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RELATIVE SINGULARITY CATEGORIES II

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Abstract

Let $\mathscr{A}$ be an abelian category with enough projective objects and $\mathscr{C}$ an additive and full subcategory of $\mathscr{A}$, and let $\mathscr{G}(\mathscr{C})$ be the Gorenstein category of $\mathscr{C}$. We study the properties of the $\mathscr{C}$-derived category $D^b_{\mathscr{C}}(\mathscr{A})$, $\mathscr{C}$-singularity category $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ and $\mathscr{G}(\mathscr{C})$-defect category $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A})$ of $\mathscr{A}$. Let $\mathscr{C}$ be admissible in $\mathscr{A}$. We show that $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) \simeq D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ if and only if $\mathscr{C} = \mathscr{G}(\mathscr{C})$; and $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) = 0$ if and only if the stable category $\underline{\mathscr{G}(\mathscr{C})}$ of $\mathscr{G}(\mathscr{C})$ is triangle-equivalent to $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$, and if and only if every object in $\mathscr{A}$ has finite $\mathscr{C}$-proper $\mathscr{G}(\mathscr{C})$-dimension. Then we apply these results to module categories. We prove that under some condition, the Gorenstein derived equivalence of artin algebras induces the Gorenstein singularity equivalence. Finally, for an artin algebra A, we establish the stability of Gorenstein defect categories of A.

1. Introduction

Throughout this paper, $\mathscr{A}$ is an abelian category with enough projective objects, all subcategories are additive, full and closed under isomorphisms, $\mathscr{P}$ is the subcategory of $\mathscr{A}$ consisting of projective objects and $C(\mathscr{A})$ is the category of complexes of objects in $\mathscr{A}$. For a subcategory $\mathscr{C}$ of $\mathscr{A}$, we use $K^*(\mathscr{C})$ to denote the homotopy category of $\mathscr{C}$ and we use $D^*(\mathscr{A})$ to denote the usual derived category of $\mathscr{A}$ by inverting the quasi-isomorphisms in $K^*(\mathscr{A})$, where $* \in \{\text{blank}, -, b\}$.

It is known that $K^b(\mathscr{P})$ is a thick triangulated subcategory of $D^b(\mathscr{A})$. So one can consider the Verdier quotient $D_{sg}(\mathscr{A}) := D^b(\mathscr{A})/K^b(\mathscr{P})$. This category was first introduced by Buchweitz in [12] under the name of *stabilized derived category* of $\mathscr{A}$. Note that $D_{sg}(\mathscr{A}) = 0$ if and only if every object of $\mathscr{A}$ has finite projective dimension. In view of this viewpoint, $D_{sg}(\mathscr{A})$ measures the homological singularity of $\mathscr{A}$, and we call it the *singularity category* of $\mathscr{A}$ after Orlov [34]. We refer to [7, 10, 12, 15, 17, 24, 31, 34, 36, 40] and references therein for more details on this topic.

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The notion of modules of G-dimension zero over left and right noetherian rings was first introduced by Auslander and Bridger in [4]. Then Enochs and Jenda generalized it in [19] to that of Gorenstein projective modules (resp. objects) over arbitrary rings (resp. abelian categories). Since then, Gorenstein projective modules and objects have been deeply studied in the relative homological theory, representation theory and algebraic geometry, see [6, 18, 19, 25, 26, 27], and so on. Denote by $\mathscr{G}(\mathscr{P})$ the subcategory of $\mathscr{A}$ consisting of Gorenstein projective objects. It is known that $\mathscr{G}(\mathscr{P})$ is a Frobenius category with $\mathscr{P}$ the subcategory of (relative) projective-injective objects and the stable category $\underline{\mathscr{G}(\mathscr{P})}$ of $\mathscr{G}(\mathscr{P})$ modulo $\mathscr{P}$ is a triangulated category, see [23]. In particular, there exists a canonical fully faithful triangle functor $F : \underline{\mathscr{G}(\mathscr{P})} \to D_{sg}(\mathscr{A})$ and we have the following

Theorem (Buchweitz [12, Theorem 4.4.1] and Happel [24, Theorem 4.6]). *The canonical functor $F : \underline{\mathscr{G}(\mathscr{P})} \to D_{sg}(\mathscr{A})$ is a triangle-equivalence provided that every object in $\mathscr{A}$ has finite Gorenstein projective dimension.*

Following Zhang [40], we call this theorem *Buchweitz-Happel Theorem*. This theorem has maken a great use in the representation theory, homological theory and singularity theory. Recently, Bergh, Oppermann and Jorgensen proved in [10, Theorem 3.6] that if A is either a left and right artin ring or a commutative noetherian local ring, $\mathscr{A} = \mathrm{mod}\, A$ (the category of finitely generated left A-modules) and $\mathscr{P} = \mathrm{proj}\, A$ (the subcategory of mod A consisting of projective modules), then the converse of the Buchweitz-Happel theorem holds true. Moreover, they called the Verdier quotient $D_{\mathscr{G}(A)\text{-}defect}(\mathrm{mod}\, A) := D_{sg}(\mathrm{mod}\, A)/\mathrm{Im}\, F$ the *Gorenstein defect category* of A, and concluded that $D_{\mathscr{G}(A)\text{-}defect}(\mathrm{mod}\, A)$ measures the Gorensteinness of A in the sense that $D_{\mathscr{G}(A)\text{-}defect}(\mathrm{mod}\, A) = 0$ if and only if A is Gorenstein. Later on, Kong and Zhang described in [30] the Gorenstein defect category $D_{\mathscr{G}(\mathscr{P})\text{-}defect}(\mathscr{A}) := D_{sg}(\mathscr{A})/\mathrm{Im}\, F$ as $D_{\mathscr{G}(\mathscr{P})\text{-}defect}(\mathscr{A}) \simeq K^{-,b}(\mathscr{P})/K^{-,b}_{\mathscr{G}}(\mathscr{P})$, where $K^{-,b}(\mathscr{P})$ is the homotopy category of upper bounded complexes of $\mathscr{P}$ with bounded cohomologies and $K^{-,b}_{\mathscr{G}}(\mathscr{P})$ is the subcategory of $K^{-,b}(\mathscr{P})$ consisting of complexes $P^{\bullet} \in K^{-,b}(\mathscr{P})$ such that there exists $n \in \mathbf{Z}$ with $H^m(P^{\bullet}) = 0$ for any $m \le n$ and $Z^n(P^{\bullet}) \in \mathscr{G}(\mathscr{P})$.

Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$ closed under direct summands. Chen introduced in [14] the relative derived category $D^{*}_{\mathscr{C}}(\mathscr{A})$ of $\mathscr{A}$ with respect to $\mathscr{C}$ for $* \in \{\text{blank}, -, b\}$. It can be viewed as a generalization of the usual derived category and the Gorenstein derived category in [22]. When $\mathscr{C}$ is admissible in $\mathscr{A}$, $D^{*}_{\mathscr{C}}(\mathscr{A})$ is the derived category of some exact category in the sense of Neeman [33]. Asadollahi, Hafezi and Vahed reconsidered in [2] this category and they pointed out that for the bounded case, $K^b(\mathscr{C})$ is a triangulated subcategory of $D^{b}_{\mathscr{C}}(\mathscr{A})$. Then it is natural to consider the Verdier quotient $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) := D^{b}_{\mathscr{C}}(\mathscr{A})/K^b(\mathscr{C})$, which was called the *$\mathscr{C}$-singularity category* of $\mathscr{A}$ in [31]. If $\mathscr{C}$ is admissible in $\mathscr{A}$, then there exists a canonical fully faithful triangle functor $\theta'_{\mathscr{C}} : \underline{\mathscr{G}(\mathscr{C})} \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$, where $\underline{\mathscr{G}(\mathscr{C})}$ is the stable category of the Gorenstein category $\mathscr{G}(\mathscr{C})$ modulo $\mathscr{C}$; furthermore, $\theta'_{\mathscr{C}}$ is a triangle-equivalence provided that every object in

$\mathscr{A}$ has finite $\mathscr{C}$-proper $\mathscr{G}(\mathscr{C})$-dimension, see [31, Section 4]. This can be viewed as a relative version of the Buchweitz-Happel theorem. As a continuation of [31], in this paper we further study the properties of the $\mathscr{C}$-derived category, $\mathscr{C}$-singularity category and so-called $\mathscr{G}(\mathscr{C})$-defect category of $\mathscr{A}$.

In Section 2, we give some terminology and some preliminary results.

In Section 3, we show that if $\mathscr{C}$ is closed under direct summands, then $K^b(\mathscr{C})$ is a thick triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$. We introduce the notion of complexes in $D^b_{\mathscr{C}}(\mathscr{A})$ having finite $\mathscr{C}$-proper $\mathscr{G}(\mathscr{C})$-dimension, and show that the subcategory $D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$ of $D^b_{\mathscr{C}}(\mathscr{A})$ consisting of complexes having finite $\mathscr{C}$-proper $\mathscr{G}(\mathscr{C})$-dimension is a triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$.

Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$ closed under direct summands. In Section 4, we first introduce the notion of $\mathscr{C}$-singularity category of $\mathscr{A}$, and then prove that if every object in $\mathscr{A}$ has finite $\mathscr{C}$-proper $\mathscr{C}$-dimension, then $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) = 0$; and the converse holds true provided that any $\mathscr{C}$-acyclic complex is acyclic. Assume further that $\mathscr{C}$ is admissible in $\mathscr{A}$. Let $\theta'_{\mathscr{C}} : \underline{\mathscr{G}(\mathscr{C})} \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ be the functor induced by the composition functor $\mathscr{G}(\mathscr{C}) \hookrightarrow \mathscr{A} \hookrightarrow D^b_{\mathscr{C}}(\mathscr{A}) \xrightarrow{\text{quotient}} D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. We call the quotient triangulated category $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) := D_{\mathscr{C}\text{-}sg}(\mathscr{A})/\operatorname{Im}\theta'_{\mathscr{C}}$ the $\mathscr{G}(\mathscr{C})$-*defect category* of $\mathscr{A}$. We first prove that $\operatorname{Im}\theta'_{\mathscr{C}} = D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}/K^b(\mathscr{C})$. Then we get the following result, in which the third assertion is a generalization of [10, Theorem 3.6].

