

# THE SOCLE OF THE LAST TERM IN THE MINIMAL INJECTIVE RESOLUTION OF A GORENSTEIN MODULE

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## Abstract

Let  $R$  be a left Noetherian ring,  $S$  a right Noetherian ring and  ${}_R U$  a Gorenstein module with  $S = \text{End}({}_R U)$ . If the injective dimensions of  ${}_R U$  and  $U_S$  are finite, then the last term in the minimal injective resolution of  ${}_R U$  has an essential socle.

## 1. Introduction

Recall that a left and right Noetherian ring is called *Gorenstein* if its left and right self-injective dimensions are finite. The following question still remains open.

QUESTION 1.1. For a Gorenstein ring  $R$ , is the socle of the last term in the minimal injective resolution of  ${}_R R$  non-zero?

The answer to this question is positive in any case of the following

- (1)  $R$  is a left and right Artinian ring.
- (2) The left and right self-injective dimensions of  $R$  are at most 2 ([7, Theorem 4.5]).
- (3)  $R$  is an Auslander-Gorenstein ring ([5, Proposition 1.1]).

Furthermore, in the case (3) above, Iwanaga and Sato showed in [12, Theorem 6] that this socle is essential in the last term. As a natural generalization of Auslander's  $n$ -Gorenstein rings, Huang introduced in [8] the notion of  $n$ -Gorenstein modules such that a left and right Noetherian ring  $R$  is Auslander's  $n$ -Gorenstein if and only if it is  $n$ -Gorenstein as an  $R$ -module. Then Huang and Wang proved in [11, Theorem 3.1] that for left and right Noetherian rings  $R$  and  $S$  and a generalized tilting module  ${}_R U$  with  $S = \text{End}({}_R U)$ , if  ${}_R U$  is  $(n - 2)$ -Gorenstein with the injective dimensions of  ${}_R U$  and  $U_S$  being  $n$  (where  $n$  is a non-negative integer), then the socle of the last term in the minimal injective resolution of  ${}_R U$  is non-zero. In this paper we extend these results and prove the following

**Theorem 1.2.** *Let  $R$  be a left Noetherian ring,  $S$  a right Noetherian ring and  ${}_R U$  a Gorenstein module with  $S = \text{End}({}_R U)$ . If the injective dimensions of  ${}_R U$  and  $U_S$  are finite, then the last term in the minimal injective resolution of  ${}_R U$  has an essential socle.*

In Section 2, we give some terminology and some preliminary results. In Section 3, we introduce the notion of Gorenstein modules. Let  $R$  be a left Noetherian ring,  $S$  a right Noetherian ring and  ${}_R U$  a Gorenstein module with  $S = \text{End}({}_R U)$  such that the injective dimensions of  ${}_R U$  and  $U_S$  are equal to  $n$ . We first prove that  $\text{Ext}_{S^{op}}^n(N, U)$  is an Artinian

left  $R$ -module for any finitely generated right  $S$ -module  $N$ . Then we get that any non-zero submodule of the last term in the minimal injective resolution of  ${}_R U$  has a non-zero Artinian submodule. Theorem 1.2 follows from this result.

**2. Preliminaries**

Let  $R$  be an arbitrary associative ring with identity, and let  $\text{Mod } R$  be the category of left  $R$ -modules and  $\text{mod } R$  the category of finitely generated left  $R$ -modules. For a module  $M$  in  $\text{Mod } R$ , we use  $\text{add}_R M$  to denote the full subcategory of  $\text{Mod } R$  consisting of modules isomorphic to direct summands of finite direct sums of copies of  ${}_R M$ , and use  $\text{pd}_R M$ ,  $\text{id}_R M$  and  $\text{fd}_R M$  to denote the projective, injective and flat dimensions of  $M$  respectively. We use  $\text{gen}^*({}_R R)$  to denote the full subcategory of  $\text{mod } R$  consisting of modules admitting a degreewise finite  $R$ -projective resolution.

DEFINITION 2.1([17, 18]). A module  ${}_R U$  is called *generalized tilting* (sometimes it is also called *Wakamatsu tilting*, see [2, 13]), if the following conditions are satisfied.

