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On F-Almost Split Sequences

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Abstract Let Λ be an Artinian algebra and F an additive subbifunctor of $\mathrm{Ext}^1_\Lambda(-,-)$ having enough projectives and injectives. We prove that the dualizing subvarieties of mod Λ closed under F-extensions have F-almost split sequences. Let T be an F-cotilting module in mod Λ and S a cotilting module over $\Gamma = \mathrm{End}(T)$. Then $\mathrm{Hom}(-,T)$ induces a duality between F-almost split sequences in ${}^{\perp_F}T$ and almost split sequences in ${}^{\perp}S$, where $\mathrm{add}_\Gamma S = \mathrm{Hom}_\Lambda(\mathscr{P}(F),T)$. Let Λ be an F-Gorenstein algebra, T a strong F-cotilting module and $0 \to A \to B \to C \to 0$ an F-almost split sequence in ${}^{\perp_F}T$. If the injective dimension of S as a Γ-module is equal to d, then $C \cong (\Omega^{-d}_{CM}\Omega^d D\mathrm{Tr}A^*)^*$, where $(-)^* = \mathrm{Hom}(-,T)$. In addition, if the F-injective dimension of A is equal to d, then $A \cong \Omega^{-d}_{CM_F} D\Omega^{-d}_{F^{\mathrm{op}}}\mathrm{Tr}C \cong \Omega^{-d}_{CM_F}\Omega^d_F D\mathrm{Tr}C$.

Keywords F-almost split sequences, almost split sequences, F-Gorenstein algebras

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1 Introduction

Throughout this paper, all algebras are Artinian algebras over a commutative Artinian ring R and D is the ordinary duality, that is, $D = \mathrm{Hom}_R(-, I(R/J(R)))$, where $J(R)$ is the Jacobson radical of R and $I(R/J(R))$ is the injective envelope of $R/J(R)$. For an algebra Λ, we use mod Λ to denote the subcategory of finitely generated left Λ-modules. A subcategory of mod Λ means a full subcategory closed under isomorphisms, finite direct sums and summands.

It is well known that the notion of almost split sequences, which was introduced by Auslander and Reiten in [1], is a very important research object in representation theory of Artinian algebras. Auslander and Smalø in [2] established the existence theorem of almost split sequences in subcategories. They proved that a subcategory $\mathscr{C}$ of mod Λ has almost split sequences provided that $\mathscr{C}$ is functorially finite and closed under extensions.

Relative homological algebra was studied by Hochschild in [3] and Butler and Horrocks in [4], and was applied to study systematically the representation theory of Artinian algebras by Auslander and Solberg in [5–8]. Auslander and Solberg's work had stimulated several further investigations (see [9–14]). Let Λ be a Gorenstein algebra. Auslander and Reiten in [15] studied

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the structure of minimal right ${}^{\perp}\Lambda$-approximations of modules, and then gave a description of the non-projective part of the first term in an almost split sequence in ${}^{\perp}\Lambda$, where ${}^{\perp}\Lambda = \{C \in \text{mod } \Lambda \mid \text{Ext}^i_{\Lambda}(C, \Lambda) = 0 \text{ for any } i \geq 1\}$. Huang in [12] generalized the first result of Auslander and Reiten mentioned above to relative homology and gave the structure of minimal right F-${}^{\perp}T$-approximations of modules over an F-Gorenstein algebra Λ, where F is an additive subbifunctor of $\text{Ext}^1_{\Lambda}(-,-) : \text{mod } \Lambda^{\text{op}} \times \text{mod } \Lambda \to \text{Ab}$ (the category of abelian groups) having enough projectives and injectives, T is an F-cotilting module and ${}^{\perp_F}T = \{C \in \text{mod } \Lambda \mid \text{Ext}^i_F(C, \Lambda) = 0 \text{ for any } i \geq 1\}$. Motivated by these results, we introduce in this paper the notion of F-almost split sequences and study the relationship between F-almost split sequences and almost split sequences. In particular, we study the structure of F-almost split sequences over F-Gorenstein algebras.

In Section 2, we give the definition of F-almost split sequences. In Section 3, we establish the existence theorem of F-almost split sequences in subcategories, and give the relationship between F-almost split sequences and almost split sequences in terms of the properties of F-cotilting modules and cotilting modules. Then in Sections 4 and 5 we apply the obtained results to studying the structure of the last term and the first term of F-almost split sequences in ${}^{\perp_F}T$ over an F-Gorenstein algebra Λ, respectively, where T is a strong F-cotilting Λ-module. In particular, we remark that because the proofs of some results in this paper are analogies to that in the standard module theory, we only list the corresponding references in the statements of these results but omit the proofs.

2 *F*-Almost Split Sequences

In this section, we introduce the notion of F-almost split sequences and give some basic properties. Recall that a morphism $f : B \to C$ is called a *split epimorphism* if the identity homomorphism of C factors through f. Dually, a morphism $g : A \to B$ is called a *split monomorphism* if the identity homomorphism of A factors through g. A morphism $f : B \to C$ is called *right almost split* if it satisfies the following conditions: (1) It is not a split epimorphism; and (2) Any morphism $X \to C$ which is not a split epimorphism factors through f. Dually, a morphism $g : A \to B$ is called *left almost split* if it satisfies the following conditions: (1) It is not a split monomorphism; and (2) Any morphism $X \to C$ which is not a split monomorphism factors through g.

From now on, for an algebra Λ, we always assume that F is an additive subbifunctor of $\text{Ext}^1_{\Lambda}(-,-)$ having enough projectives and injectives. Recall from [5] that an exact sequence $\eta : 0 \to A \to B \to C \to 0$ is called F-*exact* if $\eta \in F(C, A)$. We denote by F^{op} the subbifunctor of $\text{Ext}^1_{\Lambda^{\text{op}}}(-,-)$ such that $F^{\text{op}}(C, A) = \{\eta : 0 \to A \to B \to C \to 0 | D\eta \in F(DA, DC)\}$. It is clear that F^{op} also has enough projectives and injectives. Denote by $\mathscr{P}(F)$ and $\mathscr{I}(F)$ the classes of F-projectives and F-injectives, respectively. Then we have $\mathscr{P}(F^{\text{op}}) = D\mathscr{I}(F)$ and $\mathscr{I}(F^{\text{op}}) = D\mathscr{P}(F)$. In addition, for a module $A \in \text{mod } \Lambda$, we denote the transpose of A by $\text{Tr}A$.

Lemma 2.1 [5, Proposition 1.9] (1) *An indecomposable non-projective module* $P \in \mathscr{P}(F)$ *if and only if the almost split sequence* $0 \to D\text{Tr}P \to E \to P \to 0$ *is not* F*-exact.*

(2) *An indecomposable non-injective module $I \in \mathscr{I}(F)$ if and only if the almost split sequence $0 \to I \to E \to \mathrm{Tr}DI \to 0$ is not F-exact.*

By Lemma 2.1, we have that for an indecomposable module $C \notin \mathscr{P}(F)$, the almost split sequence ending at C is F-exact. Dually, for an indecomposable module $A \notin \mathscr{I}(F)$, the almost split sequence starting with A is F-exact.

Now we introduce the notion of F-almost split sequences as follows:

Definition 2.2 *An exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is called F-almost split if it is F-exact with g left almost split and f right almost split.*

It is clear that $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is F-almost split if and only if $0 \to DC \xrightarrow{D(f)} DB \xrightarrow{D(g)} DA \to 0$ is F^{op}-almost split.