THEOREM 1.1. *Let $\mathscr{C}$ be an admissible subcategory of $\mathscr{A}$. Then we have*
(1) *There exists a triangle-equivalence $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) \simeq D^b_{\mathscr{C}}(\mathscr{A})/D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$.*
(2) *$D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) \simeq D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ if and only if $\mathscr{C} = \mathscr{G}(\mathscr{C})$.*
(3) *$D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) = 0$ if and only if $\underline{\mathscr{G}(\mathscr{C})} \overset{\theta'_{\mathscr{C}}}{\simeq} D_{\mathscr{C}\text{-}sg}(\mathscr{A})$, and if and only if every object in $\mathscr{A}$ has finite $\mathscr{C}$-proper $\mathscr{G}(\mathscr{C})$-dimension.*

We remark that the equivalence in Theorem 1.1(2) between the defect and singularity categories is induced by the quotient functor. In Section 5, we apply the results obtained in Section 4 to module categories. Let A be an artin algebra over a commutative artin ring. We use $\mathscr{G}(A)$ to denote the subcategory of $\operatorname{mod} A$ consisting of Gorenstein projective modules. As an application of Theorem 1.1, we get that $D_{\mathscr{G}(A)\text{-}defect}(\operatorname{mod} A) = 0$ if and only if $D_{\mathscr{G}(A)\text{-}sg}(\operatorname{mod} A) = 0$, and if and only if A is Gorenstein; and $D_{\mathscr{G}(A)\text{-}defect}(\operatorname{mod} A) \simeq D_{sg}(\operatorname{mod} A)$ if and only if A is CM-free. Then we prove that under some condition, the Gorenstein derived equivalence of algebras induces the Gorenstein singularity equivalence. Finally, we establish the stability of Gorenstein defect categories as follows.

THEOREM 1.2. *There exists a triangle-equivalence*

$$D_{\mathscr{G}(\mathscr{G}(\operatorname{Mod} A))\text{-}defect}(\operatorname{Mod} A) \simeq D_{\mathscr{G}(\operatorname{Mod} A)\text{-}defect}(\operatorname{Mod} A),$$

where Mod A *is the category of left A-modules. Furthermore, if $\mathscr{G}(A)$ is contravariantly finite in* mod A, *then there exists a triangle-equivalence*

$$D_{\mathscr{G}(\mathscr{G}(A))\text{-}defect}(\operatorname{mod} A) \simeq D_{\mathscr{G}(A)\text{-}defect}(\operatorname{mod} A).$$

2. Preliminaries

Let

$$X^\bullet := \cdots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \cdots$$

be a complex in $C(\mathscr{A})$. The *n-th cycle* (resp. *boundary, homology*) of $X^\bullet$ is defined as $\operatorname{Ker} d_X^n$ (resp. $\operatorname{Im} d_X^{n-1}$, $\operatorname{Ker} d_X^n/\operatorname{Im} d_X^{n-1}$), and denoted by $Z^n(X^\bullet)$ (resp. $B^n(X^\bullet)$, $H^n(X^\bullet)$). Recall that $X^\bullet$ is called *acyclic* (or *exact*) if $H^i(X^\bullet) = 0$ for any $i \in \mathbf{Z}$ (the ring of integers). Let $f : X^\bullet \to Y^\bullet$ be a cochain map in $C(\mathscr{A})$. Then f is called a *quasi-isomorphism* if $H^i(f)$ is an isomorphism for any $i \in \mathbf{Z}$. We use $\operatorname{Con}(f)$ to denote the mapping cone of f. Then we have that f is a quasi-isomorphism if and only if $\operatorname{Con}(f)$ is acyclic.

Definition 2.1 ([5, 14, 20, 31]). Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$.

(1) A complex $X^\bullet$ in $C(\mathscr{A})$ is called *$\mathscr{C}$-acyclic* (or *$\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact*) if the complex $\operatorname{Hom}_{\mathscr{A}}(C, X^\bullet)$ is acyclic for any $C \in \mathscr{C}$. A cochain map $f : X^\bullet \to Y^\bullet$ in $C(\mathscr{A})$ is called a *$\mathscr{C}$-quasi-isomorphism* if $\operatorname{Hom}_{\mathscr{A}}(C, f)$ is a quasi-isomorphism for any $C \in \mathscr{C}$ (equivalently, $\operatorname{Con}(f)$ is $\mathscr{C}$-acyclic).

(2) Let $\mathscr{D}$ be a subcategory of $\mathscr{A}$ containing $\mathscr{C}$. A morphism $f : C \to D$ in $\mathscr{A}$ with $C \in \mathscr{C}$ and $D \in \mathscr{D}$ is called a *right $\mathscr{C}$-approximation* (or *$\mathscr{C}$-precover*) of D if f induces a $\mathscr{C}$-acyclic (but not necessarily acyclic) complex

$$0 \to \operatorname{Ker} f \to C \xrightarrow{f} D \to 0.$$

If each object of $\mathscr{D}$ has a right $\mathscr{C}$-approximation, then $\mathscr{C}$ is called *contravariantly finite* (or *precovering*) in $\mathscr{D}$.

(3) Let $\mathscr{D}$ be a subclass of objects in $\mathscr{A}$ and $M \in \mathscr{A}$. A *$\mathscr{C}$-proper $\mathscr{D}$-resolution* of M is a $\mathscr{C}$-quasi-isomorphism $f : D^\bullet \to M$, where $D^\bullet$ is a complex of objects in $\mathscr{D}$ with $D^n = 0$ for any $n > 0$, that is, it has an associated $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact complex

$$\cdots \to D^{-n} \to D^{-n+1} \to \cdots \to D^0 \xrightarrow{f} M \to 0.$$

The *$\mathscr{C}$-proper $\mathscr{D}$-dimension* of M, written $\mathscr{C}\mathscr{D}\text{-}\dim M$, is defined as the infimum of integers n such that there exists a $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact complex

$$0 \to D^{-n} \to D^{-n+1} \to \cdots \to D^0 \xrightarrow{f} M \to 0$$

in $\mathscr{A}$ with all D^i in $\mathscr{D}$. If no such an integer exists, then set $\mathscr{C}\mathscr{D}\text{-}\dim M = \infty$.

It should be remarked that all the above notions have dual versions.

Lemma 2.2 ([18, Lemma 2.4 and Proposition 2.6]).

(1) *Let $X^\bullet$ be a complex in $K(\mathscr{A})$. Then $X^\bullet$ is $\mathscr{C}$-acyclic if and only if the complex* $\operatorname{Hom}_{\mathscr{A}}(C^\bullet, X^\bullet)$ *is acyclic for any $C^\bullet \in K^-(\mathscr{C})$.*

(2) *Let* $X^\bullet$, $Y^\bullet$ *be complexes in* $K(\mathscr{A})$. *Then* $f : X^\bullet \to Y^\bullet$ *is a* $\mathscr{C}$*-quasi-isomorphism if and only if* $\mathrm{Hom}_{\mathscr{A}}(C^\bullet, f)$ *is a quasi-isomorphism for any* $C^\bullet \in K^-(\mathscr{C})$.

Lemma 2.3.
(1) *Let* $C^\bullet$ *be a complex in* $K^-(\mathscr{C})$ *and* $f : X^\bullet \to C^\bullet$ *a* $\mathscr{C}$*-quasi-isomorphism in* $C(\mathscr{A})$. *Then there exists a* $\mathscr{C}$*-quasi-isomorphism* $g : C^\bullet \to X^\bullet$ *such that* fg *is homotopic to* $\mathrm{id}_{C^\bullet}$.
(2) *Any* $\mathscr{C}$*-quasi-isomorphism between two complexes in* $K^-(\mathscr{C})$ *is a homotopy equivalence.*

Proof. (1) Consider the distinguished triangle

$$X^\bullet \xrightarrow{f} C^\bullet \to \mathrm{Con}(f) \to X^\bullet[1]$$

in $K(\mathscr{A})$ with $\mathrm{Con}(f)$ $\mathscr{C}$-acyclic. By applying the functor $\mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, -)$ to it, we get an exact sequence

$$\mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, X^\bullet) \xrightarrow{\mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, f)} \mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, C^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, \mathrm{Con}(f)).$$

By Lemma 2.2, we have

$$\mathrm{Hom}_{K(\mathscr{A})}(C^\bullet, \mathrm{Con}(f)) \cong H^0\, \mathrm{Hom}_{\mathscr{A}}(C^\bullet, \mathrm{Con}(f)) = 0.$$

So there exists a morphism $g : C^\bullet \to X^\bullet$ such that fg is homotopic to $\mathrm{id}_{C^\bullet}$. Notice that both f and fg are $\mathscr{C}$-quasi-isomorphisms, so is g.

(2) It is a consequence of (1). □

The following definition is cited from [13], see also [35] and [29].

Definition 2.4. Let $\mathscr{B}$ be an additive category. A *kernel-cokernel pair* (i, p) in $\mathscr{B}$ is a pair of composable morphisms

$$L \xrightarrow{i} M \xrightarrow{p} N$$

such that i is a kernel of p and p is a cokernel of i. Let ε be a class of kernel-cokernel pairs on $\mathscr{B}$ closed under isomorphisms, a kernel-cokernel pair (i, p) is called a *short exact sequence* (or *conflation*) if $(i, p) \in \varepsilon$, and we denote it by

$$L \overset{i}{\rightarrowtail} M \overset{p}{\twoheadrightarrow} N.$$

We call i an *admissible monic* (or *inflation*) and p an *admissible epic* (or *deflation*).

The pair $(\mathscr{B}, \varepsilon)$ (or simply $\mathscr{B}$) is called an *exact category* if it satisfies the following conditions.

[E0] For any object B in $\mathscr{B}$, the identity morphism id_B is both an admissible monic and an admissible epic.

[E1] The class of admissible monics is closed under compositions.

[E1^{op}] The class of admissible epics is closed under compositions.

[E2] The push-out of an admissible monic along an arbitrary morphism exists and yields an admissible monic.

[E2^{op}] The pull-back of an admissible epic along an arbitrary morphism exists and yields an admissible epic.

Definition 2.5 ([13]). Let $(\mathscr{B}, \varepsilon)$ be an exact category and $\mathscr{E}$ a subcategory of $\mathscr{B}$. $\mathscr{E}$ is called *closed under extensions* if for any short exact sequence

$$L \rightarrowtail M \twoheadrightarrow N$$

in $\mathscr{B}$, $L, N \in \mathscr{E}$ implies $M \in \mathscr{E}$; and $\mathscr{E}$ is called an *exact subcategory* of $\mathscr{B}$ if it is closed under extensions and equipped with the exact structure of $\mathscr{B}$ with all terms lie in $\mathscr{E}$.