- (1)  ${}_R U \in \text{gen}^*({}_R R)$ .
- (2)  $\text{Ext}_R^{\geq 1}(U, U) = 0$ , that is,  ${}_R U$  is *self-orthogonal*.
- (3) There exists an exact sequence

$$0 \rightarrow {}_R R \rightarrow U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_i \rightarrow \cdots$$

in  $\text{mod } R$  with all  $U_i$  in  $\text{add}_R U$ , such that after applying the functor  $\text{Hom}_R(-, {}_R U)$  the sequence is still exact.

Let  $R$  and  $S$  be arbitrary associative rings with identity. Recall that a bimodule  ${}_R U_S$  is called *faithfully balanced* if  $R = \text{End}(U_S)$  and  $S = \text{End}({}_R U)$ . By [18, Corollary 3.2], we have that  ${}_R U_S$  is faithfully balanced and self-orthogonal with  ${}_R U \in \text{gen}^*({}_R R)$  and  $U_S \in \text{gen}^*(S_S)$  if and only if  ${}_R U$  is generalized tilting with  $S = \text{End}({}_R U)$ , and if and only if  $U_S$  is generalized tilting with  $R = \text{End}(U_S)$ . Note that a faithfully balanced and self-orthogonal bimodule  ${}_R U_S$  with  ${}_R U \in \text{gen}^*({}_R R)$  and  $U_S \in \text{gen}^*(S_S)$  is also called a *semidualizing bimodule* (cf. [11]).

Let  ${}_R U$  be a generalized tilting module with  $S = \text{End}({}_R U)$ . We write

$$*(-) := \text{Hom}({}_R U_S, -) \text{ and } (-)^* := \text{Hom}(-, {}_R U_S).$$

We use

$$0 \rightarrow {}_R U \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_i \rightarrow \cdots$$

to denote the minimal injective resolution of  ${}_R U$  and  $K_i = \text{Ker}(E_i \rightarrow E_{i+1})$  for any  $i \geq 0$  (note:  $K_0 = {}_R U$ ), and use

$$0 \rightarrow U_S \rightarrow E'_0 \rightarrow E'_1 \rightarrow \cdots \rightarrow E'_i \rightarrow \cdots$$

to denote the minimal injective resolution of  $U_S$ . Following [8], we use  $\text{add-lim } {}_R U$  (resp.  $\text{add-lim } U_S$ ) to denote the full subcategory of  $\text{Mod } R$  (resp.  $\text{Mod } S^{op}$ ) consisting of all modules isomorphic to direct summands of a direct limit of a family of modules in which each is a finite direct sum of copies of  ${}_R U$  (resp.  $U_S$ ).

DEFINITION 2.2([8]). For a module  $M$  in  $\text{Mod } R$ , if there exists an exact sequence

$$\cdots \rightarrow U_n \rightarrow \cdots \rightarrow U_1 \rightarrow U_0 \rightarrow M \rightarrow 0$$

in  $\text{Mod } R$  with all  $U_i$  in  $\text{add-lim}_R U$ , then we define  $U\text{-lim.dim}_R M = \inf\{n \mid \text{there exists an exact sequence}$

$$0 \rightarrow U_n \rightarrow \cdots \rightarrow U_1 \rightarrow U_0 \rightarrow M \rightarrow 0$$

in  $\text{Mod } R$  with all  $U_i$  in  $\text{add-lim}_R U\}$ . We set  $U\text{-lim.dim}_R M$  infinity if no such an integer exists. For  $S^{op}$ -modules, we may define such a dimension similarly.

Let  $M$  be in  $\text{mod } R$  and  $i \geq 0$ . We say that the *grade* of  $M$  with respect to  $U$ , written  $\text{grade}_U M$ , is at least  $i$  if  $\text{Ext}_R^{0 \leq j < i}(M, U) = 0$ . We say that the *strong grade* of  $M$  with respect to  $U$ , written  $\text{s.grade}_U M$ , is at least  $i$  if  $\text{grade}_U X \geq i$  for any finitely generated  $R$ -submodule  $X$  of  $M$  (cf. [8]). The following result was proved in [8, Theorem 17.1.11] when  $R$  and  $S$  are two-sided Noetherian rings. Because the argument there remains valid in the setting here, we omit it.