Theorem 2.3 [16, Theorem 1.14] *For an F-exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$, the following are equivalent.*

(1) *The sequence is an F-almost split sequence*;

(2) *f is minimal right almost split*;

(3) *g is minimal left almost split*;

(4) *C is indecomposable and g is left almost split*;

(5) *$C \cong \mathrm{Tr}DA$ and g is left almost split*;

(6) *A is indecomposable and f is right almost split*;

(7) *$A \cong D\mathrm{Tr}C$ and f is right almost split.*

By Theorem 2.3 and Lemma 2.1, we get the following

Theorem 2.4 (1) *An indecomposable module $C \notin \mathscr{P}(F)$ if and only if there is an F-almost split sequence with C the last term.*

(2) *An indecomposable module $A \notin \mathscr{I}(F)$ if and only if there is an F-almost split sequence with A the first term.*

Having proved the existence of F-almost split sequences, we now explain in what sense they are unique.

Theorem 2.5 [17, Proposition 4.3] *The following are equivalent for two F-almost split sequences $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ and $0 \to A' \xrightarrow{g'} B' \xrightarrow{f'} C' \to 0$.*

(1) *$C \cong C'$.*

(2) *$A \cong A'$.*

(3) *The sequences are isomorphic in the sense that there is a commutative diagram*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \\ 0 & \longrightarrow & A' & \xrightarrow{g'} & B' & \xrightarrow{f'} & C' & \longrightarrow & 0\,. \end{array}$$

3 F-almost Split Sequences in Subcategories

In this section, we develop a general theory for subcategories of mod Λ having F-almost split sequences.

Let $\mathscr{C}$ be a subcategory of mod Λ which is closed under F-extensions. A module C is called F-*Extprojective* in $\mathscr{C}$ if $F(C,X)=0$ for all $X\in\mathscr{C}$. A module A is called F-*Extinjective* in $\mathscr{C}$ if $F(Y,A)=0$ for any $Y\in\mathscr{C}$. A morphism $g:B\to C$ is called a *right almost split morphism* in $\mathscr{C}$ if the following conditions are satisfied: (1) g is not a split epimorphism; and (2) whenever there is a not split epimorphism $h:X\to C$ in $\mathscr{C}$ there exists a morphism $h':X\to B$ in $\mathscr{C}$ such that $gh'=h$. Dually a morphism $f:A\to B$ is called a *left almost split morphism* in $\mathscr{C}$ if the following conditions are satisfied: (1) f is not a split monomorphism; and (2) whenever there is not a split monomorphism $h:A\to Y$ in $\mathscr{C}$ there exists a morphism $h':B\to Y$ in $\mathscr{C}$ such that $h'f=h$.

We give the definition of F-almost split sequences as follows, which is an analogy to that given in [2].

Definition 3.1 (1) *An F-exact sequence $0\to A\xrightarrow{g}B\xrightarrow{f}C\to 0$ in $\mathscr{C}$ is called F-almost split sequence if g is a left almost split morphism and f a right almost split morphism.*

(2) *A subcategory $\mathscr{C}$ of mod Λ is said to have F-almost split sequences if it satisfies the following conditions*:

(i) *If $C\in\mathscr{C}$ is indecomposable, then there are a right almost split morphism $B\to C$ and a left almost split morphism $C\to B'$ in $\mathscr{C}$.*

(ii) *If A is indecomposable and non-F-Extinjective in $\mathscr{C}$, then there is an F-almost split sequence $0\to A\to B\to C\to 0$ in $\mathscr{C}$.*

(iii) *If C is indecomposable and non-F-Extprojective in $\mathscr{C}$, then there is an F-almost split sequence $0\to A\to B\to C\to 0$ in $\mathscr{C}$.*

By Theorem 2.5, we also get the uniqueness of F-almost split sequence in $\mathscr{C}$ up to isomorphisms. Our main purpose in this section is to give a sufficient condition for a subcategory $\mathscr{C}$ of mod Λ to have F-almost split sequences. Before giving this result, we recall some facts about dualizing R-varieties from [18].

Let $G:\text{mod }\Lambda\to\text{Ab}$ be an additive functor. Then for each $C\in$mod Λ, the abelian group $G(C)$ has a natural R-structure. Define $DG:(\text{mod }\Lambda)^{\text{op}}\to\text{Ab}$ by

$$DG(C)=\text{Hom}_R(G(C),I(R/J(R))).$$

We have a contravariant functor $D:(\text{mod }\Lambda,\text{Ab})\to((\text{mod }\Lambda)^{\text{op}},\text{Ab})$, where $(\text{mod }\Lambda,\text{Ab})$ and $((\text{mod }\Lambda)^{\text{op}},\text{Ab})$ are the categories of covariant and contravariant functors from mod Λ to Ab, respectively. Similarly, we have a contravariant functor $D:((\text{mod }\Lambda)^{\text{op}},\text{Ab})\to(\text{mod }\Lambda,\text{Ab})$.

A functor $F:\text{mod }\Lambda\to\text{Ab}$ is called *finitely presented* if there is an exact sequence of functors $(C_1,-)\to(C_2,-)\to F\to 0$ where $(C_i,-)=\text{Hom}_\Lambda(C_i,-)$ for $i=1,2$. We use f.p.$(\text{mod }\Lambda,\text{Ab})$ (resp. f.p.$((\text{mod }\Lambda)^{\text{op}},\text{Ab})$) to denote the subcategories of $(\text{mod }\Lambda,\text{Ab})$ (resp. $((\text{mod }\Lambda)^{\text{op}},\text{Ab})$)) consisting of finitely presented functors. We have the following properties (see [2, p. 429]):

(1) If $0\to F_1\to F_2\to F_3\to F_4\to 0$ is an exact sequence of functors with F_2 and F_3 finitely presented, then F_1 and F_4 are finitely presented.

(2) If $0\to F_1\to F_2\to F_3\to 0$ is an exact sequence of functors with F_1 and F_3 finitely presented, then F_2 is finitely presented.

(3) A functor G is finitely presented if and only if DG is finitely presented.

(4) The induced contravariant functors $D : \text{f.p.}\,(\text{mod}\,\Lambda, \text{Ab}) \to \text{f.p.}\,((\text{mod}\,\Lambda)^{\text{op}}, \text{Ab})$ and $D : \text{f.p.}\,((\text{mod}\,\Lambda)^{\text{op}}, \text{Ab}) \to \text{f.p.}\,(\text{mod}\,\Lambda, \text{Ab})$ are dualities which are dual inverses.

Now suppose that $\mathscr{C}$ is an additive subcategory of mod Λ. Then the contravariant functors $D : (\text{mod}\,\Lambda, \text{Ab}) \to ((\text{mod}\,\Lambda)^{\text{op}}, \text{Ab})$ and $D : ((\text{mod}\,\Lambda)^{\text{op}}, \text{Ab}) \to (\text{mod}\,\Lambda, \text{Ab})$ induce contravariant functors $D : (\mathscr{C}, \text{Ab}) \to (\mathscr{C}^{\text{op}}, \text{Ab})$ and $D : (\mathscr{C}^{\text{op}}, \text{Ab}) \to (\mathscr{C}, \text{Ab})$ in an obvious way. Recall that $\mathscr{C}$ is called a *dualizing R-subvariety* of mod Λ if $G : \mathscr{C} \to \text{Ab}$ is finitely presented in $(\mathscr{C}, \text{Ab})$, if and only if $DG : \mathscr{C}^{\text{op}} \to \text{Ab}$ is finitely presented in $(\mathscr{C}^{\text{op}}, \text{Ab})$, and $H : \mathscr{C}^{\text{op}} \to \text{Ab}$ is finitely presented in $(\mathscr{C}^{\text{op}}, \text{Ab})$, if and only if $DH : \mathscr{C} \to \text{Ab}$ is finitely presented in $(\mathscr{C}, \text{Ab})$. If $\mathscr{C}$ is a dualizing R-subvariety, then $D : \text{f.p.}\,(\mathscr{C}, \text{Ab}) \to \text{f.p.}\,(\mathscr{C}^{\text{op}}, \text{Ab})$ and $D : \text{f.p.}\,(\mathscr{C}^{\text{op}}, \text{Ab}) \to \text{f.p.}\,(\mathscr{C}, \text{Ab})$ are dualities which are dual inverses.

