})}\big) \\ &\to \mathrm{Hom}_R\big(R, P^{(X_{-1})}\big) \to \mathrm{Ext}_R^1\big(\mathrm{Ker}\, f_1, P^{(X_{-1})}\big) = 0 \end{aligned}$$

is exact. Hence there exists $l \in \mathrm{Hom}_R(P_0, P^{(X_{-1})})$ such that $j = l \circ i$. Let $\theta = g_{-1} \circ l \in \mathrm{Hom}_R(P_0, \mathrm{Ker}\, g_0)$. Note that $\theta \circ i = g_{-1} \circ l \circ i = g_{-1} \circ j = h$, so that σ is epic. Now we show that $\mathrm{Ext}_R^1(X, \mathrm{Ker}\, g_{n-1}) = 0$, just proving that

$$\mathrm{Hom}_R(P_{n-1}, \mathrm{Ker}\, g_{n-1}) \to^{\sigma'} \mathrm{Hom}_R(\mathrm{Ker}\, f_{n-1}, \mathrm{Ker}\, g_{n-1}) \to 0$$

is exact. For any $h' \in \mathrm{Hom}_R(\mathrm{Ker}\, f_{n-1}, \mathrm{Ker}\, g_{n-1})$, consider the following diagram:



Since

$$\mathrm{Ext}_R^1(\mathrm{Ker}\, f_{n-1}, \mathrm{Ker}\, g_{n-2}) = 0$$

by assumption, applying the functor $\mathrm{Hom}_R(\mathrm{Ker}\, f_{n-1}, -)$ to the second row in the previous diagram, we see that the sequence

$$\mathrm{Hom}_R\big(\mathrm{Ker}\, f_{n-1}, P^{(X_{n-2})}\big) \to \mathrm{Hom}_R(\mathrm{Ker}\, f_{n-1}, \mathrm{Ker}\, g_{n-1}) \to 0$$

is exact. It follows that there exists $j' \in \mathrm{Hom}_R(\mathrm{Ker}\, f_{n-1}, P^{(X_{n-2})})$ such that $h' = g_{n-2} \circ j'$. Then the induced sequence

$$0 \to \mathrm{Hom}_R\big(X, P^{(X_{n-2})}\big) \to \mathrm{Hom}_R\big(P_{n-1}, P^{(X_{n-2})}\big) \to \mathrm{Hom}_R\big(\mathrm{Ker}\, f_{n-1}, P^{(X_{n-2})}\big)$$
$$\to \mathrm{Ext}^1_R\big(X, P^{(X_{n-2})}\big) = 0$$

is exact. Therefore there exists $l' \in \mathrm{Hom}_R(P_{n-1}, P^{(X_{n-2})})$ such that $j' = l' \circ i'$. Let $\theta' = g_{n-2} \circ l' \in \mathrm{Hom}_R(P_{n-1}, \mathrm{Ker}\, g_{n-2})$. Then $\theta' \circ i' = g_{n-2} \circ l' \circ i' = g_{n-2} \circ j' = h'$. This proves that σ' is epic. □

Remark 2. Clearly the condition $\mathrm{Gen}(P_R) \subseteq P_R^{\perp_n}$ in the previous theorem can be removed in case $n = 1$. It can also be removed in case $n = 2$. To see this, it is sufficient to show that $\mathrm{Pres}(P_R) = P_R^{\perp}$ implies $\mathrm{Gen}(P_R) \subseteq P_R^{\perp_2}$. In fact, for any $N \in \mathrm{Gen}(P_R)$, let $0 \to N \to E \to X \to 0$ bean exact sequence where E is the injective envelope of N. We have an induced exact sequence $0 \to H_P(N) \to H_P(E) \to C \to 0$ for some $C \in \text{Mod-}A$, where $A = \mathrm{End}(P_R)$. Now consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & T_P H_P(N) & \longrightarrow & T_P H_P(E) & \longrightarrow & T_P(C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_N} & & \downarrow{\scriptstyle \rho_E} & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Note that ρ_N is an epimorphism and ρ_E is an isomorphism, so that we have $T_P(C) \simeq X$. By Lemma 1.1 $X \in \mathrm{Pres}(P_R)$. Hence $\mathrm{Ext}^2_R(P, N) \simeq \mathrm{Ext}^1_R(P, X) = 0$.

In particular we can conclude that tilting modules of projective dimension $\leqslant 2$ are just $*^2$-modules which admit a finitely generated projective resolution and which present all injectives.

4. Questions

In [4], the authors studied $*_\lambda$-modules as generalizations of $*$-modules, where λ is a cardinal. Following [4], a right R-module P is a $*_\lambda$-module for some cardinal λ provided P is finitely generated and P satisfies the condition $C(k)$ for all $k < A$. Here $C(k)$ denotes the following assertion:

“For every submodule M of $P^{(k)}$, the condition $M \in \mathrm{Gen}(P)$ is equivalent to the injective of the canonical group homomorphism $\mathrm{Ext}^1_R(P, M) \to \mathrm{Ext}^1_R(P, P^{(k)})$.”

It should be noted that $C(k)$ implies $C(k')$ for all $k' \leqslant k$ [4, Lemma 2.1], and that $*_\lambda$-modules are just finitely generated modules in case $\lambda = 1$.

The following example shows that $*^n$-modules and $*_\lambda$-modules are different generalizations of $*$-modules.

Example. Let P_R be a right R-module which is finitely generated and quasi-projective. Let $A = \mathrm{End}(P_R)$. Assume that the flat dimension of ${}_AP$ is finite and that P_R is not a quasi-progenerator. Such modules exist clearly (see, for instance, [5, Example 4.6]). By [13, Corollary 3.3] P_R is a $*^n$-module for some $n \geqslant 2$. But P_R is never a $*_\lambda$-module for any $\lambda \geqslant 2$. Otherwise, we have that P_R is a self-generator since P_R is quasi-projective and P_R satisfies the condition $C(2)$. Therefore P_R must be a quasi-progenerator, which is a contradiction.

Let $\mathrm{STAR}(n)$, $\mathrm{STAR}(\lambda)$ and STAR be the class of all $*^n$-modules, all $*_\lambda$-modules and all $*$-modules respectively. We have the following question.

Question 1. Is it true that $\mathrm{STAR}(n) \cap \mathrm{STAR}(\lambda) = \mathrm{STAR}$?

As we see, there are many properties of $*^n$-modules similar to that of $*$-modules. Note that an important fact of $*$-modules is that they are finitely generated (see [10]), our second question is:

Question 2. Are all $*^n$-modules finitely generated?

Let P_R be a $*$-module and $A = \mathrm{End}(P_R)$. Then the flat dimension of ${}_AP$ is not more than 1 [9]. It seems natural to consider the following:

Question 3. Does it happen that the flat dimension of ${}_AP$ is not more than n for any $*^n$-module P_R with $A = \mathrm{End}(P_R)$?

A new result in [13] by the first author may be helpful to the third question. It claims that for any $*^n$-module P_R with $A = \mathrm{End}(P_R)$, ${}^{\perp}_{A}P := \{M_A \mid \mathrm{Tor}_i^A(M, P) = 0$ for all $i \geqslant 1\} = {}^{\perp_{1\leqslant i\leqslant n}}{}_AP := \{M_A \mid \mathrm{Tor}_i^A(M, P) = 0$ for all $1 \leqslant i \leqslant n\}$.

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