**Theorem 2.3.** *Let  $R$  be a left Noetherian ring,  $S$  a right Noetherian ring and  ${}_R U$  a generalized tilting module with  $S = \text{End}({}_R U)$ . Then for any  $n \geq 0$ , the following statements are equivalent.*

- (1)  $U\text{-lim.dim}_R E_i \leq i$  for any  $0 \leq i \leq n - 1$ .
- (2)  $\text{fd}_S {}^*E_i \leq i$  for any  $0 \leq i \leq n - 1$ .
- (3)  $\text{s.grade}_U \text{Ext}_R^i(M, U) \geq i$  for any  $M \in \text{mod } R$ .
- (4)  $\text{fd}_{R^{op}} {}^*E'_i \leq i$  for any  $0 \leq i \leq n - 1$ .
- (5)  $U\text{-lim.dim}_{S^{op}} E'_i \leq i$  for any  $0 \leq i \leq n - 1$ .
- (6)  $\text{s.grade}_U \text{Ext}_{S^{op}}^i(N, U) \geq i$  for any  $N \in \text{mod } S^{op}$ .

If one of the equivalent conditions in Theorem 2.3 is satisfied, then  ${}_R U$  (equivalent  $U_S$ ) is called *n-Gorenstein* ([8, 10]). So a left and right Noetherian ring  $R$  is (Auslander) *n-Gorenstein* ([12]) if and only if  ${}_R R$  is *n-Gorenstein*, and if and only if  $R_R$  is *n-Gorenstein*.

### 3. Main Results

In this section, we give the proof of Theorem 1.2. We begin with the following

**Lemma 3.1.** *Let  $R$  be a ring and*

$$0 \rightarrow K \xrightarrow{f} M \rightarrow N \rightarrow 0$$

*an exact sequence in  $\text{Mod } R$  with  $N \neq 0$  and  $f$  an essential monomorphism. Then  $\text{Ext}_R^1(X, K) \neq 0$  for any non-zero  $R$ -submodule  $X$  of  $N$ .*

Proof. Let  $X$  be a non-zero  $R$ -submodule of  $N$  and  $\alpha : X \hookrightarrow N$  the inclusion. Then we have the following pull-back diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \dashrightarrow & K & \dashrightarrow & W & \dashrightarrow & X & \dashrightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow \alpha & & \\
 0 & \longrightarrow & K & \xrightarrow{f} & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow & & \\
 & & & & \text{Coker } \alpha & = & \text{Coker } \alpha & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 
 \end{array}$$

We claim that the upper row does not split. Otherwise, if it splits, then  $K$  is isomorphic to a non-trivial direct summand of  $W$ . So  $K$  is not isomorphic to an essential submodule of  $W$ , and hence  $K$  is not an essential submodule of  $M$ . It contradicts that  $f$  is an essential monomorphism. The claim is proved. Thus we have  $\text{Ext}_R^1(X, K) \neq 0$ .  $\square$

From now on,  $R$  is a left Noetherian ring,  $S$  is a right Noetherian ring and  ${}_R U$  is a generalized tilting module with  $S = \text{End}({}_R U)$ . By [9, Theorem 2.7], we have that  $\text{id}_R U = \text{id}_{S^{op}} U$  provided both of them are finite.

**Lemma 3.2.** *If  $\text{id}_R U = n$ , then for any non-zero  $R$ -submodule  $X$  of  $E_n$ , we have  $\text{Ext}_R^n(X, U) \neq 0$ .*

Proof. Because  $\text{id}_R U = n$ , we have the following exact sequence

$$0 \rightarrow K_{n-1} \rightarrow E_{n-1} \rightarrow E_n \rightarrow 0$$

in  $\text{Mod } R$  with  $K_{n-1} \rightarrow E_{n-1}$  an essential monomorphism. Let  $X$  be a non-zero  $R$ -submodule of  $E_n$ . Then  $\text{Ext}_R^1(X, K_{n-1}) \neq 0$  by Lemma 3.1. Thus we have  $\text{Ext}_R^n(X, U) \cong \text{Ext}_R^1(X, K_{n-1}) \neq 0$ .  $\square$

By Lemma 3.2, we have the following

**Proposition 3.3.**

- (1) *If  $\text{id}_R U = n(\geq 1)$ , then  $E_0$  and  $E_n$  have no isomorphic non-zero direct summands.*
- (2) *If  $\text{id}_R U = \text{id}_{S^{op}} U = n$ , then  $\text{pd}_S {}^*E = n$  for any non-zero direct summand  $E$  of  $E_n$ .*