The following three lemmas are analogies to [19, Chapter II, Proposition 4.2, Lemma 4.3, Proposition 4.4], respectively. In addition, in the following three lemmas, $\mathscr{C}$ is a subcategory of mod Λ closed under F-extensions.

Lemma 3.2 [19, Chapter II, Proposition 4.2] *Let* $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ *be an* F*-exact sequence in* $\mathscr{C}$*.*

(a) *Suppose* $\text{Coker}(-, f)$ *is simple in* $(\mathscr{C}^{\text{op}}, \text{Ab})$*. Then the following are equivalent:*

(1) f *is right minimal;*

(2) g *is left almost split in* $\mathscr{C}$*;*

(3) $\text{End}(A)$ *is local.*

(b) *Suppose that* $\text{Coker}(g, -)$ *is simple in* $(\mathscr{C}^{\text{op}}, \text{Ab})$*. Then the following are equivalent:*

(1) g *is left minimal;*

(2) g *is right almost split in* $\mathscr{C}$*;*

(3) $\text{End}(C)$ *is local.*

Lemma 3.3 [19, Chapter II, Lemma 4.2] *Let* $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ *be a non-split* F*-exact sequence in* $\mathscr{C}$*.*

(1) *If* $\text{End}(A)$ *is local, then* $f : B \to C$ *is right minimal in* $\mathscr{C}$*.*

(2) *If* $\text{End}(C)$ *is local, then* $g : A \to B$ *is left minimal in* $\mathscr{C}$*.*

Lemma 3.4 [19, Chapter II, Proposition 4.4] *Let* $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ *be an* F*-exact sequence in* $\mathscr{C}$*. Then the following are equivalent:*

(1) *The sequence is an* F*-almost split sequence in* $\mathscr{C}$*;*

(2) f *is minimal right almost split in* $\mathscr{C}$*;*

(3) g *is minimal left almost split in* $\mathscr{C}$*;*

(4) $\text{End}(A)$ *is local and* f *is right almost split in* $\mathscr{C}$*;*

(5) $\text{End}(C)$ *is local and* g *is left almost split in* $\mathscr{C}$*.*

We now give the existence theorem of F-almost split sequences in dualizing R-subvarieties of mod Λ.

Theorem 3.5 *If* $\mathscr{C}$ *is a dualizing* R*-subvariety of mod* Λ *closed under* F*-extensions, then* $\mathscr{C}$ *has* F*-almost split sequences.*

Proof It is analogous to the proof of [2, Theorem 1.1] except that the phrase "closed under extensions" is systematically replaced with "closed under F-extensions". But for the sake of completeness, we give here the proof.

Since $\mathscr{C}$ is a dualizing R-subvariety of mod Λ, by [18, Proposition 3.2], all simple functors in $(\mathscr{C}, \mathrm{Ab})$ and $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ are finitely presented. In other words, if $C \in \mathscr{C}$ is indecomposable, then there are a right almost split morphism $B \to C$ in $\mathscr{C}$ and a left almost split morphism $C \to B'$ in $\mathscr{C}$.

Let $A \in \mathscr{C}$ be indecomposable and non-F-Extinjective. Then there is a non-split F-exact sequence $0 \to A \xrightarrow{u} B' \xrightarrow{v} C' \to 0$ in $\mathscr{C}$, which induces the exact sequence of functors

$$0 \to (-, A) \xrightarrow{(-,u)} (-, B') \xrightarrow{(-,v)} (-, C') \to G \to 0,$$

where $G = \mathrm{Coker}(-, v)$. Since the sequence $0 \to A \xrightarrow{u} B' \xrightarrow{v} C' \to 0$ is not splitting, $G \neq 0$. So by [18, p. 324], G contains a simple subfunctor S.

Let C be the uniquely determined indecomposable module in $\mathscr{C}$ such that $S(C) \neq 0$. Then there is a non-zero morphism $(-, C) \xrightarrow{t} S$, which is an epimorphism since S is simple. Because $(-, C)$ is projective in $(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$, there are a morphism $h : C \to C'$ and the following commutative and exact diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ (-, C) & \xrightarrow{t} & S & & \\ \downarrow{\scriptstyle (-,h)} & & \downarrow & & \\ (-, C') & \longrightarrow & G & \longrightarrow & 0, \end{array}$$

where $S \to G$ is the inclusion of S into G. Consider the following pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A & \xrightarrow{u} & B' & \xrightarrow{v} & C' & \longrightarrow & 0. \end{array}$$

Since $\mathscr{C}$ is closed under F-extensions, $B \in \mathscr{C}$. Furthermore, it is clear that the above diagram induces the following commutative and exact diagram:

$$\begin{array}{ccccccccccc} & & & & & & & & 0 & & \\ & & & & & & & & \downarrow & & \\ 0 & \longrightarrow & (-, A) & \xrightarrow{g} & (-, B) & \xrightarrow{f} & (-, C) & \xrightarrow{t} & S & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle (-,h)} & & \downarrow & & \\ 0 & \longrightarrow & (-, A) & \xrightarrow{u} & (-, B') & \xrightarrow{v} & (-, C') & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Since S is simple and $\mathrm{End}(A)$ is a local ring and $\mathscr{C}$ is closed under F-extensions, the F-exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{C}$ is F-almost split by Lemmas 3.2, 3.3 and 3.4 (3).

Similarly, there is an F-almost split sequence $0 \to A \to B \to C \to 0$ if $C \in \mathscr{C}$ is an indecomposable and non-F-Extprojective. Therefore, we conclude that $\mathscr{C}$ has F-almost split sequences. $\square$

Now we recall some notions from [5]. For any $A \in \operatorname{mod} \Lambda$, there is an F-exact sequence of Λ-modules $\cdots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$, where $P_i \in \mathscr{P}(F)$ and $0 \to \operatorname{Im} d_{i+1} \to P_i \to \operatorname{Im} d_i \to 0$ is F-exact for any $i \geq 0$. Such a sequence is called an *F-projective resolution* of A. We define that the *F-projective dimension* of A, written as $\operatorname{pd}_F A$, is the minimum (possibly infinite) such that there is an F-projective resolution $0 \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$. If all the F-exact sequences $0 \to \operatorname{Im} d_{i+1} \to P_i \to \operatorname{Im} d_i \to 0$ have the property that d_i is a right minimal homomorphism, then we denote by $\Omega_F^i A$ the i-th *F-syzygy* $\operatorname{Im} d_i$ of A. Dually we can define the notions of the *F-injective resolution*, the *F-injective dimension* $\operatorname{id}_F A$ and the i-th *F-cosyzygy* $\Omega_F^{-i} A$ of A, respectively.