Let $\mathscr{C} \subseteq \mathscr{D}$ be subcategories of $\mathscr{A}$ such that $\mathscr{C}$ is contravariantly finite in $\mathscr{D}$. Recall from [14] that $\mathscr{C}$ is called *admissible* in $\mathscr{D}$ if any right $\mathscr{C}$-approximation of $D \in \mathscr{D}$ is epic. It is trivial that if $\mathscr{C}$ is admissible in $\mathscr{A}$, then every $\mathscr{C}$-acyclic complex in $C(\mathscr{A})$ is acyclic.

Denote by ε the class of all $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact sequences of the form

$$0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$$

in $\mathscr{A}$. Note that this sequence itself is not necessarily exact. However, if $\mathscr{C}$ is admissible in $\mathscr{A}$, then any sequence in ε is exact.

Lemma 2.6. *If $\mathscr{C}$ is admissible in $\mathscr{A}$, then $(\mathscr{A}, \varepsilon)$ is an exact category.*

Proof. Both [E0] and [E1^{op}] are trivial, and [E2] and [E2^{op}] follow from [32, Lemma 2.3(1)]. Now we only need to prove [E1].

Let

$$0 \to L \xrightarrow{f} M \to N \to 0 \quad \text{and}$$
$$0 \to M \xrightarrow{g} P \to Q \to 0$$

lie in ε. Consider the following push-out diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, equal] & M \arrow[r] \arrow[d, "g"] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r, "gf"] & P \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
 & & Q \arrow[r, equal] \arrow[d] & Q \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

By [E2], the rightmost column lies in ε. For any $C \in \mathscr{C}$, applying the functor $(C,-) := \mathrm{Hom}_{\mathscr{A}}(C,-)$ to the above commutative diagram we get the following commutative diagram with exact columns and rows

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (C,L) \arrow[r, "{(C,f)}"] \arrow[d, equal] & (C,M) \arrow[r] \arrow[d, "{(C,g)}"] & (C,N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (C,L) \arrow[r, "{(C,gf)}"] & (C,P) \arrow[r] \arrow[d] & (C,K) \arrow[d] & \\
 & & (C,Q) \arrow[r, equal] \arrow[d] & (C,Q) \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

By the snake lemma, the morphism $(C,P) \to (C,K)$ is epic. Then we have that

$$0 \to L \xrightarrow{gf} P \to K \to 0$$

lies in ε, and [E1] follows. $\square$

Let $\mathscr{T}'$ be a triangulated subcategory of a triangulated category $\mathscr{T}$ and S the compatible multiplicative system determined by $\mathscr{T}'$. In the Verdier quotient $\mathscr{T}/\mathscr{T}'$ (see [39]), each morphism $f : X \to Y$ is given by an equivalence class of right fractions f/s or left fractions $s\backslash f$ as presented by $X \overset{s}{\Leftarrow} Z \overset{f}{\to} Y$ or $X \overset{f}{\to} Z \overset{s}{\Leftarrow} Y$, where the doubled arrow means $s \in S$.

3. $\mathscr{C}$-derived categories and Gorenstein categories

In this section, $\mathscr{C}$ is a subcategory of $\mathscr{A}$. It is known that $K^*(\mathscr{A})$ is a triangulated category for $* \in \{\text{blank}, -, b\}$. We use $K^*_{\mathscr{C}\text{-}ac}(\mathscr{A})$ to denote the full triangulated subcategory of $K^*(\mathscr{A})$ consisting of $\mathscr{C}$-acyclic complexes. It is easy to check that $K^*_{\mathscr{C}\text{-}ac}(\mathscr{A})$ is a thick triangulated subcategory of $K^*(\mathscr{A})$.

Definition 3.1 ([2, 14]). The Verdier quotient $D^*_{\mathscr{C}}(\mathscr{A}) := K^*(\mathscr{A})/K^*_{\mathscr{C}\text{-}ac}(\mathscr{A})$ is called the *relative derived category* of $\mathscr{A}$ with respect to $\mathscr{C}$ (the *$\mathscr{C}$-derived category* of $\mathscr{A}$ for short), where $* \in \{\text{blank}, -, b\}$.

Example 3.2.
(1) If $\mathscr{C} = \mathscr{P}$, then $D^*_{\mathscr{C}}(\mathscr{A})$ is the usual derived category $D^*(\mathscr{A})$.
(2) If $\mathscr{C} = \mathscr{G}(\mathscr{P})$ (the subcategory of $\mathscr{A}$ consisting of Gorenstein projective objects), then $D^*_{\mathscr{C}}(\mathscr{A})$ is the Gorenstein derived category $D^*_{\mathscr{G}(\mathscr{P})}(\mathscr{A})$ defined in [22].

(3) Let R be an arbitrary ring and $\mathscr{A} = \operatorname{Mod} R$ (the category of left R-modules). If $\mathscr{C} = \mathscr{P}\mathscr{P}(R)$ (the subcategory of $\operatorname{Mod} R$ consisting of pure projective modules), then $D^*_{\mathscr{C}}(\mathscr{A})$ is the pure derived category $D^*_{pur}(\mathscr{A})$ introduced and studied in [41].

Proposition 3.3 ([2, Remark 3.2]).
(1) *$D^-_{\mathscr{C}}(\mathscr{A})$ is a triangulated subcategory of $D_{\mathscr{C}}(\mathscr{A})$, and $D^b_{\mathscr{C}}(\mathscr{A})$ is a triangulated subcategory of $D^-_{\mathscr{C}}(\mathscr{A})$.*
(2) *For any $C^\bullet \in K^-(\mathscr{C})$ and $X^\bullet \in C(\mathscr{A})$, there exists an isomorphism of abelian groups*

$$\operatorname{Hom}_{K(\mathscr{A})}(C^\bullet, X^\bullet) \cong \operatorname{Hom}_{D_{\mathscr{C}}(\mathscr{A})}(C^\bullet, X^\bullet).$$

(3) *If $\mathscr{C}$ is admissible in $\mathscr{A}$, then the composition functor*

$$\mathscr{A} \to K^b(\mathscr{A}) \to D^b_{\mathscr{C}}(\mathscr{A})$$

is fully faithful, where the former functor is the inclusion functor and the latter one is the quotient functor.

Recall that an additive category $\mathscr{D}$ is called *idempotent complete* if any idempotent morphism $e : X \to X$ has a kernel.

Definition 3.4 ([1, 28]). Let $\mathscr{T}$ be a triangulated category with [1] the shift functor. A subcategory $\mathcal{S}$ of $\mathscr{T}$ is called *silting* if the following conditions are satisfied.
(1) $\operatorname{Hom}_{\mathscr{T}}(S_1, S_2[i]) = 0$ for any $S_1, S_2 \in \mathcal{S}$ and $i > 0$.
(2) $\mathscr{T} = \operatorname{thick}(\mathcal{S})$, where $\operatorname{thick}(\mathcal{S})$ denotes the smallest thick triangulated subcategory of $\mathscr{T}$ containing $\mathcal{S}$.

Lemma 3.5 ([28, Theorem 2.9]). *Let $\mathscr{T}$ be a triangulated category. If $\mathscr{T}$ has an idempotent complete silting subcategory, then $\mathscr{T}$ is idempotent complete.*

By Proposition 3.3(1)(2), we have that $K^b(\mathscr{C})$ is a triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$. It is of great interest to study whether or when $K^b(\mathscr{C})$ is a thick triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$. For triangulated categories $\mathscr{T}' \subseteq \mathscr{T}$, it is easy to see that if $\mathscr{T}'$ is idempotent complete, then $\mathscr{T}'$ is a thick triangulated subcategory of $\mathscr{T}$. However, it is still a non-trivial problem whether a triangulated category is idempotent complete or not. By Lemma 3.5, we have the following

Proposition 3.6. *If $\mathscr{C}$ is closed under direct summands, then $K^b(\mathscr{C})$ is idempotent complete; in this case, $K^b(\mathscr{C})$ is a thick triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$.*

Proof. It is clear that $\mathscr{C}$ is a silting subcategory of $K^b(\mathscr{C})$. Since $\mathscr{C}$ is closed under direct summands by assumption, we have that $\mathscr{C}$ is idempotent complete. It follows from Lemma 3.5 that $K^b(\mathscr{C})$ is also idempotent complete. $\square$

Set $K^{-,\mathscr{C}b}(\mathscr{C}) := \{X^\bullet \in K^-(\mathscr{C}) \mid \text{there exists } n \in \mathbf{Z} \text{ such that } H^i(\mathrm{Hom}_{\mathscr{A}}(C, X^\bullet)) = 0 \text{ for any } C \in \mathscr{C} \text{ and } i \leq n\}$.

Lemma 3.7 ([2, Theorem 3.5]). *If $\mathscr{C}$ is contravariantly finite in $\mathscr{A}$, then we have a triangle-equivalence $K^{-,\mathscr{C}b}(\mathscr{C}) \simeq D^b_{\mathscr{C}}(\mathscr{A})$.*

Put

$$\overline{\mathscr{C}} := \{X \in \mathscr{A} \mid X \text{ admits a } \mathscr{C}\text{-proper } \mathscr{C}\text{-resolution}\}.$$

For any $M \in \overline{\mathscr{C}}$, choose a $\mathscr{C}$-proper $\mathscr{C}$-resolution $C^\bullet \to M$ of M. Put

$$\mathrm{Ext}^n_{\mathscr{C}}(M, N) := H^n \,\mathrm{Hom}_{\mathscr{A}}(C^\bullet_M, N)$$

for any $N \in \mathscr{A}$ and $n \in \mathbf{Z}$. Note that $C^\bullet_M$ is isomorphic to M in $D_{\mathscr{C}}(\mathscr{A})$. By Proposition 3.3(1)(2), we have

$$\begin{aligned}\mathrm{Ext}^n_{\mathscr{C}}(M, N) &= H^n \,\mathrm{Hom}_{\mathscr{A}}(C^\bullet_M, N)\\ &= \mathrm{Hom}_{K(\mathscr{A})}(C^\bullet_M, N[n])\\ &\cong \mathrm{Hom}_{D_{\mathscr{C}}(\mathscr{A})}(C^\bullet_M, N[n])\\ &\cong \mathrm{Hom}_{D^b_{\mathscr{C}}(\mathscr{A})}(M, N[n]).\end{aligned}$$

The following is cited from [20, Section 8.2].