Proof. (1) Assume that  $E \neq 0$  is an indecomposable direct summand of  $E_0$  and  $E$  is isomorphic to a direct summand of  $E_n$ . Then  $E$  is the injective envelope of some finitely generated non-zero  $R$ -submodule  $Y$  of  $E$  by [14, Theorem 2.4]. Since  $U$  is an essential submodule of  $E_0$ , we have  $X := U \cap Y \neq 0$ . From the exact sequence

$$0 \rightarrow X \rightarrow U \rightarrow U/X \rightarrow 0,$$

we get the following exact sequence

$$0 = \text{Ext}_R^n(U, U) \rightarrow \text{Ext}_R^n(X, U) \rightarrow \text{Ext}_R^{n+1}(U/X, U).$$

Because  $\text{id}_R U = n$ , we have  $\text{Ext}_R^{n+1}(U/X, U) = 0$ . It induces that  $\text{Ext}_R^n(X, U) = 0$ . On the

other hand, because  $0 \neq X < E \hookrightarrow E_n$ , we have  $\text{Ext}_R^n(X, U) \neq 0$  by Lemma 3.2. It is a contradiction.

(2) Let  $E$  be a non-zero direct summand of  $E_n$ . Then  $\text{Ext}_R^n(E, U) \neq 0$  by Lemma 3.2. Because  $\text{id}_{S^{op}} U = n$ , we have  $\text{fd}_S {}^*E \leq n$  by [9, Lemma 2.6(1)]. Because  $\text{id}_R U = n$ , we have  $\text{pd}_S {}^*E \leq n$  by [9, Theorem 2.11)]. If  $\text{pd}_S {}^*E = m < n$ , then by [9, Theorem 2.9], there exists an exact sequence

$$0 \rightarrow U_m \rightarrow \cdots \rightarrow U_1 \rightarrow U_0 \rightarrow E \rightarrow 0$$

in  $\text{Mod } R$  with all  $U_i$  in  $\text{Add}_R U$ , where  $\text{Add}_R U$  is the full subcategory of  $\text{Mod } R$  consisting of modules isomorphic to direct summands of direct sums of copies of  ${}_R U$ . So  $\text{Ext}_R^n(E, U) \cong \text{Ext}_R^{n-m}(U_m, U) = 0$  by [9, Proposition 2.2(2)]. It is a contradiction. Thus we have  $\text{pd}_S {}^*E = n$ .  $\square$

**Lemma 3.4.** *The following statements are equivalent.*

- (1) *There exists  $0 \neq M \in \text{mod } R$  such that  $\text{Ext}_R^{\geq 0}(M, U) = 0$ .*
- (2) *There exists an exact sequence*

$$0 \rightarrow Q_0 \xrightarrow{k_1} Q_1 \xrightarrow{k_2} Q_3 \rightarrow \cdots$$

*in  $\text{mod } S^{op}$  with all  $Q_i$  in  $\text{add } U_S$ , such that  $\text{Ext}_{S^{op}}^1(L_1, U) \neq 0$  and  $\text{Ext}_{S^{op}}^1(L_i, U) = 0$  with  $i \geq 2$ , where  $L_i = \text{Coker } k_i$  for any  $i \geq 1$ .*

Proof. (2)  $\Rightarrow$  (1) By (2), we get the following exact sequence

$$\cdots \rightarrow Q_2^* \xrightarrow{k_2^*} Q_1^* \xrightarrow{k_1^*} Q_0^* \rightarrow M \rightarrow 0$$

in  $\text{mod } R$ , where  $M = \text{Coker } k_1^*$ . Then  $M \cong \text{Ext}_{S^{op}}^1(L_1, U) \neq 0$  by assumption. Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q_0 & \xrightarrow{k_1} & Q_1 & \xrightarrow{k_2} & Q_2 \longrightarrow \cdots \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & M^* & \longrightarrow & Q_0^{**} & \xrightarrow{k_1^{**}} & Q_1^{**} \xrightarrow{k_2^{**}} Q_2^{**} \longrightarrow \cdots \end{array}$$

Because the upper row is exact by assumption, so is the lower row. It implies  $\text{Ext}_R^{\geq 0}(M, U) = 0$ .