For any $A, C \in \operatorname{mod} \Lambda$, the right derived functors of $\operatorname{Hom}_\Lambda(C, -)$ and $\operatorname{Hom}_\Lambda(-, A)$ using the F-injective and F-projective resolutions, respectively, coincide. We denote $\operatorname{Ext}_F^i(C, -)$ the i-th right derived functor of $\operatorname{Hom}_\Lambda(C, -)$ and by $\operatorname{Ext}_F^i(-, A)$ the i-th right derived functor of $\operatorname{Hom}_\Lambda(-, A)$. It is not difficult to check that $\operatorname{pd}_F A = \inf\{n \mid \operatorname{Ext}_F^{n+1}(A, B) = 0$ for any $B \in \operatorname{mod} \Lambda\}$ and $\operatorname{id}_F A = \inf\{n \mid \operatorname{Ext}_F^{n+1}(B, A) = 0$ for any $B \in \operatorname{mod} \Lambda\}$. We use $\mathscr{P}^\infty(F)$ (resp. $\mathscr{I}^\infty(F)$) to denote the subcategory of mod Λ consisting of modules with finite F-projective (resp. injective) dimension.

For any $T \in \operatorname{mod} \Lambda$, denote by $\operatorname{add}_\Lambda T$ the full subcategory of mod Λ consisting of all modules isomorphic to direct summands of finite sums of copies of ${}_\Lambda T$, and denote by ${}^{\perp_F} T = \{X \in \operatorname{mod} \Lambda \mid \operatorname{Ext}_F^i(X, T) = 0$ for any $i \geq 1\}$ and $\widehat{\operatorname{add}_\Lambda \mathrm{T}} = \{Y \in \operatorname{mod} \Lambda \mid$ there is an F-exact sequence $0 \to T_n \to T_{n-1} \to \cdots \to T_0 \to Y \to 0$ with $T_i \in \operatorname{add}_\Lambda T$ for any $0 \leq i \leq n\}$.

Definition 3.6 [6] *A module $T \in \operatorname{mod}\Lambda$ is called F-cotilting if the following conditions are satisfied*: (1) $T \in \mathscr{I}^\infty(F)$; (2) $T \in {}^{\perp_F} T$; *and* (3) $\mathscr{I}(F) \subseteq \widehat{\operatorname{add}_\Lambda \mathrm{T}}$. *Dually, we can define the notion of F-tilting modules.*

By [18, Proposition 2.2], an additive subcategory of mod Λ is functorially finite in mod Λ if and only if it is a dualizing R-subvariety of mod Λ with a finite cover and a finite cocover. Recall from [6] that a subcategory $\mathscr{D}$ of mod Λ is called *F-resolving* if it satisfies the following conditions: (1) It is closed under F-extensions; (2) If $0 \to A \to B \to C \to 0$ is an F-exact sequence with $B, C \in \mathscr{D}$, then $A \in \mathscr{D}$; and (3) It contains $\mathscr{P}(F)$. It is shown in [6, Corollary 3.17] that ${}^{\perp_F} T$ is F-resolving functorially finite in mod Λ for an F-cotilting module T. Let S be a cotilting module over $\Gamma = \operatorname{End}_\Lambda(T)$ with $\operatorname{add}_\Gamma S = \operatorname{Hom}_\Lambda(\mathscr{P}(F), T)$. We now give the relationship between F-almost split sequences in ${}^{\perp_F} T$ and almost split sequences in ${}^\perp S$.

Theorem 3.7 *Let T be an F-cotilting module in* mod Λ *with* $\Gamma = \operatorname{End}(T)$ *and S a cotilting module over* Γ. *If* $\operatorname{add}_\Gamma S = \operatorname{Hom}_\Lambda(\mathscr{P}(F), T)$, *then* $(-)^* (= \operatorname{Hom}(-, T))$ *induces a duality between the class of F-almost split sequences in ${}^{\perp_F} T$ and the class of almost split sequences in ${}^\perp S$.*

Proof Since $(-)^*$ induces inverse dualities between ${}^{\perp_F} T$ and ${}^\perp S$ by [6, Corollary 3.6], we only need to prove the following:

(1) If $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is an F-almost split sequence in ${}^{\perp_F}T$, then the exact sequence $0 \to C^* \xrightarrow{f^*} B^* \xrightarrow{g^*} A^* \to 0$ is almost split in ${}^{\perp}S$.

(2) If $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is an almost split sequence in ${}^{\perp}S$, then the exact sequence $0 \to C^* \xrightarrow{f^*} B^* \xrightarrow{g^*} A^* \to 0$ is F-almost split in ${}^{\perp_F}T$.

Let $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ be an F-almost split sequence in ${}^{\perp_F}T$. Since $(-)^*$ is an F-exact functor in ${}^{\perp_F}T$, the sequence $0 \to C^* \xrightarrow{f^*} B^* \xrightarrow{g^*} A^* \to 0$ is non-split exact in ${}^{\perp}S$. So $A^* \in {}^{\perp}S$ is not Extprojective. Let $h : X \to A^*$ be not a split epimorphism in ${}^{\perp}S$. Then there is a non-split monomorphism $h^*\sigma_A : A \xrightarrow{\cong} A^{**} \to X^*$, where $\sigma_A : A \to A^{**}$ is defined by $\sigma_A(f)(x) = f(x)$ for any $f \in A^*$ and $x \in A$ is the canonical evaluation homomorphism. Thus we get a morphism $\eta : B \to X^*$ such that $\eta g = h^*\sigma_A$ and $\sigma_A^* h^{**}\sigma_X = g^*\eta^*\sigma_X$. Since $h = \sigma_{A^*}^{-1}h^{**}\sigma_X$ and $\sigma_A^*\sigma_{A^*} = 1_{A^*}$ by [20, Proposition 20.14], h factors through g^*. So g^* is a right almost split morphism in ${}^{\perp}S$. Since C^* is indecomposable, we get an almost split sequence $0 \to C^* \xrightarrow{f^*} B^* \xrightarrow{g^*} A^* \to 0$ in ${}^{\perp}S$ by Lemma 3.4. This proves (1).

Now, let $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ be an almost split sequence in ${}^{\perp}S$. Since ${}_\Gamma T \in \mathrm{Hom}_\Lambda(\mathscr{P}(F), T) \cong \mathrm{add}_\Gamma S$, we get an exact sequence $0 \to C^* \xrightarrow{f^*} B^* \xrightarrow{g^*} A^* \to 0$ in ${}^{\perp_F}T$. Because S is a cotilting module and the sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is almost split in ${}^{\perp}S$, $A \notin \mathrm{add}_\Gamma S$. Then $A^* \notin \mathscr{P}(F)$ since $(-)^*$ induces the inverse duality between $\mathscr{P}(F)$ and $\mathrm{add}_\Gamma S$ by [6, Corollary 3.6]. So any morphism $P \to A^*$ with $P \in \mathscr{P}(F)$ is not a split epimorphism. To prove $0 \to C^* \xrightarrow{f^*} B^* \xrightarrow{g^*} A^* \to 0$ is F-almost split, it suffices to prove any non-split epimorphism $h : X \to A^*$ factors through g^*. This can be proved by using a similar argument to that above. □

4 Applications to *F*-Gorenstein Algebras

In this section, we give some characterizations of F-Gorenstein algebras and F-self-injective algebras in terms of strong F-cotilting modules and strong F-tilting modules. Then we apply Theorem 3.7 to F-Gorenstein algebras and study the first term in an F-almost split sequence in ${}^{\perp_F}T$, where T is an F-cotilting module.