Lemma 3.8.
(1) *For any $M \in \overline{\mathscr{C}}$, the functor $\mathrm{Ext}^n_{\mathscr{C}}(M, -)$ does not depend on the choices of $\mathscr{C}$-proper $\mathscr{C}$-resolutions of M.*
(2) *For any $M \in \overline{\mathscr{C}}$ and $n < 0$, $\mathrm{Ext}^n_{\mathscr{C}}(M, -) = 0$ and there exists a natural equivalence $\mathrm{Hom}_{\mathscr{A}}(M, -) \cong \mathrm{Ext}^0_{\mathscr{C}}(M, -)$ whenever $\mathscr{C}$ is admissible in $\overline{\mathscr{C}}$.*
(3) *Let*

$$0 \to L \to M \to N \to 0$$

be a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact complex in $\overline{\mathscr{C}}$. Then it induces a long exact sequence

$$\begin{aligned}0 &\to \mathrm{Ext}^0_{\mathscr{C}}(N, -) \to \mathrm{Ext}^0_{\mathscr{C}}(M, -) \to \mathrm{Ext}^0_{\mathscr{C}}(L, -) \to \cdots \to \mathrm{Ext}^n_{\mathscr{C}}(N, -)\\ &\to \mathrm{Ext}^n_{\mathscr{C}}(M, -) \to \mathrm{Ext}^n_{\mathscr{C}}(L, -) \to \mathrm{Ext}^{n+1}_{\mathscr{C}}(N, -) \to \cdots.\end{aligned}$$

In the rest of this section, $\mathscr{C}$ is a subcategory of $\mathscr{A}$ closed under direct summands.

Definition 3.9 ([38]). The *Gorenstein category* $\mathscr{G}(\mathscr{C})$ of $\mathscr{C}$ is defined as the subcategory of $\mathscr{A}$ consisting of objects M such that there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact exact sequence

$$\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots \tag{3.1}$$

in $\mathscr{A}$ with all terms in $\mathscr{C}$ with $M \cong \operatorname{Im}(C^{-1} \to C^0)$. In this case, (3.1) is called a *complete $\mathscr{C}$-resolution* of M.

Remark 3.10.

(1) The definition above unifies the following notions: modules of Gorenstein dimension zero ([4]), Gorenstein projective modules, Gorenstein injective modules ([19]), V-Gorenstein projective modules, V-Gorenstein injective modules ([21]), and so on. In particular, if $\mathscr{C} = \mathscr{P}$, the objects in $\mathscr{G}(\mathscr{C})$ are called *Gorenstein projective*.

(2) By the definition of $\mathscr{G}(\mathscr{C})$, we have that $\mathscr{G}(\mathscr{C}) \subseteq \overline{\mathscr{C}}$ and for any $G \in \mathscr{G}(\mathscr{C})$ and $C \in \mathscr{C}$, there exists a natural equivalence $\operatorname{Ext}^0_{\mathscr{C}}(G, -) \cong \operatorname{Hom}_{\mathscr{A}}(G, -)$ and $\operatorname{Ext}^{\geq 1}_{\mathscr{C}}(G, C) = 0$.

(3) Every boundary (or cycle) of (3.1) lies in $\mathscr{G}(\mathscr{C})$.

Lemma 3.11 ([27, Theorem 4.6(2) and Proposition 4.7(5)]).

(1) *$\mathscr{G}(\mathscr{C})$ is closed under direct summands.*

(2) *Let*

$$0 \to L \to M \to N \to 0 \tag{3.2}$$

be an exact sequence in $\mathscr{A}$. If (3.2) *is both* $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$*-exact and* $\operatorname{Hom}_{\mathscr{A}}(-, \mathscr{C})$*-exact and any two of L, M and N lie in $\mathscr{G}(\mathscr{C})$, then so does the third one.*

Definition 3.12. A complex $X^\bullet \in D^b_{\mathscr{C}}(\mathscr{A})$ is said to have *finite $\mathscr{C}$-proper $\mathscr{G}(\mathscr{C})$-dimension* if $X^\bullet \cong G^\bullet$ in $D^b_{\mathscr{C}}(\mathscr{A})$ for some $G^\bullet \in K^b(\mathscr{G}(\mathscr{C}))$.

We use $D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$ to denote the subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$ consisting of complexes having finite $\mathscr{C}$-proper $\mathscr{G}(\mathscr{C})$-dimension. Inspired by [30, 6.2], we have the following description for $D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$.

Lemma 3.13. *Let $\mathscr{C}$ be admissible in $\mathscr{A}$. Then the following statements are equivalent.*

(1) *$X^\bullet \in D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$.*

(2) *There exists a $\mathscr{C}$-quasi-isomorphism $C^\bullet \to X^\bullet$ with $C^\bullet \in K^{-,\mathscr{C}b}(\mathscr{C})$, such that for some $n \in \mathbf{Z}$, $H^i(\operatorname{Hom}_{\mathscr{A}}(C, C^\bullet)) = 0$ and $Z^i(C^\bullet) \in \mathscr{G}(\mathscr{C})$ for any $C \in \mathscr{C}$ and $i \leq n$.*

Proof. (1) $\Rightarrow$ (2) Let $X^\bullet \in D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$. Then $X^\bullet \cong G^\bullet$ in $D^b_{\mathscr{C}}(\mathscr{A})$ for some $G^\bullet \in K^b(\mathscr{G}(\mathscr{C}))$. By Lemma 3.7, there exists a $\mathscr{C}$-quasi-isomorphism $C^\bullet \to X^\bullet$ with $C^\bullet \in K^{-,\mathscr{C}b}(\mathscr{C})$. So $C^\bullet \cong G^\bullet$ in $D^b_{\mathscr{C}}(\mathscr{A})$. Let $f/s : C^\bullet \overset{s}{\Leftarrow} Z^\bullet \xrightarrow{f} G^\bullet$ be an isomorphism in $D^b_{\mathscr{C}}(\mathscr{A})$, where s is a $\mathscr{C}$-quasi-isomorphism. Then f is a $\mathscr{C}$-quasi-isomorphism. By Lemma 2.3(1), there exists a $\mathscr{C}$-quasi-isomorphism $s' : C^\bullet \to Z^\bullet$. Then $fs' : C^\bullet \to G^\bullet$ is also a $\mathscr{C}$-quasi-isomorphism.

Put

$$G^\bullet := 0 \to G^m \to G^{m+1} \to \cdots \to G^n \to 0.$$

Then we get the following $\mathscr{C}$-acyclic complex

$$\mathrm{Con}(fs') := \cdots \to C^{m-1} \to C^m \to C^{m+1} \oplus G^m \to \cdots \to C^n \oplus G^{n-1} \\ \to C^{n+1} \oplus G^n \to C^{n+2} \to \cdots.$$

Note that $\mathrm{Con}(fs')$ is bounded above. Put $l := \sup\{i \mid \mathrm{Con}(fs')^i \neq 0\}$. Then the sequence

$$(3.3) \qquad 0 \to Z^{l-1}(\mathrm{Con}(fs')) \to \mathrm{Con}(fs')^{l-1} \to \mathrm{Con}(fs')^l \to 0$$

is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact. By Lemma 3.8, for any $C \in \mathscr{C}$, there exists an exact sequence

$$\mathrm{Hom}_{\mathscr{A}}(\mathrm{Con}(fs')^{l-1}, C) \to \mathrm{Hom}_{\mathscr{A}}(Z^{l-1}(\mathrm{Con}(fs')), C) \to \mathrm{Ext}^1_{\mathscr{C}}(\mathrm{Con}(fs')^l, C).$$

Because $\mathrm{Con}(fs')^l \in \mathscr{G}(\mathscr{C})$, by Remark 3.10(2) we have that $\mathrm{Ext}^1_{\mathscr{C}}(\mathrm{Con}(fs')^l, C) = 0$ and (3.3) is $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{C})$-exact. So $Z^{l-1}(\mathrm{Con}(fs')) \in \mathscr{G}(\mathscr{C})$ by Lemma 3.11(2). Iterating this process, we get that $Z^i(\mathrm{Con}(fs'))$ lies in $\mathscr{G}(\mathscr{C})$ for any $i \leq l-1$.

Note that $Z^i(\mathrm{Con}(fs')) \cong Z^{i+1}(C^\bullet)$ for any $i \leq m-2$, that is, $Z^i(C^\bullet) \in \mathscr{G}(\mathscr{C})$ for any $i \leq m-1$. Since $C^\bullet \in K^{-,\mathscr{C}b}(\mathscr{C})$, there exists $n_0 \in \mathbf{Z}$ such that $H^i(\mathrm{Hom}_{\mathscr{A}}(C, C^\bullet)) = 0$ for any $C \in \mathscr{C}$ and $i \leq n_0$. Put $n := \min\{m-1, n_0\}$. Then we have that $H^i(\mathrm{Hom}_{\mathscr{A}}(C, C^\bullet)) = 0$ and $Z^i(C^\bullet) \in \mathscr{G}(\mathscr{C})$ for any $C \in \mathscr{C}$ and $i \leq n$.

(2) $\Rightarrow$ (1) Assume that there exists a $\mathscr{C}$-quasi-isomorphism $C^\bullet \to X^\bullet$ with $C^\bullet \in K^{-,\mathscr{C}b}(\mathscr{C})$, such that for some $n \in \mathbf{Z}$, $H^i(\mathrm{Hom}_{\mathscr{A}}(C, C^\bullet)) = 0$ and $Z^i(C^\bullet) \in \mathscr{G}(\mathscr{C})$ for any $C \in \mathscr{C}$ and $i \leq n$. Then $C^\bullet$ is isomorphic to the complex

$$G^\bullet := 0 \to Z^n(C^\bullet) \to C^n \xrightarrow{d_C^n} C^{n+1} \to \cdots$$

in $D^b_{\mathscr{C}}(\mathscr{A})$. Because $G^\bullet \in K^b(\mathscr{G}(\mathscr{C}))$, we have $X^\bullet \in D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$. □

We now are in a position to prove the following result (compare with [30, Proposition 6.6]).