(1)  $\Rightarrow$  (2) Let  $0 \neq M \in \text{mod } R$  with  $\text{Ext}_R^{\geq 0}(M, U) = 0$  and let

$$\cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow M \rightarrow 0$$

be an exact sequence in  $\text{mod } R$  with all  $P_i$  projective. Then we get the following exact sequence

$$(3.1) \quad 0 \rightarrow P_0^* \xrightarrow{d_1^*} P_1^* \xrightarrow{d_2^*} P_2^* \rightarrow \cdots$$

in  $\text{mod } S^{op}$  with all  $P_i^*$  in  $\text{add } U_S$ . Notice that  $P_i \cong P_i^{**}$  naturally for any  $i \geq 0$ , so the sequence

$$\cdots \rightarrow P_2^{**} \xrightarrow{d_2^{**}} P_1^{**} \xrightarrow{d_1^{**}} P_0^{**}$$

in  $\text{mod } R$  is exact. Set  $L_i := \text{Coker } d_i^*$  for any  $i \geq 1$ . Then  $\text{Ext}_{S^{op}}^1(L_1, U) \cong M \neq 0$  and  $\text{Ext}_{S^{op}}^1(L_i, U) = 0$  for any  $i \geq 2$ , and so (3.1) is the desired exact sequence.  $\square$

Following [4], an injective resolution

$$0 \rightarrow N \xrightarrow{\varepsilon_0} I_0 \xrightarrow{\varepsilon_1} I_1 \xrightarrow{\varepsilon_2} I_2 \rightarrow \dots$$

of a module  $N$  in  $\text{Mod } S^{op}$  is said to *have a redundant image* if some  $\text{Im } \varepsilon_n = \bigoplus_{j=1}^m W_j$  such that each  $W_j$  is isomorphic to a direct summand of some  $\text{Im } \varepsilon_{i_j}$  with  $i_j \neq n$ . It is clear that the minimal injective resolution of  $N_S$  has a redundant image if  $\text{id}_{S^{op}} N < \infty$ .

**Lemma 3.5.** *If  $U_S$  has an injective resolution with a redundant image, then  $M = 0$  for any  $M \in \text{mod } R$  with  $\text{Ext}_R^{\geq 0}(M, U) = 0$ .*

Proof. Let  $M \in \text{mod } R$  with  $\text{Ext}_R^{\geq 0}(M, U) = 0$ . If  $M \neq 0$ , then by Lemma 3.4 and its proof, we get an exact sequence

$$0 \rightarrow Q_0 \xrightarrow{k_1} Q_1 \xrightarrow{k_2} Q_3 \rightarrow \dots$$

in  $\text{mod } S^{op}$  with all  $Q_i$  in  $\text{add } U_S$  such that  $\text{Ext}_{S^{op}}^1(L_1, U) \cong M \neq 0$  and  $\text{Ext}_{S^{op}}^1(L_i, U) = 0$  for any  $i \geq 2$ , where  $L_i = \text{Coker } k_i$  for any  $i \geq 1$ . It yields that  $\text{Ext}_{S^{op}}^{\geq 2}(L_1, U) = 0$  and  $\text{Ext}_{S^{op}}^{\geq 1}(L_i, U) = 0$  for any  $i \geq 2$ .

Because  $U_S$  has an injective resolution with a redundant image, there exists an injective resolution

$$0 \rightarrow U_S \xrightarrow{\alpha_0} I_0 \xrightarrow{\alpha_1} I_1 \xrightarrow{\alpha_2} I_2 \rightarrow \dots$$

of  $U_S$  in  $\text{Mod } S^{op}$  with some  $\text{Im } \alpha_n = \bigoplus_{j=1}^m W_j$  such that each  $W_j$  is isomorphic to a direct summand of some  $\text{Im } \alpha_{i_j}$  with  $i_j \neq n$ . Then we have

$$\text{Ext}_{S^{op}}^1(L_1, U) \cong \text{Ext}_{S^{op}}^{n+1}(L_{n+1}, U) \cong \text{Ext}_{S^{op}}^1(L_{n+1}, \text{Im } \alpha_n) \cong \bigoplus_{j=1}^m \text{Ext}_{S^{op}}^1(L_{n+1}, W_j).$$