Let $\mathscr{A}$ be a full subcategory of mod Λ. Denote by $\mathscr{A}^{\perp_F}$ the subcategory of mod Λ consisting of the modules M such that $\mathrm{Ext}_F^i(A, M) = 0$ for any $A \in \mathscr{A}$ and $i \geq 1$. Recall from [8] that an algebra Λ is called *F-Gorenstein* if $\mathscr{I}^\infty(F) = \mathscr{P}^\infty(F)$. Moreover, Λ is called *F-selfinjective* if $\mathscr{P}(F) = \mathscr{I}(F)$ and $\mathscr{P}(F)$ is of finite type. An F-cotilting module T is called *strong F-cotilting* if $({}^{\perp_F}T)^{\perp_F} = \mathscr{I}^\infty(F)$. Dually, an F-tilting module is called *strong F-tilting* if ${}^{\perp_F}(T^{\perp_F}) = \mathscr{P}^\infty(F)$.

Theorem 4.1 [8, Proposition 3.6] *Let $\Gamma = \mathrm{End}(T)$ for some F-cotilting module T over an algebra Λ. Then the following are equivalent:*

(1) Γ *is Gorenstein*;

(2) T *is an F-tilting module*;

(3) Λ *is F-Gorenstein.*

Propositon 4.2 *Let Λ be an F-Gorenstein algebra. If T is an F-cotilting module with $\mathrm{id}_F T = d$, then the following are equivalent:*

(1) *T is a strong F-cotilting module*;

(2) $\mathscr{P}(F) \subseteq \mathrm{add}_\Lambda T$;

(3) $\mathscr{P}(F) = \mathrm{add}_\Lambda T$.

Proof (1) $\Rightarrow$ (2) Since Λ is an F-Gorenstein algebra and T is a strong F-cotilting module, $\mathscr{P}(F) \subseteq \mathscr{P}^\infty(F) = \mathscr{I}^\infty(F) = \widehat{\mathrm{add}_\Lambda T}$. So there is a split F-exact sequence $0 \to K \to T_0 \to P \to 0$ with $T_0 \in \mathrm{add}_\Lambda T$ for any $P \in \mathscr{I}^\infty(F)$, and hence $\mathscr{P}(F) \subseteq \mathrm{add}_\Lambda T$.

(2) $\Rightarrow$ (3) Since $\mathrm{id}_F T = d < \infty$ and Λ is F-Gorenstein, there is an F-projective resolution of T: $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to T \to 0$. Let $K = \mathrm{Ker}(P_0 \to T)$. Then by (2), $K \in T^{\perp_F}$ since $T \in T^{\perp_F}$. So T is a direct summand of P_0, and hence it is F-projective.

(3) $\Rightarrow$ (1) Since $\widehat{\mathrm{add}_\Lambda T} \subseteq \mathscr{I}^\infty(F)$ by [6, Theorem 3.2], we only need to prove $\mathscr{I}^\infty(F) \subseteq \widehat{\mathrm{add}_\Lambda T}$. For any $M \in \mathscr{I}^\infty(F)$, we have an F-projective resolution of M: $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ since $\mathscr{I}^\infty(F) = \mathscr{P}^\infty(F)$. So $M \in \widehat{\mathrm{add}_\Lambda T}$. □

For an F-Gorenstein algebra Λ, we can get the dual results of the above results for a strong F-tilting module T'. It is natural to ask a question: Is there a module M which is both strong F-tilting and strong F-cotilting module in mod Λ?

Corollary 4.3 *Let Λ be an F-Gorenstein algebra. Then there is a module $T \in \mathrm{mod}\ \Lambda$ such that it is both strong F-tilting and strong F-cotilting module if and only if Λ is F-selfinjective.*

Proof By Proposition 4.2 and its dual version for F-tilting modules. □

Let Λ be a Gorenstein artin algebra with self-injective dimension d. By [15], for any $C \in {}^\perp\Lambda$, there is an exact sequence $0 \to C \to P_{d-1} \to \cdots \to P_0 \to A \to 0$, where P_i is projective and $A \in {}^\perp\Lambda$ and has no non-zero projective summands if $d \geq 1$. We denote this uniquely determined (up to isomorphisms) module by $A = \Omega_{CM}^{-d} C$. We use $\Omega^i M$ (resp. $\Omega^{-i} M$) to denote the usual i-th syzygy (resp. cosyzygy) of $M \in \mathrm{mod}\ \Lambda$ (see [21]).

Theorem 4.4 [15, Theorem 3.7] *Let Λ be a Gorenstein algebra with self-injective dimension d and $0 \to \tau C \to B \to C \to 0$ an almost split sequence in ${}^\perp\Lambda$. Then $\tau C \cong \Omega_{CM}^{-d} D\Omega^{-d}\mathrm{Tr}C \cong \Omega_{CM}^{-d}\Omega^d D\mathrm{Tr}C$.*

The following is the main result in this section.

Theorem 4.5 *Let Λ be an F-Gorenstein algebra and T a strong F-cotilting module with $\Gamma = \mathrm{End}_\Lambda(T)$, and let $0 \to A \to B \to \rho A \to 0$ be an F-almost split sequence in ${}^{\perp_F}T$. If S is a cotilting Γ- module with injective dimension d and $\mathrm{add}_\Gamma S = \mathrm{Hom}_\Lambda(\mathscr{P}(F), T)$, then $\rho A \cong (\Omega_{CM}^{-d}\Omega^d D\mathrm{Tr}A^*)^*$, where $(-)^* = \mathrm{Hom}(-, T)$.*

Proof To prove ${}^\perp S = {}^\perp\Gamma$, it suffices to prove that S and Γ have the same non-isomorphic direct summands. Since $\Gamma \in \mathrm{add}_\Gamma S$, Γ is a direct summand of S^m for some $m \geq 1$. Let n_1 and n_2 be the numbers of non-isomorphic direct summands for cotilting modules S and Γ, respectively, and n_3 be the number of non-isomorphic simple modules of Γ. Then we have $n_1 = n_3 = n_2$ by [22, p. 141]. Now our assertion follows from Theorems 3.7 and 4.4. □

5 The First Term of an *F*-almost Split Sequence in ${}^{\perp_F}T$

In this section, we give an F-version of Theorem 4.4. Assume that Λ is an F-Gorenstein algebra and T is a strong F-cotilting module with $\mathrm{id}_F T = n$ and $\Gamma = \mathrm{End}_\Lambda(T)$, and S is a cotilting module with $\mathrm{Hom}(\mathscr{P}(F), T) = \mathrm{add}_\Gamma S$. We know from Theorem 3.5 that ${}^{\perp_F}T$ has F-almost split sequences. Let $0 \to \tau C \to B \to C \to 0$ be an F-almost split sequence in ${}^{\perp_F}T$. We denote $\mathrm{Hom}(-, T)$ by $(-)^*$.

Let $\mathscr{X}$ be a subcategory of mod Λ. A *right F-$\mathscr{X}$-approximation* of a module $C \in$mod Λ is an F-exact sequence $0 \to Y \to X \xrightarrow{f} C \to 0$ with $X \in \mathscr{X}$ such that $\mathrm{Hom}_\Lambda(\mathscr{X}, X) \to \mathrm{Hom}_\Lambda(\mathscr{X}, C) \to 0$ is exact. The approximation is called *minimal* if f is a right minimal homomorphism (see [5]). In addition, the definitions of contravariantly finite subcategories and functorially finite subcategories are referred to [22].