THEOREM 3.14. *Let $\mathscr{C}$ be admissible in $\mathscr{A}$. Then $D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$ is a triangulated subcategory of $D^b_{\mathscr{C}}(\mathscr{A})$.*

Proof. Clearly, $D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$ is closed under isomorphisms and shift functors [1] and [−1]. Now let

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1]$$

be a distinguished triangle in $D^b_{\mathscr{C}}(\mathscr{A})$ with $X^\bullet, Y^\bullet \in D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$. By Lemma 3.13, there exist $\mathscr{C}$-quasi-isomorphisms $C_X^\bullet \xrightarrow{f_X} X^\bullet$ and $C_Y^\bullet \xrightarrow{f_Y} Y^\bullet$ with $C_X^\bullet, C_Y^\bullet \in K^{-,\mathscr{C}b}(\mathscr{C})$ such that for some $n \in \mathbf{Z}$, we have that

$$H^i(\mathrm{Hom}_{\mathscr{A}}(C, C_X^\bullet)) = 0 = H^i(\mathrm{Hom}_{\mathscr{A}}(C, C_Y^\bullet))$$

and $Z^i(C_X^\bullet), Z^i(C_Y^\bullet) \in \mathscr{G}(\mathscr{C})$ for any $C \in \mathscr{C}$ and $i \leq n$. Since there exists a triangle-equivalence $F : K^{-,\mathscr{C}b}(\mathscr{C}) \to D^b_{\mathscr{C}}(\mathscr{A})$, there exists $f \in \mathrm{Hom}_{K(\mathscr{A})}(C_X^\bullet, C_Y^\bullet)$

such that $u = F(f)$. Embed f into a distinguished triangle

$$C_X^\bullet \xrightarrow{f} C_Y^\bullet \to \mathrm{Con}(f) \to X^\bullet[1] \tag{3.4}$$

in $K(\mathscr{A})$. Then $\mathrm{Con}(f) \in K^{-,\mathscr{C}b}(\mathscr{C})$ and

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1]$$

is the image of (3.4) under F. So $\mathrm{Con}(f) \cong Z^\bullet$ in $D^b_{\mathscr{C}}(\mathscr{A})$. Then by an argument similar to that in the proof of Lemma 3.13, there exists a $\mathscr{C}$-quasi-isomorphism $f_Z : \mathrm{Con}(f) \to Z^\bullet$.

In the following, we will show that there exists $n' \in \mathbf{Z}$ such that $H^i(\mathrm{Hom}_{\mathscr{A}}(C, \mathrm{Con}(f))) = 0$ and $Z^i(\mathrm{Con}(f)) \in \mathscr{G}(\mathscr{C})$ for any $C \in \mathscr{C}$ and $i \le n'$. Note that $f : C_X^\bullet \to C_Y^\bullet$ induces a morphism

$$\begin{array}{ccccccccccc}
G_X^\bullet := & \cdots & \longrightarrow & C_X^{n-2} & \longrightarrow & C_X^{n-1} & \longrightarrow & Z^n(C_X^\bullet) & \longrightarrow & 0 \\
\Big\downarrow \tilde{f} & & & \Big\downarrow f^{n-2} & & \Big\downarrow f^{n-1} & & \Big\downarrow \tilde{f}^n & & \\
G_Y^\bullet := & \cdots & \longrightarrow & C_Y^{n-2} & \longrightarrow & C_Y^{n-1} & \longrightarrow & Z^n(C_Y^\bullet) & \longrightarrow & 0
\end{array}$$

in $K(\mathscr{A})$, where $\tilde{f}^n$ is the restriction of f^n on $Z^n(C_X^\bullet)$. Since both $G_X^\bullet$ and $G_Y^\bullet$ are $\mathscr{C}$-acyclic and $Z^n(C_X^\bullet), Z^n(C_Y^\bullet) \in \mathscr{G}(\mathscr{C})$, we have that $\tilde{f}$ is a $\mathscr{C}$-quasi-isomorphism. We have the following $\mathscr{C}$-acyclic complex

$$\mathrm{Con}(\tilde{f}) := \cdots \to C_X^{n-1} \oplus C_Y^{n-2} \to Z^n(C_X^\bullet) \oplus C_Y^{n-1} \to Z^n(C_Y^\bullet) \to 0.$$

Since $\mathscr{C}$ is admissible, $\mathrm{Con}(\tilde{f})$ is acyclic. By Lemmas 3.8 and 3.11, we have $Z^i(\mathrm{Con}(\tilde{f})) \in \mathscr{G}(\mathscr{C})$ for any $i \le n-1$. Because $Z^i(\mathrm{Con}(\tilde{f}))$ coincides with $Z^i(\mathrm{Con}(f))$ for any $i \le n-2$, by putting $n' = n-2$ we have that $H^i(\mathrm{Hom}_{\mathscr{A}}(C, \mathrm{Con}(f))) = 0$ and $Z^i(\mathrm{Con}(f)) \in \mathscr{G}(\mathscr{C})$ for any $C \in \mathscr{C}$ and $i \le n'$. $\square$

Remind that ε is the class of all $\mathrm{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact sequences $0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$ in $\mathscr{A}$. We use ε' to denote the subclass of ε with all terms in $\mathscr{G}(\mathscr{C})$.

Proposition 3.15. *Let $\mathscr{C}$ be admissible in $\mathscr{A}$. Then $(\mathscr{G}(\mathscr{C}), \varepsilon')$ is an exact subcategory of $(\mathscr{A}, \varepsilon)$ with $\mathscr{C}$ the subcategory of (relative) projective and injective objects. In other words, $(\mathscr{G}(\mathscr{C}), \varepsilon')$ is a Frobenius category.*

Proof. By Lemma 2.6, we have that $(\mathscr{A}, \varepsilon)$ is an exact category. Now let

$$0 \to G' \xrightarrow{i} G \xrightarrow{p} G'' \to 0 \tag{3.5}$$

lie in ε. Then by Lemma 3.8, for any $C \in \mathscr{C}$ we have the following exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(G'', C) &\xrightarrow{\mathrm{Hom}_{\mathscr{A}}(p, C)} \mathrm{Hom}_{\mathscr{A}}(G, C) \\ &\xrightarrow{\mathrm{Hom}_{\mathscr{A}}(i, C)} \mathrm{Hom}_{\mathscr{A}}(G', C) \longrightarrow \mathrm{Ext}^1_{\mathscr{C}}(G'', C). \end{aligned} \tag{3.6}$$

If $G', G'' \in \mathcal{G}(\mathcal{C})$, then $\mathrm{Ext}^1_{\mathcal{C}}(G'', C) = 0$ and (3.5) is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{C})$-exact. It follows from Lemma 3.11(2) that $G \in \mathcal{G}(\mathcal{C})$ and $(\mathcal{G}(\mathcal{C}), \varepsilon')$ is an exact subcategory of $(\mathcal{A}, \varepsilon)$.

It is trivial that $\mathcal{C}$ is the subcategory of (relative) projective objects of $(\mathcal{G}(\mathcal{C}), \varepsilon')$. Now assume that (3.5) lies in ε'. Then for any morphism $f : G' \to C$, from (3.6) we get that there exists $g : G \to C$ such that $f = gi$. It implies that $\mathcal{C}$ is the subcategory of (relative) injective objects of $(\mathcal{G}(\mathcal{C}), \varepsilon')$. $\square$

We use $\underline{\mathcal{G}(\mathcal{C})}$ to denote stable category of $\mathcal{G}(\mathcal{C})$ modulo $\mathcal{C}$.

Remark 3.16. Let $\mathcal{C}$ be admissible in $\mathcal{A}$. By [23], we have that $\underline{\mathcal{G}(\mathcal{C})}$ is a triangulated category and every short exact sequence in ε' induces a distinguished triangle of $\underline{\mathcal{G}(\mathcal{C})}$ as follows.

Let

$$0 \to G' \xrightarrow{i} G \xrightarrow{p} G'' \to 0$$

be in ε'. Then we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G' \arrow[r, "i"] \arrow[d, equal] & G \arrow[r, "p"] \arrow[d, dashed] & G'' \arrow[r] \arrow[d, dashed, "w"] & 0 \\
0 \arrow[r] & G' \arrow[r] & C_{G'} \arrow[r] & TG' \arrow[r] & 0,
\end{tikzcd}$$

where

$$0 \to G' \to C_{G'} \to TG' \to 0$$

lies in ε' with $C_{G'} \in \mathcal{C}$ and $TG' \in \mathcal{G}(\mathcal{C})$. It induces a distinguished triangle

$$G' \xrightarrow{i} G \xrightarrow{p} G'' \xrightarrow{-w} TG''$$

in $\underline{\mathcal{G}(\mathcal{C})}$. For more details, we refer to [23, Chapter I].

4. $\mathcal{C}$-singularity categories and $\mathcal{G}(\mathcal{C})$-defect categories

In this section, $\mathcal{C}$ is a subcategory of $\mathcal{A}$ closed under direct summands. By Proposition 3.6, we have that $K^b(\mathcal{C})$ is a thick triangulated subcategory of $D^b_{\mathcal{C}}(\mathcal{A})$. It is of interest to consider the Verdier quotient $D^b_{\mathcal{C}}(\mathcal{A})/K^b(\mathcal{C})$. Compare the following definition with [31, Definition 4.1], in which A is a finite-dimensional algebra over a field, $\mathcal{A} = \mathrm{mod}\, A$ and $\mathcal{C} \subseteq \mathcal{A}$ is a subcategory which is admissible in $\mathcal{A}$ and closed under direct summands.

Definition 4.1. We call the Verdier quotient

$$D_{\mathcal{C}\text{-}sg}(\mathcal{A}) := D^b_{\mathcal{C}}(\mathcal{A})/K^b(\mathcal{C})$$

the *relative singularity category* of $\mathcal{A}$ with respect to $\mathcal{C}$ (the *$\mathcal{C}$-singularity category* of $\mathcal{A}$ for short).

Example 4.2.

(1) If $\mathscr{C} = \mathscr{P}$, then the $\mathscr{C}$-singularity category of $\mathscr{A}$ is the classical singularity category $D_{sg}(\mathscr{A}) = D^b(\mathscr{A})/K^b(\mathscr{P})$ introduced in [12].

(2) If $\mathscr{C} = \mathscr{G}(\mathscr{P})$, then the $\mathscr{C}$-singularity category of $\mathscr{A}$ is $D^b_{\mathscr{G}(\mathscr{P})}(\mathscr{A})/K^b(\mathscr{G}(\mathscr{P}))$, which is called the *Gorenstein singularity category* of $\mathscr{A}$ in [7].