Since

$$\text{Ext}_{S^{op}}^1(L_{n+1}, \text{Im } \alpha_{i_j}) \cong \text{Ext}_{S^{op}}^{i_j+1}(L_{n+1}, U) = 0$$

by the above argument, we have  $\text{Ext}_{S^{op}}^1(L_{n+1}, W_j) = 0$  for any  $1 \leq j \leq m$ . It follows that  $\text{Ext}_{S^{op}}^1(L_1, U) = 0$ , a contradiction. Consequently we conclude that  $M = 0$ .  $\square$

We introduce the notion of Gorenstein modules as follows.

**DEFINITION 3.6.** We called  ${}_R U$  (resp.  $U_S$ ) *Gorenstein* if  $U\text{-lim.dim}_R E_i \leq i$  (resp.  $U\text{-lim.dim}_{S^{op}} E'_i \leq i$ ) for any  $i \geq 0$ .

By Theorem 2.3, we have that  ${}_R U$  is Gorenstein if and only if  $U_S$  is Gorenstein, and if and only if  ${}_R U$  (equivalent  $U_S$ ) is  $n$ -Gorenstein for all  $n$ . Following Theorem 2.3 and [16, Theorem 3.6], we know that the notion defined above is a non-commutative version of that of Gorenstein modules in [16]. The following proposition is useful in proving the main result.

**Proposition 3.7.** *Let  ${}_R U$  be a Gorenstein module with  $\text{id}_R U = \text{id}_{S^{op}} U = n$ . Then  $\text{Ext}_{S^{op}}^n(N, U) \in \text{mod } R$  is Artinian for any  $N \in \text{mod } S^{op}$ .*

Proof. Let  $X \subseteq Y$  be left  $R$ -submodules of  $\text{Ext}_{S^{op}}^n(N, U)$ . Then by Theorem 2.3, we have

$$\text{Ext}_R^{0 \leq i < n}(X, U) = 0 = \text{Ext}_R^{0 \leq i < n}(Y, U).$$

So from the exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Y/X \rightarrow 0,$$

we get  $\text{Ext}_R^{0 \leq i < n}(Y/X, U) = 0$  and the following exact sequence

$$0 \rightarrow \text{Ext}_R^n(Y/X, U) \rightarrow \text{Ext}_R^n(Y, U) \rightarrow \text{Ext}_R^n(X, U) \rightarrow 0.$$

Put  $A_0 := \text{Ext}_{S^{op}}^n(N, U)$  and let

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

be a descending chain of left  $R$ -submodules of  $A_0$ . Then from the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{i+1} & \longrightarrow & A_0 & \longrightarrow & A/A_{i+1} & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & A_i & \longrightarrow & A_0 & \longrightarrow & A/A_i & \longrightarrow & 0, \end{array}$$

we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ext}_R^n(A_0/A_i, U) & \longrightarrow & \text{Ext}_R^n(A_0, U) & \longrightarrow & \text{Ext}_R^n(A_i, U) & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & \text{Ext}_R^n(A_0/A_{i+1}, U) & \longrightarrow & \text{Ext}_R^n(A_0, U) & \longrightarrow & \text{Ext}_R^n(A_{i+1}, U) & \longrightarrow & 0. \end{array}$$

We can regard  $\text{Ext}_R^n(A_0/A_i, U)$  as a right  $S$ -submodule of  $\text{Ext}_R^n(A_0, U)$ . Then the following sequence is the ascending chain of right  $S$ -submodules of  $\text{Ext}_R^n(A_0, U)$ :

$$(3.2) \quad \text{Ext}_R^n(A_0/A_1, U) \subseteq \text{Ext}_R^n(A_0/A_2, U) \subseteq \cdots \subseteq \text{Ext}_R^n(A_0, U).$$