Lemma 5.1 [2, Lemma 3.2] *For an algebra Λ, let $\mathscr{X}$ be a contravariantly finite F-resolving subcategory of* mod Λ, *and let $C \in$ mod Λ be indecomposable and not F-projective. Then we have the following*:

(1) *τC is a direct summand of the minimal right F-$\mathscr{X}$-approximation $X_{D\mathrm{Tr}C}$ of $D\mathrm{Tr}C$*;

(2) *If $X_{D\mathrm{Tr}C} = A \oplus B$, where A is indecomposable and B is F-Extinjective in $\mathscr{X}$, that is,* $\mathrm{Ext}^1_F(\mathscr{X}, B) = 0$, *then $\tau C = A$*;

(3) *If $X_{D\mathrm{Tr}C}$ is indecomposable, then $\tau C = A$.*

For any $i \geq 0$, we use $\Omega^i_F(\mathrm{mod}\ \Lambda)$ to denote the subcategory of mod Λ consisting of the i-th F-syzygy modules. The following result is an F-version of [15, Proposition 3.1].

Propositon 5.2 *Let Λ be an F-Gorenstein algebra. If $T \in$ mod Λ is a strong F-cotilting module with $\mathrm{id}_F T = d$ and $\Gamma = \mathrm{End}(T)$, then we have the following*:

(1) *$(-)^* : {}^{\perp_F}T \to {}^{\perp}S$ is a duality, where S is a cotilting Γ-module such that $\mathrm{add}_\Gamma S = \mathscr{P}(F)^*$.*

(2) *${}^{\perp_F}T = \Omega^d_F(\mathrm{mod}\ \Lambda)$.*

(3) *${}^{\perp_F}T \cap \mathscr{P}^\infty(F) = \mathrm{add}_\Lambda T = \mathscr{P}(F)$.*

Proof (1) By [6, Propositions 3.6 and 3.13].

(2) Since F has enough projectives, for any $M \in \mathrm{mod}\Lambda$, we have an F-projective resolution in mod Λ:

$$0 \to \Omega^d_F M \to P_{d-1} \to P_{d-2} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

Since $\mathrm{id}_F T = d$, $\mathrm{Ext}^i_F(\Omega^d_F M, T) \cong \mathrm{Ext}^{i+d}_F(M, T) = 0$ for any $i \geq 0$ and $\Omega^d_F(M) \subseteq {}^{\perp_F}T$.

Conversely, for any $C \in {}^{\perp_F}T$, let $\cdots \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to C^* \to 0$ be a minimal projective resolution in mod Γ. Since $C^* \in {}^{\perp}S$ and ${}_\Gamma T \cong \mathrm{Hom}_\Lambda(\Lambda, {}_\Gamma T) \in \mathrm{add}_\Gamma S$, $\mathrm{Ext}^i_\Gamma(C^*, T) = 0$ for any $i \geq 1$. So the induced sequence $0 \to C^{**} \to P_0^* \to P_1^* \to \cdots \to P_{d-1}^* \to P_d^* \to \cdots$ is F-exact, where $P_i^* \in \mathrm{add}_\Lambda T = \mathscr{P}(F)$. Since $(-)^* : {}^{\perp_F}T \to {}^{\perp}S$ is a duality, $C \in \Omega^d_F(\mathrm{mod}\ \Lambda)$.

(3) Since T is a strong F-cotilting module, $\widehat{\mathrm{add}_\Lambda T} = \mathscr{I}^\infty(F)$. Since Λ is F-Gorenstein, $\mathscr{P}^\infty(F) = \mathscr{I}^\infty(F)$. In addition, T is a cotilting module, so ${}^{\perp_F}T \cap \widehat{\mathrm{add}_\Lambda T} = \mathrm{add}_\Lambda T = \mathscr{P}(F)$. □

Denote by $\underline{\mathrm{mod}\ \Lambda}$ (resp., $\overline{\mathrm{mod}\ \Lambda}$) and $\underline{\mathscr{C}}$ (resp., $\overline{\mathscr{C}}$) the categories mod Λ and $\mathscr{C}$ modulo F-projectives (resp., injectives), respectively. We denote the image of a Λ-module M in $\underline{\mathrm{mod}\ \Lambda}$ (resp., $\overline{\mathrm{mod}\ \Lambda}$) by $\underline{M}$ (resp., $\overline{M}$). We use $\underline{f}$ (resp., $\overline{f}$) to denote the image of a morphism f in $\underline{\mathrm{mod}\ \Lambda}$ (resp., $\overline{\mathrm{mod}\ \Lambda}$) (see [17]).

In [15], Auslander and Reiten showed that $\Omega^1 : \underline{{}^{\perp}\Lambda} \to \underline{{}^{\perp}\Lambda}$ is an equivalence of categories over a Gorenstein algebra Λ. We give an F-version of this result as follows.

Propositon 5.3 *Let Λ be an F-Gorenstein algebra. If $T \in \operatorname{mod}\Lambda$ is a strong F-cotilting module with $\operatorname{id}_F T = d$ and $\Gamma = \operatorname{End}(T)$, then $\Omega_F^1 : \underline{{}^{\perp_F}T} \to \underline{{}^{\perp_F}T}$ is an equivalence of categories.*

Proof We divide the proof into three steps:

(1) We show that Ω_F^1 is a functor from $\underline{{}^{\perp_F}T}$ to $\underline{{}^{\perp_F}T}$. We only need to prove that for any $M \in \underline{{}^{\perp_F}T}$, $\Omega_F^1 M \in \underline{{}^{\perp_F}T}$. Assume that $\Omega_F^1 M = M_1 \oplus M_2$ with $M_1 = \underline{\Omega_F^1 M}$ and $M_2 \in \mathscr{P}(F)$. Then we have the following commutative diagram with the bottom row splitting:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_F^1 M & \xrightarrow{i_1} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & {\scriptstyle i}\uparrow\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g} & & \| & & \\
0 & \longrightarrow & M_2 & \underset{\pi_2}{\overset{i_2}{\rightleftarrows}} & E & \longrightarrow & M & \longrightarrow & 0,
\end{array}$$

where $P_0 \to M$ is right minimal, $\pi i = 1_{M_2} = \pi_2 i_2$ and the composition of sequences of morphisms $M_2 \xrightarrow{i} \Omega_F^1 M \xrightarrow{i_1} P_0 \xrightarrow{g} E \xrightarrow{\pi_2} M_2$ is the identity homomorphism of M_2. Then M_2 is isomorphic to a direct summand of P_0. But $P_0 \to M$ is right minimal, so $M_2 = 0$ and hence $\Omega_F^1 M \in \underline{{}^{\perp_F}T}$.

(2) We show that Ω_F^1 is fully faithful. That is, $\underline{\operatorname{Hom}_F}(A, B) \cong \underline{\operatorname{Hom}_F}(\Omega_F^1 A, \Omega_F^1 B)$ for any $A, B \in \underline{{}^{\perp_F}T}$.

We first define the morphism $\Omega_F^1 : \underline{\operatorname{Hom}_F}(A, B) \to \underline{\operatorname{Hom}_F}(\Omega_F^1 A, \Omega_F^1 B)$ via $\underline{f} \to \Omega_F^1(\underline{f}) = \underline{\Omega_F^1(f)}$ satisfying the following commutative and F-exact diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_F^1 A & \longrightarrow & P_0 & \xrightarrow{f_A} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Omega_F^1(f)} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \Omega_F^1 B & \longrightarrow & Q_0 & \xrightarrow{f_B} & B & \longrightarrow & 0.
\end{array}$$

It is not difficult to prove that $\Omega_F^1(\underline{f})$ does not depend on the choice of g. If $\underline{f} = 0$, then $\Omega_F^1(\underline{f}) = 0$. Obviously, f factors through Q_0. Assume that $f = f_B h$ with $h : A \to Q_0$ and $f_B : Q_0 \to B$. Put $g = h f_A$. Then we have the following commutative diagram with F-exact rows and the middle row splitting:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_F^1 A & \longrightarrow & P_0 & \xrightarrow{f_A} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Omega_F^1(f)} & & \downarrow{\scriptstyle i_{Q_0} g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & \Omega_F^1 B & \longrightarrow & E & \underset{i_{Q_0}}{\overset{\pi_{Q_0}}{\rightleftarrows}} & Q_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \pi_{Q_0}} & & \downarrow{\scriptstyle f_B} & & \\
0 & \longrightarrow & \Omega_F^1 B & \longrightarrow & Q_0 & \xrightarrow{f_B} & B & \longrightarrow & 0,
\end{array}$$

where $\pi_{Q_0} i_{Q_0} = 1_{Q_0}$. Since the middle row splits, $\Omega_F^1(f)$ factors through P_0 and $\Omega_F^1(\underline{f}) = 0$.