Given a complex $X^\bullet$ and an integer $i \in \mathbf{Z}$, we use $\sigma^{\geq i}X^\bullet$ to denote the complex with X^j in the jth degree whenever $j \geq i$ and 0 elsewhere, and set $\sigma^{>i}X^\bullet := \sigma^{\geq i+1}X^\bullet$. Dually, $\sigma^{\leq i}X^\bullet$ and $\sigma^{<i}X^\bullet$ are defined. Recall that the *width* of $X^\bullet$, written $\omega(X^\bullet)$, is defined to be the cardinal of the set $\{X^i \neq 0 \mid i \in \mathbf{Z}\}$.

It is known that $D_{sg}(\mathscr{A})$ reflects the homological singularity of $\mathscr{A}$ in sense that $D_{sg}(\mathscr{A}) = 0$ if and only if every object in $\mathscr{A}$ has finite projective dimension. Analogically, we have the following result for $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$.

Proposition 4.3. *If $\mathscr{C}\mathscr{C}$-dim $M < \infty$ for any $M \in \mathscr{A}$, then $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) = 0$. The converse holds true provided that any $\mathscr{C}$-acyclic complex is acyclic.*

Proof. Assume that $\mathscr{C}\mathscr{C}$-dim $M < \infty$ for any $M \in \mathscr{A}$. We claim that for every $X^\bullet \in K^b(\mathscr{A})$, there exists a $\mathscr{C}$-quasi-isomorphism $C^\bullet_X \to X^\bullet$ such that $C^\bullet_X \in K^b(\mathscr{C})$. We proceed by induction on the width $\omega(X^\bullet)$ of $X^\bullet$.

If $\omega(X^\bullet) = 1$, then the claim follows directly.

Let $\omega(X^\bullet) \geq 2$ with $X^j \neq 0$ and $X^i = 0$ for any $i < j$. Put

$$X^\bullet_1 := X^j[-j-1], \quad X^\bullet_2 := \sigma^{>j}X^\bullet \quad \text{and} \quad g = d^j_X[-j-1].$$

We have a distinguished triangle

$$X^\bullet_1 \xrightarrow{g} X^\bullet_2 \to X^\bullet \to X^\bullet_1[1]$$

in $K^b(\mathscr{A})$. By the induction hypothesis, there exist $\mathscr{C}$-quasi-isomorphisms $f_{X_1} : C^\bullet_{X_1} \to X^\bullet_1$ and $f_{X_2} : C^\bullet_{X_2} \to X^\bullet_2$ with $C^\bullet_{X_1}, C^\bullet_{X_2} \in K^b(\mathscr{C})$. Then by Lemma 2.2, f_{X_2} induces an isomorphism

$$\mathrm{Hom}_{K^b(\mathscr{A})}(C^\bullet_{X_1}, C^\bullet_{X_2}) \cong \mathrm{Hom}_{K^b(\mathscr{A})}(C^\bullet_{X_1}, X^\bullet_2).$$

So there exists a morphism $f^\bullet : C^\bullet_{X_1} \to C^\bullet_{X_2}$, which is unique up to homotopy, such that $f_{X_2}f^\bullet = gf_{X_1}$. Put $C^\bullet_X := \mathrm{Con}(f^\bullet)$. We have the following distinguished triangle

$$C^\bullet_{X_1} \xrightarrow{f^\bullet} C^\bullet_{X_2} \to C^\bullet_X \to C^\bullet_{X_1}[1]$$

in $K^b(\mathscr{C})$. Then there exists a morphism $f_X : C^\bullet_X \to X^\bullet$ such that the following diagram commutes

$$\begin{tikzcd}
C^\bullet_{X_1} \arrow[r, "f^\bullet"] \arrow[d, "f_{X_1}"] & C^\bullet_{X_2} \arrow[r] \arrow[d, "f_{X_2}"] & C^\bullet_X \arrow[r] \arrow[d, dashed, "f_X"] & C^\bullet_{X_1}[1] \arrow[d, "f_{X_1}[1]"] \\
X^\bullet_1 \arrow[r, "g"] & X^\bullet_2 \arrow[r] & X^\bullet \arrow[r] & X^\bullet_1[1].
\end{tikzcd}$$

For any $C \in \mathscr{C}$ and $n \in \mathbf{Z}$, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
(C, C^\bullet_{X_1}[n]) \arrow[r] \arrow[d, "{(C, f_{X_1}[n])}"] & (C, C^\bullet_{X_2}[n]) \arrow[r] \arrow[d, "{(C, f_{X_2}[n])}"] & (C, C^\bullet_{X}[n]) \arrow[r] \arrow[d, dashed, "{(C, f_{X}[n])}"] & (C, C^\bullet_{X_1}[n+1]) \arrow[r] \arrow[d, "{(C, f_{X_1}[n+1])}"] & (C, C^\bullet_{X_2}[n+1]) \arrow[d, "{(C, f_{X_2}[n+1])}"] \\
(C, X_1^\bullet[n]) \arrow[r] & (C, X_2^\bullet[n]) \arrow[r] & (C, X^\bullet[n]) \arrow[r] & (C, X_1^\bullet[n+1]) \arrow[r] & (C, X_2^\bullet[n+1]),
\end{tikzcd}$$

where $(C, -)$ denotes the functor $\mathrm{Hom}_{K(\mathscr{A})}(C, -)$. Since f_{X_1} and f_{X_2} are $\mathscr{C}$-quasi-isomorphisms, $(C, f_{X_1}[n])$ and $(C, f_{X_2}[n])$ are isomorphisms, and hence so is $(C, f_X[n])$ for each n, that is, f_X is a $\mathscr{C}$-quasi-isomorphism. The claim is proved.

It follows from the claim that every object $X^\bullet$ of $D^b_{\mathscr{C}}(\mathscr{A})$ is isomorphic to some $C^\bullet_X$ of $K^b(\mathscr{C})$ in $D^b_{\mathscr{C}}(\mathscr{A})$. Thus $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) = 0$.

Conversely, assume that $D_{\mathscr{C}\text{-}sg}(\mathscr{A}) = 0$ and any $\mathscr{C}$-acyclic complex is acyclic. Let $M \in \mathscr{A}$. Then $M = 0$ in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ and $M \cong C^\bullet$ in $D^b_{\mathscr{C}}(\mathscr{A})$ for some $C^\bullet \in K^b(\mathscr{C})$. Let $f/s : C^\bullet \overset{s}{\Leftarrow} Z^\bullet \overset{f}{\rightarrow} M$ be an isomorphism in $D^b_{\mathscr{C}}(\mathscr{A})$, where s is a $\mathscr{C}$-quasi-isomorphism. Then f is a $\mathscr{C}$-quasi-isomorphism. By Lemma 2.3(1), there exists a $\mathscr{C}$-quasi-isomorphism $s' : C^\bullet \to Z^\bullet$. So $fs' : C^\bullet \to M$ is also a $\mathscr{C}$-quasi-isomorphism, and hence $H^i \mathrm{Hom}_{\mathscr{A}}(C, C^\bullet) = 0$ whenever $C \in \mathscr{C}$ and $i \neq 0$. Consider the truncation

$$C'^\bullet := \cdots \to C^{-2} \to C^{-1} \to Z^0(C^\bullet) \to 0$$

of $C^\bullet$. Then the composition

$$C'^\bullet \hookrightarrow C^\bullet \overset{fs'}{\rightarrow} M$$

is a $\mathscr{C}$-quasi-isomorphism. Notice that $C^\bullet \in K^b(\mathscr{C})$, so we may suppose $C^n \neq 0$ and $C^i = 0$ whenever $i > n$. Then we have a $\mathscr{C}$-acyclic complex

$$C''^\bullet := 0 \to Z^0(C^\bullet) \to C^0 \overset{d_C^0}{\rightarrow} C^1 \to \cdots \to C^n \to 0$$

with all C^i in $\mathscr{C}$. By assumption, $C''^\bullet$ is acyclic and hence split exact. Because $\mathscr{C}$ is closed under direct summands, $Z^0(C^\bullet) \in \mathscr{C}$ and $\mathscr{C}\mathscr{C}\text{-dim}\, M < \infty$. □

In the rest of this section, assume further that $\mathscr{C}$ is admissible in $\mathscr{A}$ to make sure that $\underline{\mathscr{G}(\mathscr{C})}$ is a triangulated category.

Let $\theta_{\mathscr{C}} : \mathscr{G}(\mathscr{C}) \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ be the composition of following three functors: the embedding functors: $\mathscr{G}(\mathscr{C}) \hookrightarrow \mathscr{A}$, $\mathscr{A} \hookrightarrow D^b_{\mathscr{C}}(\mathscr{A})$ and the quotient functor $D^b_{\mathscr{C}}(\mathscr{A}) \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. Then $\theta_{\mathscr{C}}$ induces a functor $\theta'_{\mathscr{C}} : \underline{\mathscr{G}(\mathscr{C})} \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. The following result is a slight generalization of [31, Propositions 4.9 and 4.10 and Theorem 4.12].

THEOREM 4.4.
(1) *$\theta'_{\mathscr{C}} : \underline{\mathscr{G}(\mathscr{C})} \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ is a fully faithful triangle functor.*
(2) *If $\mathscr{C}\mathscr{G}(\mathscr{C})$-dim $M < \infty$ for any $M \in \mathscr{A}$, then*

$$\theta'_{\mathscr{C}} : \underline{\mathscr{G}(\mathscr{C})} \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$$

is a triangle-equivalence.

Proof. The arguments in [31] remain valid in the setting here, so we omit them. $\square$

Put

$$\operatorname{Im}\theta'_{\mathscr{C}} := \{Y \in D_{\mathscr{C}\text{-}sg}(\mathscr{A}) \mid Y \cong \theta'_{\mathscr{C}}(X) \text{ for } X \in \underline{\mathscr{G}(\mathscr{C})}\}.$$

By Theorem 4.4, $\theta'_{\mathscr{C}}$ is full and $\operatorname{Im}\theta'_{\mathscr{C}}$ is a triangulated subcategory of $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. Motivated by Bergh, Oppermann and Jorgensen [10], we introduce the following

Definition 4.5. We call the quotient triangulated category

$$D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) := D_{\mathscr{C}\text{-}sg}(\mathscr{A})/\operatorname{Im}\theta'_{\mathscr{C}}$$

the $\mathscr{G}(\mathscr{C})$*-defect category* of $\mathscr{A}$.

By Theorem 4.4(2), we have that if $\mathscr{C}\mathscr{G}(\mathscr{C})\text{-dim}\, M < \infty$ for any $M \in \mathscr{A}$, then $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) = 0$. In general, we have the following result (cf. [30, Remark 6.8]).