Since  $A_0 \in \text{mod } R$ ,  $\text{Ext}_R^n(A_0, U) \in \text{mod } S^{op}$  is Noetherian. So (3.2) terminates at some step  $m$ . Then from the exact sequence

$$0 \rightarrow A_m/A_{m+1} \rightarrow A_0/A_{m+1} \rightarrow A_0/A_m \rightarrow 0,$$

we get the following exact sequence

$$\text{Ext}_R^n(A_0/A_m, U) \xrightarrow{\cong} \text{Ext}_R^n(A_0/A_{m+1}, U) \rightarrow \text{Ext}_R^n(A_m/A_{m+1}, U) \rightarrow 0$$

and  $\text{Ext}_R^n(A_m/A_{m+1}, U) = 0$ . Then by the above argument, we have that  $\text{Ext}_R^{0 \leq i \leq n}(A_m/A_{m+1}, U) = 0$ . Because  $\text{id}_R U = n$ , we have  $\text{Ext}_R^{\geq 0}(A_m/A_{m+1}, U) = 0$ . Because  $\text{id}_{S^{op}} U = n$ , by Lemma 3.5 we have that  $A_m/A_{m+1} = 0$  and  $A_m = A_{m+1}$ . Thus  $A_0 (= \text{Ext}_{S^{op}}^n(N, U))$  is an Artinian left  $R$ -module.  $\square$

By Proposition 3.7, we get the following

**Corollary 3.8.** *Let  ${}_R U$  be a Gorenstein module with  $\text{id}_R U = \text{id}_{S^{op}} U = n$  and  $N \in \text{mod } S^{op}$ . Then we have*

- (1)  $\text{Ext}_{S^{op}}^n(N, U)$  embeds in  $E_n^{(t)}$  for some  $t \geq 1$ .

(2) If  $M$  is a non-zero  $R$ -submodule of  $\text{Ext}_{S^{op}}^n(N, U)$ , then  $\text{Ext}_R^n(M, U) \neq 0$ .

Proof. (1) Let  $N \in \text{mod } S^{op}$ . Then  $\text{Ext}_{S^{op}}^n(N, U) \in \text{mod } R$  embeds in a finite direct sum of copies of  $\bigoplus_{i=0}^n E_i$  by [8, Lemma 17.2.5]. On the other hand, by [3, Proposition VI.5.3] and Theorem 2.4, we have

$$\text{Hom}_R(\text{Ext}_{S^{op}}^n(N, U), E_i) \cong \text{Tor}_n^S(N, {}^*E_i) = 0$$

for any  $0 \leq i \leq n-1$ . So there exists some  $t \geq 1$  such that  $\text{Ext}_{S^{op}}^n(N, U)$  embeds in  $E_n^{(t)}$ .

(2) Let  $M$  be a non-zero  $R$ -submodule of  $\text{Ext}_{S^{op}}^n(N, U)$ . Because we have the following exact sequence

$$0 \rightarrow K_{n-1}^{(t)} \rightarrow E_{n-1}^{(t)} \rightarrow E_n^{(t)} \rightarrow 0$$

in  $\text{Mod } R$  with  $K_{n-1}^{(t)} \rightarrow E_{n-1}^{(t)}$  an essential monomorphism, by (1) and Lemma 3.1 we have that  $\text{Ext}_R^n(M, U)^{(t)} \cong \text{Ext}_R^1(M, K_{n-1}^{(t)})^{(t)} \cong \text{Ext}_R^1(M, K_{n-1}^{(t)}) \neq 0$  and  $\text{Ext}_R^n(M, U) \neq 0$ .  $\square$

We now are in a position to prove the following

**Theorem 3.9.** *Let  ${}_R U$  be a Gorenstein module with  $\text{id}_R U = \text{id}_{S^{op}} U = n$ . Then any non-zero submodule of  $E_n$  has a non-zero Artinian submodule.*

Proof. Let  $V$  be a non-zero submodule of  $E_n$  and  $E$  a non-zero indecomposable direct summand of the injective envelope  $Q$  of  $V$ . Then  $E$  is the injective envelope of some finitely generated non-zero  $R$ -submodule  $X$  of  $E$  by [14, Theorem 2.4]. By Lemma 3.2, we have  $\text{Ext}_R^n(X, U) \neq 0$ . Let  $I$  be an injective cogenerator for  $\text{Mod } S^{op}$ . Then by [3, Proposition VI.5.3], we have

$$\text{Tor}_n^R({}^*I, X) \cong \text{Hom}_{S^{op}}(\text{Ext}_R^n(X, U), I) \neq 0.$$