We next prove that Ω_F^1 is epic. That is, for any $\underline{h} \in \underline{\operatorname{Hom}_F}(\Omega_F^1 A, \Omega_F^1 B)$, there is an $\underline{f} \in \underline{\operatorname{Hom}_F}(A, B)$ such that $\Omega_F^1(\underline{f}) = \underline{h}$.

Since $A \in {}^{\perp_F}T$, $0 \to A^* \to P_0^* \to (\Omega_F^1 A)^* \to 0$ is exact. Since $\mathscr{P}(F) = \mathrm{add}_\Lambda T$, we have the following commutative diagram with F-exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_F^1 A & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \Omega_F^1 B & \longrightarrow & Q_0 & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

So $\Omega_F^1(f) = h$ and hence $\Omega_F^1(\underline{f}) = \underline{h}$.

Finally, we show that Ω_F^1 is monic. By the above argument, for any $\underline{h}$, there is an $\underline{f}$ such that $\Omega_F^1(\underline{f}) = \underline{h}$. So we only need to prove that $\underline{h} = 0$ implies that $\underline{f} = 0$. It is easy to see that $\underline{f}$ does not depend on the choice of g. $\underline{h} = 0$ implies clearly that $h : \Omega_F^1 A \to \Omega_F^1 B$ factors through P_0. Put $g = i_B \rho$. Then we have the following commutative diagram with F-exact rows and the middle row splitting:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_F^1 A & \xrightarrow{i_A} & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle i_{P_0}} & & \| & & \\ 0 & \longrightarrow & P_0 & \underset{\pi_{P_0}}{\overset{i_{P_0}}{\rightleftarrows}} & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle g\pi_{P_0}} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \Omega_F^1 B & \xrightarrow{i_B} & Q_0 & \longrightarrow & B & \longrightarrow & 0, \end{array}$$

where $\pi_{P_0} i_{P_0} = 1_{P_0}$. Since the middle row splits, $\underline{f} = 0$.

(3) We show that Ω_F^1 is dense. If $A \in \underline{{}^{\perp_F}T}$, then A is an F-syzygy by the proof of Proposition 5.2 (2). So ${}^{\perp_F}T$ is functorially finite F-resolving in mod Λ for an F-cotilting module by [6, Corollary 3.17]. Then for each indecomposable and non-F-projective module $C \in {}^{\perp_F}T$, we have an F-almost split sequence $0 \to A \to B \to C \to 0$. □

Remark By using a similar argument to that above, we have

$$\underline{\mathrm{Hom}_F}(A, B) \cong \underline{\mathrm{Hom}_F}(\Omega_F^i A, \Omega_F^i B)$$

for any $1 \leq i \leq n$, $A \in \underline{{}^{\perp_{nF}}\mathscr{P}(F)}$ and $B \in$mod Λ, where

$${}^{\perp_{nF}}\mathscr{P}(F) = \{A \in \text{mod } \Lambda \mid \mathrm{Ext}_F^i(A, C) = 0 \text{ for any } C \in \mathscr{P}(F) \text{ and } 1 \leq i \leq n\}.$$

Lemma 5.4 [6, Theorem 3.2] *Let Λ be an algebra and T an F-cotilting module. Then, for any $M \in$ mod Λ, there is a minimal right F-${}^{\perp_F}T$-approximation of M :*

$$0 \to Y_M \to X_M \to M \to 0$$

with $X_M \in {}^{\perp_F}T$ and $Y_M \in \widehat{\mathrm{add}_\Lambda T}$.

The following Lemmas 5.5, 5.6 and 5.7 are the F-versions of [15, Lemmas 3.3, 3.4 and 3.5], respectively.

Lemma 5.5 *Let Λ be an F-Gorenstein algebra and T be a strong F-cotilting module. For any $M \in$ mod Λ, X_M is F-projective if and only if $M \in \mathscr{P}^\infty(F)$.*

Proof Since Λ is F-Gorenstein and T is a strong F-cotilting module, $\widehat{\mathrm{add}_\Lambda T} = \mathscr{P}^\infty(F)$. So, by Lemma 5.4, we have an F-exact sequence $0 \to Y_M \to X_M \to M \to 0$ with $Y_M \in \mathscr{P}^\infty(F)$.

If X_M is F-projective, then $\mathrm{pd}_F M \leq \mathrm{pd}_F Y_M + 1 < \infty$. Conversely, if $M \in \mathscr{P}^\infty(F)$, then $\mathrm{pd}_F X_M \leq \max\{\mathrm{pd}_F Y_M, \mathrm{pd}_F M\}$, and hence $M \in \mathscr{P}(F)$ by Proposition 5.2 (3). □

Lemma 5.6 *Let Λ be an F-Gorenstein algebra, T a strong F-cotilting module and $0 \to A \to B \to C \to 0$ an F-exact sequence with $B \in \mathscr{P}^\infty(F)$. Then we have the following*:

(1) $\underline{\Omega_F^1 X_C} \cong \underline{X_A}$;

(2) $\underline{X_A}$ *is indecomposable if and only if* $\underline{X_C}$ *is indecomposable.*

Proof Because (2) follows from (1) and Proposition 5.3, we only need to prove (1).

Since ${}^{\perp_F}T$ is F-resolving, by [6, Proposition 2.7], we have the following commutative and F-exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_A & \longrightarrow & Y & \longrightarrow & Y_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_A & \longrightarrow & X & \longrightarrow & X_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with $X \in {}^{\perp_F}T$ and $Y \in \mathscr{P}^\infty(F)$. Then the map $X \to B$ is a right F-${}^{\perp_F}T$-approximation. Since ${}^{\perp_F}T \cap \mathscr{P}^\infty(F) = \mathscr{P}(F)$, $X \in \mathscr{P}(F)$. □

Lemma 5.7 *Let Λ be an F-Gorenstein algebra, and T be a strong F-cotilting module with* $\mathrm{id}_F T = d$. *Then we have the following*:

(1) $\underline{\Omega_F^d X_{\Omega_F^{-d} A}} \cong \underline{X_A}$;

(2) $\underline{X_A}$ *is indecomposable if and only if* $\underline{X_{\Omega_F^{-d}(A)}}$ *is indecomposable*;

(3) *If $A \in {}^{\perp_F}T$ is indecomposable and not F-projective, then* $\underline{X_{\Omega_F^{-d}A}}$ *and* $\overline{\Omega_F^{-d}A}$ *are indecomposable.*

Proof Both (1) and (2) follow from the iterative applications of Lemma 5.6 since each term in a minimal F-injective resolution of A has a finite F-projective dimension. So we only need to prove (3).