Proposition 4.6. $\operatorname{Im}\theta'_{\mathscr{C}} = D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}/K^b(\mathscr{C})$.

Proof. It is clear that $\theta_{\mathscr{C}}(G) \in D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$ for any $G \in \mathscr{G}(\mathscr{C})$. So we have

$$\operatorname{Im}\theta'_{\mathscr{C}} \subseteq D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}/K^b(\mathscr{C}).$$

Now let $X^{\bullet} \in D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$. Then by Lemma 3.13, there exists a $\mathscr{C}$-quasi-isomorphism $C^{\bullet} \to X^{\bullet}$ with $C^{\bullet} \in K^{-,\mathscr{C}b}(\mathscr{C})$ such that for some $n \in \mathbf{Z}$, $H^i(\operatorname{Hom}_{\mathscr{A}}(C, C^{\bullet})) = 0$ and $Z^i(C^{\bullet}) \in \mathscr{G}(\mathscr{C})$ for any $C \in \mathscr{C}$ and $i \le n$. So $C^{\bullet}$ is isomorphic to the complex

$$0 \to Z^n(C^{\bullet}) \to C^n \to C^{n+1} \to \cdots$$

in $D^b_{\mathscr{C}}(\mathscr{A})$. It induces a distinguished triangle in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ of the following form

$$Z^n(C^{\bullet})[-n] \to \sigma^{\ge n}(C^{\bullet}) \to C^{\bullet} \to Z^n(C^{\bullet})[-n+1].$$

Then $C^{\bullet} \cong Z^n(C^{\bullet})[-n+1]$ in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. We may assume $n \ll 0$. Because $Z^n(C^{\bullet}) \in \mathscr{G}(\mathscr{C})$, we have a $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C}, -)$-exact exact sequence

$$0 \to Z^n(C^{\bullet}) \to C^0 \to C^1 \to \cdots \to C^{-n} \to G \to 0$$

with $G \in \mathscr{G}(\mathscr{C})$ and all $C^i \in \mathscr{C}$. It follows that $G \cong Z^n(C^{\bullet})[-n+1]$ and

$$X^{\bullet} \cong C^{\bullet} \cong Z^n(C^{\bullet})[-n+1] \cong G$$

in $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. So $X^{\bullet} \cong \theta'_{\mathscr{C}}(G)$ and

$$D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}/K^b(\mathscr{C}) \subseteq \operatorname{Im}\theta'_{\mathscr{C}}. \qquad \square$$

By Proposition 4.6 and Theorem 3.14, we have the following result, which can be viewed as a generalization of [30, Theorem 6.7(1)].

THEOREM 4.7. *There exists a commutative diagram*

$$\begin{tikzcd}
\underline{\mathscr{G}(\mathscr{C})} \arrow[r, "\theta'_{\mathscr{C}}"] \arrow[d, "\simeq"] & D_{\mathscr{C}\text{-}sg}(\mathscr{A}) \arrow[r, "\text{quotient}"] \arrow[d, equal] & D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) \arrow[d, "\simeq"] \\
D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}/K^b(\mathscr{C}) \arrow[r, "\text{embedding}"] & D_{\mathscr{C}\text{-}sg}(\mathscr{A}) \arrow[r, "\text{quotient}"] & D^b_{\mathscr{C}}(\mathscr{A})/D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}
\end{tikzcd}$$

of triangulated categories with all columns triangle-equivalences.

By definition, we have that $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A})$ is a Verdier quotient of $D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. The extreme case is $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) \simeq D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ or $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) = 0$. In the following, we will characterize when one of these two extreme cases happens. We first give the following

THEOREM 4.8. *$D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) \simeq D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ if and only if $\mathscr{C} = \mathscr{G}(\mathscr{C})$.*

Proof. The sufficiency is trivial. Conversely, let $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) \simeq D_{\mathscr{C}\text{-}sg}(\mathscr{A})$. It follows that $\operatorname{Im}\theta'_{\mathscr{C}} = 0$. As $\theta'_{\mathscr{C}}$ is fully faithful, we get $\underline{\mathscr{G}(\mathscr{C})} \simeq \operatorname{Im}\theta'_{\mathscr{C}} = 0$, and hence $\mathscr{C} = \mathscr{G}(\mathscr{C})$ as desired. □

To characterize when $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) = 0$, we need the following

LEMMA 4.9. *Let $M \in \mathscr{A}$ be an object. Then $M \in D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$ if and only if $\mathscr{C}\mathscr{G}(\mathscr{C})$-dim $M < \infty$.*

Proof. The sufficiency is trivial. Conversely, let $M \in D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$. Then $M \cong G^\bullet$ in $D^b_{\mathscr{C}}(\mathscr{A})$ for some $G^\bullet \in K^b(\mathscr{G}(\mathscr{C}))$. Let $C^\bullet \to M$ be a $\mathscr{C}$-proper $\mathscr{C}$-resolution of M. Then $C^\bullet \cong G^\bullet$ in $D^b_{\mathscr{C}}(\mathscr{A})$ and there exists a $\mathscr{C}$-quasi-isomorphism $f : C^\bullet \to G^\bullet$. By an argument similar to that in the proof of Lemma 3.13, there exists $n > 0$ such that $Z^{-n+1}(C^\bullet) \in \mathscr{G}(\mathscr{C})$. Thus the complex

$$0 \to Z^{-n+1}(C^\bullet) \to C^{-n+1} \to \cdots \to C^{-1} \to C^0 \to M \to 0$$

is $\mathscr{C}$-acyclic, and therefore $\mathscr{C}\mathscr{G}(\mathscr{C})$-dim $M < \infty$. □

We now are in a position to prove the following result, which means that the converse of Theorem 4.4(2) holds true.

THEOREM 4.10. *The following statements are equivalent.*
(1) *$D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) = 0$.*
(2) *$\theta'_{\mathscr{C}} : \underline{\mathscr{G}(\mathscr{C})} \to D_{\mathscr{C}\text{-}sg}(\mathscr{A})$ is a triangle-equivalence.*
(3) *$\mathscr{C}\mathscr{G}(\mathscr{C})$-dim $M < \infty$ for any $M \in \mathscr{A}$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is trivial. The implication (3) $\Rightarrow$ (1) follows from Theorem 4.4(2). In the following, we prove (1) $\Rightarrow$ (3).

Let $M \in \mathscr{A}$. Because $D_{\mathscr{G}(\mathscr{C})\text{-}defect}(\mathscr{A}) = 0$ by (1), we have $M \in D^b_{\mathscr{C}}(\mathscr{A})_{f\mathscr{C}\mathscr{G}(\mathscr{C})d}$ by Theorem 4.7. Then we have $\mathscr{C}\mathscr{G}(\mathscr{C})$-dim $M < \infty$ by Lemma 4.9. □

For an object $M \in \mathscr{A}$, the $\mathscr{G}(\mathscr{C})$-*dimension* of M, denoted by $\mathscr{G}(\mathscr{C})$-dim M, is defined as the infimum of integers n such that there exists an exact sequence

$$0 \to G^{-n} \to G^{-n+1} \to \cdots \to G^0 \to M \to 0$$

in $\mathscr{A}$ with all G^i in $\mathscr{G}(\mathscr{C})$. Set $\mathscr{G}(\mathscr{C})$-dim $M = \infty$ if no such an integer exists. As an immediate consequence of Theorem 4.10, we get the Buchweitz-Happel theorem and its converse.

Corollary 4.11 ([12, Theorem 4.4.1], [24, Theorem 4.6] and [10, Theorem 3.6]). *The following statements are equivalent.*

(1) $D_{\mathscr{G}(\mathscr{P})\text{-}defect}(\mathscr{A}) = 0$.

(2) $\theta'_{\mathscr{P}} : \underline{\mathscr{G}(\mathscr{P})} \to D_{sg}(\mathscr{A})$ *is a triangle-equivalence.*

(3) $\mathscr{G}(\mathscr{P})$-dim $M < \infty$ *for any* $M \in \mathscr{A}$.

For an object M in $\mathscr{A}$, we use $\mathscr{G}\mathscr{G}(\mathscr{C})$-dim M to denote the $\mathscr{G}(\mathscr{C})$-proper $\mathscr{G}(\mathscr{C})$-dimension of M. The following result generalizes [6, Proposition 4.8], which states that under some condition, the $\mathscr{C}$-proper $\mathscr{G}(\mathscr{C})$-dimension, $\mathscr{G}(\mathscr{C})$-proper $\mathscr{G}(\mathscr{C})$-dimension and $\mathscr{G}(\mathscr{C})$-dimension of an object in $\mathscr{A}$ are identical.

Proposition 4.12. *If $\mathscr{C}$ is self-orthogonal, then for any $M \in \mathscr{A}$, we have*

$$\mathscr{C}\mathscr{G}(\mathscr{C})\text{-dim } M = \mathscr{G}\mathscr{G}(\mathscr{C})\text{-dim } M = \mathscr{G}(\mathscr{C})\text{-dim } M.$$

Proof. Let $M \in \mathscr{A}$. Because $\mathscr{C} \subseteq \mathscr{G}(\mathscr{C})$, we have $\mathscr{C}\mathscr{G}(\mathscr{C})$-dim $M \leq \mathscr{G}\mathscr{G}(\mathscr{C})$-dim M. Notice that $\mathscr{C}$ is admissible, so any $\mathscr{C}$-acyclic complex is acyclic. Thus we have $\mathscr{G}(\mathscr{C})$-dim $M \leq \mathscr{C}\mathscr{G}(\mathscr{C})$-dim M.

In the following, we will prove $\mathscr{G}\mathscr{G}(\mathscr{C})$-dim $M \leq \mathscr{G}(\mathscr{C})$-dim M. Without loss of generality, assume $\mathscr{G}(\mathscr{C})$-dim $M = n < \infty$. Then by [27, Theorem 5.8], there exists an exact sequence

$$0 \to C^{-n} \to C^{-n+1} \to \cdots \to C^{-1} \to G^0 \to M \to 0 \tag{4.2}$$

in $\mathscr{A}$ with $G^0 \in \mathscr{G}(\mathscr{C})$ and all $C^i \in \mathscr{C}$. Because $\mathscr{C}$ is self-orthogonal by assumption, it follows from [27, Lemma 5.7] that (4.2) is $\mathscr{G}(\mathscr{C})$-acyclic and $\mathscr{G}\mathscr{G}(\mathscr{C})$-dim $M \leq n$. □

5. Some applications to module categories

In this section, A is an artin algebra over a commutative artin ring, Mod A is the category of left A-modules and mod A is the category of finitely generated left A-modules. We use Proj A (resp. proj A) to denote the subcategory of Mod A (resp. mod A) consisting of projective modules. For the sake of simplicity, we write $\mathscr{G}(A) := \mathscr{G}(\text{mod } A)$.