Because  $\text{fd}_{R^{op}} {}^*I = \text{id}_R U = n$  by [9, Lemma 2.6(2)] and assumption, the inclusion  $X \hookrightarrow E$  induces a monomorphism  $\text{Tor}_n^R({}^*I, X) \hookrightarrow \text{Tor}_n^R({}^*I, E)$ . It yields  $\text{Tor}_n^R({}^*I, E) \neq 0$ . By [11, Lemmas 5.1(c) and 4.1], we have that  $\text{Tor}_{\geq 1}^R({}^*I, U) = 0$  and both  $I$  and  $E$  are in the Bass class with respect to  ${}_R U_S$ . So we get the following isomorphisms

$$\begin{aligned} & \text{Tor}_n^S(I, {}^*E) \\ & \cong \text{Tor}_n^S({}^*I \otimes_R U, {}^*E) \text{ (by [11, Lemma 4.1])} \\ & \cong \text{Tor}_n^R({}^*I, E) \text{ (by [11, Theorem 6.4(c)]),} \end{aligned}$$

and hence  $\text{Tor}_n^S(I, {}^*E) \neq 0$ . Let  $\{N_i\}$  be the set of all finitely generated right  $S$ -submodules of  $I$ . Because  $I = \varinjlim N_i$  and the functor  $\text{Tor}$  commutes with  $\varinjlim$  by [15, Example 5.32(iii) and Proposition 7.8], there exists some  $N_i$  such that  $\text{Tor}_n^S(N_i, {}^*E) \neq 0$ . Then by [3, Proposition VI.5.3] again, we have

$$\text{Hom}_R(\text{Ext}_{S^{op}}^n(N_i, U), E) \cong \text{Tor}_n^S(N_i, {}^*E) \neq 0.$$

So there exists a non-zero homomorphism  $f : \text{Ext}_{S^{op}}^n(N_i, U) \rightarrow E$  in  $\text{Mod } R$ . By Proposition 3.7,  $\text{Ext}_{S^{op}}^n(N_i, U) \in \text{mod } R$  is Artinian. Thus as an  $R$ -quotient module of  $\text{Ext}_{S^{op}}^n(N_i, U)$ ,  $\text{Im } f$  is a non-zero Artinian  $R$ -submodule of  $E(< Q)$ . Because  $V$  is essential in  $Q$ , we have that  $V \cap \text{Im } f$  is a non-zero Artinian  $R$ -submodule of  $V$ .  $\square$



Theorem 1.2 is a special case of the following result.

**Corollary 3.10.** *Let  ${}_R U$  be a Gorenstein module with  $\text{id}_R U = \text{id}_{S^{\text{op}}} U = n$ . Then any non-zero  $R$ -submodule of  $E_n$  has an essential socle.*

Proof. By Theorem 3.9, we have that any non-zero  $R$ -submodule of  $E_n$  has a non-zero Artinian  $R$ -submodule. Now the assertion follows from [1, Corollary 9.10].  $\square$

The second assertion in the following result is a supplement to Proposition 3.3(1).

**Corollary 3.11.** *Let  ${}_R U$  be a Gorenstein module with  $\text{id}_R U = \text{id}_{S^{\text{op}}} U = n$ . Then we have*

- (1) *Any non-zero direct summand of  $E_n$  is a direct sum of the injective envelopes of some simple left  $R$ -modules.*
- (2) *If  $n \geq 1$  and  $S$  is further a right Artinian ring, then  $\bigoplus_{i=0}^{n-1} E_i$  and  $E_n$  have no isomorphic non-zero direct summands.*

Proof. (1) By Corollary 3.10 and [14, Proposition 2.1].

(2) If  $S$  is a right Artinian ring, then  $\text{pd}_S^*(\bigoplus_{i=0}^{n-1} E_i) = \text{fd}_S^*(\bigoplus_{i=0}^{n-1} E_i) \leq n - 1$  by Theorem 2.3. So  $\text{pd}_S^* E' \leq n - 1$  for any direct summand  $E'$  of  $\bigoplus_{i=0}^{n-1} E_i$ . On the other hand, we have  $\text{pd}_S^* E = n$  for any non-zero direct summand  $E$  of  $E_n$  by Proposition 3.3(2). Thus the assertion follows.  $\square$

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