Because $A \in {}^{\perp_F}T$, $X_A = A$. In addition, $A \notin \mathscr{P}(F)$, so $\underline{X_{\Omega_F^{-d}A}}$ is indecomposable by (2).

Let $\mathrm{add}T' = \mathscr{I}(F)$. It is clear that T' is an F-cotilting module. Since Λ is F-Gorenstein, T' is a strong F-tilting module by the dual version of Proposition 4.2. By [8, Proposition 3.4], we have $\mathrm{pd}_F T' = \mathrm{id}_F T = d$. Then DT' is a strong F^{op}-cotilting module with $\mathrm{id}_{F^{\mathrm{op}}} DT' = d$. Since $D\Omega_F^{-d}A \in \Omega_{F^{\mathrm{op}}}^d(\mathrm{mod}\,\Lambda^{\mathrm{op}})$, any direct summand of $D\Omega_F^{-d}A$ with finite F^{op}-projective dimension is in $\mathscr{P}(F^{\mathrm{op}})$. Hence any direct summand of $\Omega_F^{-d}A$ with finite F-projective dimension is in $\mathscr{I}(F)$. Put $\Omega_F^{-d}A = B \oplus C$, where B and C are non-zero and not F-injective. Then $\underline{X_B}$ and $\underline{X_C}$ are non-zero by Lemma 5.5, which contradicts that $\underline{X_{\Omega_F^{-d}A}}$ is indecomposable. Hence we conclude that $\overline{\Omega_F^{-d}A}$ is indecomposable. $\square$

By Proposition 5.2, for any $C \in {}^{\perp_F}T$, there is an F-exact sequence $0 \to C \to P_{d-1} \to \cdots \to P_0 \to A \to 0$, where all P_i are F-projective and $A \in {}^{\perp_F}T$. Denote $A = \Omega_{CM_F}^{-d}C$. The following result is cited from [12], which gives the construction of the non-F-projective part of minimal right F-${}^{\perp_F}T$-approximations.

Lemma 5.8 [12, Corollary 2] *Let Λ be an F-Gorenstein algebra and T a strong F-cotilting module with* $\mathrm{id}_F T = d$. *Then* $\underline{X_C} \cong \Omega_{CM_F}^{-d}\Omega_F^d C$ *for any* $C \in \mathrm{mod}\ \Lambda$.

For any $B \in \mathrm{mod}\ \Lambda^{\mathrm{op}}$, we use $\mathrm{Tor}_i^F(B,-)$ to denote the left derived functor of $B \otimes_\Lambda -$ using F-projective resolutions in mod Λ (see [5]). We now are in a position to give the main result in this section, which is the F-version of [15, Theorem 3.7].

Theorem 5.9 *Let Λ be an F-Gorenstein algebra and T a strong F-cotilting module with* $\mathrm{id}_F T = d$. *If* $0 \to \tau C \to B \to C \to 0$ *is an F-almost split sequence in* ${}^{\perp_F}T$, *then*

$$\tau C \cong \Omega_{CM_F}^{-d} D\Omega_{F^{\mathrm{op}}}^{-d}\mathrm{Tr}C \cong \Omega_{CM_F}^{-d}\Omega_F^d D\mathrm{Tr}C.$$

Proof The case for $d = 0$ is trivial. Now suppose $d \geq 1$. Since the F-Extinjective objects in ${}^{\perp_F}T$ are the F-projective modules, $\tau C \notin \mathscr{P}(F)$. By Lemma 5.8, there is an F-projective module Q, such that

$$X_{D\mathrm{Tr}C} \cong \Omega_{CM_F}^{-d}\Omega_F^d D\mathrm{Tr}C \oplus Q \cong \Omega_{CM_F}^{-d} D\Omega_{F^{\mathrm{op}}}^{-d}\mathrm{Tr}C \oplus Q,$$

since $\Omega_F^d DX \cong D\Omega_{F^{\mathrm{op}}}^{-d}X$ clearly.

Let $0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to T' \to 0$ be a minimal F-projective resolution of the strong tilting module T'. By applying the functor $\mathrm{Hom}_\Lambda(C,-)$ to the following F-exact sequences:

$$0 \to P_d \to P_{d-1} \to \Omega_F^{d-1}T' \to 0,$$
$$0 \to \Omega_F^{d-1}T' \to P_{d-2} \to \Omega_F^{d-2}T' \to 0,$$
$$\cdots\cdots$$
$$0 \to \Omega_F^2 T' \to P_1 \to \Omega_F^1 T' \to 0,$$
$$0 \to \Omega_F^1 T' \to P_0 \to T' \to 0,$$

we have that $\mathrm{Ext}_F^i(C, \Omega_F^j T') = 0$ for any $i \geq 1$ and $1 \leq j \leq d-1$. On the other hand, by applying the functor $\mathrm{Tr}C \otimes_\Lambda -$ to the above F-exact sequences, we have that $\mathrm{Tor}_{i+1}^F(\mathrm{Tr}C, T') \cong \mathrm{Tor}_1^F(\mathrm{Tr}C, \Omega_F^i T')$ since $D\mathrm{Tor}_l^F(\mathrm{Tr}C, P_k) \cong \mathrm{Ext}_{F^{\mathrm{op}}}^l(\mathrm{Tr}C, DP_k) = 0$ for any $l \geq 1$ and $k \geq 0$ by [5, Lemma 2.1]. So

$$\mathrm{Ext}_{F^{\mathrm{op}}}^i(\mathrm{Tr}C, DT') \cong D\mathrm{Tor}_i^F(\mathrm{Tr}C, T') \cong D\mathrm{Tor}_1^F(\mathrm{Tr}C, \Omega_F^{i-1}T') \cong D\underline{\mathrm{Hom}}_{\mathscr{P}(F)}(C, \Omega_F^{i-1}T')$$

by [5, Lemmas 2.1 and 2.2]. In addition, given any morphism $f: C \to \Omega_F^{d-1}T'$, we get the following commutative F-exact diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_d & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & P_d & \longrightarrow & P_{d-1} & \longrightarrow & \Omega_F^{d-1}T' & \longrightarrow & 0.
\end{array}$$

Since $C \in {}^{\perp_F}T$, the first row splits and f factors through P_{d-1}. For the same reason, we have

$$\underline{\mathrm{Hom}_{\mathscr{P}(F)}}(C, \Omega_F^{i-1}T') = 0$$

for any $1 \leq i \leq d$. So

$$\mathrm{Tr}C \in {}^{\perp_{F^{\mathrm{op}}}}DT'.$$

It is clear that $\mathrm{Tr}C \notin \mathscr{P}(F^{\mathrm{op}})$. Otherwise, the F-almost split sequence $0 \to D\mathrm{Tr}C \to E \to C \to 0$ splits, a contradiction. So $\overline{\Omega_{F^{\mathrm{op}}}^{-d}\mathrm{Tr}C}$ is indecomposable by Lemma 5.7. It follows that $\Omega_F^d D\mathrm{Tr}C \cong D\Omega_{F^{\mathrm{op}}}^{-d}\mathrm{Tr}C$ is indecomposable up to an F-projective summand and is in ${}^{\perp_F}T$, so $\Omega_{CM_F}^{-d} D\Omega_{F^{\mathrm{op}}}^{d}\mathrm{Tr}C$ is also indecomposable by Lemma 5.6. Consequently,

$$\tau C \cong \Omega_{CM_F}^{-d} D\Omega_{F^{\mathrm{op}}}^{d}\mathrm{Tr}C \cong \Omega_{CM_F}^{-d}\Omega_F^d D\mathrm{Tr}C$$

by Lemma 5.1. □

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