Recall from [10, 7] that

$$D_{\mathscr{G}(A)\text{-}defect}(\mathrm{mod}\,A) := D_{sg}(\mathrm{mod}\,A)/\mathrm{Im}\,\theta'_{\mathrm{proj}\,A} \quad \text{and}$$
$$D_{\mathscr{G}(\mathrm{Mod}\,A)\text{-}defect}(\mathrm{Mod}\,A) := D_{sg}(\mathrm{Mod}\,A)/\mathrm{Im}\,\theta'_{\mathrm{Proj}\,A}$$

are called the *Gorenstein defect categories* of A. Recall that A is called *Gorenstein* if the left and right self-injective dimensions of A are finite. For a module $M \in \mathrm{mod}\,A$, we use $\mathscr{G}\mathscr{G}(A)$-dim M to denote the $\mathscr{G}(A)$-proper $\mathscr{G}(A)$-dimension of M. The following result gives some equivalent characterizations for A being Gorenstein.

Corollary 5.1. *The following statements are equivalent.*
(1) *A is Gorenstein.*
(2) $D_{\mathscr{G}(A)\text{-}sg}(\mathrm{mod}\,A) = 0$.
(3) $D_{\mathscr{G}(A)\text{-}defect}(\mathrm{mod}\,A) = 0$.
(4) $\theta'_{\mathrm{proj}\,A} : \underline{\mathscr{G}(A)} \to D_{sg}(\mathrm{mod}\,A)$ *is a triangle-equivalence.*

Proof. It follows from [26, Theorem] that A is Gorenstein if and only if $\mathscr{G}(A)$-dim $M < \infty$ for any $M \in \mathrm{mod}\,A$. So by Corollary 4.11, we have (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

By [6, Proposition 4.8] (cf. Proposition 4.12), we have $\mathscr{G}(A)$-dim $M =$ $\mathscr{G}\mathscr{G}(A)$-dim M for any $M \in \mathrm{mod}\,A$. Then it follows from [26, Theorem] again that A is Gorenstein if and only if $\mathscr{G}\mathscr{G}(A)$-dim $M < \infty$ for any $M \in \mathrm{mod}\,A$. Because any $\mathscr{G}(A)$-acyclic complex is acyclic, the equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 4.3. □

Recall from [16] that A is called *CM-free* if $\mathscr{G}(A) = \mathrm{proj}\,A$. We have the following

Corollary 5.2. $D_{\mathscr{G}(A)\text{-}defect}(\mathrm{mod}\,A) \simeq D_{sg}(\mathrm{mod}\,A)$ *if and only if A is CM-free.*

Proof. Putting $\mathscr{A} = \mathrm{mod}\,A$ and $\mathscr{C} = \mathrm{proj}\,A$ in Theorem 4.8, the assertion follows. □

Recall from [9] that A is said to be of *finite Cohen-Macaulay type*, *CM-finite* for short, if $\mathscr{G}(A)$ is of finite representation type. Let A be CM-finite and $\{G_1, G_2, \dots, G_n\}$ the set of all pairwise non-isomorphic indecomposable Gorenstein projective modules in mod A. Put

$${}_AG := G_1 \oplus G_2 \oplus \cdots \oplus G_n.$$

Then $\mathscr{G}(A) = \mathrm{add}\,{}_AG$ (the full subcategory of mod A consisting of direct summands of finite direct sums of ${}_AG$). Recall from [2] that the opposite of the endomorphism algebra $\Gamma_A := \mathrm{End}({}_AG)^{op}$ of ${}_AG$ is called the *Gorenstein Auslander algebra* of A; and two artin algebras A and B are called *Gorenstein derived equiv-*

alent if there exists a triangle-equivalence

$$D^b_{\mathscr{G}(A)}(\text{mod } A) \simeq D^b_{\mathscr{G}(B)}(\text{mod } B).$$

The following result shows that under some condition, the Gorenstein derived equivalence of algebras induces the Gorenstein singularity equivalence.

COROLLARY 5.3. *Let A and B be two CM-finite artin algebras. If A and B are Gorenstein derived equivalent, then there exists a triangle-equivalence*

$$D_{\mathscr{G}(A)\text{-}sg}(\text{mod } A) \simeq D_{\mathscr{G}(B)\text{-}sg}(\text{mod } B).$$

In this case, we call A and B Gorenstein singularity equivalent.

Proof. Let A and B be two CM-finite artin algebras. Then there exist ${}_AG \in \mathscr{G}(A)$ and ${}_BH \in \mathscr{G}(B)$ such that $\mathscr{G}(A) = \text{add }{}_AG$ and $\mathscr{G}(B) = \text{add }{}_BH$. Let Γ_A and Γ_B be the Gorenstein Auslander algebras of A and B respectively. If A and B are Gorenstein derived equivalent, then we have that Γ_A and Γ_B are derived equivalent by [2, Corollary 3.7]. It follows from [37, Theorem 6.4] that there exists a triangle-equivalence

$$K^b(\text{proj } \Gamma_A) \simeq K^b(\text{proj } \Gamma_B).$$

Note that

$$\text{add }{}_AG \simeq \text{proj } \Gamma_A \quad \text{and} \quad \text{add }{}_BH \simeq \text{proj } \Gamma_B$$

are equivalences of additive categories by [3, Chapter VI, Lemma 3.1(b)]. So there exists a triangle-equivalence

$$K^b(\mathscr{G}(A)) \simeq K^b(\mathscr{G}(B)),$$

and hence there exists a triangle-equivalence

$$D_{\mathscr{G}(A)\text{-}sg}(\text{mod } A) \simeq D_{\mathscr{G}(B)\text{-}sg}(\text{mod } B). \qquad \square$$

As an application of Corollaries 5.1 and 5.3, we have the following

COROLLARY 5.4 ([2, Proposition 5.1.3]). *Let A and B be two CM-finite artin algebras. If A and B are Gorenstein derived equivalent, then A is Gorenstein if and only if so is B.*

Proof. Let A and B be two CM-finite artin algebras. If A and B are Gorenstein derived equivalent, then there exists a triangle-equivalence

$$D_{\mathscr{G}(A)\text{-}sg}(\text{mod } A) \simeq D_{\mathscr{G}(B)\text{-}sg}(\text{mod } B)$$

by Corollary 5.3. Now the assertion follows from Corollary 5.1. $\square$

Putting $\mathscr{C} = \mathscr{G}(A)$ and $\mathscr{C} = \mathscr{G}(\text{Mod } A)$ in Definition 4.5 respectively, then we have

$$D_{\mathscr{G}(\mathscr{G}(A))\text{-}defect}(\mathrm{mod}\ A) = D_{\mathscr{G}(A)\text{-}sg}(\mathrm{mod}\ A)/\mathrm{Im}\ \theta'_{\mathscr{G}(A)} \quad \text{and}$$
$$D_{\mathscr{G}(\mathscr{G}(\mathrm{Mod}\ A))\text{-}defect}(\mathrm{Mod}\ A) = D_{\mathscr{G}(\mathrm{Mod}\ A)\text{-}sg}(\mathrm{Mod}\ A)/\mathrm{Im}\ \theta'_{\mathscr{G}(\mathrm{Mod}\ A)}.$$

Inspired by the stability of Gorenstein categories (see [27, Theorem 4.1]), we get the stability of Gorenstein defect categories as follows.

THEOREM 5.5.
(1) *There exists a triangle-equivalence*

$$D_{\mathscr{G}(\mathscr{G}(\mathrm{Mod}\ A))\text{-}defect}(\mathrm{Mod}\ A) \simeq D_{\mathscr{G}(\mathrm{Mod}\ A)\text{-}defect}(\mathrm{Mod}\ A).$$

(2) *If* $\mathscr{G}(A)$ *is contravariantly finite in* mod A, *then there exists a triangle-equivalence*

$$D_{\mathscr{G}(\mathscr{G}(A))\text{-}defect}(\mathrm{mod}\ A) \simeq D_{\mathscr{G}(A)\text{-}defect}(\mathrm{mod}\ A).$$

Proof. Because $\mathscr{G}(\mathscr{G}(\mathrm{Mod}\ A)) = \mathscr{G}(\mathrm{Mod}\ A)$ and $\mathscr{G}(\mathscr{G}(A)) = \mathscr{G}(A)$ by [27, Theorem 4.1], we have

$$D_{\mathscr{G}(\mathscr{G}(\mathrm{Mod}\ A))\text{-}defect}(\mathrm{Mod}\ A) \simeq D_{\mathscr{G}(\mathrm{Mod}\ A)\text{-}sg}(\mathrm{Mod}\ A) \quad \text{and}$$
$$D_{\mathscr{G}(\mathscr{G}(A))\text{-}defect}(\mathrm{mod}\ A) \simeq D_{\mathscr{G}(A)\text{-}sg}(\mathrm{mod}\ A)$$

by Theorem 4.8. On the other hand, note that $\mathscr{G}(\mathrm{Mod}\ A)$ is contravariantly finite in Mod A by [8, Theorem 3.5] (or cf. [11, Proposition 8.10]). So there exist triangle-equivalences

$$D_{\mathscr{G}(\mathrm{Mod}\ A)\text{-}sg}(\mathrm{Mod}\ A) \simeq D_{\mathscr{G}(\mathrm{Mod}\ A)\text{-}defect}(\mathrm{Mod}\ A) \quad \text{and}$$
$$D_{\mathscr{G}(A)\text{-}sg}(\mathrm{mod}\ A) \simeq D_{\mathscr{G}(A)\text{-}defect}(\mathrm{mod}\ A)$$

by [7, Theorem 4.3] and assumption, and hence we have

$$D_{\mathscr{G}(\mathscr{G}(\mathrm{Mod}\ A))\text{-}defect}(\mathrm{mod}\ A) \simeq D_{\mathscr{G}(\mathrm{Mod}\ A)\text{-}defect}(\mathrm{Mod}\ A) \quad \text{and}$$
$$D_{\mathscr{G}(\mathscr{G}(A))\text{-}defect}(\mathrm{mod}\ A) \simeq D_{\mathscr{G}(A)\text{-}defect}(\mathrm{mod}\ A). \qquad \square$$

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